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ENTROPY-TYPE CONDITIONS FOR RIEMANN SOLVERS AT NODES

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Abstract. This paper deals with conservation laws on networks, represented by graphs. Entropy-type conditions are considered to determine dynamics at nodes. Since entropy dispersion is a local concept, we consider a network composed by a single node J with n incoming and m outgoing arcs. We extend at J the classical Kružkov entropy obtaining two conditions, denoted by (E1) and (E2), the first requiring entropy condition for all Kružkov entropies, the second only for the value corresponding to a sonic point. First we show that in the case $n \neq m$, no Riemann solver can satisfy the strongest condition. Then we characterize all the Riemann solvers at J satisfying the strongest condition (E1), in the case of nodes with at most two incoming and two outgoing arcs. Finally we focus three different Riemann solvers, introduced in previous papers. In particular, we show that the Riemann solver introduced for data networks is the only one always satisfying (E2).

1. Introduction

Nonlinear hyperbolic conservation laws on networks have recently attracted a lot of interest in various fields: car traffic [6, 16, 17, 22], gas dynamics [1, 2, 3, 7, 8, 9, 10, 11, 12, 13], irrigation channels [4, 19, 20, 24] and supply chains [5, 21]. A network is modeled by a graph: a finite collection of arcs connected together by vertices. On each arc we consider a scalar

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conservation law. For instance one may think of the Lighthill-Whitham-Richards model for car traffic [25, 27]. However, our results apply to the other application domains.

It is easy to check that the dynamic at nodes is not uniquely determined by imposing the conservation of mass through vertices. Then, to completely describe the network load evolution, the first step is to appropriately define the concept of solution at a vertex.

As in the classical theory of conservation laws, this problem is equivalent to giving the solution to Riemann problems (now at vertices). More precisely, a Riemann problem at a vertex is simply a Cauchy problem with constant initial conditions in each arc of the vertex. The map, which associates the solution to each Riemann problem at a vertex J, is called a Riemann solver at J. Similarly to the case of a real line, one has to resort to the concept of weak solutions in the sense of distributions and there are infinitely many Riemann solvers producing weak solutions. First one uses entropy-type conditions inside arcs as for the real line. Then, in order to select a particular solution (i.e., a Riemann solver) at the vertex, one has to impose some additional conditions. In [6], for example, the authors required some rules about the distribution of the fluxes in the arcs and a maximization condition; see also [14, 26]. It is then natural to ask if entropy-like conditions can be imposed also at the vertex and not only inside arcs.

In this paper, we focus on a single vertex J, composed of n incoming and m outgoing arcs and we extend the Kružkov [23] entropy-type conditions. More precisely, we propose two different entropy conditions for admissibility of solutions, called, respectively, (E1) and (E2). The condition (E1) is stronger than (E2); indeed, the first asks for Kružkov entropy condition to be verified for all entropies, while the second asks only for the precise Kružkov entropy corresponding to a sonic point. It is interesting to note that the entropy condition (E1) imposes strong restrictions both on Riemann solvers and on the geometry of the vertex. Indeed, Riemann solvers satisfying (E1) can exist only in the case of vertices with the same number of incoming and outgoing arcs.

We then test our conditions on Riemann solvers considered in the literature. First we can prove that the Riemann solver, introduced in [14] for data networks, satisfies (E2) and, in special situations, also (E1). Then we show that the Riemann solvers defined in [6, 26] do not satisfy (E2). However, at least for the Riemann solver in [6], the entropy condition and the maximization procedure agree on some particular set, over which the maximization is taken. Roughly speaking the solver respects the entropy condition once traffic distribution is imposed.

The paper is organized as follows. Section 2 introduces the basic definitions of networks and of solutions. Section 3 deals with the solution to the Riemann problem at the vertex J. Moreover, we introduce the entropy conditions (E1) and (E2) for Riemann solvers at J. In Section 4, we determine which Riemann solvers satisfy the entropy condition (E1). The paper ends with Section 5, which considers the Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$, introduced respectively in [6, 14, 26], and analyzes what entropy conditions these Riemann solvers satisfy.

2. Basic Definitions and Notation

Consider a node J with n incoming arcs $I_1, \dots, I_n$ and m outgoing arcs $I_{n+1}, \dots, I_{n+m}$. We model each incoming arc I_i ($i \in \{1, \dots, n\}$) of the node with the real interval $I_i = (-\infty, 0]$ and each outgoing arc I_j ($j \in \{n+1, \dots, n+m\}$) of the node with the real interval $I_j = [0, +\infty)$. On each arc I_l ($l \in \{1, \dots, n+m\}$), the traffic evolution is given by

$$(\rho_l)_t + f(\rho_l)_x = 0, \tag{2.1}$$

where $\rho_l = \rho_l(t, x) \in [0, \rho_{max}]$ is the *density*, $v_l = v_l(\rho_l)$ is the *average velocity*, and $f(\rho_l) = v_l(\rho_l)\,\rho_l$ is the *flux*. Hence the network load is described by a finite collection of functions ρ_l defined on $[0, +\infty) \times I_l$. For simplicity, we put $\rho_{max} = 1$. On the flux f we make the following assumption:

($\mathcal{F}$) $f : [0, 1] \to \mathbb{R}$ is a piecewise-smooth concave function satisfying
   (1) $f(0) = f(1) = 0$;
   (2) there exists a unique $\sigma \in (0, 1)$ such that f is strictly increasing in $[0, \sigma)$ and strictly decreasing in $(\sigma, 1]$.

