Advances in Differential Equations Volume 16, Numbers 1–2 (2011), 165–198

SINGULAR BEHAVIOR OF THE SOLUTION OF THE HELMHOLTZ EQUATION IN WEIGHTED L^p-SOBOLEV SPACES

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Abstract. We study the Helmholtz equation

$$(1) \quad -\Delta u + zu = g \text{ in } \Omega,$$

with Dirichlet boundary conditions in a polygonal domain Ω, where z is a complex number. Here g belongs to $L^p_\mu(\Omega) = \{v \in L^p_{loc}(\Omega) : r^\mu v \in L^p(\Omega)\}$, with a real parameter μ and $r(x)$ the distance from x to the set of corners of Ω. We give sufficient conditions on μ, p, and Ω that guarantee that problem (1) has a unique solution $u \in H^1_0(\Omega)$ that admits a decomposition into a regular part in weighted L^p-Sobolev spaces and an explicit singular part. We further obtain some estimates where the explicit dependence on $|z|$ is given.

1. Introduction

This paper is the first one of a large program of research devoted to the study of the (nonlinear) heat equation in nonsmooth domains in weighted L^p-Sobolev spaces. Our final goal requires precise information about the solution of the linear heat equation, in particular its decomposition into a regular part and an explicit singular part. Although this theory is well developed in weighted L^2-Sobolev spaces [8, 11, 10, 3] or in L^p-Sobolev spaces [9], to our best knowledge such a result does not exist in the framework of weighted L^p-Sobolev spaces. For maximal regularity-type results in weighted L^p-Sobolev spaces, we refer to [5, 15, 20, 16, 19].

According to the approach of [9], the study of the linear heat equation in non-Hilbertian Sobolev spaces can be performed with the help of the theory of sums of operators. This theory requires in a first step to obtain uniform

Accepted for publication: September 2010.
AMS Subject Classifications: 35J05, 35B65.

estimates of the solution of the Helmholtz equation. Hence the goal of this paper is to make this analysis in $L^p_\mu(\Omega)$ for a large range of values of μ and p. Our results extend the ones from [8] to the $L^p_\mu(\Omega)$ setting.

For the sake of simplicity we have restricted ourselves to the two-dimensional situation. The results of this paper can be easily extended to the case of domains with conical points.

The paper is organized as follows: In section 2 after recalling some information, we prove some embeddings and some a priori estimates in H^1_0 and in L^p_μ norms. Section 3 is devoted to the proof of the decomposition of the solution of (1) with uniform estimates with respect to the parameter z.

2. Some preliminary results

2.1. **Some notation and definitions.** In this paper we consider polygonal domains of $\mathbb{R}^2$ in the following sense.

Definition 2.1. *Let Ω be a bounded domain of $\mathbb{R}^2$. We say that Ω is a* polygonal domain *if its boundary is the union of a finite number of line segments $\bar{\Gamma}_j$, $j \in \{1, \dots, J\}$ (Γ_j being supposed to be open). Hence, we do not assume that Ω is a Lipschitz domain; that is, we include the presence of cracks.*

Denote by S_j, $j = 1, \dots, J$ the vertices of $\partial\Omega$ enumerated clockwise. Without loss of generality we may assume that $B(S_j, 1) \cap \Omega$ does not contain any other vertex of Ω. For $j \in \{1, 2, \dots, J\}$, let ψ_j be the interior angle of Ω at the vertex S_j, $\lambda_j = \frac{\pi}{\psi_j}$ and (r_j, θ_j) the polar coordinates centered at S_j such that

$$B(S_j, 1) \cap \Omega = \{(r_j \cos\theta_j, r_j \sin\theta_j) : 0 < r_j < 1,\ 0 < \theta_j < \psi_j\} =: D_j.$$

For $\vec{\mu} = (\mu_j)_{j=1}^J$, we define the spaces $L^p_{\vec{\mu}}(\Omega) = \{f \in L^p_{loc}(\Omega) : wf \in L^p(\Omega)\}$ with

$$w = 1 + \sum_{j=1}^{J} \eta_j (r_j^{\mu_j} - 1), \tag{2.1}$$

where $r_j(x)$ is the distance from x to the vertex S_j and $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ are such that

$$\eta_j \equiv 1 \text{ in } D_j(1/2), \quad \eta_j \equiv 0 \text{ on } \Omega \setminus D_j(1),$$

where $D_j(r)$ is the truncated cone $D_j(r) = \Omega \cap B(S_j, r)$. Note that the weight w satisfies

$$w = r_j^{\mu_j} \text{ on } D_j(1/2) \quad \text{ and } w = 1 \text{ on } \Omega \setminus \cup_{j=1}^J D_j(1).$$

The space $L^p_{\vec{\mu}}(\Omega)$ is a Banach space for the norm

$$\|f\|_{L^p_{\vec{\mu}}(\Omega)} = \Big(\int_\Omega |f(x)|^p \, w^p(x) \; dx \Big)^{1/p}.$$

$V^{k,p}_{\vec{\mu}}(\Omega)$ is now defined as the closure of

$$\mathcal{C}^\infty_S(\Omega) = \{ v \in \mathcal{C}^\infty(\overline{\Omega}) : S_j \notin \text{ supp } v \}$$

with respect to the norm

$$\|u\|_{V^{k,p}_{\vec{\mu}}(\Omega)} = \Big(\sum_{|\gamma| \le k} \int_\Omega |D^\gamma u(x)|^p \, w^p(x) \, r^{(|\gamma|-k)p}(x) \, dx \Big)^{1/p}.$$

We denote

$$|u|_{V^{k,p}_{\vec{\mu}}(\Omega)} = \Big(\sum_{|\gamma| = k} \int_\Omega |D^\gamma u(x)|^p \, w^p(x) \, r^{(|\gamma|-k)p}(x) \, dx \Big)^{1/p}.$$

In $H^1_0(\Omega)$ we will denote the norms in the following way:

$$|u|^2_{H^1_0} = \int_\Omega |\nabla u|^2 \quad \text{and} \quad \|u\|^2_{H^1_0} = \int_\Omega (|\nabla u|^2 + |u|^2).$$

For $\vec{\mu}$ and $\vec{\gamma}$, we write $\vec{\mu} > \vec{\gamma}$ in case, for all $j \in \{1, \dots, J\}$, $\mu_j > \gamma_j$.

2.2. **Embeddings and consequences.**

Lemma 2.1. *Let Ω be a polygonal domain of $\mathbb{R}^2$, $p \ge 2$ and $\vec{\mu}$ satisfy*

$$\begin{aligned} \forall j = 1, \dots, J, \quad \mu_j &< \tfrac{2p-2}{p}, \quad \text{if } p > 2, \\ \mu_j &\le 1, \quad\quad \text{if } p = 2. \end{aligned} \tag{2.2}$$

Then the next continuous embeddings hold:

1. $L^p_{\vec{\mu}}(\Omega) \hookrightarrow L^2_{\vec{1}}(\Omega)$,
2. $L^2_{-\vec{1}}(\Omega) \hookrightarrow (L^p_{\vec{\mu}}(\Omega))' = L^q_{-\vec{\mu}}(\Omega)$, *with* $q = \frac{p}{p-1}$,
3. $H^1_0(\Omega) \hookrightarrow L^q_{-\vec{\mu}}(\Omega)$.

Proof. The first assertion follows from the identity

$$r_j f = r_j^{\mu_j} f r_j^{1-\mu_j},$$

Hölder's inequality and the fact that $r_j^{1-\mu_j}$ belongs to $L^q(\Omega)$ if $1 - \mu_j > -\frac{2}{q}$.

The second assertion is a consequence of the first one by using duality.

For the last assertion, in case $p = 2$, it is known (see [4, Theorem 14.5.5] or Lemma 2.11 below) that $H^1_0(\Omega)$ is continuously embedded in $L^2_{-\vec{1}}(\Omega)$. We then conclude observing that, for $\mu_j \le 1$, we have $r_j^{2(-\mu_j+1)} \in L^\infty(\Omega)$.

In the case $p > 2$, we use the embedding $H_0^1(\Omega) \hookrightarrow L^2_{-\vec{1}}(\Omega)$ and the second assertion. □

Lemma 2.2. *Let Ω be a polygonal domain of $\mathbb{R}^2$, $p \geq 2$ and $\vec{\mu}$ satisfy, for all $j = 1, \ldots, J$,*

$$\begin{aligned} \mu_j &> \tfrac{-2}{p}, \quad \text{if } p > 2, \\ \mu_j &\geq -1, \quad \text{if } p = 2. \end{aligned}$$

Then $H_0^1(\Omega)$ is continuously embedded in $L^p_{\vec{\mu}}(\Omega)$.

Proof. We have the continuous embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for all $p > 1$ and $r_j^{\mu p} \in L^s(\Omega)$ for all $s > 1$ satisfying $\mu p s + 2 > 0$. The result follows by Hölder's inequality as we have by assumption $s > 1$ such that $\mu p s + 2 > 0$ and, for $f \in H_0^1(\Omega)$,

$$\int_\Omega r_j^{\mu p} |f|^p \leq \| \, |f|^p \|_{L^{s'}} \| r_j^{\mu p} \|_{L^s},$$

with $\frac{1}{s} + \frac{1}{s'} = 1$. □

Lemma 2.3. *Let Ω be a polygonal domain of $\mathbb{R}^2$, $p \geq 2$, $\vec{\mu}$ satisfy* (2.2) *and $\theta_A \in (0, \pi)$. Then for all $g \in L^p_{\vec{\mu}}(\Omega)$ and all $z \in \mathbb{C}$ with $|\arg z| \leq \theta_A$, the problem*

$$\forall \varphi \in H_0^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla \overline{\varphi} + z \int_\Omega u \overline{\varphi} = \int_\Omega g \overline{\varphi}, \tag{2.3}$$

has a unique solution $u \in H_0^1(\Omega)$.

Proof. For all $g \in L^p_{\vec{\mu}}(\Omega)$, we know by Lemma 2.1 that the mapping

$$H : H_0^1(\Omega) \to \mathbb{C} : u \mapsto \int_\Omega g \, \bar{u}$$

is linear and continuous. Observe that, for all $z \in \mathbb{C}$ with $|\arg z| \leq \theta_A$, there exists $\theta \in [0, 2\pi]$ such that $\cos\theta > 0$ and $\Re(z e^{i\theta}) \geq 0$. Hence the conclusion follows from the Lax-Milgram Lemma as, for all $u \in H_0^1(\Omega)$,

$$\Re[e^{i\theta} \int_\Omega (|\nabla u|^2 + z|u|^2)] \geq \cos\theta \int_\Omega |\nabla u|^2 \geq \alpha \|u\|^2_{H_0^1},$$

for some $\alpha > 0$, due to Poincaré's inequality. □

Remark 2.1 The unique solution $u \in H_0^1(\Omega)$ of (2.3) is called a weak solution of the problem

$$\begin{cases} -\Delta u + z u = g, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

since $-\Delta u + zu = g$ in the distributional sense.

2.3. **Some inequalities.** For $R > 0$ and $\theta_A \in (\frac{\pi}{2}, \pi)$ fixed, we define

$$\pi^+ = \{z \in \mathbb{C} : \Re(z) \geq 0\}, \quad S_A = \{z \in \mathbb{C} : |z| \geq R \text{ and } |\arg z| \leq \theta_A\}. \tag{2.4}$$

For a solution $u \in H_0^1(\Omega)$ of (2.3) we will use the following notation: $\|u\| \lesssim \|g\|$ means the existence of a positive constant C, which is independent of the quantities $\|u\|$, $\|g\|$, and z under consideration such that $\|u\| \leq C\|g\|$ and $\|u\| \sim \|g\|$ means $\|u\| \lesssim \|g\|$ and $\|g\| \lesssim \|u\|$.

Lemma 2.4. *Let $R > 0$ and $\theta_A \in (\frac{\pi}{2}, \pi)$ be fixed. Let Ω be a polygonal domain of $\mathbb{R}^2$, $p \geq 2$, $\vec{\mu}$ satisfy* (2.2)*, $z \in \pi^+ \cup S_A$, and $u \in H_0^1(\Omega)$ be the solution of* (2.3)*. Then u satisfies the inequality*

$$|u|_{H_0^1(\Omega)} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}. \tag{2.5}$$

Proof. Applying (2.3) with $w = u$, we have

$$|u|^2_{H_0^1(\Omega)} + z \int_\Omega |u|^2 = \int_\Omega g\bar{u}. \tag{2.6}$$

By Lemma 2.1, taking the real and the imaginary parts of (2.6), we obtain

$$|u|^2_{H_0^1(\Omega)} + \Re(z) \int_\Omega |u|^2 \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)} |u|_{H_0^1(\Omega)} \tag{2.7}$$

and

$$|\Im(z)| \int_\Omega |u|^2 \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)} |u|_{H_0^1(\Omega)}. \tag{2.8}$$

Case 1: $\Re(z) \geq 0$. In that case the result can be directly deduced from (2.7).

