Advances in Differential Equations Volume 16, Numbers 1–2 (2011), 145–164

ASYMPTOTIC NON-DEGENERACY OF THE MULTIPLE BLOW-UP SOLUTIONS TO THE GEL'FAND PROBLEM IN TWO SPACE DIMENSIONS

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(Submitted by: Jean-Michel Coron)

Abstract. We consider a sequence of solutions u_n of the problem

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $\lambda = \{\lambda_n\}_{n\in\mathbf{N}}$ and blowing up at m points $\kappa_1, \ldots, \kappa_m$ in Ω. Under some non-degeneracy assumption on some suitable finite-dimensional function (related to $\kappa_1, \ldots, \kappa_m$) we show that u_n is non-degenerate for n large enough.

1. Introduction

This paper is concerned with the *the Gel'fand problem*:

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. Let $\{\lambda_n\}_{n\in\mathbf{N}}$ be a sequence satisfying $\lambda_n \downarrow 0$ and $u_n = u_n(x)$ be a solution to (1.1) for $\lambda = \lambda_n$. We focus our attention on the asymptotic behavior of u_n as $n \longrightarrow \infty$ and establish the property of *the asymptotic non-degeneracy* of u_n.

Accepted for publication: August 2010.
AMS Subject Classifications: 35B40, 35J08, 35J20, 35J60, 35Q35, 76B47, 76F99.
[1]Supported by Grant-in-Aid for Scientific Research (C) (No.19540222 and 22540231), Japan Society for the Promotion of Science

The fundamental facts concerning the asymptotic behavior of u_n are established by the pioneering work of Nagasaki and Suzuki:

Theorem 1.1 ([10])**.** *Let $\Sigma_n = \lambda_n \int_\Omega e^{u_n}$. Then $\{\Sigma_n\}$ accumulate to Σ_∞, which is either*

(i) 0,
(ii) $8\pi m$ *for some positive integer* m,
(iii) $+\infty$.

According to these cases, the (sub-)sequence of solutions $\{u_n\}$ behave as follows:

(i) *uniform convergence to* 0,
(ii) *m-point blow-up, that is, there is a blow-up set $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\} \subset \Omega$ of distinct m-points such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for every $\omega \subset\subset \overline{\Omega}\backslash\mathcal{S}$ and $\{u_n(x)\}$ have a limit for $x \in \overline{\Omega}\backslash\mathcal{S}$ while $u_n|_{\mathcal{S}} \longrightarrow +\infty$,*
(iii) *entire blow-up, that is, $u_n(x) \longrightarrow +\infty$ for every $x \in \Omega$.*

Moreover, in the case (ii)*, the limiting function u_∞ has the form*

$$u_\infty(x) = 8\pi \sum_{j=1}^{m} G(x, \kappa_j), \tag{1.2}$$

where $G(x,y)$ is the Green's function of $-\Delta$ under the Dirichlet condition, that is,

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Omega, \qquad G(\cdot, y) = 0 \quad \text{on } \partial\Omega.$$

Next, the blow-up points κ_i $(i = 1, \dots, m)$ satisfy the relations

$$\nabla \Big[K(x, \kappa_j) + \sum_{i \neq j} G(x, \kappa_i) \Big] \Big|_{x=\kappa_j} = 0 \quad (1 \leqq j \leqq m), \tag{1.3}$$

where $K(x,y) = G(x,y) - \frac{1}{2\pi} \log |x - y|^{-1}$.

Here we introduce *the Hamiltonian* function

$$H^m(x_1, \dots, x_m) = \frac{1}{2} \sum_{i=1}^{m} R(x_i) + \frac{1}{2} \sum_{\substack{1 \leq i,j \leq m \\ i \neq j}} G(x_i, x_j),$$

where $R(x) = K(x,x)$ is the Robin function of Ω. Then the relation (1.3) means that $\mathcal{S} \in \Omega^m$ is *a critical point* of the $2m$-valued function H^m. Therefore *the limit function of $\{u_n\}$ blows up at the critical point of the Hamiltonian H^m*. The main result of this paper shows a deeper link between H^m and $\{u_n\}$.

Theorem 1.2 (Main Theorem)**.** *Assume* (ii) *of Theorem* 1.1 *and suppose that* $\mathcal{S}$ *is a* non-degenerate *critical point of* H^m*. Then the associated* u_n *is* non-degenerate *for* $n \gg 1$*; that is, the linearized problem of* (1.1) *around* u_n*,*

$$-\Delta v = \lambda_n e^{u_n} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

has no non-trivial solution.

The above theorem has been already established by Gladiali and Grossi [6] for the case $m = 1$. There, the conclusion is obtained using a contradiction argument and some “Pohozaev-type” identities. If $m > 1$ the problem is much more complicated and we need new ideas to derive the claim. In this paper we localize the relations around each point of $\mathcal{S}$ and then integrate them. This flexible method seems to enable us to handle other problems relative to (1.1). Note that this approach provides a simpler proof of the result in [6].

Remark 1.3. Our problem has a variational structure. Let us introduce the functional

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega e^u dx$$

over $H_0^1(\Omega)$, which leads (1.1) as an Euler-Lagrange equation. Then Theorem 1.2 says the following:

> Assume (ii) of Theorem 1.1 and suppose that $\mathcal{S}$ is a non-degenerate critical point of H^m. Then u_n is a non-degenerate critical point of F_{λ_n} for n large enough.

Recall that a critical point u of a C^2 functional $F(u)$ over a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ is non-degenerate if $D^2F(u)[\cdot,\cdot]$ is a non-degenerate bilinear form over H. This property is equivalent to the invertibility of the self-adjoint operator A_u over H defined by $D^2F(u)[\varphi,\psi] = \langle A_u\varphi,\psi\rangle$. In our case $H = H_0^1(\Omega)$, $\langle u, v\rangle = \int_\Omega \nabla u\cdot\nabla v\, dx$, and $A_u\varphi = \varphi - \lambda G(e^u\varphi)$, where $G = (-\Delta)^{-1} : H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)$. This A_u has the form $I -$ (compact) provided with the closed range. Therefore the invertibility of A_u is reduced to $\operatorname{Ker} A_u = \{0\}$, which is equivalent to the non-existence of non-trivial solutions for (1.4) described above.

Concerning the existence of the solutions satisfying the assumption of Theorem 1.2, we recall the following result:

Theorem 1.4 ([1])**.** *For every non-degenerate critical point* $\mathcal{S}$ *of* H^m*, there exist* $\lambda_0 > 0$ *and a one-parameter family of solutions* $\{u_\lambda\}_{0<\lambda\le\lambda_0}$ *for* (1.1) *such that* $\lim_{\lambda\longrightarrow 0} u_\lambda = u_\infty$ *in* $C^{2,\alpha}_{\mathrm{loc}}(\Omega\backslash\mathcal{S})$*, where* u_∞ *is defined by* (1.2).

Recently such a one-parameter family of solutions was constructed under a weaker condition [4, 5]. It would be interesting to find some relationships between the topological structure of critical points of F_{λ_n} around u_n, and the corresponding one of H^m around $\mathcal{S}$.