Definition 2.1. *Let $\tau : [0, 1] \to [0, 1]$ be the map such that*
(1) $f(\tau(\rho)) = f(\rho)$ *for every* $\rho \in [0, 1]$;
(2) $\tau(\rho) \neq \rho$ *for every* $\rho \in [0, 1] \setminus \{\sigma\}$.

Definition 2.2. *A function $\rho_l \in C([0, +\infty); L^1_{loc}(I_l))$ is an entropy-admissible solution to* (2.1) *in the arc I_l if for every $k \in [0, \rho_{max}]$ and every $\tilde{\varphi} : [0, +\infty) \times I_l \to \mathbb{R}$ smooth, positive with compact support in $(0, +\infty) \times (I_l \setminus \{0\})$,*

$$\int_0^{+\infty} \int_{I_l} \left(|\rho_l - k| \frac{\partial \tilde{\varphi}}{\partial t} + sgn(\rho_l - k)(f(\rho_l) - f(k)) \frac{\partial \tilde{\varphi}}{\partial x} \right) dx\, dt \geq 0. \tag{2.2}$$

Definition 2.3. *A collection of functions $\rho_l \in C([0, +\infty); L^1_{loc}(I_l))$, ($l \in \{1, \dots, n+m\}$) is a weak solution at J if*

(1) *for every* $l \in \{1, \dots, n+m\}$, *the function* ρ_l *is an entropy-admissible solution to* (2.1) *in the arc* I_l;
(2) *for every* $l \in \{1, \dots, n+m\}$ *and for almost every* $t > 0$, *the function* $x \mapsto \rho_l(t, x)$ *has a version with bounded total variation;*
(3) *for almost every* $t > 0$, *it holds that*

$$\sum_{i=1}^{n} f(\rho_i(t, 0-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, 0+)), \tag{2.3}$$

where ρ_l *stands for the version with bounded total variation.*

3. The Riemann Problem at J

Given $\rho_{1,0}, \dots, \rho_{n+m,0} \in [0, 1]$, a Riemann problem at J is a Cauchy problem at J with constant initial data on each arc, i.e.,

$$\begin{cases} \dfrac{\partial}{\partial t}\rho_l + \dfrac{\partial}{\partial x} f(\rho_l) = 0, \\ \rho_l(0, \cdot) = \rho_{0,l}, \end{cases} \quad l \in \{1, \dots, n+m\}. \tag{3.1}$$

Now, we give some definitions for later use. The first one is the definition of Riemann solver, which is a map giving a solution to the Riemann problem (3.1).

Definition 3.1. *A Riemann solver* $\mathcal{RS}$ *is a function*

$$\begin{array}{rccc} \mathcal{RS}: & [0,1]^{n+m} & \longrightarrow & [0,1]^{n+m} \\ & (\rho_{1,0}, \dots, \rho_{n+m,0}) & \longmapsto & (\bar{\rho}_1, \dots, \bar{\rho}_{n+m}) \end{array}$$

satisfying

(1) $\sum_{i=1}^{n} f(\bar{\rho}_i) = \sum_{j=n+1}^{n+m} f(\bar{\rho}_j)$;
(2) *for every* $i \in \{1, \dots, n\}$, *the classical Riemann problem*

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \begin{cases} \rho_{i,0}, & \text{if } x < 0, \\ \bar{\rho}_i, & \text{if } x > 0, \end{cases} \end{cases}$$

is solved with waves with negative speed;
(3) *for every* $j \in \{n+1, \dots, n+m\}$, *the classical Riemann problem*

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \begin{cases} \bar{\rho}_j, & \text{if } x < 0, \\ \rho_{j,0}, & \text{if } x > 0, \end{cases} \end{cases}$$

is solved with waves with positive speed.

We introduce the concepts of equilibrium and consistency for Riemann solvers. The fixed points of a Riemann solver are called equilibria, while a Riemann solver has the consistency condition when its image is contained in the equilibria.

Definition 3.2. *We say that* $(\rho_{1,0}, \dots, \rho_{n+m,0})$ *is an equilibrium for the Riemann solver* $\mathcal{RS}$ *if* $\mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\rho_{1,0}, \dots, \rho_{n+m,0})$.

Definition 3.3. *We say that a Riemann solver* $\mathcal{RS}$ *satisfies the consistency condition if, for every* $(\rho_{1,0}, \dots, \rho_{n+m,0}) \in [0,1]^{n+m}$*, then* $\mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0})$ *is an equilibrium for* $\mathcal{RS}$*.*

We introduce now the concepts of entropy functions and admissible entropy conditions (E1) and (E2) for Riemann solvers. We are essentially extending the Kružkov entropy condition to the case of a node; see [23].

Definition 3.4. *The function* $\mathcal{F} : [0,1]^{n+m} \times [0,1] \to \mathbb{R}$*, defined by*

$$\mathcal{F}(\rho_1, \dots, \rho_{n+m}, k) = \sum_{i=1}^{n} sgn(\rho_i - k)\,(f(\rho_i) - f(k)) - \sum_{j=n+1}^{n+m} sgn(\rho_j - k)\,(f(\rho_j) - f(k)), \tag{3.2}$$

is called an entropy-flux function.