Case 2: $\Re(z) < 0$. As $z \in S_A \cup \pi^+$, we have $\Re(z) = \rho\cos\theta$ and $\Im(z) = \rho\sin\theta$, with $\rho > R$ and $|\Im(z)| > \rho \sin\theta_A$. Hence we deduce from (2.8) that

$$\int_\Omega |u|^2 \lesssim \frac{1}{\rho} \|g\|_{L^p_\mu(\Omega)} |u|_{H_0^1(\Omega)}. \tag{2.9}$$

Then (2.7) together with $\Re(z) < 0$ gives

$$\begin{aligned} |u|^2_{H_0^1(\Omega)} &\lesssim \|g\|_{L^p_\mu(\Omega)} |u|_{H_0^1(\Omega)} - \Re(z) \int_\Omega |u|^2 \\ &\lesssim \|g\|_{L^p_\mu(\Omega)} |u|_{H_0^1(\Omega)} - \Re(z) \frac{1}{\rho} \|g\|_{L^p_\mu(\Omega)} |u|_{H_0^1(\Omega)}. \end{aligned}$$

We conclude using the inequality $-\Re(z) \leq \rho$. $\square$

Corollary 2.5. *Let $R > 0$ and $\theta_A \in (\frac{\pi}{2}, \pi)$ be fixed. Let Ω be a polygonal domain of $\mathbb{R}^2$, $p \geq 2$, $\vec{\mu}$ satisfy* (2.2), *$g \in L^p_{\vec{\mu}}(\Omega)$, $z \in \pi^+ \cup S_A$, and $u \in H^1_0(\Omega)$ be the solution of* (2.3)*. Then we have the inequalities*

$$|u|^2_{L^2(\Omega)} \lesssim \frac{1}{|z|} \|g\|_{L^p_{\vec{\mu}}(\Omega)} |u|_{H^1_0(\Omega)} \lesssim \frac{1}{|z|} \|g\|^2_{L^p_{\vec{\mu}}(\Omega)} \tag{2.10}$$

and

$$(1 + |z|^{1/2}) |u|_{L^2(\Omega)} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}. \tag{2.11}$$

Proof. By (2.6), we have

$$z \int_\Omega |u|^2 = \int_\Omega g\bar{u} - |u|^2_{H^1_0(\Omega)},$$

from which we deduce, using Lemmas 2.1 and 2.4,

$$|z| \int_\Omega |u|^2 \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)} |u|_{H^1_0(\Omega)} + |u|^2_{H^1_0(\Omega)} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)} |u|_{H^1_0(\Omega)}.$$

We then prove (2.10) by a second application of Lemma 2.4, and (2.11) is obtained by (2.10), Lemma 2.4, and Poincaré's inequality. □

Corollary 2.6. *Let $R > 0$ and $\theta_A \in (\frac{\pi}{2}, \pi)$ be fixed. Let Ω be a bounded domain of $\mathbb{R}^2$, $g \in L^2(\Omega)$, $z \in \pi^+ \cup S_A$, and $u \in H^1_0(\Omega)$ be the solution of* (2.3)*. Then we have the inequality*

$$(1 + |z|) \|u\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}.$$

Proof. As in Lemma 2.4, we obtain (2.6).

Case 1: $\Re(z) \geq 0$. In that case, we also have

$$|u|^2_{H^1_0(\Omega)} \leq \Re\Big(\int_\Omega g\overline{u} \Big) \leq \|g\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \tag{2.12}$$

and

$$z \int_\Omega |u|^2 = \int_\Omega g\overline{u} - \int_\Omega |\nabla u|^2.$$

Using (2.12), this implies

$$|z| \int_\Omega |u|^2 \leq \Big| \int_\Omega g\overline{u} \Big| + |u|^2_{H^1_0(\Omega)} \lesssim \|g\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)},$$

from which we deduce

$$|z| \, \|u\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}.$$

By Poincaré's inequality and (2.12), we have

$$\|u\|^2_{L^2(\Omega)} \lesssim |u|^2_{H^1_0(\Omega)} \leq \|g\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)},$$

and the result follows.

Case 2: $\Re(z) < 0$. In that case $z \in S_A$. As in the proof of Lemma 2.4, we have

$$\rho \|u\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)},$$

where $\rho = |z| > R$. Hence we deduce

$$(1 + |z|)\|u\|_{L^2(\Omega)} \lesssim |z| \, \|u\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)},$$

which concludes the proof. □

Lemma 2.7. *Let Ω be a polygonal domain of $\mathbb{R}^2$, $p \geq 2$, $\vec{\mu}$ satisfy (2.2), $f \in L^p_{\vec{\mu}}(\Omega)$, and $u \in H^1_0(\Omega)$ be the solution of*

$$\forall \varphi \in H^1_0(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla \overline{\varphi} = \int_\Omega f \, \overline{\varphi}. \tag{2.13}$$

Then, for all $\vec{\gamma} \geq \vec{\mu}$ with $\vec{\gamma} > 2 - \frac{2}{p} - \vec{\lambda}$ and $\vec{\lambda} = (\lambda_j)_{j=1}^J = (\frac{\pi}{\psi_j})_{j=1}^J$, $u \in V^{2,p}_{\vec{\gamma}}(\Omega)$, and, in particular,

$$r_j^{\gamma_j - 2} u \in L^p(D_j), \qquad r_j^{\gamma_j - 1} \nabla u \in (L^p(D_j))^2, \qquad r_j^{\gamma_j} \Delta u \in L^p(D_j).$$

If moreover $\vec{\mu} > -\vec{\lambda}$, then we have in particular $u \in V^{2,p}_{\vec{\mu}+2-\frac{2}{p}}(\Omega)$, and hence,

$$r_j^{\mu_j - \frac{2}{p}} u \in L^p(D_j), \qquad r_j^{\mu_j + 1 - \frac{2}{p}} \nabla u \in (L^p(D_j))^2, \qquad r_j^{\mu_j + 2 - \frac{2}{p}} \Delta u \in L^p(D_j).$$

Proof. Using regularity results far from the corners of Ω for the Laplace equation with Dirichlet boundary conditions (see [6, 7]), $u \in W^{2,p}(\tilde{\Omega})$, where, for some $\delta > 0$, $\tilde{\Omega}$ is a subdomain of $\overline{\Omega}$ with a smooth boundary, its boundary being the same as Ω except in $\bigcup_{j=1}^J B(S_j, \delta)$, with the estimate

$$\|u\|_{W^{2,p}(\tilde{\Omega})} \lesssim \|f\|_{L^p_{\vec{\mu}}(\Omega)}.$$

It then remains to look at the behaviour of u near the corners. For any $j = 1, \ldots, J$, consider the cut-off function $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ used in (2.1). For shortness we write $D = D_j(1)$ and drop the index j. Let us set $\tilde{u} = \eta u$. This function satisfies

$$\begin{cases} -\Delta \tilde{u} = \tilde{h}, & \text{in} \quad D, \\ \tilde{u} = 0, & \text{on} \quad \partial D, \end{cases} \tag{2.14}$$

where $\tilde{h} = \eta h - 2\nabla\eta \cdot \nabla u - u\Delta\eta$. Moreover, due to the above results (regularity far from the corners), $\tilde{h}$ belongs to $L^p_\mu(D)$ and

$$\|\tilde{h}\|_{L^p_\mu(D)} \lesssim \|h\|_{L^p_{\vec{\mu}}(\Omega)}.$$

For $\gamma \geq \mu$ we have $L^p_\mu(D) \subset L^p_\gamma(D)$. As, by [12, Remark 9.11], $L^p_\gamma(D) = V^{0,p}_\gamma(D)$, applying [13, Lemma 11.2 (ii)] (as in [13, Example 11.3]) we prove that $u \in V^{2,2}_1(D)$. Let γ_1 be such that $\mu \leq \gamma_1 \leq \gamma$, $\gamma_1 > 2 - \frac{2}{p} - \lambda$ and $\gamma_1 \leq 2 - \frac{2}{p}$. As $f \in L^p_{\gamma_1}(D)$, by [13, Theorem 9.3] we have $u \in V^{2,p}_{\gamma_1}(D)$ if

$$0 \leq 2 - \tfrac{2}{p} - \gamma_1 < \lambda,$$

which is equivalent to $2 - \frac{2}{p} - \lambda < \gamma_1 \leq 2 - \frac{2}{p}$. This gives the first part of the result observing that $V^{2,p}_{\gamma_1}(D) \subset V^{2,p}_\gamma(D)$ if $\gamma_1 \leq \gamma$.

To prove the second part of the result, we just have to observe that if $\mu > -\lambda$, then $\mu - \frac{2}{p} + 2 \geq \mu$ and $\mu + 2 - \frac{2}{p} > -\lambda + 2 - \frac{2}{p}$. □

Corollary 2.8. *Let Ω be a polygonal domain, $p \geq 2$, $\vec{\mu} > -\vec{\lambda}$ satisfy* (2.2)*, $f \in L^p_{\vec{\mu}}(\Omega)$, and $u \in H^1_0(\Omega)$ be the solution of* (2.13)*, and define $v = wu$ with w defined by* (2.1)*. Then we have*

$$v \in V^{2,p}_{2-\frac{2}{p}}(\Omega), \quad r^{-\frac{2}{p}} v \in L^p(\Omega), \quad r^{1-\frac{2}{p}} \nabla v \in (L^p(\Omega))^2, \quad r^{2-\frac{2}{p}} \Delta v \in L^p(\Omega).$$

Proof. This can be easily deduced from Lemma 2.7. □

Definition 2.2. *For Ω a polygonal domain, $p \geq 2$, and $\vec{\mu} \in \mathbb{R}^J$, we define*

$$D(\Delta_{p,\vec{\mu}}) = \{u \in H^1_0(\Omega) : \Delta u \in L^p_{\vec{\mu}}(\Omega)\}.$$

Before going on let us show that Lemma 2.7 furnishes an existence result in $D(\Delta_{p,\vec{\mu}})$ for problem (2.3).

Lemma 2.9. *Let $\theta_A \in (0, 2\pi)$, $p \geq 2$, $\vec{\mu}$ satisfy* (2.2) *as well as $\vec{\mu} > -\vec{\lambda} - \frac{2}{p}$, $z \in \mathbb{C}$ with $|\arg z| \leq \theta_A$, and $u \in H^1_0(\Omega)$ be the solution of* (2.3) *with $g \in L^p_{\vec{\mu}}(\Omega)$. Then $u \in D(\Delta_{p,\vec{\mu}})$.*

Proof. We only need to show that

$$-\Delta u \in L^p_{\vec{\mu}}(\Omega). \tag{2.15}$$

Using regularity results far from the corners of Ω for the Laplace equation with Dirichlet boundary conditions (see [6, 7]), $u \in W^{2,p}(\tilde{\Omega})$, where $\tilde{\Omega}$ was introduced in the proof of Lemma 2.7. We directly deduce that $u \in W^{2,p}(\tilde{\Omega})$. Hence it remains to show (2.15) near each corner S_j. For a fixed $j \in \{1, 2, \ldots, J\}$, we then set $u_j = \eta_j u \in H^1_0(D_j)$ with η_j the cut-off function used in (2.1). The function u_j is a weak solution of

$$\begin{cases} -\Delta u_j + z\, u_j = g_j, & \text{in} \quad D_j, \\ u_j = 0, & \text{on} \quad \partial D_j, \end{cases} \tag{2.16}$$

where $g_j = \eta_j\, g - 2\nabla\eta_j \cdot \nabla u - u\, \Delta\eta_j$. By the regularity $u \in W^{2,p}(\tilde{\Omega})$ mentioned before, we have

$$\nabla u \in W^{1,p}(\tilde{\Omega})^2, \quad u \in L^p(\tilde{\Omega}),$$

and consequently $g_j \in L^p_{\mu_j}(D_j)$ with

$$\|g_j\|_{L^p_{\mu_j}(D_j)} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}. \tag{2.17}$$

Hence we can concentrate on (2.16) and prove that

$$-\Delta u_j \in L^p_{\mu_j}(D_j). \tag{2.18}$$

For the rest of the proof we drop the index j and write D for D_j, μ for $\mu_j, \ldots$.

As Lemma 2.2 guarantees that $H^1_0(D) \hookrightarrow L^p_{\mu'}(D)$ for all $\mu' > -\frac{2}{p}$, we distinguish different cases:

Case 1. $\mu > -\frac{2}{p}$. In that case $-\Delta u = g - zu \in L^p_\mu(D)$.

Case 2. $-2 - \frac{2}{p} < \mu \le -\frac{2}{p}$. In that case, we take $\mu_1 = \mu + 2$ if $\mu < -\frac{2}{p}$ and $\mu_1 = 2 - \frac{2}{p} - \epsilon$, with $\epsilon \in (0, 2)$ if $\mu = -\frac{2}{p}$. Since $\mu_1 > -\frac{2}{p}$, $u \in H^1_0(\Omega)$ is a solution of

$$-\Delta u = g - zu \in L^p_{\mu_1}(D).$$

This implies by Lemma 2.7 that $u \in V^{2,p}_{\mu_1}(D)$. Accordingly, $r^{\mu_1 - 2} u \in L^p(D)$, which implies that

$$u \in L^p_\mu(D), \tag{2.19}$$

due to $\mu_1 - 2 \le \mu$. This again guarantees (2.18) because $-\Delta u = g - zu$.

Case 3. $-4 - \frac{2}{p} < \mu \le -2 - \frac{2}{p}$. Then $\mu_2 = \mu + 2$ enters in the framework of Case 2, and therefore by Case 2, we obtain by (2.19) that $u \in L^p_{\mu_2}(D)$. Now we use the same argument as before: look at $u \in H^1_0(D)$ as a solution of

$$-\Delta u = g - zu \in L^p_{\mu_2}(\Omega).$$

This implies by Lemma 2.7 that $u \in V^{2,p}_{\mu_2}(\Omega)$. Accordingly, $r^{\mu_2 - 2} u \in L^p(\Omega)$, and we conclude that (2.19) holds.

The general case follows by induction. □

Corollary 2.10. *Let* $\theta_A \in (0, 2\pi)$, $D = \{(r\cos\theta, r\sin\theta) : 0 < r < 1, 0 < \theta < \psi\}$, $\lambda = \frac{\pi}{\psi}$, $p \geq 2$, $\mu \in \mathbb{R}$, $f \in L^p_\mu(D)$, $z \in \mathbb{C}$ *with* $|\arg z| \leq \theta_A$, *and* $u \in D(\Delta_{p,\mu})$ *be the solution of*

$$\begin{cases} -\Delta u + zu = f, & in \quad D, \\ u = 0, & on \quad \partial D. \end{cases} \tag{2.20}$$

Assume $v = r^\mu u \in V^{2,p}_{2-\frac{2}{p}}(D)$. *Then* v *satisfies*

(*a*) $\Re\Big(\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v}\Big) = 0$;

(*b*) $$\frac{p}{2}\int_D |\nabla v|^2|v|^{p-2} + \frac{p-2}{2}\int_D |v|^{p-4}\overline{v}^2(\nabla v)^2 - \mu^2\int_D r^{-2}|v|^p + 2\mu\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v} + z\int_D |v|^p = \int_D r^\mu f|v|^{p-2}\overline{v}.$$

Remark 2.2 Recall that, by Corollary 2.8, if $\mu > -\lambda$ satisfies (2.2), then $v \in V^{2,p}_{2-\frac{2}{p}}(D)$ as $u \in D(\Delta_{p,\mu})$ is a solution of (2.20).