2. Preliminaries

Similarly to [6], we prove the main theorem arguing by contradiction. For this purpose we assume the existence of a sequence $\{v_n\}$ of non-trivial solutions of (1.4) as $n \longrightarrow \infty$. Without loss of generality we may assume

$$\|v_n\|_{L^\infty(\Omega)} \equiv 1 \tag{2.1}$$

and we will get the contradiction $\|v_n\|_{L^\infty(\Omega)} \longrightarrow 0$ after several calculations.

Fix a sufficiently small positive number $\bar{R} > 0$ and choose a sequence $\{x_{j,n}\}$ for each $\kappa_j \in \mathcal{S}$ satisfying

$$x_{j,n} \longrightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_{\bar{R}}(x_{j,n})} u_n(x) \longrightarrow \infty$$

as $n \longrightarrow \infty$; see [9]. Then we re-scale u_n and v_n around $x_{j,n}$ as follows:

$$\begin{aligned} \widetilde{u}_{j,n}(\widetilde{x}) &= u_n(\delta_{j,n}\widetilde{x} + x_{j,n}) - u_n(x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \\ \widetilde{v}_{j,n}(\widetilde{x}) &= v_n(\delta_{j,n}\widetilde{x} + x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \end{aligned} \tag{2.2}$$

where the scaling parameter $\delta_{j,n}$ is determined by

$$\lambda_n e^{u_n(x_{j,n})} \delta_{j,n}^2 = 1.$$

Then $\widetilde{u}_{j,n}$ and $\widetilde{v}_{j,n}$ satisfy

$$\begin{cases} -\Delta \widetilde{u}_{j,n} = e^{\widetilde{u}_{j,n}}, & \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \\ \widetilde{u}_{j,n} \leqq \widetilde{u}_{j,n}(0) = 0, & \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \end{cases}$$

$$\begin{cases} -\Delta \widetilde{v}_{j,n} = e^{\widetilde{u}_{j,n}} \widetilde{v}_{j,n}, & \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \\ \|\widetilde{v}_{j,n}\|_{L^\infty(B_{\frac{\bar{R}}{\delta_{j,n}}}(0))} \leqq 1. \end{cases}$$

Using standard arguments ([6]) based on the classification results in [3, 1], we are able to get $\boldsymbol{a}_j = (a_{j,1}, a_{j,2}) \in \mathbf{R}^2$, $b_j \in \mathbf{R}$ for each j, and subsequences of $\widetilde{u}_{j,n}$ and $\widetilde{v}_{j,n}$ (denoted by the same symbol) satisfying

$$\widetilde{u}_{j,n} \longrightarrow U(\widetilde{x}) = \log \frac{1}{(1 + \frac{|\widetilde{x}|^2}{8})^2},$$

$$\widetilde{v}_{j,n} \longrightarrow \frac{\boldsymbol{a}_j \cdot \widetilde{x}}{8+|\widetilde{x}|^2} + b_j \frac{8-|\widetilde{x}|^2}{8+|\widetilde{x}|^2} = \boldsymbol{a}_j \cdot \nabla(-\frac{1}{4}U) + \frac{b_j}{2}\{\widetilde{\boldsymbol{x}} \cdot \nabla U + 2\},$$

locally uniformly in $\mathbf{R}^2$. During the proof of Theorem 1.2, we shall prove $\boldsymbol{a}_j = 0$ and $b_j = 0$ for every j.

Under the above preparations, we get the following asymptotic formula which is one of key lemmas in our argument:

Lemma 2.1 (cf. [6, (3.14)]). *There exist $C_j > 0$ $(j = 1, \ldots, m)$ and a subsequence of v_n satisfying*

$$\frac{v_n}{\lambda_n^{\frac{1}{2}}} \longrightarrow 2\pi \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, \kappa_j) \quad \text{in } C^1(\overline{\Omega} \setminus \cup_{j=1}^m B_{2\bar{R}}(\kappa_j)). \tag{2.3}$$

Here we note that the C_j's are the same constants appearing in Corollary 4.3. We shall prove Lemma 2.1 in Section 4.

Next we show the following fact:

Lemma 2.2. *If $\mathcal{S}$ is a non-degenerate critical point of H^m, we have $\boldsymbol{a}_j = 0$ for every $j = 1, \ldots, m$ in Lemma* 2.1.

This lemma is obtained by the asymptotic formula (2.3) and the Rellich-Pohozaev-type identity concerning the Green's function stated below.

Proposition 2.3. *Take $z_k \in \Omega$ $(k = 1, 2, 3)$ and $R > 0$ such that $B_R(z_1) \subset\subset \Omega$ and $z_2, z_3 \notin \overline{B_R(z_1)}$. Set*

$$I_{ij}(z_1, z_2, z_3) := \int_{\partial B_R(z_1)} \left\{ \frac{\partial}{\partial \nu_x} G_{x_i}(x, z_2) G_{y_j}(x, z_3) - G_{x_i}(x, z_2) \frac{\partial}{\partial \nu_x} G_{y_j}(x, z_3) \right\} d\sigma_x$$

for $i, j = 1, 2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, it holds that

$$I_{ij}(z_1, z_2, z_3) = \begin{cases} 0 & (z_1 \neq z_2,\ z_1 \neq z_3) \\ \frac{1}{2} R_{x_i x_j}(z_1) & (z_1 = z_2 = z_3) \\ G_{x_i y_j}(z_1, z_3) & (z_1 = z_2,\ z_1 \neq z_3) \\ G_{x_i x_j}(z_1, z_2) & (z_1 \neq z_2,\ z_1 = z_3). \end{cases} \tag{2.4}$$

It is not difficult to see that the definition of I_{ij} is independent of R (Remark 3.1).

The second case of (2.4) is a *localized* version of the known identity

$$-\int_{\partial\Omega} G_{x_i}(x, y) \frac{\partial}{\partial \nu_x} G_{y_j}(x, y) d\sigma_x = \frac{1}{2} R_{x_i x_j}(y)$$

(see [6, Lemma 7] for example), while the other cases describe the correlation of the singularities of the Green's function between $\{z_1, z_2, z_3\}$. We shall prove Proposition 2.3 in Section 3. Here we show Lemma 2.2 assuming Lemma 2.1 and Proposition 2.3.

Proof of Lemma 2.2. Fix $\kappa_j \in \mathcal{S}$ and $R > 2\bar{R} > 0$ satisfying $B_R(\kappa_j) \subset\subset \Omega$ and $\overline{B_R(\kappa_j)} \cap \mathcal{S} = \kappa_j$. Differentiating equation (1.1) with respect to x_i, we get the identity

$$-\Delta u_{x_i} = \lambda e^u u_{x_i}, \tag{2.5}$$

which means that u_{x_i} ($i = 1$, 2) is a solution to (1.4) except for the boundary condition. Therefore, using Green's identity we have

$$\begin{aligned}&\int_{\partial B_R(\kappa_j)} \Big(\frac{\partial}{\partial \nu}(u_n)_{x_i} v_n - (u_n)_{x_i} \frac{\partial}{\partial \nu} v_n\Big) d\sigma \\ &= \int_{B_R(\kappa_j)} (\Delta (u_n)_{x_i} v_n - (u_n)_{x_i} \Delta v_n)\, dx = 0.\end{aligned}$$