Definition 3.5. *A Riemann solver* $\mathcal{RS}$ *satisfies the entropy condition* (E1) *if, for every initial condition* $(\rho_{1,0}, \dots, \rho_{n+m,0})$ *and for every* $k \in [0,1]$*, we have*

$$\mathcal{F}(\bar{\rho}_1, \dots, \bar{\rho}_{n+m}, k) \geq 0, \tag{3.3}$$

where $(\bar{\rho}_1, \dots, \bar{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0})$.

Remark 1. If $k = 0$, then equation (3.3) becomes

$$\sum_{i=1}^{n} f(\bar{\rho}_i) \geq \sum_{j=n+1}^{n+m} f(\bar{\rho}_j).$$

If $k = 1$, then equation (3.3) becomes

$$\sum_{i=1}^{n} f(\bar{\rho}_i) \leq \sum_{j=n+1}^{n+m} f(\bar{\rho}_j).$$

Therefore, the entropy condition (E1) implies the conservation identity

$$\sum_{i=1}^{n} f(\bar\rho_i) = \sum_{j=n+1}^{n+m} f(\bar\rho_j).$$

Definition 3.6. *A Riemann solver $\mathcal{RS}$ satisfies the entropy condition* (E2) *if, for every initial condition $(\rho_{1,0},\dots,\rho_{n+m,0})$, we have*

$$\mathcal{F}(\bar\rho_1,\dots,\bar\rho_{n+m},\sigma) \geq 0, \tag{3.4}$$

where $(\bar\rho_1,\dots,\bar\rho_{n+m}) = \mathcal{RS}(\rho_{1,0},\dots,\rho_{n+m,0})$.

Remark 2. The entropy condition (3.3) can be deduced in the following way: fix, for every $l \in \{1,\dots,n+m\}$, a smooth function $\varphi_l : [0,+\infty)\times I_l \to [0,+\infty)$ with support contained in $[0,+\infty)\times[-M,M]$ for some $M>0$ and assume that $\varphi_{l'}(t,0) = \varphi_{l''}(t,0)$ for every $t \geq 0$ and $l', l'' \in \{1,\dots,n+m\}$. Applying the divergence theorem to the inequality

$$\sum_{l=1}^{n+m}\int_0^{+\infty}\int_{I_l}\left[|\bar\rho_l - k|\,\varphi_{l,t} + \mathrm{sgn}(\bar\rho_l - k)\,(f(\bar\rho_l) - f(k))\,\varphi_{l,x}\right] dx\,dt \geq 0,$$

where $(\bar\rho_1,\dots,\bar\rho_{n+m})$ is an equilibrium at J, we deduce (3.3).

Obviously, these kinds of entropies are not justified by physical considerations.

Finally, let us introduce sets Ω_l and Φ_l, related to the points 2 and 3 of Definition 3.1.

(1) For every $i \in \{1,\dots,n\}$ define

$$\Omega_i = \begin{cases} [0, f(\rho_{i,0})], & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [0, f(\sigma)], & \text{if } \sigma \leq \rho_{i,0} \leq 1, \end{cases} \tag{3.5}$$

and

$$\Phi_i = \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1], & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [\sigma, 1], & \text{if } \sigma \leq \rho_{i,0} \leq 1. \end{cases} \tag{3.6}$$

(2) For every $j \in \{n+1,\dots,n+m\}$ define

$$\Omega_j = \begin{cases} [0, f(\sigma)], & \text{if } 0 \leq \rho_{j,0} \leq \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \leq \rho_{j,0} \leq 1, \end{cases} \tag{3.7}$$

and

$$\Phi_j = \begin{cases} [0, \sigma], & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \le \rho_{j,0} \le 1. \end{cases} \tag{3.8}$$

The following Proposition links the previous sets with Definition 3.1.

Proposition 3.1. *The following statements hold.*

(1) *For every $i \in \{1, \ldots, n\}$, an element $\bar{\gamma}$ belongs to Ω_i if and only if there exists $\bar{\rho}_i \in [0, 1]$ such that $f(\bar{\rho}_i) = \bar{\gamma}$ and point* (2) *of Definition* 3.1 *is satisfied.*
(2) *For every $j \in \{n+1, \ldots, n+m\}$, an element $\bar{\gamma}$ belongs to Ω_j if and only if there exists $\bar{\rho}_j \in [0, 1]$ such that $f(\bar{\rho}_j) = \bar{\gamma}$ and point* (3) *of Definition* 3.1 *is satisfied.*

The proof is trivial and hence omitted. The main result of this section is that, if $n \neq m$, then every Riemann solver $\mathcal{RS}$ at J does not satisfy the entropy condition (E1). We first need the following result.

Proposition 3.2. *Fix a node J with n incoming arcs and m outgoing arcs and a Riemann solver $\mathcal{RS}$ satisfying the entropy condition* (E1). *Denote with $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m})$ the image through $\mathcal{RS}$ of the initial condition $(\rho_{1,0}, \ldots, \rho_{n+m,0})$.*

(1) *If $n > m$, then $\min\{\bar{\rho}_1, \ldots, \bar{\rho}_n\} = 0$.*
(2) *If $n < m$, then $\max\{\bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}\} = 1$.*

Proof. Consider first the case $n > m$. Suppose for the sake of contradiction that $\min\{\bar{\rho}_1, \ldots, \bar{\rho}_n\} > 0$. Define the set $J = \{j \in \{n+1, \ldots, n+m\} : \bar{\rho}_j = 0\}$ and fix $0 < k < \min\{\bar{\rho}_l : l \in \{1, \ldots, n+m\} \setminus J\}$. Thus, the entropy inequality $\mathcal{F}(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}, k) \ge 0$ becomes

$$\sum_{i=1}^{n} [f(\bar{\rho}_i) - f(k)] \ge \sum_{j \in \{n+1, \ldots, n+m\} \setminus J} [f(\bar{\rho}_j) - f(k)] + \sum_{j \in J} f(k).$$

By point (1) of Definition 3.1, we deduce that

$$-nf(k) \ge -(m - \#(J))f(k) + \#(J)f(k),$$

where $\#(J)$ denotes the cardinality of J; thus $(m - n - 2\#(J))f(k) \ge 0$, which is a contradiction.