Proof. Recall that $V^{2,p}_{2-\frac{2}{p}}(D) = \overline{\mathcal{C}^\infty_S(D)}$ with $\mathcal{C}^\infty_S(D) = \{v \in \mathcal{C}^\infty(\overline{D}) : S \notin \text{supp } v\}$.

Step 1: Proof of (a). For $v \in \mathcal{C}^\infty_S(D)$, we have

$$\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\bar{v}\, r\, dr\, d\theta = -\int_D v\frac{\partial}{\partial r}(|v|^{p-2}\bar{v})\, dr\, d\theta + \int_0^\psi |v(1,\theta)|^p d\theta$$

and hence

$$\begin{aligned}
&\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\bar{v}\, r dr\, d\theta \\
&= \int_0^\psi |v(1,\theta)|^p d\theta - \int_D v[\frac{\partial}{\partial r}((v\,\bar{v})^{\frac{p-2}{2}})\bar{v} + |v|^{p-2}\frac{\partial \bar{v}}{\partial r}]\, dr\, d\theta \\
&= \int_0^\psi |v(1,\theta)|^p d\theta - \int_D v(\frac{p-2}{2}|v|^{p-4}(\bar{v})^2\frac{\partial v}{\partial r} \\
&\qquad + \frac{p-2}{2}|v|^{p-2}\frac{\partial \bar{v}}{\partial r} + |v|^{p-2}\frac{\partial \bar{v}}{\partial r})\, dr\, d\theta \\
&= \int_0^\psi |v(1,\theta)|^p d\theta - \int_D \frac{p-2}{2}|v|^{p-2}\bar{v}\frac{\partial v}{\partial r}\, dr\, d\theta - \int_D \frac{p}{2}|v|^{p-2}v\frac{\partial \bar{v}}{\partial r}\, dr\, d\theta.
\end{aligned}$$

We obtain then

$$\frac{p}{2}\Big(\int_D r^{-1}|v|^{p-2}\bar{v}\frac{\partial v}{\partial r}\, r\, dr\, d\theta + \int_D r^{-1}|v|^{p-2}v\frac{\partial \bar{v}}{\partial r}\, r\, dr\, d\theta\Big) = \int_0^\psi |v(1,\theta)|^p d\theta.$$

By density and [17, Theorem 1.31, p. 27 and Definition 1.9, p. 15], we can pass to the limit, and we see that the previous equality is also valid for $v \in V^{2,p}_{2-2/p}(D)$. As $v = r^\mu u$ with $u \in H^1_0(D)$ we have $\gamma_0 v = 0$ on ∂D which gives the equality

$$\frac{p}{2}\Big(\int_D r^{-1}|v|^{p-2}\bar{v}\frac{\partial v}{\partial r} + \int_D r^{-1}|v|^{p-2}v\frac{\partial \bar{v}}{\partial r}\Big) = 0. \tag{2.21}$$

Then (a) follows by taking the real part of (2.21).

Step 2: Proof of (b). Observe that v satisfies

$$\begin{cases} -\Delta v - \mu^2 r^{-2} v + 2\mu r^{-1}\frac{\partial v}{\partial r} + zv = g, & \text{in} \quad D, \\ v = 0, & \text{on} \quad \partial D, \end{cases} \tag{2.22}$$

with $g = r^\mu f \in L^p(D)$. As $v \in V^{2,p}_{2-\frac{2}{p}}(D)$, it is meaningful to multiply (2.22) by $|v|^{p-2}\bar{v}$ and integrate. We then obtain

$$\begin{aligned} -\int_D \Delta v\, |v|^{p-2}\bar{v} - \mu^2\int_D r^{-2}|v|^p + 2\mu\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\bar{v} + z\int_D |v|^p \\ = \int_D g\,|v|^{p-2}\bar{v}. \end{aligned} \tag{2.23}$$

For $v \in \mathcal{C}^\infty_S(D)$ we have

$$\begin{aligned} &-\int_D \Delta v\,|v|^{p-2}\bar{v} = \int_D \nabla v\cdot\nabla(|v|^{p-2}\bar{v}) + \int_{\partial D}\gamma_0(|v|^{p-2}\bar{v})\,\gamma_0(\nabla v)\cdot\nu \\ &= \int_D |\nabla v|^2|v|^{p-2} + \frac{p-2}{2}\Big(\int_D |\nabla v|^2|v|^{p-2} + \int_D |v|^{p-4}\bar{v}^2(\nabla v)^2\Big) \\ &+ \int_{\partial D}\gamma_0(|v|^{p-2}\bar{v})\,\gamma_0(\nabla v)\cdot\nu. \end{aligned}$$

As in Step 1, we show that we can pass to the limit and the previous equality is also valid for $v \in V^{2,p}_{2-2/p}(D)$. Again the boundary term is equal to zero and we obtain the equality

$$-\int_D \Delta v\,|v|^{p-2}\bar{v} = \frac{p}{2}\int_D |\nabla v|^2|v|^{p-2} + \frac{p-2}{2}\int_D |v|^{p-4}\bar{v}(\nabla v)^2. \tag{2.24}$$

The result follows then from (2.23) and (2.24). □

Lemma 2.11. *Let $D = \{(r\cos\theta, r\sin\theta) : 0 < r < 1,\, 0 < \theta < \psi\}$. For all $w \in H^1_0(D)$, we have*

$$\int_D |\nabla w|^2 \geq \frac{\pi^2}{\psi^2}\int_D \frac{1}{r^2}w^2.$$

Proof. First observe that

$$\int_D |\nabla w|^2 = \int_D ((\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial \theta})^2).$$

Moreover, for all $r \in (0,1)$ we have $w(r,\cdot) \in H_0^1(0,\psi)$ and hence, by applying Poincaré's inequality in $(0,\psi)$,

$$\int_0^\psi (\frac{\partial w}{\partial \theta})^2(r,\theta)\, d\theta \geq \frac{\pi^2}{\psi^2}\int_0^\psi w^2(r,\theta)\, d\theta.$$

We deduce from this inequality that

$$\int_0^1 \int_0^\psi ((\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial \theta})^2)\, r\, d\theta\, dr \geq \int_0^1 \int_0^\psi (\frac{\partial w}{\partial \theta})^2 d\theta \frac{1}{r} dr$$
$$\geq \frac{\pi^2}{\psi^2}\int_0^1 \int_0^\psi w^2(r,\theta)\, d\theta \frac{1}{r} dr = \frac{\pi^2}{\psi^2}\int_D w^2\, \frac{1}{r^2}.$$ □

Remark 2.3 Observe that for $w(r,\theta) = r^\beta(1-r)^\alpha \sin(\lambda\theta)$ with $\beta > 0$ and $\alpha > 1/2$ we have $w \in H_0^1(D)$ and

$$\int_D |\nabla w|^2 = (\frac{\pi^2}{\psi^2} + \frac{\beta(\alpha+\beta)}{2\alpha - 1})\int_D \frac{1}{r^2} w^2,$$

which proves the optimality of the previous inequality.

Corollary 2.12. *Let* $D = \{(r\cos\theta, r\sin\theta) : 0 < r < 1,\, 0 < \theta < \psi\}$, $\lambda = \frac{\pi}{\psi}$, $p \geq 2$, *and* $\mu \in \mathbb{R}$ *satisfy*

$$4(p-1)\lambda^2 - \mu^2 p^2 > 0. \tag{2.25}$$

Let $f \in L^p_\mu(D)$, $z \in \mathbb{C}$ *with* $\Re(z) \geq 0$, *and* $u \in D(\Delta_{p,\mu})$ *be the solution of*

$$\begin{cases} -\Delta u + zu = f, & in \quad D, \\ u = 0, & on \quad \partial D. \end{cases} \tag{2.26}$$

Assume $v = r^\mu u \in V^{2,p}_{2-\frac{2}{p}}(D)$. *Then we have the inequalities*

$$\Re(z)\, \|u\|_{L^p_\mu(D)} \leq \|f\|_{L^p_\mu(D)} \qquad and \qquad |\Im(z)|\, \|u\|_{L^p_\mu(D)} \lesssim \|f\|_{L^p_\mu(D)}.$$

Proof. By Corollary 2.10, $v = r^\mu u$ satisfies

$$\begin{aligned} \frac{p}{2}\int_D |\nabla v|^2 |v|^{p-2} + \frac{p-2}{2}\int_D |v|^{p-4}\overline{v}^2(\nabla v)^2 - \mu^2 \int_D r^{-2}|v|^p \\ + 2\mu \int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v} + z\int_D |v|^p = \int_D g\, |v|^{p-2}\overline{v}, \end{aligned} \tag{2.27}$$

with $g = r^\mu f$ and

$$\Re(\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v}) = 0. \tag{2.28}$$

Writing $v = v_1 + i\, v_2$ with $v_i : D \to \mathbb{R}$, (2.27) becomes

$$\begin{aligned}
&\frac{p}{2}\int_D [(\frac{\partial v_1}{\partial r})^2 + (\frac{\partial v_2}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial v_1}{\partial \theta})^2 + \frac{1}{r^2}(\frac{\partial v_2}{\partial \theta})^2](v_1^2 + v_2^2)|v|^{p-4} \\
&\quad + \frac{p-2}{2}\int_D |v|^{p-4}(v_1^2 - v_2^2 - 2iv_1v_2)\,[(\frac{\partial v_1}{\partial r})^2 - (\frac{\partial v_2}{\partial r})^2 + 2i\frac{\partial v_1}{\partial r}\frac{\partial v_2}{\partial r} \\
&\qquad + \frac{1}{r^2}(\frac{\partial v_1}{\partial \theta})^2 - \frac{1}{r^2}(\frac{\partial v_2}{\partial \theta})^2 + \frac{2i}{r^2}\frac{\partial v_1}{\partial \theta}\frac{\partial v_2}{\partial \theta}] \\
&\quad - \mu^2\int_D r^{-2}|v|^p + 2\mu\int_D r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v} + z\int_D |v|^p = \int_D g\,|v|^{p-2}\overline{v}.
\end{aligned} \tag{2.29}$$

By taking the real part of (2.29) and using (2.28) we obtain

$$\begin{aligned}
&\int_D |v|^{p-4}\{(p-1)[(v_1\frac{\partial v_1}{\partial r} + v_2\frac{\partial v_2}{\partial r})^2 + (v_1\frac{1}{r}\frac{\partial v_1}{\partial \theta} + v_2\frac{1}{r}\frac{\partial v_2}{\partial \theta})^2] \\
&\qquad + [(v_2\frac{\partial v_1}{\partial r} - v_1\frac{\partial v_2}{\partial r})^2 + (v_2\frac{1}{r}\frac{\partial v_1}{\partial \theta} - v_1\frac{1}{r}\frac{\partial v_2}{\partial \theta})^2]\} \\
&\qquad - \mu^2\int_D r^{-2}|v|^p + \Re(z)\int_D |v|^p = \Re(\int_D g\,|v|^{p-2}\overline{v}).
\end{aligned} \tag{2.30}$$

Denoting $w = (v_1^2 + v_2^2)^{p/4} = |v|^{p/2}$, (2.30) gives

$$\begin{aligned}
\frac{4(p-1)}{p^2}\int_D [(\frac{\partial w}{\partial r})^2 + (\frac{1}{r}\frac{\partial w}{\partial \theta})^2] - \mu^2\int_D r^{-2}w^2 \\
+ \Re(z)\int_D w^2 \le \Re(\int_D g\,|v|^{p-2}\overline{v}).
\end{aligned} \tag{2.31}$$

By Lemma 2.11, we have

$$\frac{4(p-1)}{p^2}\int_D |\nabla w|^2 - \mu^2\int_D r^{-2}w^2 \ge (\frac{4(p-1)\lambda^2}{p^2} - \mu^2)\int_D r^{-2}w^2.$$

Hence, we deduce from (2.31) that

$$\Re(z)\,\|v\|_{L^p}^p \le \|g\|_{L^p}\,\|v\|_{L^p}^{p-1},$$

in case $\frac{4(p-1)\lambda^2}{p^2} - \mu^2 \ge 0$. This gives the first inequality.