From (ii) of Theorem 1.1 and the elliptic estimate, we have that

$$(u_n)_{x_i} \longrightarrow 8\pi \sum_{j=1}^m G_{x_i}(x, \kappa_j) \quad \text{in } C^1(\overline{\omega})$$

for every $\omega \subset\subset \overline{\Omega} \backslash \mathcal{S}$. Then the relation (2.3) implies

$$\begin{aligned}0 &= \int_{\partial B_R(\kappa_j)} \Big\{\frac{\partial}{\partial \nu}(u_n)_{x_i} \cdot \frac{v_n}{\lambda_n^{\frac{1}{2}}} - (u_n)_{x_i} \frac{\partial}{\partial \nu}\Big(\frac{v_n}{\lambda_n^{\frac{1}{2}}}\Big)\Big\} d\sigma \\ &\longrightarrow \int_{\partial B_R(\kappa_j)} \Big\{\frac{\partial}{\partial \nu} 8\pi \sum_{k=1}^m G_{x_i}(x, \kappa_k) \cdot 2\pi \sum_{l=1}^m C_l \boldsymbol{a}_l \cdot \nabla_y G(x, \kappa_l) \\ &\qquad - 8\pi \sum_{k=1}^m G_{x_i}(x, \kappa_k) \frac{\partial}{\partial \nu} 2\pi \sum_{l=1}^m C_l \boldsymbol{a}_l \cdot \nabla_y G(x, \kappa_l)\Big\} d\sigma_x \\ &= 16\pi^2 \sum_{1 \le k, l \le m} C_l \sum_{i'=1}^2 a_{l,i'} I_{ii'}(\kappa_j, \kappa_k, \kappa_l) \\ &= 16\pi^2 \sum_{\substack{1 \le l \le m \\ i'=1,2}} \Big(\sum_{1 \le k \le m} I_{ii'}(\kappa_j, \kappa_k, \kappa_l)\Big) C_l a_{l,i'}.\end{aligned}$$

Here we use Proposition 2.3 and get

$$\begin{aligned}
&\sum_{1\le k\le m} I_{ii'}(\kappa_j,\kappa_k,\kappa_l)\\
&=\begin{cases} \frac{1}{2}R_{x_ix_{i'}}(\kappa_j)+\sum\limits_{\substack{1\le k\le m\\ k\ne j}} G_{x_ix_{i'}}(\kappa_j,\kappa_k), & (j=l)\\ G_{x_iy_{i'}}(\kappa_j,\kappa_l), & (j\ne l)\end{cases}\\
&=H^m_{x_{j,i}x_{l,i'}}(x_1,\ldots,x_m)\Big|_{(x_1,\ldots,x_m)=(\kappa_1,\ldots,\kappa_m)},
\end{aligned}$$

which implies

$$0=16\pi^2 \text{ Hess } (H^m)|_{(x_1,\ldots,x_m)=(\kappa_1,\ldots,\kappa_m)} \cdot^t (C_1\boldsymbol{a}_1,\ldots,C_m\boldsymbol{a}_m)$$

where Hess (H^m) denotes the Hessian of H^m. Since $(\kappa_1,\ldots,\kappa_m)$ is a non-degenerate critical point of H^m, this Hess(H^m) is invertible. Then we conclude $\boldsymbol{a}_j=0$ for every $j=1,\ldots,m$ by $C_j>0$. □

The rest of this paper is organized as follows: In Section 3 we prove Proposition 2.3, that is, the localized Rellich-Pohozaev-type identity for Green's function and some variants. In Section 4 the asymptotic formula for v_n (Lemma 2.1) is proved. The proof of the main theorem, Theorem 1.2, is completed in Section 5.

3. Rellich-Pohozaev-type identities for the Green's functions

Proof of Proposition 2.3. We divide the proof into four cases.

Case 1: $z_1\ne z_2$ and $z_1\ne z_3$. In this case, $G_{x_i}(\cdot,z_2)$ and $G_{y_j}(\cdot,z_3)$ are harmonic in $B_R(z_1)$ and we get $I_{ij}=0$ from Green's formula.

Remark 3.1. Using a similar argument, we are able to see that the value of I_{ij} is independent of the choice of small R.

Case 2: $z_1=z_2=z_3$. By Remark 3.1, it holds that

$$\begin{aligned}
&I_{ij}(z_1,z_1,z_1)\\
&=\lim_{r\longrightarrow 0}\int_{\partial B_r(z_1)}\left\{\frac{\partial}{\partial\nu_x}G_{x_i}(x,z_1)G_{y_j}(x,z_1)-G_{x_i}(x,z_1)\frac{\partial}{\partial\nu_x}G_{y_j}(x,z_1)\right\}d\sigma_x.
\end{aligned}$$

Writing $|x-z_1|=r$ and $\nu=(\nu_1,\nu_2)=(\cos\theta,\sin\theta)$, we have $x-z_1=r\nu$ and

$$G_{x_i}(x,z_1)=-\frac{1}{2\pi}\frac{\nu_i}{r}+K_{x_i}(x,z_1),\ \frac{\partial}{\partial\nu_x}G_{x_i}(x,z_1)=\frac{1}{2\pi}\frac{\nu_i}{r^2}+\frac{\partial}{\partial\nu_x}K_{x_i}(x,z_1),$$

$$G_{y_j}(x,z_1)=\frac{1}{2\pi}\frac{\nu_j}{r}+K_{y_j}(x,z_1),\ \frac{\partial}{\partial\nu_x}G_{y_j}(x,z_1)=-\frac{1}{2\pi}\frac{\nu_i}{r^2}+\frac{\partial}{\partial\nu_x}K_{y_j}(x,z_1).$$

Therefore,

$$\begin{aligned}&\int_{\partial B_r(z_1)}\left\{\frac{\partial}{\partial\nu_x}G_{x_i}(x,z_1)G_{y_j}(x,z_1)-G_{x_i}(x,z_1)\frac{\partial}{\partial\nu_x}G_{y_j}(x,z_1)\right\}d\sigma_x\\&=\frac{1}{2\pi r}\int_0^{2\pi}\left\{\nu_iK_{y_j}(x,z_1)+K_{x_i}(x,z_1)\nu_j\right\}d\theta\\&\quad+\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial}{\partial\nu_x}K_{x_i}(x,z_1)\nu_j+\nu_i\frac{\partial}{\partial\nu_x}K_{y_j}(x,z_1)\right\}d\theta+o(1)\end{aligned}$$

as $r\longrightarrow 0$. Since, from the mean value theorem,

$$K_{y_j}(x,z_1)=K_{y_j}(z_1,z_1)+(x-z_1)\cdot\nabla_xK_{y_j}(z_1+\xi(x-z_1),z_1)$$

for some $\xi=\xi(x)\in[0,1]$, it holds that

$$\begin{aligned}&\frac{1}{2\pi r}\int_0^{2\pi}\nu_iK_{y_j}(x,z_1)d\theta=\frac{1}{2\pi r}K_{y_j}(z_1,z_1)\int_0^{2\pi}\nu_i\,d\theta\\&+\frac{1}{2\pi}\int_0^{2\pi}\nu_i\left\{\nu_1K_{x_1y_j}(z_1+\xi(x-z_1),z_1)+\nu_2K_{x_2y_j}(z_1+\xi(x-z_1),x_1)\right\}d\theta\\&\longrightarrow\frac{1}{2\pi}\left\{K_{x_1y_j}(z_1,z_1)\int_0^{2\pi}\nu_i\nu_1\,d\theta+K_{x_2y_j}(z_1,z_1)\int_0^{2\pi}\nu_i\nu_2\,d\theta\right\}\\&=\frac{1}{2}K_{x_iy_j}(z_1,z_1)\end{aligned}$$

as $r\longrightarrow 0$. Similarly, we get

$$\frac{1}{2\pi r}\int_0^{2\pi}\nu_jK_{x_i}(x,z_1)d\theta\longrightarrow\frac{1}{2}K_{x_jx_i}(z_1,z_1).$$