Consider now the situation $n < m$. For the sake of contradiction we assume that $\max\{\bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}\} < 1$. Define the set $I = \{i \in \{1, \ldots, n\} :$

$\bar{\rho}_i = 1\}$ and fix $\max\{\bar{\rho}_l : l \in \{1, \ldots, n+m\} \setminus I\} < k < 1$. Thus, the entropy inequality $\mathcal{F}(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}, k) \geq 0$ becomes

$$\sum_{i \in \{1,\ldots,n\} \setminus I} [f(k) - f(\bar{\rho}_i)] - \sum_{i \in I} f(k) \geq \sum_{j=n+1}^{n+m} [f(k) - f(\bar{\rho}_j)] .$$

By point (1) of Definition 3.1, we deduce that $(n - 2\#(I) - m) f(k) \geq 0$, which is a contradiction. □

Theorem 3.1. *Fix a node J with n incoming arcs and m outgoing arcs and suppose that $n \neq m$. Every Riemann solver $\mathcal{RS}$ at J does not satisfy the entropy condition* (E1).

Proof. Suppose, for the sake of contradiction, that there exists a Riemann solver $\mathcal{RS}$ at J satisfying the entropy condition (E1).

Assume $n > m$ and consider an initial condition $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ satisfying $\rho_{i,0} \neq 0$ for every $i \in \{1, \ldots, n\}$. If $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \ldots, \rho_{n+m,0})$, then, by Proposition 3.2, there exists $i_1 \in \{1, \ldots, n\}$ such that $\bar{\rho}_{i_1} = 0$, which is a contradiction since the wave $(\rho_{i_1,0}, \bar{\rho}_{i_1})$ has non-negative speed.

Assume now $n < m$ and consider an initial condition $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ satisfying $\rho_{j,0} \neq 1$ for every $j \in \{n+1, \ldots, n+m\}$. By Proposition 3.2, if $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \ldots, \rho_{n+m,0})$, then there exists $j_1 \in \{n+1, \ldots, n+m\}$ such that $\bar{\rho}_{j_1} = 1$, which is a contradiction since the wave $(\bar{\rho}_{j_1}, \rho_{j_1,0})$ has non-positive speed. □

4. RIEMANN SOLVERS SATISFYING (E1)

In this section we determine which Riemann solver satisfies the entropy condition (E1), in the sense of Definition 3.5, for nodes with $n = m \in \{1, 2\}$. In the case $n \neq m$, Theorem 3.1 implies that every Riemann solver does not satisfy (E1). Moreover, if $n = m = 1$, then there exists exactly one Riemann solver at J satisfying (E1), while if $n = m = 2$, then there exist infinitely many Riemann solvers satisfying (E1); see Sections 4.1 and 4.2. We do not treat the case $n = m > 2$, for the huge number of different situations.

4.1. Nodes with $n = m = 1$.

In this subsection, we fix a node J with one incoming and one outgoing arc. The following result holds.

Proposition 4.1. *A Riemann solver $\mathcal{RS}$ at J satisfies the entropy condition* (E1) *if and only if, for every initial datum $(\rho_{1,0}, \rho_{2,0})$, the image $(\bar{\rho}_1, \bar{\rho}_2) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0})$ satisfies either*

$$\bar{\rho}_1 = \bar{\rho}_2 \tag{4.1}$$

or

$$\bar{\rho}_1 < \bar{\rho}_2 \quad \textit{and} \quad f(\bar{\rho}_1) = f(\bar{\rho}_2). \tag{4.2}$$

Proof. Consider first a Riemann solver $\mathcal{RS}$ satisfying the entropy condition (E1). By 1 of Definition 3.1, it is clear that $f(\bar{\rho}_1) = f(\bar{\rho}_2)$. Assume for the sake of contradiction that $\bar{\rho}_1 > \bar{\rho}_2$. Since $f(\bar{\rho}_1) = f(\bar{\rho}_2)$, we easily deduce that $\bar{\rho}_2 < \sigma < \bar{\rho}_1$. Putting $k = \sigma$ in equation (3.3) we derive

$$f(\bar{\rho}_1) - f(\sigma) \geq f(\sigma) - f(\bar{\rho}_2),$$

which is, by the assumptions, equivalent to $f(\bar{\rho}_1) \geq f(\sigma)$, and so we get a contradiction.

Consider now a Riemann solver $\mathcal{RS}$ such that, for every initial datum $(\rho_{1,0}, \rho_{2,0})$, the image $(\bar{\rho}_1, \bar{\rho}_2) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0})$ satisfies either (4.1) or (4.2). It is trivial to prove that (E1) holds. □

Theorem 4.1. *There exists a unique Riemann solver $\mathcal{RS}$ at J satisfying the entropy condition* (E1). *This Riemann solver satisfies the consistency condition and coincides with the Riemann solver introduced in* [6] *for traffic or with the Riemann solver introduced in* [14].