As $\frac{4(p-1)\lambda^2}{p^2} - \mu^2 > 0$ and $\Re(z) \ge 0$, we also have

$$\int_D w^2\frac{1}{r^2} = \int_D |v|^p\frac{1}{r^2} \lesssim \|g\|_{L^p}\,\|v\|_{L^p}^{p-1}, \tag{2.32}$$

$$\int_D |\nabla w|^2 = \frac{p^2}{4}\int_D |v|^{p-4}[(v_1\frac{\partial v_1}{\partial r} + v_2\frac{\partial v_2}{\partial r})^2 + (v_1\frac{1}{r}\frac{\partial v_1}{\partial \theta} + v_2\frac{1}{r}\frac{\partial v_2}{\partial \theta})^2] \lesssim \|g\|_{L^p}\|v\|_{L^p}^{p-1}, \tag{2.33}$$

and

$$\int_D |v|^{p-4}[(v_2\frac{\partial v_1}{\partial r} - v_1\frac{\partial v_2}{\partial r})^2 + (v_2\frac{1}{r}\frac{\partial v_1}{\partial \theta} - v_1\frac{1}{r}\frac{\partial v_2}{\partial \theta})^2] \lesssim \|g\|_{L^p}\|v\|_{L^p}^{p-1}. \tag{2.34}$$

By taking the imaginary part of (2.29) we obtain

$$\begin{aligned}(p-2)\int_D |v|^{p-4}[v_1^2\frac{\partial v_1}{\partial r}\frac{\partial v_2}{\partial r} - v_2^2\frac{\partial v_1}{\partial r}\frac{\partial v_2}{\partial r} - v_1v_2(\frac{\partial v_1}{\partial r})^2 + v_1v_2(\frac{\partial v_2}{\partial r})^2 \\ + v_1^2\frac{1}{r}\frac{\partial v_1}{\partial \theta}\frac{1}{r}\frac{\partial v_2}{\partial \theta} - v_2^2\frac{1}{r}\frac{\partial v_1}{\partial \theta}\frac{1}{r}\frac{\partial v_2}{\partial \theta} - v_1v_2(\frac{1}{r}\frac{\partial v_1}{\partial \theta})^2 + v_1v_2(\frac{1}{r}\frac{\partial v_2}{\partial \theta})^2] \\ + 2\mu\int_D r^{-1}|v|^{p-2}(v_1\frac{\partial v_2}{\partial r} - v_2\frac{\partial v_1}{\partial r}) + \Im(z)\int_D |v|^p = \Im(\int_D g|v|^{p-2}\overline{v}).\end{aligned}$$

Hence, we deduce from (2.32)-(2.33)-(2.34) that

$$|\Im(z)|\int_D |v|^p \lesssim \|g\|_{L^p}\|v\|_{L^p}^{p-1}$$

and hence $|\Im(z)|\|v\|_{L^p} \lesssim \|g\|_{L^p}$. The conclusion follows by observing that $\|v\|_{L^p} = \|u\|_{L^p_\mu}$ and $\|g\|_{L^p} = \|f\|_{L^p_\mu}$. □

Before going on let us show that this result is mainly optimal.

Lemma 2.13. *Let D and $\lambda = \frac{\pi}{\psi}$ be as in Corollary 2.12, and let $p \geq 2$ and $\mu \in \mathbb{R}$. If*

$$\lambda^2 < 2(1-\frac{1}{p})^2, \tag{2.35}$$

then, for all $\mu \in (\sqrt{2(1-\frac{1}{p})^2+\lambda^2}, 2-\frac{2}{p})$, there exist $f \in L^p_\mu(D)$, $z \in \mathbb{R}_+$ and $u \in D(\Delta_{p,\mu})$ a solution of (2.26) such that $r^\mu u \in V^{2,p}_{2-\frac{2}{p}}(D)$ and

$$z\|u\|_{L^p_\mu(D)} > \|f\|_{L^p_\mu(D)}. \tag{2.36}$$

Proof. Take $p \geq 2$, $z \in \mathbb{R}_+$, and $u \in D(\Delta_{p,\mu})$ such that $r^\mu u \in V^{2,p}_{2-\frac{2}{p}}(D)$, $u > 0$ on D and $\|u\|_{L^p_\mu(D)} = 1$. Let us consider the function $q : \mathbb{R}_0^+ \to \mathbb{R}^+$ defined by

$$q(X) = \frac{\|-X\Delta u + u\|^p_{L^p_\mu(D)}}{\|u\|^p_{L^p_\mu(D)}},$$

in order that

$$\frac{\|-\Delta u+zu\|^p_{L^p_\mu(D)}}{z^p\,\|u\|^p_{L^p_\mu(D)}}=q(1/z).$$

Clearly, we have $q(0)=1$, and therefore if q satisfies $q'(0)<0$, then there exists a positive real number X close enough to 0 (and then a positive real number z large enough) such that $q(X)<1$, which will show (2.36).

In view of the definition of q, we directly see that

$$q'(0)=p\int_D u^{p-1}(-\Delta u)r^{\mu p}\,dx.$$

Hence we are reduced to finding a nonzero positive function $u\in D(\Delta_{p,\mu})$ such that

$$\int_D u^{p-1}(-\Delta u)r^{\mu p}\,dx<0. \tag{2.37}$$

For that purpose, according to the proof of the previous Corollary and to Remark 2.3 we take

$$u(r,\theta)=\varphi(r)\sin(\lambda\theta),$$

with $\varphi(r)=r^{-\mu+\beta}(1-r)^\alpha$. Then we see that $u\in H^1_0(D)$ if

$$\mu<\beta \quad \text{and} \quad \alpha>1/2. \tag{2.38}$$

On the other hand by direct calculation we check that $\Delta u\in L^p_\mu(D)$ if

$$\beta>2-\frac{2}{p} \quad \text{and} \quad \alpha>2-\frac{1}{p}, \tag{2.39}$$

and $r^\mu u\in V^{2,p}_{2-\frac{2}{p}}(D)$ if $\beta>0$ and $\alpha>2-\frac{1}{p}$.

Now we come back to (2.37). By direct calculation, (2.37) reduces to

$$\begin{aligned} I:=&-[(\beta-\mu)^2-\lambda^2]\int_0^1 r^{\beta p-2}(1-r)^{\alpha p}\,r\,dr\\ &+[2(\beta-\mu)+1]\alpha\int_0^1 r^{\beta p-1}(1-r)^{\alpha p-1}\,r\,dr\\ &-[\alpha(\alpha-1)]\int_0^1 r^{\beta p}(1-r)^{\alpha p-2}\,r\,dr\\ <&\ 0. \end{aligned} \tag{2.40}$$

By using the definition of the Beta function, we get that

$$\begin{aligned} I=-\mathrm{B}(p\beta,p\alpha+1)[(\beta-\mu)^2-\lambda^2]+\mathrm{B}(p\beta+1,p\alpha)[2(\beta-\mu)+1]\alpha\\ -\mathrm{B}(p\beta+2,p\alpha-1)[\alpha(\alpha-1)]. \end{aligned}$$

Finally, the relation $\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ and the recurrence relation $\Gamma(z+1) = z\,\Gamma(z)$ lead to

$$I = \alpha p \frac{\Gamma(p\beta)\Gamma(p\alpha-1)}{\Gamma(p\alpha+p\beta+1)}\Big(\{-[(\beta-\mu)^2-\lambda^2] + [2(\beta-\mu)+1]\beta\}(p\alpha-1)-(\alpha-1)(p\beta+1)\beta\Big).$$

In conclusion we are looking for parameters α, β, and μ satisfying

$$\beta > \max(\mu, 2-\tfrac{2}{p}), \qquad \alpha > 2-\tfrac{1}{p},$$

and

$$[(p-1)\beta + p(\lambda^2-\mu^2)]\alpha + (p-1)\beta^2 - (\lambda^2-\mu^2) < 0.$$

Hence, if $2(p-1)^2 + p^2(\lambda^2-\mu^2) < 0$, then for $\beta > 2-\frac{2}{p}$ close enough to $2-\frac{2}{p}$, $\mu \le 2-\frac{2}{p}$ and α large, the above conditions are satisfied. □

Remark 2.4 Observe that, in the notation of the previous proof, if $2(p-1)^2 + p^2(\lambda^2-\mu^2) \ge 0$, $\beta > 2-\frac{2}{p}$ and $\alpha > 2-\frac{1}{p}$ then we have

$$[(p-1)\beta + p(\lambda^2-\mu^2)]\alpha + (p-1)\beta^2 - (\lambda^2-\mu^2) \ge 0.$$

In order to go further, we need estimates like in Corollary 2.12 for z in a larger part of the complex plane.

Corollary 2.14. *Let $D = \{(r\cos\theta, r\sin\theta) : 0 < r < 1,\ 0 < \theta < \psi\}$, $\lambda = \frac{\pi}{\psi}$, $p \ge 2$, $\mu > -\lambda$ satisfy* (2.2) *and* (2.25)*. Then there exists $\theta_A \in (\frac{\pi}{2}, \pi)$ such that, for all $f \in L^p_\mu(D)$, all $z \in \mathbb{C}$ with $|\arg z| \le \theta_A$, and $u \in D(\Delta_{p,\mu})$ a solution of* (2.26)*, we have the estimate*

$$|z|\,\|u\|_{L^p_\mu(D)} \lesssim \|f\|_{L^p_\mu(D)}.$$

Proof. Observe that by Lemma 2.7, we have that $D(\Delta_{p,\mu}) \subset L^p_\mu(D)$. Hence we can define the operator

$$A : D(\Delta_{p,\mu}) \subset L^p_\mu(D) \to L^p_\mu(D) : u \mapsto -\Delta u.$$

First, Lemma 2.9 guarantees that, for all $z > 0$, the range of $A + zI$ is $L^p_\mu(D)$. Second, by Corollary 2.12 we have, for all $z > 0$,

$$z\,\|(A+zI)^{-1}\| \le 1.$$

By [18, Theorem I-4.2], this implies that $-A$ is dissipative. As $L^p_\mu(D)$ is reflexive, we have by [18, Theorem I-4.6] that $D(\Delta_{p,\mu})$ is dense in $L^p_\mu(D)$. Hence by the Hille-Yosida Theorem [18, Theorem I-3.1], $-A$ is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$ for $t \ge 0$.

By Corollary 2.12, we have also a constant $C > 0$ such that, for all $\sigma \geq 0$ and all $\tau \in \mathbb{R}$

$$|\tau| \, \|(A + (\sigma + i\tau)I)^{-1}\| \leq C;$$

hence, by [18, Theorem II-5.2], there exists $\delta \in (0, \frac{\pi}{2})$ and $M > 0$ such that

$$\rho(-A) \supset \Sigma := \{z \in \mathbb{C} : |\arg z| < \tfrac{\pi}{2} + \delta\} \cup \{0\},$$

and, for all $z \in \Sigma$,

$$|z| \, \|(A + zI)^{-1}\| \leq M.$$

This proves the required inequality. □

Lemma 2.15. *Let $\tilde{\Omega}$ be a bounded domain with a smooth boundary and $p \geq 2$. Then there exists $\tilde{\theta}_A \in (\frac{\pi}{2}, \pi)$ such that, for all $h \in L^p(\tilde{\Omega})$, all $z \in \mathbb{C}$ with $|\arg z| \leq \tilde{\theta}_A$, and $v \in H_0^1(\tilde{\Omega})$ a weak solution of*

$$\begin{cases} -\Delta v + z\, v = h, & \text{in} \quad \tilde{\Omega}, \\ \quad v = 0, & \text{on} \quad \partial\tilde{\Omega}, \end{cases}$$

we have

$$(1 + |z|)\|v\|_{L^p(\tilde{\Omega})} \lesssim \|h\|_{L^p(\tilde{\Omega})}.$$

Proof. Recall that v satisfies

$$\forall \varphi \in H_0^1(\tilde{\Omega}), \qquad \int_{\tilde{\Omega}} \nabla v \cdot \nabla \overline{\varphi} + z \int_{\tilde{\Omega}} v \overline{\varphi} = \int_{\tilde{\Omega}} h \overline{\varphi}.$$

Step 1: $\Re z \geq 0$. By [1, Theorem 1.6], we have

$$|z| \, \|v\|_{L^p(\tilde{\Omega})} \lesssim \|h\|_{L^p(\tilde{\Omega})}.$$

As moreover

$$\|v\|_{L^p(\tilde{\Omega})} \lesssim \|v\|_{H_0^1(\tilde{\Omega})} \lesssim \|h\|_{L^p(\tilde{\Omega})},$$

we obtain the result.

Step 2. Extension. The extension to $\{z \in \mathbb{C} : |\arg z| \leq \tilde{\theta}_A\}$ can be made as in Corollary 2.14. □

Corollary 2.16. *Under the assumptions of Corollary 2.14, for all $f \in L^q_{-\vec{\mu}}(\Omega)$, there exists a unique $\phi \in L^q_{-\vec{\mu}}(\Omega)$ solution of*

$$\forall v \in D(\Delta_{p,\vec{\mu}}), \qquad \int_\Omega \phi \left((-\Delta + \bar{z})\bar{v}\right) = \int_\Omega f \, \bar{v}.$$

Moreover, ϕ satisfies

$$|z| \, \|\phi\|_{L^q_{-\vec{\mu}}(\Omega)} \lesssim \|f\|_{L^q_{-\vec{\mu}}(\Omega)}.$$

Proof. By Corollary 2.14 and Lemma 2.15, the linear operator

$$A : D(\Delta_{p,\vec{\mu}}) \subset L^p_{\vec{\mu}}(\Omega) \to L^p_{\vec{\mu}}(\Omega) : u \mapsto -\Delta u$$

satisfies, for $z \in \mathbb{C}$ with $|\arg(z)| \leq \theta_A$,

$$|z| \, \|(A + z\, I)^{-1} f\|_{L^p_{\vec{\mu}}(\Omega)} \lesssim \|f\|_{L^p_{\vec{\mu}}(\Omega)}.$$

Moreover, $D(\Delta_{p,\vec{\mu}})$ is dense in $L^p_{\vec{\mu}}(\Omega)$ (because $\mathcal{D}(\Omega) \subset D(\Delta_{p,\vec{\mu}})$). Hence we can define the adjoint A^* of the operator A

$$A^* : D(A^*) \subset (L^p_{\vec{\mu}}(\Omega))^* \to (L^p_{\vec{\mu}}(\Omega))^*,$$

where

$$D(A^*) = \{x^* \in (L^p_{\vec{\mu}}(\Omega))^* : \exists y^* \in (L^p_{\vec{\mu}}(\Omega))^*, \forall x \in D(A), \int_\Omega x^* \overline{Ax} = \int_\Omega y^* \overline{x}\}$$
$$A^* x^* = y^*, \text{ for } x^* \in D(A^*).$$