On the other hand,

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\nu_x}K_{x_i}(x,z_1)\nu_j\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left\{\nu_1K_{x_1x_i}(x,z_1)+\nu_2K_{x_2x_i}(x,z_1)\right\}\nu_j\,d\theta\\&\longrightarrow\frac{1}{2\pi}K_{x_jx_i}(z_1,z_1)\int_0^{2\pi}\nu_j^2\,d\theta=\frac{1}{2}K_{x_jx_i}(z_1,z_1)\end{aligned}$$

and similarly

$$\frac{1}{2\pi}\int_0^{2\pi}\nu_i\frac{\partial}{\partial\nu_x}K_{y_j}(x,z_1)d\theta\longrightarrow\frac{1}{2}K_{x_iy_j}(z_1,z_1).$$

Combining these relations, we get

$$\begin{aligned} I_{ij} =& \frac{1}{2}K_{x_iy_j}(z_1,z_1) + \frac{1}{2}K_{x_jx_i}(z_1,z_1) + \frac{1}{2}K_{x_jx_i}(z_1,z_1) + \frac{1}{2}K_{x_iy_j}(z_1,z_1) \\ =& \frac{1}{2}R_{x_ix_j}(z_1) \end{aligned}$$

by $K(x,y) = K(y,x)$.

Case 3: $z_1 = z_2$ and $z_2 \neq z_3$. Similarly to Case 2, it holds that

$$\begin{aligned} & I_{ij}(z_1,z_1,z_3) \\ &= \lim_{r\longrightarrow 0}\int_{\partial B_r(z_1)}\left\{\frac{\partial}{\partial\nu_x}G_{x_i}(x,z_1)G_{y_j}(x,z_3) - G_{x_i}(x,z_1)\frac{\partial}{\partial\nu_x}G_{y_j}(x,z_3)\right\}d\sigma_x \\ &= \lim_{r\longrightarrow 0}\left[\frac{1}{2\pi r}\int_0^{2\pi}\nu_i G_{y_j}(x,z_3)d\theta + \frac{1}{2\pi}\int_0^{2\pi}\nu_i\frac{\partial}{\partial\nu_x}G_{y_j}(x,z_3)d\theta + o(1)\right] \end{aligned}$$

because $G(x,z_3)$ is regular in $B_r(z_1)$. In the same way we get

$$\begin{aligned} &\frac{1}{2\pi r}\int_0^{2\pi}\nu_i G_{y_j}(x,z_3)d\theta \longrightarrow \frac{1}{2}G_{x_iy_j}(z_1,z_3) \\ &\frac{1}{2\pi}\int_0^{2\pi}\nu_i\frac{\partial}{\partial\nu_x}G_{y_j}(x,z_3)d\theta \longrightarrow \frac{1}{2}G_{x_iy_j}(z_1,z_3), \end{aligned}$$

and consequently we get

$$I_{ij} = \frac{1}{2}G_{x_iy_j}(z_1,z_3) + \frac{1}{2}G_{x_iy_j}(z_1,z_3) = G_{x_iy_j}(z_1,z_3).$$

Case 4: $z_1 \neq z_2$ and $z_1 = z_3$. Similarly to Cases 2 and 3, it holds that

$$\begin{aligned} & I_{ij}(z_1,z_2,z_1) \\ &= \lim_{r\longrightarrow 0}\int_{\partial B_r(z_1)}\left\{\frac{\partial}{\partial\nu_x}G_{x_i}(x,z_2)G_{y_j}(x,z_1) - G_{x_i}(x,z_2)\frac{\partial}{\partial\nu_x}G_{y_j}(x,z_1)\right\}d\sigma_x \\ &= \lim_{r\longrightarrow 0}\left[\frac{1}{2\pi r}\int_0^{2\pi}G_{x_i}(x,z_2)\nu_j\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\nu_x}G_{x_i}(x,z_2)\nu_j\,d\theta + o(1)\right] \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2\pi r}\int_0^{2\pi}G_{x_i}(x,z_2)\nu_j d\theta \longrightarrow \frac{1}{2}G_{x_jx_i}(z_1,z_2) \\ &\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\nu_x}G_{x_i}(x,z_2)\nu_j\,d\theta \longrightarrow \frac{1}{2}G_{x_jx_i}(z_1,z_2). \end{aligned}$$

Consequently we get

$$I_{ij} = \frac{1}{2}G_{x_j x_i}(z_1, z_3) + \frac{1}{2}G_{x_j x_i}(z_1, z_2) = G_{x_i x_j}(z_1, z_2). \qquad \square$$

Similar identities formulated in Proposition 2.3 will be used in Lemma 2.1 in the next section. We collect them in the following proposition. They are localized versions of known identities for the Green's function [6, Lemma 3]. The proof is actually almost the same as in the previous proposition and it will be omitted.

Proposition 3.2. *Take* $z_i \in \Omega$ $(i = 1, 2, 3)$ *and* $R > 0$ *such that* $B_R(z_1) \subset\subset \Omega$ *and* $z_2, z_3 \notin \overline{B_R(z_1)}$. *Set*

$$J(z_1, z_2, z_3) := \int_{\partial B_R(z_1)} \left[\frac{\partial}{\partial \nu_x} \{(x - z_1) \cdot \nabla_x G(x, z_2)\} G(x, z_3) - \{(x - z_1) \cdot \nabla_x G(x, z_2)\} \frac{\partial}{\partial \nu_x} G(x, z_3) \right] d\sigma_x,$$

$$L_i(z_1, z_2, z_3) := \int_{\partial B_R(z_1)} \left[\frac{\partial}{\partial \nu_x} \{(x - z_1) \cdot \nabla_x G(x, z_2)\} \frac{\partial}{\partial y_l} G(x, z_3) - \{(x - z_1) \cdot \nabla_x G(x, z_2)\} \frac{\partial}{\partial \nu_x} \frac{\partial}{\partial y_l} G(x, z_3) \right] d\sigma_x.$$

Then we have

(1)

$$J(z_1, z_2, z_3) = \begin{cases} -\frac{1}{2\pi} & (z_1 = z_2 = z_3), \\ 0 & (else). \end{cases}$$

(2)

$$L_i(z_1, z_2, z_3) = \begin{cases} K_{x_i}(z_1, z_1) & (z_1 = z_2 = z_3), \\ G_{x_i}(z_1, z_2) & (z_1 \neq z_2,\ z_1 = z_3), \\ 0 & (else). \end{cases}$$

4. ASYMPTOTIC FORMULA

We collect several known facts used in the proof of Lemma 2.1.