Proof. Fix an initial datum $(\rho_{1,0}, \rho_{2,0})$. We show that there exists a unique $(\bar{\rho}_1, \bar{\rho}_2)$, which is the image of an entropy admissible Riemann solver.

If $\rho_{1,0} = \rho_{2,0}$, then we claim that $\bar{\rho}_1 = \bar{\rho}_2 = \rho_{1,0}$. Assume for the sake of contradiction that $\bar{\rho}_1 \neq \bar{\rho}_2$. In this case either $\bar{\rho}_1 < \sigma < \bar{\rho}_2$ or $\bar{\rho}_2 < \sigma < \bar{\rho}_1$. By Proposition 4.1, the only possibility is $\bar{\rho}_1 < \sigma < \bar{\rho}_2$. By Proposition 3.1, either $\bar{\rho}_1 = \rho_{1,0}$ or $\bar{\rho}_2 = \rho_{2,0}$. In the first case $\bar{\rho}_2 = \tau(\rho_{2,0})$, while in the second one $\bar{\rho}_1 = \tau(\rho_{1,0})$. It is not possible.

Assume now that $\rho_{1,0} \neq \rho_{2,0}$. We have some different possibilities.

(1) $\max\{\rho_{1,0}, \rho_{2,0}\} \leq \sigma$. By Proposition 3.1, we deduce that $\bar{\rho}_2 \in [0, \sigma]$. Moreover, by Proposition 4.1, we deduce that $\bar{\rho}_1 = \rho_{1,0}$; hence $\bar{\rho}_2 = \bar{\rho}_1 = \rho_{1,0}$. This solution respects all the properties of Definition 3.1 and the entropy condition (3.3).

(2) $\min\{\rho_{1,0}, \rho_{2,0}\} \geq \sigma$. By Proposition 3.1, we deduce that $\bar{\rho}_1 \in [\sigma, 1]$. Moreover, by Proposition 4.1, we deduce that $\bar{\rho}_2 = \rho_{2,0}$; hence $\bar{\rho}_2 = \bar{\rho}_1 = \rho_{2,0}$. This solution respects all the properties of Definition 3.1 and the entropy condition (3.3).

(3) $\rho_{1,0} < \sigma < \rho_{2,0}$. By Proposition 3.1, we deduce that $\bar{\rho}_1 = \rho_{1,0}$ or $\bar{\rho}_1 > \sigma$ and that $\bar{\rho}_2 = \rho_{2,0}$ or $\bar{\rho}_2 < \sigma$. If $f(\rho_{1,0}) = f(\rho_{2,0})$, then, by Proposition 4.1, the only possibility is that $\bar{\rho}_1 = \rho_{1,0}$ and $\bar{\rho}_2 = \rho_{2,0}$. If $f(\rho_{1,0}) > f(\rho_{2,0})$, then, by Proposition 4.1, the only possibility is that $\bar{\rho}_1 = \bar{\rho}_2 = \rho_{2,0}$. Finally, if $f(\rho_{1,0}) < f(\rho_{2,0})$, then,

by Proposition 4.1, the only possibility is that $\bar{\rho}_1 = \bar{\rho}_2 = \rho_{1,0}$. In all the cases, the solution respects all the properties of Definition 3.1 and the entropy condition (3.3).

(4) $\rho_{2,0} < \sigma < \rho_{1,0}$. By Proposition 3.1, we deduce that $\bar{\rho}_1 \geq \sigma$ and $\bar{\rho}_2 \leq \sigma$. By Proposition 4.1, the only possibility is that $\bar{\rho}_1 = \bar{\rho}_2 = \sigma$. The solution respects all the properties of Definition 3.1 and the entropy condition (3.3).

The proof is completed. □

Remark 3. In [15], the authors described all the Riemann solvers, with suitable properties, for nodes J with $n = m = 1$. The unique Riemann solver $\mathcal{RS}$ satisfying (E1) corresponds to the Riemann solver generated by the set $X = \{f(\sigma)\}$ and described in Section 3.1 of [15].

Remark 4. One can try to generalize the entropy condition (E1), at least for nodes with $n = m = 1$, to the case of fluxes depending on the arcs. Unfortunately this is not a trivial problem. Consider indeed the following example. Let $f_1 : [0, 1] \to \mathbb{R}$ and $f_2 : [0, 1] \to \mathbb{R}$ be two fluxes satisfying $(\mathcal{F})$ and assume that

(1) f_1 is the flux in the arc I_1;
(2) f_2 is the flux in the arc I_2;
(3) $\sigma = \frac{1}{2}$ is the point of maximum for both f_1 and f_2;
(4) $f_1(\rho) < f_2(\rho)$ for every $\rho \in (0, 1)$.

Choose $0 < \bar{\rho}_2 < \bar{\rho}_1 < \frac{1}{2}$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and take $k \in [\bar{\rho}_2, \bar{\rho}_1]$; see Figure 1. Then, the entropy condition (3.3) becomes

$$f_1(\bar{\rho}_1) - f_1(k) \geq f_2(k) - f_2(\bar{\rho}_2),$$

which is equivalent to $f_1(k) + f_2(k) \leq f_1(\bar{\rho}_1) + f_2(\bar{\rho}_2)$. The last inequality does not hold for $k = \bar{\rho}_1$ and for all $k \in [\bar{\rho}_2, \bar{\rho}_1]$ near $\bar{\rho}_1$.