For $z \in \mathbb{C}$ with $|\arg(z)| \leq \theta_A$, we have $z \in \rho(A)$ and by [18, Lemma I-10.2], $\bar{z} \in \rho(A^*)$ and

$$(\bar{z}\, I + A^*)^{-1} = ((z\, I + A)^{-1})^*.$$

The result follows from [18, Lemma I-10.1] observing that A^* is such that $u \in D(A^*) \subset L^q_{-\vec{\mu}}(\Omega)$ if and only if there exists $g \in L^q_{-\vec{\mu}}(\Omega)$ such that

$$\forall v \in D(\Delta_{p,\vec{\mu}}), \qquad \int_\Omega g\, \bar{v} = -\int_\Omega u \overline{\Delta v},$$

and, for $u \in D(A^*)$, in the above notation, $A^*(u) = g$. □

3. Uniform decomposition

Theorem 3.1. *Let $R > 0$ be fixed. Let $p \geq 2$, $\epsilon > 0$ fixed, and Ω be a bounded polygonal domain of $\mathbb{R}^2$. Denote by S_j, $j = 1, \ldots, J$, the vertices of $\partial\Omega$. Set*

$$\Omega_\epsilon = \{x \in \Omega : \forall j = 1, \ldots, J, \quad \mathit{dist}\ (x, S_j) > \epsilon\},$$

and let $\vec{\mu} \in \mathbb{R}^J$. Then, there exists $\theta_A \in (\frac{\pi}{2}, \pi)$ such that, for all $g \in L^p_{\vec{\mu}}(\Omega)$, all $z \in \pi^+ \cup S_A$, the solution $u \in H^1_0(\Omega)$ of (2.3) satisfies $u \in W^{2,p}(\Omega_\epsilon)$ and

$$|u|_{W^{2,p}(\Omega_\epsilon)} + (1 + |z|^{1/2})\, |u|_{W^{1,p}(\Omega_\epsilon)} + (1 + |z|)\, |u|_{L^p(\Omega_\epsilon)} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}. \quad (3.1)$$

Proof. Let θ_A be given by Corollary 2.15.
Step 1: Regularity H^2. Let us fix η a cut-off function such that $\eta = 0$ on $\bigcup_{j=1,\ldots,J} B(S_j, \epsilon/2)$ and $\eta = 1$ on Ω_ϵ. Let $\tilde{\Omega}$ be a regular domain such that

$\Omega_\epsilon \subset \tilde{\Omega} \subset \overline{\Omega}$, the boundary of $\tilde{\Omega}$ satisfying $\partial\tilde{\Omega} = \partial\Omega$ except near the corner, and $v = \eta u \in H_0^1(\tilde{\Omega})$. In that case v is a weak solution of

$$\begin{cases} -\Delta v + z\, v = \eta g - 2\nabla\eta \cdot \nabla u - \Delta\eta\, u =: h, & \text{in} \quad \tilde{\Omega}, \\ v = 0, & \text{on} \quad \partial\tilde{\Omega}. \end{cases} \tag{3.2}$$

By Lemma 2.4 and Poincaré's inequality, we have $h \in L^2(\tilde{\Omega})$ and satisfies

$$\|h\|_{L^2(\tilde{\Omega})} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)} + \|\nabla u\|_{L^2(\tilde{\Omega})} + \|u\|_{L^2(\tilde{\Omega})} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}. \tag{3.3}$$

By Lemma 2.4 and Corollary 2.6 we obtain

$$|v|_{H_0^1(\tilde{\Omega})} + (1+|z|)\, \|v\|_{L^2(\tilde{\Omega})} \lesssim \|h\|_{L^2(\tilde{\Omega})}. \tag{3.4}$$

We can also consider $v \in H_0^1(\tilde{\Omega})$ as a weak solution of

$$\begin{cases} -\Delta v + v = h + (1-z)\, v =: h_1, & \text{in} \quad \tilde{\Omega}, \\ v = 0, & \text{on} \quad \partial\tilde{\Omega}, \end{cases}$$

where $h_1 \in L^2(\tilde{\Omega})$ and, by (3.4),

$$\|h_1\|_{L^2(\tilde{\Omega})} \lesssim \|h\|_{L^2(\tilde{\Omega})} + (1+|z|)\, \|v\|_{L^2(\tilde{\Omega})} \lesssim \|h\|_{L^2(\tilde{\Omega})} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}.$$

By the strong ellipticity of $-\Delta$ and the fact that the boundary conditions cover $-\Delta$ we have (see [4]) $v \in H^2(\tilde{\Omega})$ and

$$\|v\|_{H^2(\tilde{\Omega})} \lesssim \|h_1\|_{L^2(\tilde{\Omega})} \lesssim \|h\|_{L^2(\tilde{\Omega})}. \tag{3.5}$$

As $v \in H^2(\tilde{\Omega})$ we have $\Delta v \in L^2(\tilde{\Omega})$. Multiplying (3.2) by $-\Delta\bar{v}$ and integrating, we obtain

$$z \int_{\tilde{\Omega}} |\nabla v|^2 = -z \int_{\tilde{\Omega}} v\Delta\bar{v} = -\int_{\tilde{\Omega}} h\, \Delta\bar{v} - \int_{\tilde{\Omega}} |\Delta v|^2.$$

This implies, using (3.5),

$$|z| \int_{\tilde{\Omega}} |\nabla v|^2 \le \|h\|_{L^2(\tilde{\Omega})} \|\Delta\bar{v}\|_{L^2(\tilde{\Omega})} + \|\Delta\bar{v}\|^2_{L^2(\tilde{\Omega})} \lesssim \|h\|^2_{L^2(\tilde{\Omega})}.$$

Combining (3.3), (3.4), and (3.5) we prove the inequality

$$|v|_{H^2(\tilde{\Omega})} + (1+\sqrt{|z|})\, |v|_{H^1(\tilde{\Omega})} + (1+|z|)\, |v|_{L^2(\tilde{\Omega})} \lesssim \|h\|_{L^2(\tilde{\Omega})} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}.$$

As $v = u$ on Ω_ϵ, this implies $u \in H^2(\Omega_\epsilon)$ and

$$|u|_{H^2(\Omega_\epsilon)} + (1+\sqrt{|z|})\, |u|_{H^1(\Omega_\epsilon)} + (1+|z|)\, |u|_{L^2(\Omega_\epsilon)} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}.$$

Step 2: Regularity $W^{2,p}(\Omega_\epsilon)$. As in Step 1, let η_1 be a cut-off function such that $\eta_1 = 0$ near the corners and $\eta_1 = 1$ on $\tilde{\Omega}$. Let Ω_1 be a regular domain such that $\tilde\Omega \subset \Omega_1 \subset \overline{\Omega}$, the boundary of Ω_1 satisfying $\partial\Omega_1 = \partial\Omega$ except near the corner, and $w = \eta_1 u \in H^1_0(\Omega_1)$. In that case w is a weak solution of

$$\begin{cases} -\Delta w + z\, w = \eta_1 g - 2\nabla\eta_1\cdot\nabla u - \Delta\eta_1\, u =: \tilde h_1, & \text{in} \quad \Omega_1,\\ w = 0, & \text{on} \quad \partial\Omega_1. \end{cases}$$

By Step 1, we have that $w \in H^2(\Omega_1)$ and

$$|w|_{H^2(\Omega_1)} + (1+\sqrt{|z|})\, |w|_{H^1(\Omega_1)} + (1+|z|)\, |w|_{L^2(\Omega_1)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}.$$

As $w = u$ on $\tilde\Omega$ this shows $u \in H^2(\tilde\Omega)$ and

$$|u|_{H^2(\tilde\Omega)} + (1+\sqrt{|z|})\, |u|_{H^1(\tilde\Omega)} + (1+|z|)\, |u|_{L^2(\tilde\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}. \tag{3.6}$$

This implies that

$$h := \eta g - 2\nabla\eta\cdot\nabla u - \Delta\eta\, u \in L^p(\tilde\Omega)$$

with

$$\|h\|_{L^p(\tilde\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)} + |u|_{W^{1,p}(\tilde\Omega)} + |u|_{L^p(\tilde\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}, \tag{3.7}$$

where we have used Sobolev inequality and (3.6).

We can now proceed in the same way as before but with a given function $h \in L^p(\tilde\Omega)$. Hence $v \in H^1_0(\tilde\Omega)$ is a weak solution of

$$\begin{cases} -\Delta v + v = h + (1-z)\, v =: h_1, & \text{in} \quad \tilde\Omega,\\ w = 0, & \text{on} \quad \partial\tilde\Omega. \end{cases}$$

By the elliptic regularity, for all $\tilde p \in [2,p]$ we have $v \in W^{2,\tilde p}(\tilde\Omega)$ and

$$\|v\|_{W^{2,\tilde p}(\tilde\Omega)} \lesssim \|h_1\|_{L^{\tilde p}(\tilde\Omega)}.$$

Using Corollary 2.6, we obtain

$$(1+|z|)\|v\|_{L^2(\tilde\Omega)} \lesssim \|h\|_{L^2(\tilde\Omega)},$$

which allows us to prove in particular

$$\|v\|_{H^2(\tilde\Omega)} \lesssim \|h_1\|_{L^2(\tilde\Omega)} \le \|h\|_{L^2(\tilde\Omega)} + (1+|z|)\|v\|_{L^2(\tilde\Omega)} \lesssim \|h\|_{L^2(\tilde\Omega)} \le \|h\|_{L^p(\tilde\Omega)}.$$

By Corollary 2.15, we deduce that

$$(1+|z|)\, \|v\|_{L^p(\tilde\Omega)} \lesssim \|h\|_{L^p(\tilde\Omega)}$$

and hence

$$\|h_1\|_{L^p(\tilde\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}.$$

By interpolation (see [7, Theorem 1.4.3.3]) we obtain

$$\|v\|_{W^{1,p}(\tilde\Omega)} \le \epsilon\|v\|_{W^{2,p}(\tilde\Omega)} + K\epsilon^{-1}\|v\|_{L^p(\tilde\Omega)}.$$

Applying this inequality with $\epsilon = \frac{1}{1+\sqrt{|z|}}$, this gives

$$|v|_{W^{2,p}(\tilde{\Omega})} + (1 + \sqrt{|z|})\, |v|_{W^{1,p}(\tilde{\Omega})} + (1 + |z|)\, |v|_{L^p(\tilde{\Omega})} \lesssim \|g\|_{L^p_{\vec{\mu}}(\Omega)}.$$

The result follows as $v = u$ on Ω_ϵ. □

Lemma 3.2. *Let $R > 0$ and $\theta_A \in (\frac{\pi}{2}, \pi)$ be fixed. Let C be the cone with interior angle ψ and $\lambda = \frac{\pi}{\psi}$. For $z \in \pi^+ \cup S_A$ and $u \in V^{2,p}_\mu(C) \cap H^1_0(C)$ with $u = 0$ for $r > r_0$, if μ satisfies* (2.2) *and, for all $k \in \mathbb{Z}^*$, $2 - \frac{2}{p} - \mu \neq k\lambda$, then we have*

$$|u|_{V^{2,p}_\mu(C)} + |u|_{V^{1,p}_{\mu-1}(C)} + \|u\|_{L^p_{\mu-2}(C)} + |z|^{1/2} |u|_{V^{1,p}_\mu(C)} + |z|\, \|u\|_{L^p_\mu(C)} \lesssim \|(-\Delta + z)u\|_{L^p_\mu(C)} + |z|\, \|u\|_{L^p_\mu(C)}.$$

Proof. By [14, Theorem 6.2], we have, for $\theta \in [-\theta_A, \theta_A]$ and $u \in E^{2,p}_\mu(C)$,

$$\|u\|_{E^{2,p}_\mu(C)} \lesssim \|(-\Delta + e^{i\theta})u\|_{L^p_\mu(C)} + \|u\|_{L^p(S)}, \tag{3.8}$$

where $S = \{x \in \mathbb{R}^2 : \delta_1 < |x| < \delta_2\} \cap C$ with $0 < \delta_1 < \delta_2$, $E^{2,p}_\mu(C) = \mathrm{adh}_{\|\cdot\|_{E^{2,p}_\mu}}(\mathcal{C}^\infty_0(\overline{C} \setminus \{0\})$ where

$$\|u\|_{E^{2,p}_\mu} = \Big(\int_C r^{p\mu} \sum_{|\alpha|=0}^{2} (r^{p(|\alpha|-2)} + 1)\, |D^\alpha_y u(y)|^p \Big)^{1/p}.$$

As $\|u\|_{L^p(S)} \lesssim \|u\|_{L^p_\mu(C)}$, the result can be deduced from a change of variable. □

Theorem 3.3. *Let $R > 0$, $p \geq 2$, and Ω be a bounded polygonal domain of $\mathbb{R}^2$. Denote by S_j, $j = 1, \ldots, J$, the vertices of $\partial\Omega$ enumerated clockwise and, for $j \in \{1, 2, \ldots, J\}$, let ψ_j be the interior angle of Ω at the vertex S_j, $\lambda_j = \frac{\pi}{\psi_j}$ and $\vec{\lambda} = (\lambda_j)_{1 \leq j \leq J}$. Let $\vec{\mu} > -\vec{\lambda}$ satisfy* (2.2), (2.25), *and, for all $k \in \mathbb{Z}^*$ and all $j \in \{1, 2, \ldots, J\}$, $2 - \frac{2}{p} - \mu_j \neq k\lambda_j$. Then, there exists $\theta_A \in (\frac{\pi}{2}, \pi)$ such that, for all $g \in L^p_{\vec{\mu}}(\Omega)$, all $z \in \pi^+ \cup S_A$, the unique solution $u \in D(\Delta_{p,\vec{\mu}})$ of*

$$\begin{cases} -\Delta u + zu = g, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

admits the decomposition

$$u = u_R + \sum_{j=1}^{J} \eta_j(r) \sum_{\substack{0 < \lambda'_j < 2 - \frac{2}{p} - \mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} c_{\lambda'_j}(z)\, P_{j,\lambda'_j}(r\sqrt{z}) e^{-r\sqrt{z}}\, r^{\lambda'_j} \sin(\lambda'_j \theta), \tag{3.9}$$

with $u_R \in V^{2,p}_{\vec\mu}(\Omega)$, $c_{\lambda'_j}(z) \in \mathbb{C}$, $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ *is a cut-off function such that*

$$\eta_j \equiv 1 \text{ in } D_j(1/2), \quad \eta_j \equiv 0 \text{ on } \Omega \setminus D_j(1),$$

and $P_{j,\lambda'_j}(s) = \sum_{i=0}^{l_{j,\lambda'_j}-1} \frac{s^i}{i!}$ *with* $l_{j,\lambda'_j} > 2 - \mu_j - \frac{2}{p} - \lambda'_j$.