Theorem 4.1 ([8])**.** *For every fixed* $0 < R \ll 1$, *there exists a constant* C *independent of* j *and* $n \gg 1$ *such that*

$$\left| u_n(x) - \log \frac{e^{u_n(x_{j,n})}}{\left(1 + \frac{\lambda_n}{8} e^{u_n(x_{j,n})} |x - x_{j,n}|^2\right)^2} \right| \leq C \quad \forall x \in B_R(x_{j,n}).$$

The following consequences are immediate; see (2.2):

Corollary 4.2. *For fixed R, there exists a constant C satisfying*

$$\left|\widetilde{u}_{j,n}(\widetilde{x}) - \log\frac{1}{\left(1+\frac{1}{8}|\widetilde{x}|^2\right)^2}\right| \le C \quad \forall \widetilde{x}\in B_{\frac{R}{\delta_{j,n}}}(0)$$

for every j.

Corollary 4.3. *For each j there exists a constant $C_j > 0$ and a subsequence of $\delta_{j,n}$ satisfying*

$$\delta_{j,n} = C_j\lambda_n^{\frac{1}{2}} + o(\lambda_n^{\frac{1}{2}}) \quad \textit{as } n\longrightarrow\infty.$$

Proof. Since $\{u_n\}$ is locally uniformly bounded on $\overline{\Omega}\backslash\mathcal{S}$, we have

$$\left|u_n(x) - \log\frac{e^{u_n(x_{j,n})}}{\left(1+\frac{\lambda_n}{8}e^{u_n(x_{j,n})}R^2\right)^2}\right| = O(1)$$

for $x\in\partial B_R(x_{j,n})$ and $0 < R \ll 1$. Since

$$\frac{e^{u_n(x_{j,n})}}{\left(1+\frac{\lambda_n}{8}e^{u_n(x_{j,n})}R^2\right)^2} = \frac{1}{\left(e^{-\frac{1}{2}u_n(x_{j,n})}+\frac{1}{8}\lambda_n^{\frac{1}{2}}\delta_{j,n}^{-1}R^2\right)^2}$$

and $e^{-\frac{1}{2}u_n(x_{j,n})} \longrightarrow 0$ as $n\longrightarrow\infty$, the above relation implies $\lambda_n^{\frac{1}{2}}\delta_{j,n}^{-1}\sim 1$. Taking a subsequence if necessary, we get $C_j > 0$ and hence the claim follows. □

To prove Lemma 2.1 we adopt the argument used in the proof of [7, Proposition 6.4]. First, we take a cut-off function $\xi\in C_0^\infty([0,\infty))$ satisfying

$$\xi \equiv \begin{cases} 1, & (0\le x\le 1/2) \\ 0, & (1\le x)\end{cases}, \qquad 0\le\xi\le 1.$$

Then it follows that

$$\begin{aligned} v_n(x) &= \int_\Omega G(x,y)\lambda_n e^{u_n(y)}v_n(y)dy \\ &= \sum_{j=1}^m\int_\Omega G(x,y)\lambda_n e^{u_n(y)}v_n(y)\xi\Big(\frac{|y-x_{j,n}|}{\bar{R}}\Big)dy \\ &\quad + \int_\Omega G(x,y)\lambda_n e^{u_n(y)}v_n(y)\Big\{1-\sum_{j=1}^m\xi\Big(\frac{|y-x_{j,n}|}{\bar{R}}\Big)\Big\}dy \end{aligned}$$

$$=: \sum_{j=1}^{m} \psi_{j,n} + \psi_{0,n}.$$

Recall that outside of $\kappa_1, \ldots, \kappa_m$ we have that u_n is bounded, and then we derive $\|\psi_{0,n}\|_{L^\infty(\Omega)} = O(\lambda_n)$ and hence

$$\frac{\psi_{0,n}}{\lambda_n^{\frac{1}{2}}} = O(\lambda_n^{\frac{1}{2}}) = o(1) \quad \text{uniformly in } \overline{\Omega}. \tag{4.1}$$

Proposition 4.4. *For each j,*

$$\psi_{j,n}(x) = G(x, x_{j,n})\gamma_{j,n} + 2\pi \boldsymbol{a}_j \cdot \nabla_y G(x, x_{j,n})\delta_{j,n} + o(\delta_{j,n})$$

uniformly for all $x \in \overline{\Omega} \backslash B_{\bar{R}}(x_{j,n})$, where

$$\gamma_{j,n} = \int_\Omega \lambda_n e^{u_n(y)} v_n(y) \xi\Big(\frac{|y - x_{j,n}|}{\bar{R}}\Big) dy.$$

Proof. For simplicity, we shall omit j in several characters, e.g., ψ_n as $\psi_{j,n}$, $\widetilde{u}_n$ as $\widetilde{u}_{j,n}$, ... Without loss of generality, furthermore, we may assume $\kappa_j = 0$.

Taking sufficiently small $\rho > 0$ prescribed later, we divide ψ_n into two parts:

$$\begin{aligned}\psi_n(x) &= \int_\Omega G(x, y)\lambda_n e^{u_n(y)} v_n(y) \xi\Big(\frac{|y - x_n|}{\bar{R}}\Big) dy \\ &= \int_{\Omega \backslash B_\rho(x_n)} + \int_{B_\rho(x_n)} =: I_1 + I_2.\end{aligned}$$

Corollary 4.2 implies

$$0 \le e^{u_n(y)} \xi\Big(\frac{|y - x_n|}{\bar{R}}\Big) \le \frac{C e^{u_n(x_n)}}{\big(1 + \frac{\lambda_n}{8} e^{u_n(x_n)} |y - x_n|^2\big)^2} \quad \text{in } \Omega, \tag{4.2}$$

since

$$\operatorname{supp} \xi\Big(\frac{|y - x_n|}{\bar{R}}\Big) \subset B_{\bar{R}}(x_n).$$

Then it follows that

$$|I_1| \le \int_{\Omega \backslash B_\rho(x_n)} |G(x, y)| \lambda_n \frac{C e^{u_n(x_n)}}{\big(1 + \frac{\lambda_n}{8} e^{u_n(x_n)} |y - x_n|^2\big)^2} dy \le \frac{C' \delta_n^{-2}}{\big(1 + \frac{1}{8}\frac{\rho^2}{\delta_n^2}\big)^2}$$

for $C' = C \sup_{x \in \Omega} \int_\Omega |G(x, y)| dy$, and hence

$$|I_1| \quad \le \quad \frac{C'}{\big(\delta_n + \frac{1}{8}\frac{\rho^2}{\delta_n}\big)^2} = \delta_n \frac{C'}{\big(\delta_n^{\frac{3}{2}} + \frac{1}{8}\rho^2 \delta_n^{-\frac{1}{2}}\big)^2} = o(\delta_n)$$

if we choose $\rho_n = \delta_n^k$ for some k satisfying $2k - \frac{1}{2} < 0$, that is, $0 < k < \frac{1}{4}$. Henceforth such k is fixed.