4.2. **Nodes with $n = m = 2$.** Consider a Riemann solver $\mathcal{RS}$ for a node J with two incoming and two outgoing arcs. In this subsection, we assume that $(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4)$ denotes an equilibrium for $\mathcal{RS}$. Recall that the equilibrium must satisfy $f(\bar{\rho}_1) + f(\bar{\rho}_2) = f(\bar{\rho}_3) + f(\bar{\rho}_4)$. By symmetry, we may assume also that

(H1): $\bar{\rho}_1 \leq \bar{\rho}_2$ and $\bar{\rho}_3 \leq \bar{\rho}_4$.

The results of this subsection are summarized in Table 1.

Proposition 4.2. *Assume* (H1) *and that every* $\bar{\rho}_l$ $(l \in \{1, 2, 3, 4\})$ *is a good datum.*

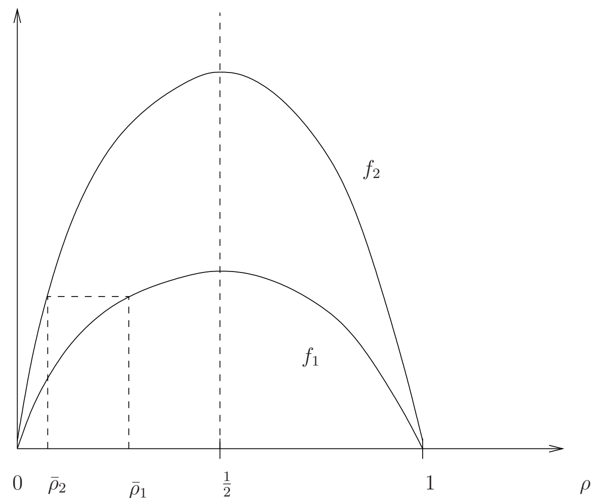

FIGURE 1. The situation in the example of Remark 4.

(1) *If $\mathcal{RS}$ satisfies the entropy condition* $(E1)$, *then* $\bar{\rho}_1 = \bar{\rho}_2 = \bar{\rho}_3 = \bar{\rho}_4 = \sigma$.
(2) *If* $\bar{\rho}_1 = \bar{\rho}_2 = \bar{\rho}_3 = \bar{\rho}_4 = \sigma$, *then* $\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, k) = 0$, *for every* $k \in [0, 1]$.

Proof. Since all the data are good, then $\bar{\rho}_3 \leq \bar{\rho}_4 \leq \sigma \leq \bar{\rho}_1 \leq \bar{\rho}_2$.

If $k \in [\bar{\rho}_3, \bar{\rho}_4]$, then the entropy condition (E1) becomes

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

which is equivalent to $f(k) \leq f(\bar{\rho}_3)$. This implies that $f(\bar{\rho}_4) = f(\bar{\rho}_3)$ and so $\bar{\rho}_3 = \bar{\rho}_4$.

If $k \in [\bar{\rho}_1, \bar{\rho}_2]$, then in the same way we deduce that $\bar{\rho}_1 = \bar{\rho}_2$.

Finally, if $k \in [\bar{\rho}_4, \bar{\rho}_1]$, then (3.3), coupled with the previous results, becomes

$$2f(\bar{\rho}_1) - 2f(k) \geq 2f(k) - 2f(\bar{\rho}_4),$$

which is equivalent to $f(k) \leq f(\bar{\rho}_1)$. Therefore, $\bar{\rho}_1 = \sigma$ and the conclusion follows. □

Proposition 4.3. *Assume* (H1) *and that the equilibrium* $(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4)$ *for* $\mathcal{RS}$ *is composed of three good data and one bad datum.*

(1) *Assume that the bad datum is in an incoming arc, say* $\bar{\rho}_1 < \sigma$. *If* $\mathcal{RS}$ *satisfies* (E1), *then* $\bar{\rho}_2 = \sigma$ *and both* $\bar{\rho}_3$ *and* $\bar{\rho}_4$ *belong to* $[\bar{\rho}_1, \sigma]$. *If*

$\bar{\rho}_2 = \sigma$ *and both* $\bar{\rho}_3$ *and* $\bar{\rho}_4$ *belong to* $[\bar{\rho}_1, \sigma]$, *then* $\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, k) \geq 0$ *for every* $k \in [0,1]$.

(2) *Assume that the bad datum is in an outgoing arc, say* $\bar{\rho}_4 > \sigma$. *If* $\mathcal{RS}$ *satisfies* (E1), *then* $\bar{\rho}_3 = \sigma$ *and both* $\bar{\rho}_1$ *and* $\bar{\rho}_2$ *belong to* $[\sigma, \bar{\rho}_4]$. *If* $\bar{\rho}_3 = \sigma$ *and both* $\bar{\rho}_1$ *and* $\bar{\rho}_2$ *belong to* $[\sigma, \bar{\rho}_4]$, *then* $\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, k) \geq 0$ *for every* $k \in [0,1]$.

Proof. First assume that the bad datum is in an incoming arc and the Riemann solver satisfies the entropy condition (E1). Without loss of generality, suppose that $\bar{\rho}_1 < \sigma$, $\bar{\rho}_2 \geq \sigma$, and $\bar{\rho}_3 \leq \bar{\rho}_4 \leq \sigma$. We have three possibilities.