Moreover, the following inequalities are satisfied:

(a) $|u_R|_{V^{2,p}_{\vec\mu}(\Omega)} + |u_R|_{V^{1,p}_{\vec\mu-1}(\Omega)} + |u_R|_{L^p_{\vec\mu-2}(\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$;

(b) $|u_R|_{V^{2,p}_{\vec\mu}(\Omega)} + |z|^{1/2}\, |u_R|_{V^{1,p}_{\vec\mu}(\Omega)} + |z|\, |u_R|_{L^p_{\vec\mu}(\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$;

(c) $\displaystyle\sum_{j=1}^{J} \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N},\lambda'_j=k\lambda_j}} |c_{\lambda'_j}(z)|\, (1+|z|^{1-\frac{1}{p}-\frac{\mu_j+\lambda'_j}{2}}) \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$.

Proof. Let θ_A be the minimum of the ones obtained by Theorem 3.1 and by Corollary 2.14. By Theorem 3.1, we have the required result in the interior of Ω. It remains to prove the regularity near the corners.

Let $j \in \{1, 2, \ldots, J\}$ be fixed and observe that $v = \eta_j\, u$ is the solution of

$$\begin{cases} -\Delta v + z\, v = g_1, & \text{in} \quad D_j, \\ v = 0, & \text{on} \quad \partial D_j, \end{cases} \tag{3.10}$$

with $D_j = \{(r_j \cos\theta_j, r_j \sin\theta_j) : 0 < r_j < 1, 0 < \theta_j < \psi_j\}$ and $g_1 = \eta_j\, g - 2\nabla\eta_j \cdot \nabla u - \Delta\eta_j\, u$. By Theorem 3.1, we have

$$\nabla u \in W^{1,p}(\Omega_\epsilon), \qquad u \in L^p(\Omega_\epsilon),$$

and $g_1 \in L^p_{\mu_j}(D_j)$ with

$$\|g_1\|_{L^p_{\mu_j}(D_j)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}. \tag{3.11}$$

Hence we can concentrate on (3.10). For the rest of the proof we drop the index j and write D for D_j, μ for $\mu_j, \ldots$.

Step 1: $v \in V^{2,2}_1(D)$ **and**

$$|v|_{V^{2,2}_1(D)} + |v|_{V^{1,2}_0(D)} + |v|_{V^{0,2}_{-1}(D)} \lesssim \|g_1\|_{L^2_1(D)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}. \tag{3.12}$$

Using Lemma 2.4 and Corollary 2.5, we have

$$\begin{aligned} \|g_1\|_{L^2_1(D)} &\lesssim \|rg\|_{L^2(D)} + \|u\|_{H^1(\Omega)} \\ &\lesssim \Big(\int_D |r|^{\frac{2(1-\mu)p}{p-2}}\Big)^{\frac{p-2}{2p}} \|r^\mu g\|_{L^p(D)} + \|u\|_{H^1(\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}. \end{aligned}$$

This proves

$$\|g_1\|_{L^2_1(D)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}. \tag{3.13}$$

Moreover, v can be considered as a solution of

$$\begin{cases} -\Delta v = g_1 - z\,v =: g_2, & \text{in} \quad D, \\ v = 0, & \text{on} \quad \partial D. \end{cases} \tag{3.14}$$

By Lemma 2.7, $v \in L^p_\mu(D)$ and by Corollary 2.14, we obtain

$$\|g_2\|_{L^p_\mu(D)} \lesssim \|g_1\|_{L^p_\mu(D)} + |z|\|v\|_{L^p_\mu(D)} \lesssim \|g_1\|_{L^p_\mu(D)} \lesssim \|g\|_{L^p_{\bar\mu}(\Omega)}. \tag{3.15}$$

Applying [13, Lemma 11.2(ii)] (as in [13, Example 11.3]), this allows us to conclude $v \in V^{2,2}_1(D)$ and

$$\|v\|_{V^{2,2}_1(D)} = |v|_{V^{2,2}_1(D)} + |v|_{V^{1,2}_0(D)} + |v|_{V^{0,2}_{-1}(D)} \lesssim \|g_2\|_{L^2_1(D)} \lesssim \|g\|_{L^p_{\bar\mu}(\Omega)}.$$

The case $p = 2$ and $\mu = 1$ is proved.

Step 2: L^p-decomposition in case $p > 2$ or in case $p = 2$ and $\mu < 1$. As $\mu < \frac{2p-2}{p}$ and, by Step 1, $v \in V^{2,2}_1(D)$, applying [13, Corollary (iv) of Theorem 10.2 and Theorem 10.3], we have

$$v = v^{(1)}_R + \sum_{\substack{0<\lambda'<2-\frac{2}{p}-\mu \\ \exists k\in\mathbb{N}, \lambda'=k\lambda}} c_{\lambda'}(z)\eta\, r^{\lambda'} \sin(\lambda'\theta) =: v^{(1)}_R + v_S \tag{3.16}$$

with $v^{(1)}_R \in V^{2,p}_\mu(D)$ satisfying, by (3.15),

$$\|v^{(1)}_R\|_{V^{2,p}_\mu(D)} \lesssim \|g_2\|_{L^p_\mu(D)} \lesssim \|g_1\|_{L^p_\mu(D)} \lesssim \|g\|_{L^p_{\bar\mu}(\Omega)}. \tag{3.17}$$

As $\mu < \frac{2p-2}{p}$, by [13, section 0.10], we have $V^{2,p}_\mu(D) \hookrightarrow V^{2,2}_1(D)$ and $L^p_\mu(D) \hookrightarrow L^2_1(D)$. Hence, using (3.12), (3.15), and (3.17), we obtain

$$\|v_S\|_{V^{2,2}_1(D)} \lesssim \|v\|_{V^{2,2}_1(D)} + \|v^{(1)}_R\|_{V^{2,p}_\mu(D)} \lesssim \|g_1\|_{L^p_\mu(D)} \lesssim \|g\|_{L^p_{\bar\mu}(\Omega)}.$$

The space $V_S = \operatorname{span}\{r^{\lambda'}\sin(\lambda'\theta) : 0 < \lambda' < 2 - \frac{2}{p} - \mu \text{ and } \exists k \in \mathbb{N}, \lambda' = k\lambda\}$ being of finite dimension, we have

$$\sum_{\substack{0<\lambda'<2-\frac{2}{p}-\mu \\ \exists k\in\mathbb{N}, \lambda'=k\lambda}} |c_{\lambda'}(z)| \lesssim \|v_S\|_{V^{2,2}_1(D)} \lesssim \|g_1\|_{L^p_\mu(D)}. \tag{3.18}$$

Now in order to have uniform estimates with respect to $|z|$, we choose another decomposition. We rewrite (3.16) in the following way:

$$v = v_R + \sum_{\substack{0<\lambda'<2-\frac{2}{p}-\mu \\ \exists k\in\mathbb{N}, \lambda'=k\lambda}} c_{\lambda'}(z)\,\eta(r)\,\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\sin(\lambda'\theta), \tag{3.19}$$

where $v_R = v_R^{(1)} + \sum_{\substack{0<\lambda'<2-\frac{2}{p}-\mu \\ \exists k\in\mathbb{N},\lambda'=k\lambda}} c_{\lambda'}(z)\,\eta(r)\,(1-\psi_{\lambda'}(r\sqrt{z}))\,r^{\lambda'}\sin(\lambda'\theta)$ and $\psi_{\lambda'}(s) = P_{\lambda'}(s)e^{-s}$ with $P_{\lambda'}(s) = \sum_{i=0}^{l_{\lambda'}-1}\frac{s^i}{i!}$ and $l_{\lambda'} > 2-\mu-\frac{2}{p}-\lambda'$ is chosen in such a way that $(1-\psi_{\lambda'}(r\sqrt{z}))\,r^{\lambda'}\sin(\lambda'\theta) \in V_\mu^{2,p}(D)$.

Step 3: Estimates (c) **on** $|c_{\lambda'}(z)|$**.** By (3.18) we know that, for all $z \in \pi^+ \cup S_A$, the mapping $L_\mu^p(D) \to \mathbb{C} : g_1 \mapsto c_{\lambda'}(z)$ is linear and continuous. Hence by the Riesz theorem, there exists a unique $w_z^{\lambda'} \in L_{-\mu}^q(D) = (L_\mu^p(D))'$ such that

$$c_{\lambda'}(z) = \int_D g_1 \bar{w}_z^{\lambda'}\,dx.$$

Claim: *The function* $w_z^{\lambda'}$ *is characterized by*

(i) *for all* $v \in V_\mu^{2,p}(D) \cap D(\Delta_{p,\mu})$, $\int_D (-\Delta v + z\,v)\,\bar{w}_z^{\lambda'} = 0$;

(ii) *for all* $\xi \in (0, 2-\frac{2}{p}-\mu)$ *such that* $\exists k \in \mathbb{N}$, $\lambda' = k\lambda$, *we have*

$$\int_D (-\Delta + z)(\eta(r)\,r^\xi \sin(\xi\theta))\,\bar{w}_z^{\lambda'} = \delta_{\xi\lambda'}.$$

This can be easily deduced from (3.16) as this implies

$$v \in V_\mu^{2,p}(D) \cap D(\Delta_{p,\mu}) \iff \forall \lambda' \in (0, 2-\tfrac{2}{p}-\mu) \text{ with } \exists k \in \mathbb{N}, \lambda' = k\lambda,\ c_{\lambda'}(z) = 0.$$

and

$$\int_D (-\Delta + z)(\eta(r)\,r^\xi \sin(\xi\theta))\,\bar{w}_z^{\lambda'} = \delta_{\xi\lambda'}.$$

The reversed implications can be deduced by uniqueness of $w_z^{\lambda'}$.

To obtain the good estimate on $c_{\lambda'}(z)$, we will decompose $w_z^{\lambda'}$ in an appropriate way. To this aim define the function

$$\psi^{\lambda'} = A_{\lambda'} e^{-r\sqrt{\bar{z}}}\,Q(\sqrt{\bar{z}}r)\,r^{-\lambda'}\sin(\lambda'\theta), \tag{3.20}$$

with $A_{\lambda'} = \frac{1}{\lambda'\psi}$, $Q(s) = \sum_{i=0}^m \alpha_i s^i$, with m fixed below and α_i such that

$$\begin{aligned} &\alpha_0 = \alpha_1 = 1,\\ \forall i \in \{1,\dots,m-1\},\qquad &(2\lambda'-1-2i)\,\alpha_i = (i+1)\,(2\lambda'-i-1)\,\alpha_{i+1}. \end{aligned}$$

The positive integer m is fixed as follows: if $\lambda'-1 \geq 1$, then we take $m > \lambda'-1 \geq m-1$; hence, we have $m \geq 1$ and for $i \in \{1,\dots,m-1\}$ we have $i+1 \leq m \leq \lambda'$ and hence $i+1 \neq 2\lambda'$. If $\lambda'-1 < 1$, then we take $m = 1$.

The above choice leads to

$$(-\Delta+\bar z)(\eta\psi^{\lambda'}) = -\eta\, A_{\lambda'}\,(2\lambda'-1-2m)\alpha_m\, r^{m-\lambda'-1}\,\bar z^{\frac{m+1}{2}}\, e^{-r\sqrt{\bar z}}\,\sin(\lambda'\theta) \\ +(-2\nabla\psi^{\lambda'}\cdot\nabla\eta-\psi^{\lambda'}\Delta\eta).$$

Let $t>1$ be such that $(m-\lambda'-1)t+2>0$; by the above computation, we see that $(-\Delta+\bar z)(\eta\psi^{\lambda'})\in L^t(D)$. Hence we can define $\phi^{\lambda'}\in H_0^1(D)$ such that

$$\forall v\in H_0^1(D),\qquad \int_D(\nabla\phi^{\lambda'}\cdot\nabla\bar v+\bar z\phi^{\lambda'}\,\bar v) = \int_D(-\Delta+\bar z)(\eta\psi^{\lambda'})\,\bar v. \tag{3.21}$$

Let us show that

$$w_z^{\lambda'} = \eta\psi^{\lambda'}-\phi^{\lambda'}. \tag{3.22}$$

To this aim, let us prove that $\eta\psi^{\lambda'}-\phi^{\lambda'}$ satisfies the conditions (i) and (ii) above.