For every $x \in \overline{\Omega} \backslash B_R(x_n)$ and $y \in B_R(x_n)$, Taylor's theorem guarantees

$$G(x,y) = G(x,x_n) + \nabla_y G(x,x_n)(y - x_n) + s(x,\eta,y - x_n)$$

with

$$s(x,\eta,y-x_n) = \frac{1}{2} \sum_{1 \le k,l \le 2} G_{y_k y_l}(x,\eta)(y_k - x_{n,k})(y_l - x_{n,l})$$

and $\eta = \eta(n,y) \in B_{\bar{R}}(x_n)$. Since $\rho = \delta_n^k \longrightarrow 0$ as $n \longrightarrow \infty$, we have $B_\rho(x_n) \subset\subset B_{\bar{R}}(x_n)$ for $n \gg 1$, and hence we can apply this formula to I_2:

$$\begin{aligned} I_2 =& G(x,x_n) \int_{B_\rho(x_n)} \lambda_n e^{u_n(y)} v_n(y) \xi\Big(\frac{|y-x_n|}{\bar{R}}\Big) dy \\ &+ \nabla_y G(x,x_n) \cdot \int_{B_\rho(x_n)} (y-x_n) \lambda_n e^{u_n(y)} v_n(y) \xi\Big(\frac{|y-x_n|}{\bar{R}}\Big) dy \\ &+ \int_{B_\rho(x_n)} s(x,\eta,y-x_n) \lambda_n e^{u_n(y)} v_n(y) \xi\Big(\frac{|y-x_n|}{\bar{R}}\Big) dy \\ =:& I_{2,1} + I_{2,2} + I_{2,3} \end{aligned}$$

Using the estimate of I_1, we obtain

$$I_{2,1} = G(x,x_n)\left\{\gamma_n + o(\delta_n)\right\} = G(x,x_n)\gamma_n + o(\delta_n)$$

since $x \in \overline{\Omega} \backslash B_R(x_n)$. Similarly,

$$I_{2,2} = \nabla_y G(x,x_n) \int_{B_{\frac{\rho}{\delta_n}}(0)} \delta_n \widetilde{y} e^{\widetilde{u}_n(\widetilde{y})} \widetilde{v}_n(\widetilde{y}) \xi\Big(\frac{|y-x_n|}{\bar{R}}\Big) d\widetilde{y},$$

by (2.2). Therefore, we obtain

$$\left| e^{\widetilde{u}_n(\widetilde{y})} \widetilde{v}_n(\widetilde{y}) \xi\Big(\frac{|y-x_n|}{\bar{R}}\Big)\right| \le \frac{C}{\big(1 + \frac{|\widetilde{y}|^2}{8}\big)^2} \quad \text{in } \mathbf{R}^2 \tag{4.3}$$

by Corollary 4.2. Here $\rho/\delta_n = \delta_n^{k-1} \longrightarrow \infty$ as $n \longrightarrow \infty$ since $k < \frac{1}{4}$. Thus we can apply the dominated convergence theorem:

$$\begin{aligned} &\int_{B_{\frac{\rho}{\delta_n}}(0)} \widetilde{y}_j e^{\widetilde{u}_n(\widetilde{y})} \widetilde{v}_n(\widetilde{y}) \xi\Big(\frac{|\delta_n \widetilde{y}|}{\bar{R}}\Big) d\widetilde{y} \\ &\longrightarrow \int_{\mathbf{R}^2} \widetilde{y}_j \Big\{ \boldsymbol{a} \cdot \nabla\left(-\tfrac{1}{4}e^U\right) + b \,\mathrm{div}\, \left(\tfrac{1}{2}\widetilde{y} e^U\right) \Big\} d\widetilde{y} = 2\pi a_j, \end{aligned}$$

which implies

$$I_{2,2} = 2\pi \boldsymbol{a} \cdot \nabla_y G(x, x_n)\delta_n + o(\delta_n).$$

Finally, we use

$$\sup_{\substack{x \notin B_{\bar{R}}(0),\, \eta \in B_\rho(0) \\ 1 \le k,l \le 2}} |G_{y_k y_l}(x, \eta)| \le C < \infty$$

for some constant C independent of $\rho \ll 1$ to estimate

$$\begin{aligned} |I_{2,3}| &\leqq C\lambda_n \int_{B_\rho(x_n)} |y - x_n|^2 e^{u_n(y)} \xi\Big(\frac{|y - x_n|}{\bar{R}}\Big) dy \\ &\le C\rho\delta_n \int_{B_{\frac{\rho}{\delta_n}}(0)} |\widetilde{y}| e^{\widetilde{u}_n(\widetilde{y})} \xi\Big(\frac{|\delta_n \widetilde{y}|}{\bar{R}}\Big) dy. \end{aligned}$$

Using Corollary 4.2 again, we get the following limit:

$$\int_{B_{\frac{\rho}{\delta_n}}(0)} |\widetilde{y}| e^{\widetilde{u}_n(\widetilde{y})} \xi\Big(\frac{|\delta_n \widetilde{y}|}{\bar{R}}\Big) d\widetilde{y} \longrightarrow \int_{\mathbf{R}^2} |\widetilde{y}| e^{U(\widetilde{y})} d\widetilde{y} < \infty.$$

Consequently we get $I_{2,3} = o(\delta_n)$ and the conclusion. □

Since $B_{2\bar{R}}(\kappa_j) \supset B_{\bar{R}}(x_{j,n})$ for every j and $n \gg 1$, Proposition 4.4 and Corollary 4.3 imply the following *pre*-asymptotic formula:

$$v_n(x) = \sum_{j=1}^m \gamma_{j,n} G(x, x_{j,n}) + 2\pi\lambda_n^{\frac{1}{2}} \sum_{j=1}^m C_j \boldsymbol{a}_j \cdot \nabla_y G(x, x_{j,n}) + o(\lambda_n^{\frac{1}{2}}) \quad (4.4)$$

uniformly in $x \in \overline{\Omega} \setminus \cup_{j=1}^m B_{2\bar{R}}(\kappa_j)$ and consequently in $C^1(\overline{\Omega} \setminus \cup_{j=1}^m B_{2\bar{R}}(\kappa_j))$ from the elliptic regularity theory.

To get the finer asymptotic formula (Lemma 2.1) we need to get

$$\gamma_{j,n} = o(\lambda_n^{\frac{1}{2}}) \quad (4.5)$$

for some subsequence. Now we complete the proof of Lemma 2.1.

Proof of Lemma 2.1. We argue by contradiction. If (4.5) does not hold then there exists j satisfying

$$\limsup_{n \longrightarrow \infty} \frac{\lambda_n^{\frac{1}{2}}}{|\gamma_{j,n}|} < \infty.$$

Without loss of generality we may assume

$$r_j := \lim_{n \longrightarrow \infty} \frac{\gamma_{j,n}}{\gamma_{1,n}}, \quad c := \lim_{n \longrightarrow \infty} \frac{\lambda_n^{\frac{1}{2}}}{\gamma_{1,n}}$$

and $1 = r_1 \geq r_2 \geq \cdots \geq r_m \geq -1$ for some subsequence. Then we get

$$\frac{v_n(x)}{\gamma_{1,n}} \longrightarrow \sum_{j=1}^{m} r_j G(x,\kappa_j) + 2\pi c \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x,\kappa_j) \tag{4.6}$$

uniformly in $x \in \Omega \setminus \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)$.