(a) $\bar{\rho}_1 \leq \bar{\rho}_3 \leq \bar{\rho}_4$.
(b) $\bar{\rho}_3 \leq \bar{\rho}_1 \leq \bar{\rho}_4$.
(c) $\bar{\rho}_3 \leq \bar{\rho}_4 \leq \bar{\rho}_1$.

Consider the case **(a)**. If $k \in [\bar{\rho}_1, \bar{\rho}_3]$, then the entropy condition (E1) becomes

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar{\rho}_1)$, which is true.

If $k \in [\bar{\rho}_4, \bar{\rho}_2]$, then the entropy condition (E1) becomes

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq 2f(k) - f(\bar{\rho}_3) - f(\bar{\rho}_4),$$

equivalent to $f(\bar{\rho}_2) \geq f(k)$, which implies that $\bar{\rho}_2 = \sigma$.

If $k \in [\bar{\rho}_3, \bar{\rho}_4]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\sigma) \geq f(\bar{\rho}_4)$, which is true.

Consider the case **(b)**. If $k \in [\bar{\rho}_3, \bar{\rho}_1]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar{\rho}_3) \geq f(k)$. This implies that $\bar{\rho}_3 = \bar{\rho}_1$ and so we are in the case **(a)**.

Consider the case **(c)**. If $k \in [\bar{\rho}_3, \bar{\rho}_4]$, then the entropy condition (E1) becomes

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar{\rho}_3) \geq f(k)$. This implies that $\bar{\rho}_3 = \bar{\rho}_4$.
If $k \in [\bar{\rho}_4, \bar{\rho}_1]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq 2f(k) - 2f(\bar{\rho}_4);$$

i.e., $f(\bar{\rho}_4) \geq f(k)$. This implies that $\bar{\rho}_4 = \bar{\rho}_1$ and so we have a contradiction since, by case **(a)**, $\bar{\rho}_1 = \bar{\rho}_3 = \bar{\rho}_4 < \sigma = \bar{\rho}_2$ and so $f(\bar{\rho}_1) + f(\bar{\rho}_2) \neq f(\bar{\rho}_3) + f(\bar{\rho}_4)$.

The second statement in the case the bad datum is in an incoming arc easily follows.

Assume now that the bad datum is in an outgoing arc and that the Riemann solver satisfies the entropy condition (E1). Without loss of generality, suppose that $\bar{\rho}_3 \leq \sigma$, $\bar{\rho}_4 > \sigma$, and $\sigma \leq \bar{\rho}_1 \leq \bar{\rho}_2$. We have three possibilities.

(a) $\bar{\rho}_1 \leq \bar{\rho}_2 \leq \bar{\rho}_4$.
(b) $\bar{\rho}_1 \leq \bar{\rho}_4 \leq \bar{\rho}_2$.
(c) $\bar{\rho}_4 \leq \bar{\rho}_1 \leq \bar{\rho}_2$.

Consider the case **(a)**. If $k \in [\bar{\rho}_3, \bar{\rho}_1]$, then the entropy condition (E1) becomes

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3);$$

i.e., $f(\bar{\rho}_3) \geq f(k)$. This implies that $\bar{\rho}_3 = \sigma$.

If $k \in [\bar{\rho}_2, \bar{\rho}_4]$, then (3.3) becomes

$$2f(k) - f(\bar{\rho}_1) - f(\bar{\rho}_2) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(k) \geq f(\bar{\rho}_4)$, which is true.

If $k \in [\bar{\rho}_1, \bar{\rho}_2]$, then (3.3) becomes

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\sigma) = f(\bar{\rho}_3) \geq f(\bar{\rho}_1)$, which is true.

Consider the case **(b)**. If $k \in [\bar{\rho}_4, \bar{\rho}_2]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq 2f(k) - f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

which is equivalent to $f(\bar{\rho}_2) \geq f(k)$. Thus we deduce that $\bar{\rho}_2 = \bar{\rho}_4$ and so we are in the case **(a)**.

Consider the case **(c)**. If $k \in [\bar{\rho}_1, \bar{\rho}_2]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq 2f(k) - f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar{\rho}_2) \geq f(k)$. This implies that $\bar{\rho}_1 = \bar{\rho}_2$.
If $k \in [\bar{\rho}_3, \bar{\rho}_4]$, then (3.3) reads

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3);$$

i.e., $f(\bar{\rho}_3) \geq f(k)$ and so $\bar{\rho}_3 = \sigma$. Therefore $\bar{\rho}_1 = \bar{\rho}_2 = \bar{\rho}_3 = \bar{\rho}_4 = \sigma$, which is a contradiction.