The condition (i) can be deduced from the Green's formula; see Lemma 3.5 below. Consider then the condition (ii). By the Green's formula (apply [17, Corollary 1.42] and remindng the reader that $H_0^1(D)\hookrightarrow L^2_{-1}(D)$ (see Lemma 2.11)) we deduce

$$\int_D(-\Delta+z)(\eta(r)r^\xi\sin(\xi\theta))\,\bar\phi^{\lambda'} \\ = \int_D\nabla(\eta(r)r^\xi\sin(\xi\theta))\cdot\nabla\bar\phi^{\lambda'}+z\int_D\eta(r)r^\xi\sin(\xi\theta)\bar\phi^{\lambda'}.$$

By the definition of $\phi^{\lambda'}$ we obtain

$$\int_D\nabla(\eta(r)r^\xi\sin(\xi\theta))\cdot\nabla\bar\phi^{\lambda'}+z\int_D\eta(r)r^\xi\sin(\xi\theta)\bar\phi^{\lambda'} \\ = \int_D\overline{(-\Delta+\bar z)(\eta\psi^{\lambda'})}\,\eta(r)r^\xi\sin(\xi\theta).$$

Hence we have

$$\int_D(-\Delta+z)(\eta(r)r^\xi\sin(\xi\theta))\,(\eta\bar\psi^{\lambda'}-\bar\phi^{\lambda'}) \\ = \int_D(-\Delta+z)(\eta(r)r^\xi\sin(\xi\theta))\,\eta\bar\psi^{\lambda'} \\ -\int_D\overline{(-\Delta+\bar z)(\eta\psi^{\lambda'})}\,\eta(r)r^\xi\sin(\xi\theta) \\ = -\int_D\Delta(\eta(r)r^\xi\sin(\xi\theta))\,\eta\bar\psi^{\lambda'}+\int_D\Delta(\eta\bar\psi^{\lambda'})\,\eta(r)r^\xi\sin(\xi\theta).$$

Let us denote $D_\epsilon = D \setminus B(0,\epsilon)$; then we can write by the Green's formula

$$\begin{aligned}
&-\int_D \Delta(\eta(r) r^\xi \sin(\xi\theta))\, \eta\bar{\psi}^{\lambda'} + \int_D \Delta(\eta\bar{\psi}^{\lambda'})\, \eta(r) r^\xi \sin(\xi\theta) \\
&= \lim_{\epsilon\to 0} \Big[-\int_{D_\epsilon} \Delta(\eta(r) r^\xi \sin(\xi\theta))\, \eta\bar{\psi}^{\lambda'} + \int_{D_\epsilon} \Delta(\eta\bar{\psi}^{\lambda'})\, \eta(r) r^\xi \sin(\xi\theta) \Big] \\
&= \lim_{\epsilon\to 0} \Big[-\int_{\partial D_\epsilon} \frac{\partial}{\partial\nu}(\eta(r) r^\xi \sin(\xi\theta))\, \eta\bar{\psi}^{\lambda'} + \int_{\partial D_\epsilon} \frac{\partial}{\partial\nu}(\eta\bar{\psi}^{\lambda'})\, \eta(r) r^\xi \sin(\xi\theta) \Big] \\
&= \lim_{\epsilon\to 0} \epsilon \int_0^\psi \Big[\frac{\partial}{\partial r}(\eta(r) r^\xi \sin(\xi\theta))\, \eta\bar{\psi}^{\lambda'} - \frac{\partial}{\partial r}(\eta\bar{\psi}^{\lambda'})\, \eta(r) r^\xi \sin(\xi\theta) \Big]\Big|_{r=\epsilon} d\theta.
\end{aligned}$$

By definition of $\psi^{\lambda'}$ we deduce that, for $\xi \neq \lambda'$,

$$\int_D (-\Delta + z)(\eta(r) r^\xi \sin(\xi\theta))\, \bar{w}_z^{\lambda'} = 0,$$

as $\int_0^\psi \sin(\xi\theta)\sin(\lambda'\theta) d\theta = 0$ if $\xi \neq \lambda'$.

In the case $\xi = \lambda'$ we have

$$\begin{aligned}
&\int_D (-\Delta + z)(\eta(r) r^{\lambda'} \sin(\lambda'\theta))\, (\eta\bar{\psi}^{\lambda'} - \bar{\phi}^{\lambda'}) \\
&\qquad = A_{\lambda'} \frac{\psi}{2} \lim_{\epsilon\to 0} \Big[2\lambda' e^{-\epsilon\sqrt{z}}\, Q(\sqrt{z}\epsilon) + \sqrt{z} e^{-\epsilon\sqrt{z}} \epsilon\, (Q(\sqrt{z}\epsilon) - Q'(\sqrt{z}\epsilon)) \Big] \\
&\qquad = A_{\lambda'} \psi \lambda' = 1.
\end{aligned}$$

This proves that $w_z^{\lambda'} = \eta\psi^{\lambda'} - \phi^{\lambda'}$.

Estimate on $\|w_z^{\lambda'}\|_{L^q_{-\mu}(D)}$. Observe that, as $z \in \pi^+ \cup S_A$, we can write $z = |z| e^{i\theta}$ with $\theta \in [-\theta_A, \theta_A]$, and we have $|e^{-qr\sqrt{\bar{z}}}| = e^{-qr|z|^{1/2}\cos(\theta/2)} \lesssim e^{-\gamma qr|z|^{1/2}}$ with $\gamma = \cos(\theta_A/2) > 0$. Hence we obtain

$$\|\psi^{\lambda'}\|^q_{L^q_{-\mu}(D)} \lesssim \int_0^1 e^{-\gamma qr|z|^{1/2}}\, |Q(\sqrt{\bar{z}} r)|^q\, r^{(-\lambda'-\mu)q+1}\, dr.$$

Making the change of variable $s = r|z|^{1/2}$ we obtain

$$\|\psi^{\lambda'}\|^q_{L^q_{-\mu}(D)} \lesssim |z|^{-\frac{1}{2}((-\mu-\lambda')q+2)} \int_0^{+\infty} e^{-\gamma qs} s^{(-\lambda'-\mu)q+1} |Q(s)|^q\, ds,$$

where the integral

$$\int_0^{+\infty} e^{-\gamma qs} s^{(-\lambda'-\mu)q+1} |Q(s)|^q\, ds < +\infty$$

as $(-\lambda' - \mu)q + 2 > 0$. Hence we have

$$\|\psi^{\lambda'}\|_{L^q_{-\mu}(D)} \lesssim |z|^{-\frac{1}{q}+\frac{\mu+\lambda'}{2}}. \tag{3.23}$$

Recall that

$$(-\Delta + \bar{z})(\eta\psi^{\lambda'}) = -\eta\, A_{\lambda'}\,(2\lambda' - 1 - 2m)\alpha_m\, r^{m-\lambda'-1}\, \bar{z}^{\frac{m+1}{2}}\, e^{-r\sqrt{\bar{z}}}\, \sin(\lambda'\theta) \\ +(-2\nabla\psi^{\lambda'}\cdot\nabla\eta - \psi^{\lambda'}\Delta\eta).$$

Hence, we see that $(-\Delta+\bar{z})(\eta\psi^{\lambda'}) \in L^q_{-\mu}(D)$ as $\lambda' < -\mu + 2 - \frac{2}{p}$ and $m \geq 1$. Making the change of variables $s = r|z|^{1/2}$ we obtain

$$\|e^{-r\sqrt{\bar{z}}} r^{m-\lambda'-1}\sin(\lambda'\theta)\|^q_{L^q_{-\mu}(D)} \lesssim \int_0^1 |e^{-qr\sqrt{\bar{z}}}| r^{(m-\mu-\lambda'-1)q+1}\, dr$$

$$\lesssim \Big(\int_0^{+\infty} e^{-\gamma q s} s^{(m-\mu-\lambda'-1)q+1} ds\Big)|z|^{-\frac{q}{2}(m-\mu-\lambda'-1)-1},$$

where the integral $\int_0^{+\infty} e^{-\gamma q s}\, s^{(m-\mu-\lambda'-1)q+1} ds < +\infty$. Hence we obtain

$$\|e^{-r\sqrt{\bar{z}}} r^{m-\lambda'-1}\, \bar{z}^{\frac{m+1}{2}}\, \sin(\lambda'\theta)\|_{L^q_{-\mu}(D)} \lesssim |z|^{-\frac{1}{2}((-\mu-\lambda'-2)+\frac{2}{q})}. \tag{3.24}$$

Let us denote by $r_{\lambda'} := (-2\nabla\psi^{\lambda'}\cdot\nabla\eta - \psi^{\lambda'}\Delta\eta)$. For $|z| \geq 1$, we have

$$\begin{aligned}
\|r_{\lambda'}\|^q_{L^q_{-\mu}(D)} \;&\lesssim\; \int_{r_0}^1 |e^{-rq\sqrt{\bar{z}}}|\, r^{(-\lambda'-\mu)q+1}|Q(r\sqrt{\bar{z}})|^q\, dr \\
&\quad + \int_{r_0}^1 |e^{-rq\sqrt{\bar{z}}}|\, r^{(-\lambda'-\mu-1)q+1}|Q(r\sqrt{\bar{z}})|^q\, dr \\
&\quad + \int_{r_0}^1 |e^{-rq\sqrt{\bar{z}}}|\, r^{(-\lambda'-\mu)q+1}|Q(r\sqrt{\bar{z}})|^q|z|^{\frac{q}{2}}\, dr \\
&\quad + \int_{r_0}^1 |e^{-rq\sqrt{\bar{z}}}|\, r^{(-\lambda'-\mu-2)q+1}|Q(r\sqrt{\bar{z}})|^q\, dr \\
&\quad + \int_{r_0}^1 |e^{-rq\sqrt{\bar{z}}}|\, r^{(-\lambda'-\mu)q+1}|Q'(r\sqrt{\bar{z}})|^q|z|^{\frac{q}{2}}\, dr \\
&\lesssim\; \int_{r_0}^1 e^{-\gamma r q|z|^{\frac{1}{2}}}\, |z|^{\frac{q}{2}}(|Q(r\sqrt{\bar{z}})|^q + |Q'(r\sqrt{\bar{z}})|^q)\, dr \\
&\quad + \int_{r_0}^1 e^{-\gamma r q|z|^{\frac{1}{2}}}\, |Q(r\sqrt{\bar{z}})|^q\, dr \\
&\lesssim\; e^{-\gamma r_0 q|z|^{\frac{1}{2}}}\, (|z|^{\frac{(l+1)q}{2}} + 1).
\end{aligned}$$

Hence we obtain

$$\|r_{\lambda'}\|_{L^q_{-\mu}(D)} \lesssim e^{-|z|^{1/2}\tilde{\gamma}}, \tag{3.25}$$

for some $\tilde{\gamma} > 0$.

We then deduce from (3.24) and (3.25) that

$$\|(-\Delta + \bar{z})(\eta\psi^{\lambda'})\|_{L^q_{-\mu}(D)} \lesssim |z|^{1-\frac{1}{q}+\frac{\mu+\lambda'}{2}}.$$

As $(-\Delta+\bar{z})(\eta\psi^{\lambda'}) \in L^q_{-\mu}(D)$, by Lemma 3.4 below, $\phi^{\lambda'} \in H^1_0(D) \hookrightarrow L^q_{-\mu}(D)$ satisfies the assumptions of Corollary 2.16, hence it satisfies

$$|z|\,\|\phi^{\lambda'}\|_{L^q_{-\mu}(D)} \lesssim \|(-\Delta + \bar{z})(\eta\psi^{\lambda'})\|_{L^q_{-\mu}(D)} \lesssim |z|^{1-\frac{1}{q}+\frac{\mu+\lambda'}{2}},$$

i.e.,

$$\|\phi^{\lambda'}\|_{L^q_{-\mu}(D)} \lesssim |z|^{-\frac{1}{q}+\frac{\mu+\lambda'}{2}}. \tag{3.26}$$

By (3.23) and (3.26) we deduce

$$\|w^{\lambda'}\|_{L^q_{-\mu}(D)} \lesssim |z|^{-\frac{1}{q}+\frac{\mu+\lambda'}{2}}$$

and by Hölder's inequality

$$|c_{\lambda'}| = \Big|\int_D g_1 \bar{w}^{\lambda'}\Big| \le \|g_1\|_{L^p_\mu(D)}\,\|w^{\lambda'}\|_{L^q_{-\mu}(D)} \lesssim |z|^{-\frac{1}{q}+\frac{\mu+\lambda'}{2}}\|g_1\|_{L^p_\mu(D)}.$$

Hence, using (3.11), we conclude that, for $|z| \ge 1$,

$$|c_{\lambda'}| \lesssim |z|^{-\frac{1}{q}+\frac{\mu+\lambda'}{2}}\|g\|_{L^p_\mu(\Omega)}, \tag{3.27}$$

which proves (c).