We take $R > 2\bar{R}$ satisfying $B_R(\kappa_j) \subset\subset \Omega$, $B_R(\kappa_j) \cap B_R(\kappa_k) = \emptyset$ $(j \neq k)$. We also note that

$$\bar{u}_n := (x-p) \cdot \nabla u_n + 2$$

satisfies (1.4) except for the boundary condition where $p \in \mathbf{R}^2$ is arbitrary. Taking $p = x_{1,n}$ and using Green's formula again, we get

$$\begin{aligned}
0 &= \int_{\partial B_R(x_{1,n})} \Big\{ \frac{\partial}{\partial \nu} \bar{u}_n \cdot \frac{v_n}{\gamma_{1,n}} - \bar{u}_n \frac{\partial}{\partial \nu}\Big(\frac{v_n}{\gamma_{1,n}}\Big) \Big\} d\sigma \\
&\longrightarrow \int_{\partial B_R(\kappa_1)} \Big[\frac{\partial}{\partial \nu} \Big\{ (x-\kappa_1) \cdot \nabla \sum_{k=1}^{m} 8\pi G(x,\kappa_k) \Big\} \\
&\qquad \times \Big\{ \sum_{l=1}^{m} r_l G(x,\kappa_l) + 2\pi c \sum_{l=1}^{m} C_l \boldsymbol{a}_l \cdot \nabla_y G(x,\kappa_l) \Big\} \\
&\qquad - \Big\{ (x-\kappa_1) \cdot \nabla \sum_{k=1}^{m} 8\pi G(x,\kappa_k) + 2 \Big\} \\
&\qquad \times \frac{\partial}{\partial \nu} \Big\{ \sum_{l=1}^{m} r_l G(x,\kappa_l) + 2\pi c \sum_{l=1}^{m} C_l \boldsymbol{a}_l \cdot \nabla_y G(x,\kappa_l) \Big\} \Big] d\sigma_x \\
&= 8\pi \sum_{1 \leq k,l \leq m} r_l J(\kappa_1,\kappa_k,\kappa_l) + 16\pi^2 c \sum_{1 \leq k,l \leq m} C_l \sum_{i=1}^{2} \boldsymbol{a}_{l,i} L_i(\kappa_1,\kappa_k,\kappa_l) \\
&\quad - 2 \sum_{l=1}^{m} r_l \int_{\partial B_R(\kappa_1)} \frac{\partial}{\partial \nu} G(x,\kappa_l) - 4\pi c \sum_{l=1}^{m} C_l \frac{\partial}{\partial \nu} \{ \boldsymbol{a}_l \cdot \nabla_y G(x,\kappa_l) \} \, d\sigma_x.
\end{aligned} \tag{4.7}$$

Here we apply Proposition 3.2. It holds that

$$\sum_{1 \leq k,l \leq m} r_l J(\kappa_1,\kappa_k,\kappa_l) = r_1 J(\kappa_1,\kappa_1,\kappa_1) = -\frac{r_1}{2\pi}$$

and

$$\sum_{1 \leq k,l \leq m} C_l \sum_{i=1}^{2} \boldsymbol{a}_{l,i} L_i(\kappa_1,\kappa_k,\kappa_l) = C_1 \sum_{1 \leq k \leq m} \sum_{i=1}^{2} \boldsymbol{a}_{1,i} L_i(\kappa_1,\kappa_k,\kappa_1)$$

$$= C_1 \boldsymbol{a}_1 \cdot \nabla_{x_1} H^m(\kappa_1, \ldots, \kappa_m) = 0$$

because $\mathcal{S} = \{\kappa_1, \ldots, \kappa_m\}$ is a critical point of H^m; see Theorem 1.1. On the other hand, by elementary calculations we get

$$\sum_{l=1}^m r_l \int_{\partial B_R(\kappa_1)} \frac{\partial}{\partial \nu} G(x, \kappa_l) d\sigma_x = r_1 \int_{\partial B_R(\kappa_1)} \frac{\partial}{\partial \nu} G(x, \kappa_1) d\sigma_x$$

$$= r_1 \lim_{r \longrightarrow 0} \int_{\partial B_r(\kappa_1)} \frac{\partial}{\partial \nu} G(x, \kappa_1) d\sigma_x = -r_1$$

$$\sum_{l=1}^m C_l \int_{\partial B_R(\kappa_1)} \frac{\partial}{\partial \nu} \{\boldsymbol{a}_l \cdot \nabla_y G(x, \kappa_l)\} \, d\sigma_x$$

$$= C_1 \int_{\partial B_R(\kappa_1)} \frac{\partial}{\partial \nu} \{\boldsymbol{a}_1 \cdot \nabla_y G(x, \kappa_1)\} \, d\sigma_x$$

$$= C_1 \lim_{r \longrightarrow 0} \int_{\partial B_r(\kappa_1)} \frac{\partial}{\partial \nu} \{\boldsymbol{a}_1 \cdot \nabla_y G(x, \kappa_1)\} \, d\sigma_x = 0.$$

Putting these relations into (4.7), we obtain

$$0 = 8\pi \left(-\frac{r_1}{2\pi} \right) + 0 - 2(-r_1) + 0, \quad \text{that is,} \quad r_1 = 0.$$

This conclusion contradicts $r_1 = 1$ and we obtain the claim. □

5. Proof of the Main Theorem

We continue the argument in Section 2. First, we show $b_j = 0$ for $j = 1, \ldots, m$.

Step 1, $b_j = 0$ *for every* j. Fix j and $R > 2\bar{R}$. We use Green's formula to u_n and v_n over $B_R(x_{j,n})$:

$$\int_{\partial B_R(x_{j,n})} \left(\frac{\partial u_n}{\partial \nu} v_n - u_n \frac{\partial v_n}{\partial \nu} \right) \, d\sigma = \int_{B_R(x_{j,n})} \{(\Delta u_n) v_n - u_n \Delta v_n\} \, dx$$

$$= \int_{B_R(x_{j,n})} [-\lambda_n e^{u_n} v_n + \lambda_n e^{u_n} v_n \{u_v - u_n(x_{j,n})\} + \lambda_n e^{u_n} v_n u_n(x_{j,n})] \, dx$$

$$= (u_n(x_{j,n}) - 1) \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n dx + \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\widetilde{u}_n} \widetilde{v}_n \widetilde{u}_n dx. \tag{5.1}$$

Then Lemma 2.1 and Theorem 1.1 guarantee that

$$\int_{\partial B_R(x_{j,n})} \left(\frac{\partial u_n}{\partial \nu} v_n - u_n \frac{\partial v_n}{\partial \nu} \right) \, d\sigma = o(1)$$

as $n \longrightarrow \infty$. It also holds that

$$\int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n dx = -\int_{B_R(x_{j,n})} (\Delta v_n) dx = -\int_{\partial B_R(x_{j,n})} \frac{\partial v_n}{\partial \nu} d\sigma = O(\lambda_n^{\frac{1}{2}})$$

by Lemma 2.1. On the other hand, Corollary 4.3 implies

$$1 = \lambda_n e^{u_n(x_{j,n})} \delta_{j,n}^2 = (C_j + o(1)) \lambda_n^2 e^{u_n(x_{j,n})};$$

that is,

$$u_n(x_{j,n}) = -2 \log \lambda_n + O(1),$$

and hence

$$(u_n(x_{j,n}) - 1) \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n \ dx = (-2 \log \lambda_n + O(1)) O(\lambda_n^{\frac{1}{2}}) = o(1)$$

as $n \longrightarrow \infty$. Concerning the last term in (5.1) we have

$$\int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\widetilde{u}_n} \widetilde{v}_n \widetilde{u}_n \longrightarrow \int_{\mathbf{R}^2} \left\{ \boldsymbol{a}_j \cdot \nabla \Big(-\frac{1}{4} e^U \Big) + b_j \,\mathrm{div} \left(\frac{1}{2} x e^U \right) \right\} U = 8\pi b_j$$

by Corollary 4.2. Putting these relations into (5.1), we obtain $b_j = 0$. □

The following fact is necessary to finish the proof:

Proposition 5.1. $v_n \longrightarrow 0$ *locally uniformly in* $\overline{\Omega} \backslash \mathcal{S}$.

Proof. Taking a subsequence if necessary, we have $v_\infty \in L^\infty(\Omega)$ such that

$$v_n \longrightarrow v_\infty \quad \text{weakly} * \text{ in } L^\infty(\Omega).$$

On the other hand, for some subsequence, $\lambda_n e^{u_n} \longrightarrow 0$ locally uniformly in $\overline{\Omega} \backslash \mathcal{S}$ (see Theorem 1.1) and also

$$v_n \longrightarrow v_\infty \quad \text{locally uniformly in } \overline{\Omega} \backslash \mathcal{S}$$

by the elliptic regularity. Moreover, v_∞ satisfies

$$-\Delta v_\infty = 0 \quad \text{in } \overline{\Omega} \backslash \mathcal{S}, \qquad v_\infty = 0 \quad \text{in } \partial\Omega.$$

Here $\# \mathcal{S} < \infty$ and $\|v_\infty\|_{L^\infty(\Omega)} \leq 1$. Hence, $\mathcal{S}$ is composed of removable singular points from the standard property of harmonic functions. Consequently, we have $v_\infty \equiv 0$ and the uniqueness of the limit implies the convergence of the full sequence. □

Step 2, $a_j=0, b_j=0$ *for all* $1 \le j \le m$ *is impossible.* Since Proposition 5.1 has been established, all we have to do is to show

$$v_n \longrightarrow 0 \quad \text{uniformly in } B_R(\kappa_j) \tag{5.2}$$

for each j, where $0 < R \ll 1$. In the sequel, we abbreviate j in several characters and assume $\kappa_j = 0$ as before.

Since $\boldsymbol{a} = 0$ and $b = 0$, we have readily obtained

$$\widetilde{v}_n \longrightarrow 0 \quad \text{locally uniformly in } \mathbf{R}^2. \tag{5.3}$$

By contradiction, if (5.2) does not hold, we get

$$\limsup_{n\longrightarrow\infty} \max_{x\in\overline{B_R(0)}} |v_n(x)| = M > 0,$$

and therefore, up to subsequences (denoted by the same symbol), we have

$$|\widetilde{v}_n(\widetilde{x}_n)| = \max_{x\in\overline{B_R(0)}} |v_n| \longrightarrow M, \quad |\widetilde{x}_n| \longrightarrow \infty.$$

We take the Kelvin transformation of $\widetilde{u}_n$ and $\widetilde{v}_n$

$$\hat{u}_n(x) = \widetilde{u}_n\Big(\frac{x}{|x|^2}\Big), \quad \hat{v}_n(x) = \widetilde{v}_n\Big(\frac{x}{|x|^2}\Big),$$

which satisfies

$$-\Delta\hat{v}_n = \frac{1}{|x|^4} e^{\hat{u}_n}\hat{v}_n \quad \text{in } B_{\frac{\delta_n}{R}}(0)^c.$$

Note that

$$\hat{x}_n := \frac{\widetilde{x}_n}{|\widetilde{x}_n|^2} \longrightarrow 0, \quad \hat{v}_n(\hat{x}_n) = \widetilde{v}_n(\widetilde{x}_n) \longrightarrow M. \tag{5.4}$$

Next we take $w_n \in H_0^1(B_1(0))$ such that

$$-\Delta w_n = f_n := \begin{cases} \frac{1}{|x|^4} e^{\hat{u}_n}\hat{v}_n, & \text{in } B_1(0)\backslash\overline{B_{\frac{\delta_n}{R}}(0)}, \\ 0, & \text{in } B_{\frac{\delta_n}{R}}(0) \end{cases} \qquad w_n = 0 \quad \text{on } \partial B_1(0).$$

Using Corollary 4.2, we get

$$0 \le \frac{1}{|x|^4} e^{\hat{u}_n(x)} = \frac{1}{|x|^4} e^{\widetilde{u}_n\left(\frac{x}{|x|^2}\right)} \le \frac{1}{|x|^4}\frac{e^C}{(1+\frac{1}{8|x|^2})^2} \le C' < \infty$$

in $B_1(0)\backslash\overline{B_{\frac{\delta_n}{R}}(0)}$, where C and C' are constants independent of n. On the other hand, $\hat{v}_n(x) = \widetilde{v}(\frac{x}{|x|^2}) \longrightarrow 0$ for every $x \in B_R(0)\backslash\{0\}$ by (5.3), and therefore,

$$\|f_n\|_{L^p(B_1(0))} \longrightarrow 0 \quad \text{for each } p \in [1, \infty)$$

by the dominated convergence theorem. Consequently,

$$w_n \longrightarrow 0 \quad \text{uniformly in } B_1(0)$$

follows from the elliptic regularity.

Now we consider the difference $\hat{v}_n - w_n$. This function is harmonic on $B_1(0)\backslash\overline{B_{\frac{\delta_n}{R}}(0)}$ and the maximum principle for harmonic functions guarantees

$$\begin{aligned}\|\hat{v}_n - w_n\|_{L^\infty(B_1(0)\backslash\overline{B_{\frac{\delta_n}{R}}(0)})} &\le \|\hat{v}_n - w_n\|_{L^\infty(\partial B_1(0))} + \|\hat{v}_n - w_n\|_{L^\infty(\partial B_{\frac{\delta_n}{R}}(0))}\\ &\le \|\hat{v}_n\|_{L^\infty(\partial B_1(0))} + \|\hat{v}_n\|_{L^\infty(\partial B_{\frac{\delta_n}{R}}(0))} + \|w_n\|_{L^\infty(\partial B_{\frac{\delta_n}{R}}(0))},\end{aligned}$$

where

$$\begin{aligned}&\|\hat{v}_n\|_{L^\infty(\partial B_1(0))} = \|\widetilde{v}_n\|_{L^\infty(\partial B_1(0))} = o(1)\\ &\|\hat{v}_n\|_{L^\infty(\partial B_{\frac{\delta_n}{R}}(0))} = \|\widetilde{v}_n\|_{L^\infty(\partial B_{\frac{R}{\delta_n}}(0))} = \|v_n\|_{L^\infty(\partial B_R(x_n))} = o(1)\end{aligned}$$

follow by (5.3) and Proposition 5.1. Hence it follows that

$$\|\hat{v}_n\|_{L^\infty(B_1(0)\backslash\overline{B_{\frac{\delta_n}{R}}(0)})} \le \|w_n\|_{L^\infty(B_1(0))} + \|\hat{v}_n - w_n\|_{L^\infty(B_1(0)\backslash\overline{B_{\frac{\delta_n}{R}}(0)})} = o(1),$$

which contradicts (5.4). □

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