Step 4: Estimates (a) **and** (b) **on** v_R**.** By (3.19) we know that $v_R \in V^{2,p}(C) \cap H^1_0(C)$ with C the cone with interior angle ψ. By Lemma 3.2, to obtain the estimates (a) and (b), it remains to estimate $\|(-\Delta+z)v_R\|_{L^p_\mu(C)} + |z|\,\|v_R\|_{L^p_\mu(C)}$. We have

$$\begin{aligned}
&\|(-\Delta + z)v_R\|_{L^p_\mu(C)} + |z|\,\|v_R\|_{L^p_\mu(C)} \\
&\qquad \lesssim \|(-\Delta + z)v\|_{L^p_\mu(D)} + |z|\,\|v\|_{L^p_\mu(D)} \\
&\qquad + \sum_{\substack{0<\lambda'<2-\frac{2}{p}-\mu \\ \exists k\in\mathbb{N},\lambda'=k\lambda}} |c_{\lambda'}(z)|\,\|(-\Delta + z)(\eta(r)\,\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta))\|_{L^p_\mu(D)} \\
&\qquad + |z| \sum_{\substack{0<\lambda'<2-\frac{2}{p}-\mu \\ \exists k\in\mathbb{N},\lambda'=k\lambda}} |c_{\lambda'}(z)|\,\|\eta(r)\,\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta)\|_{L^p_\mu(D)}.
\end{aligned}$$

Recall that v is a solution of (3.10) and hence, by Corollary 2.14 and (3.11),

$$\|(-\Delta + z)v\|_{L^p_\mu(D)} + |z|\,\|v\|_{L^p_\mu(D)} \lesssim \|g_1\|_{L^p_\mu(D)} \lesssim \|g\|_{L^p_\mu(\Omega)}.$$

It remains to estimate the two last terms. Since $\lambda' p + \mu p + 2 > 0$, we have

$$\begin{aligned}
&\|\eta(r)\,\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta)\|^p_{L^p_\mu(D)}\\
&\lesssim \int_0^1 |P_{\lambda'}(r\sqrt{z})\,e^{-\gamma r|z|^{\frac{1}{2}}}\,r^{\lambda'}\sin(\lambda'\theta)|^p r^{\mu p+1}\,dr\\
&\lesssim \int_0^\infty |P_{\lambda'}(s)|^p\,e^{-\gamma sp}\,s^{\lambda' p+\mu p+1}\,ds\,|z|^{-\frac{1}{2}(\lambda' p+\mu p+2)} \lesssim |z|^{-\frac{1}{2}(\lambda' p+\mu p+2)}.
\end{aligned}$$

We then deduce from (3.27) that

$$\begin{aligned}
&|z| \sum_{0<\lambda'<2-\frac{2}{p}-\mu} |c_{\lambda'}(z)|\,\|\eta(r)\,\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta)\|_{L^p_\mu(D)}\\
&\lesssim \sum_{0<\lambda'<2-\frac{2}{p}-\mu} |z|^{1-\frac{1}{q}+\frac{\lambda'+\mu}{2}}\,\|g\|_{L^p_\mu(\Omega)}\,|z|^{-\frac{1}{2}(\lambda'+\mu)-\frac{1}{p}} \lesssim \|g\|_{L^p_\mu(\Omega)}.
\end{aligned}$$

Moreover, if $l_{\lambda'} \geq 2$, we have

$$\|(-\Delta+z)(\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta))\|^p_{L^p_\mu(D)} = \|z\,e^{-r\sqrt{z}}\,r^{\lambda'}\,\sin(\lambda'\theta)\tilde{P}(r\sqrt{z})\|^p_{L^p_\mu(D)}$$

with $\tilde{P}$ a polynomial function of degree $l_{\lambda'}-2$. As before we obtain

$$\|(-\Delta+z)(\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta))\|^p_{L^p_\mu(D)} \lesssim |z|^{-\frac{1}{2}(\lambda' p+\mu p+2-2p)}$$

since $\lambda' p + \mu p + 2 > 0$. Hence we have

$$\|(-\Delta+z)(\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta))\|_{L^p_\mu(D)} \lesssim |z|^{1-\frac{1}{p}-\frac{\lambda'+\mu}{2}}. \tag{3.28}$$

If $l_{\lambda'} = 1$, then we have

$$(-\Delta+z)(\psi_{\lambda'}(r\sqrt{z})\,r^{\lambda'}\,\sin(\lambda'\theta)) = (2\lambda'+1)\sqrt{z}\,e^{-r\sqrt{z}}\,r^{\lambda'-1}\,\sin(\lambda'\theta),$$

and the same arguments as before yield (3.28) since here $\lambda' - 1 + \mu + \frac{2}{p} > 0$ (reminding the reader that if $l_{\lambda'} = 1$, then $2 - \mu - \frac{2}{p} - \lambda' < 1$).

As the function η does not interfere the result follows from (3.27). $\square$

Lemma 3.4. *Let $p \geq 2$, $D = \{(r\cos\theta, r\sin\theta) : 0 < r < 1,\ 0 < \theta < \psi\}$ and μ satisfy (2.2). Let $\phi \in H^1_0(D)$ and $v \in D(\Delta_{p,\mu})$; then we have*

$$\int_D (\Delta v\,\bar{\phi} + \nabla v \cdot \nabla\bar{\phi}) = 0. \tag{3.29}$$

Proof. For all $\epsilon \in (0,1)$, we set $D_\epsilon = \{x \in D\ :\ r(x) > \epsilon\}$. Since v is regular far from the origin, we can apply Green's formula on D_ϵ and find

$$\int_{D_\epsilon} (\Delta v\bar{\phi} + \nabla v \cdot \nabla\bar{\phi}) = -\epsilon \int_{r=\epsilon} \partial_r v\bar{\phi}\,d\theta. \tag{3.30}$$

Since Hölder's inequality and Lemma 2.1 guarantee that the integrand of the left-hand side of (3.30) is integrable on the whole domain D, Lebesgue's convergence theorem allows us to conclude that the left-hand side of (3.30) tends to the left-hand side of (3.29) as $\epsilon \to 0$. Hence it remains to show that the right-hand side of (3.30) tends to 0.

Now denote by $\hat{A}$ the annulus defined as follows:

$$\hat{A} = \{(r\cos\theta, r\sin\theta) : 1/2 < r < 1,\ 0 < \theta < \psi\}.$$

For any $\hat{v} \in H^1(\hat{A})$ such that $\Delta u \in L^p(\hat{A})$ and any $\hat{\phi} \in H^1(\hat{A})$ such that $\hat{\phi} = 0$ on $\theta = 0$ and $\theta = \psi$, let us show that

$$\Big|\int_{\hat{r}=1} \partial_r \hat{v}\overline{\hat{\phi}}\, d\hat{\theta}\Big| \lesssim |\hat{\phi}|_{H^1(\hat{A})}|\hat{v}|_{H^1(\hat{A})} + \|\hat{\phi}\|_{L^q_{-\mu}(\hat{A})}\|\Delta\hat{v}\|_{L^p_\mu(\hat{A})}. \tag{3.31}$$

Indeed, by taking an arbitrary cut-off function $\eta \in \mathcal{D}(\mathbb{R})$ such that $\eta(1) = 1$ and $\eta(1/2) = 0$, Green's formula yields

$$\int_{\hat{r}=1} \partial_r \hat{v}\overline{\hat{\phi}}\, d\hat{\theta} = \int_{\hat{A}} (\Delta\hat{v}(\eta\overline{\hat{\phi}}) + \nabla\hat{v}\cdot\nabla(\eta\overline{\hat{\phi}})).$$

By Hölder's inequality we obtain

$$\Big|\int_{\hat{r}=1} \partial_r \hat{v}\overline{\hat{\phi}}\, d\hat{\theta}\Big| \lesssim \|\hat{\phi}\|_{H^1(\hat{A})}|\hat{v}|_{H^1(\hat{A})} + \|\hat{\phi}\|_{L^q(\hat{A})}\|\Delta\hat{v}\|_{L^p(\hat{A})}.$$

Since Poincaré's inequality guarantees that

$$\|\hat{\phi}\|_{H^1(\hat{A})} \lesssim |\hat{\phi}|_{H^1(\hat{A})},$$

and since $\hat{r}$ is equivalent to 1 on $\hat{A}$, the previous estimate leads to (3.31).

Now for all $j \in \mathbb{N}$, we set

$$A_j = \{(r\cos\theta, r\sin\theta) : \epsilon 2^{-(j+1)} < r < \epsilon 2^{-j},\ 0 < \theta < \psi\}.$$

We then see that the linear mapping $\hat{x} \to x = \epsilon 2^{-j}\hat{x}$, is a bijection from $\hat{A}$ into A_j.

Now we perform the change of variables $x = \epsilon 2^{-j}\hat{x}$ and using (3.31), we obtain

$$\Big|\int_{r=\epsilon 2^{-j}} \partial_r v\bar{\phi} r\, d\theta\Big| \lesssim |\phi|_{H^1(\hat{A}_j)}|v|_{H^1(\hat{A}_j)} + \|\phi\|_{L^q_{-\mu}(A_j)}\|\Delta v\|_{L^p_\mu(A_j)}. \tag{3.32}$$

Since the measure of A_j tends to zero as j goes to infinity, we deduce from this estimate that

$$I_j(\epsilon) := \int_{r=\epsilon 2^{-j}} \partial_r v\bar{\phi} r\, d\theta \to 0 \text{ as } j \to \infty.$$

Moreover, Green's formula on $B_j = \{(r\cos\theta, r\sin\theta) : \epsilon 2^{-j} < r < \epsilon,\, 0 < \theta < \psi\}$ yields

$$I_0(\epsilon) = \int_{B_j} (\Delta v \bar{\phi} + \nabla v \cdot \nabla \bar{\phi}) + I_j(\epsilon).$$

Accordingly, passing to the limit in j, we obtain that

$$I_0(\epsilon) = \int_{r=\epsilon} \partial_r v \bar{\phi} r \, d\theta = \int_{D(\epsilon)} (\Delta v \bar{\phi} + \nabla v \cdot \nabla \bar{\phi}),$$

where we recall that $D(\epsilon) = \{(r\cos\theta, r\sin\theta) : 0 < r < \epsilon,\, 0 < \theta < \psi\}$. Passing to the limit as ϵ goes to zero, by Lebesgue's bounded convergence theorem, we find that $I_0(\epsilon) \to 0$ as $\epsilon \to 0$. This proves (3.29) thanks to (3.30). □

Lemma 3.5. *With the notation of the proof of Theorem* 3.3, *the function* $\eta\psi^{\lambda'} - \phi^{\lambda'}$ *(with* $\psi^{\lambda'}$ *and* $\phi^{\lambda'}$ *defined by* (3.20) *and* (3.21)) *satisfies, for all* $v \in V_\mu^{2,p}(D) \cap D(\Delta_{p,\mu})$,

$$\int_D (-\Delta v + z\, v)\,(\eta \bar{\psi}^{\lambda'} - \bar{\phi}^{\lambda'}) = 0. \tag{3.33}$$

Proof. For brevity we now skip the indices z and λ' and we use the notation of the proof of Lemma 3.4. Thanks to (3.29), we may write

$$\int_D (-\Delta v + z\, v)\bar{\phi} = \int_D (\nabla v \cdot \nabla \bar{\phi} + z v \bar{\phi}),$$

and using (3.21), we obtain

$$\int_D (-\Delta v + z\, v)\bar{\phi} = \int_D v(-\Delta + z)(\eta\bar{\psi}).$$

Consequently, if we prove that

$$\int_D (-\Delta v + z\, v)\eta\bar{\psi} = \int_D v(-\Delta + z)(\eta\bar{\psi}), \tag{3.34}$$

then the difference between these two last identities yields (3.33).

It remains to prove (3.34). For that purpose, we again apply Green's formula in $D_\epsilon = \{x \in D : r(x) > \epsilon\}$, and find

$$\int_{D_\epsilon} (-\Delta v + z\, v)\eta\bar{\psi} = \int_{D_\epsilon} v(-\Delta + z)(\eta\bar{\psi}) + J_\epsilon,$$

where

$$J_\epsilon = \epsilon \int_{r=\epsilon} (v \partial_r \bar{\psi} - \partial_r v \bar{\psi})\, d\theta.$$

Hence as before it suffices to show that

$$J_\epsilon \to 0 \text{ as } \epsilon \to 0. \tag{3.35}$$

For that last purpose we remark that

$$\partial_r\psi = A[\sqrt{\bar z}r\,\frac{d}{ds}Q(\sqrt{\bar z}r) - \sqrt{\bar z}r\,Q(\sqrt{\bar z}r) - \lambda'\,Q(\sqrt{\bar z}r)]\,e^{-\sqrt{\bar z}r}r^{-\lambda'-1}\sin(\lambda'\theta);$$

consequently,

$$|J_\epsilon| \lesssim \epsilon^{-\lambda'}\int_{r=\epsilon}|v(\epsilon,\theta)|\,d\theta + \epsilon^{-\lambda'+1}\int_{r=\epsilon}|\partial_r v(\epsilon,\theta)|\,d\theta. \tag{3.36}$$

Now we estimate each term of this right-hand side separately. For the first one, we perform the change of variables $x = \epsilon\hat x$ that maps $\hat A$ into A_0, and by setting $\hat v(\hat x) = v(x)$, we obtain

$$\int_{r=\epsilon}|v(\epsilon,\theta)|\,d\theta = \int_{\hat r=1}|\hat v(1,\hat\theta)|\,d\hat\theta.$$

On $\hat A$, we notice that $\hat v$ belongs to $W^{1,p}(\hat A)$, and by a standard trace theorem, we deduce that

$$\int_{\hat r=1}|\hat v(1,\hat\theta)|\,d\hat\theta \lesssim (\int_{\hat r=1}|\hat v(1,\hat\theta)|^p\,d\hat\theta)^{\frac1p} \lesssim \|\hat v\|_{W^{1,p}(\hat A)}.$$

Since $\hat v = 0$ on a part of the boundary of $\hat A$, $\|\hat v\|_{W^{1,p}(\hat A)} \lesssim |\hat v|_{W^{1,p}(\hat A)}$, and therefore we obtain

$$\int_{\hat r=1}|\hat v(1,\hat\theta)|\,d\hat\theta \lesssim |\hat v|_{W^{1,p}(\hat A)}.$$

Finally, as $\hat r$ is equivalent to 1 on $\hat A$, we arrive at

$$\int_{\hat r=1}|\hat v(1,\hat\theta)|\,d\hat\theta \lesssim \Big(\int_{\hat A}\hat r^{(\mu-1)p}|\nabla\hat v|^p\Big)^{\frac1p}.$$

Going back to A_0, we have proved that

$$\int_{r=\epsilon}|v(\epsilon,\theta)|\,d\theta \lesssim \epsilon^{2-\frac2p-\mu}\Big(\int_{A_0}r^{(\mu-1)p}|\nabla v|^p\Big)^{\frac1p} \lesssim \epsilon^{2-\frac2p-\mu}\|v\|_{V^{2,p}_\mu(D)}.$$

In a similar way we show that

$$\int_{r=\epsilon}|\partial_r v(\epsilon,\theta)|\,d\theta \lesssim \epsilon^{1-\frac2p-\mu}\Big(\int_{A_0}r^{\mu p}|\nabla\partial_r v|^p\Big)^{\frac1p} \lesssim \epsilon^{1-\frac2p-\mu}\|v\|_{V^{2,p}_\mu(D)}.$$

These two estimates in (3.36) lead to

$$|J_\epsilon| \lesssim \epsilon^{2-\frac2p-\mu-\lambda'}\|v\|_{V^{2,p}_\mu(D)}.$$

Since $2 - \frac{2}{p} - \mu - \lambda'$ is positive, we obtain (3.35). $\square$

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