The second statement of the item (2) of the Proposition easily follows. The proof is finished. □

Proposition 4.4. *Assume* (H1) *and that the equilibrium* $(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4)$ *for* $\mathcal{RS}$ *is composed of two good and two bad data.*

(1) *Assume that* $\bar\rho_2 < \sigma$*; i.e., the bad data are both in the incoming arcs. If the Riemann solver* $\mathcal{RS}$ *satisfies the entropy condition* (E1)*, then* $\bar\rho_1 \le \bar\rho_3 \le \bar\rho_4 \le \bar\rho_2$*. If* $\bar\rho_1 \le \bar\rho_3 \le \bar\rho_4 \le \bar\rho_2 < \sigma$*, then* $\mathcal{F}(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4, k) \ge 0$ *for every* $k \in [0, 1]$*.*
(2) *Assume that* $\bar\rho_3 > \sigma$*; i.e., the bad data are in the outgoing arcs. If the Riemann solver* $\mathcal{RS}$ *satisfies the entropy condition* (E1)*, then* $\bar\rho_3 \le \bar\rho_1 \le \bar\rho_2 \le \bar\rho_4$*. If* $\sigma < \bar\rho_3 \le \bar\rho_1 \le \bar\rho_2 \le \bar\rho_4$*, then* $\mathcal{F}(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4, k) \ge 0$ *for every* $k \in [0, 1]$*.*
(3) *Assume that* $\bar\rho_1 < \sigma < \bar\rho_4$*; i.e., the bad data are in the arcs* I_1 *and* I_4*. If the Riemann solver* $\mathcal{RS}$ *satisfies the entropy condition* (E1)*, then* $\bar\rho_1 \le \bar\rho_3 \le \sigma \le \bar\rho_2 \le \bar\rho_4$*. If* $\bar\rho_1 \le \bar\rho_3 \le \sigma \le \bar\rho_2 \le \bar\rho_4$*, then* $\mathcal{F}(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4, k) \ge 0$ *for every* $k \in [0, 1]$*.*

Proof. Assume that $\bar\rho_2 < \sigma$ and that the Riemann solver satisfies the entropy condition (E1). Since there are exactly two bad data, then $\bar\rho_4 \le \sigma$. The conservation of mass at J implies that we have the following possibilities.

(a) $\bar\rho_1 \le \bar\rho_3 \le \bar\rho_4 \le \bar\rho_2$.
(b) $\bar\rho_3 \le \bar\rho_1 \le \bar\rho_2 \le \bar\rho_4$.

Consider the case **(a)**. If $k \in [\bar\rho_1, \bar\rho_3]$, then the entropy condition (E1) reads

$$f(\bar\rho_2) - f(\bar\rho_1) \ge f(\bar\rho_3) + f(\bar\rho_4) - 2f(k),$$

equivalent to $f(k) \ge f(\bar\rho_1)$, which is true. If $k \in [\bar\rho_3, \bar\rho_4]$, then (3.3) becomes

$$f(\bar\rho_2) - f(\bar\rho_1) \ge f(\bar\rho_4) - f(\bar\rho_3),$$

which clearly holds. If $k \in [\bar\rho_4, \bar\rho_2]$, then the entropy condition (E1) reads

$$f(\bar\rho_2) - f(\bar\rho_1) \ge 2f(k) - f(\bar\rho_3) - f(\bar\rho_4),$$

equivalent to $f(\bar\rho_2) \ge f(k)$, which is true.

Consider the case **(b)**. If $k \in [\bar\rho_3, \bar\rho_1]$, then the entropy condition (E1) reads

$$f(\bar\rho_1) + f(\bar\rho_2) - 2f(k) \ge f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(\bar\rho_3) \ge f(k)$. This implies that $\bar\rho_1 = \bar\rho_3$ and consequently $\bar\rho_2 = \bar\rho_4$. The second statement of item 1 of the Proposition easily follows.

Assume now that $\bar\rho_3 > \sigma$ and the Riemann solver satisfies the entropy condition (E1). Consequently $\bar\rho_1 \ge \sigma$. Since $f(\bar\rho_1) + f(\bar\rho_2) = f(\bar\rho_3) + f(\bar\rho_4)$, we have the following possibilities.

(a) $\bar\rho_3 \le \bar\rho_1 \le \bar\rho_2 \le \bar\rho_4$.

(b) $\bar{\rho}_1 \leq \bar{\rho}_3 \leq \bar{\rho}_4 \leq \bar{\rho}_2$.

Consider the case **(a)**. If $k \in [\bar{\rho}_3, \bar{\rho}_1]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar{\rho}_3) \geq f(k)$, which is true. If $k \in [\bar{\rho}_1, \bar{\rho}_2]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

which clearly holds. If $k \in [\bar{\rho}_2, \bar{\rho}_4]$, then (3.3) reads

$$2f(k) - f(\bar{\rho}_1) - f(\bar{\rho}_2) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(k) \geq f(\bar{\rho}_4)$, which is true.

Consider the case **(b)**. If $k \in [\bar{\rho}_1, \bar{\rho}_3]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar{\rho}_1)$. This implies $\bar{\rho}_1 = \bar{\rho}_3$ and so $\bar{\rho}_2 = \bar{\rho}_4$. The second statement of the item 2 of the Proposition easily follows.

Assume now $\bar{\rho}_1 < \sigma < \bar{\rho}_4$, i.e., the bad data are in the arcs I_1 and I_4, and that the Riemann solver satisfies the entropy condition (E1). We have the following possibilities.

(a) $\bar{\rho}_1 \leq \bar{\rho}_3 \leq \sigma \leq \bar{\rho}_2 \leq \bar{\rho}_4$.

(b) $\bar{\rho}_3 \leq \bar{\rho}_1 < \sigma < \bar{\rho}_4 \leq \bar{\rho}_2$.

Consider the case **(a)**. If $k \in [\bar{\rho}_1, \bar{\rho}_3]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar{\rho}_1)$, which is true. If $k \in [\bar{\rho}_3, \bar{\rho}_2]$, then (3.3) becomes

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar{\rho}_3) \geq f(\bar{\rho}_1)$, which is true. If $k \in [\bar{\rho}_2, \bar{\rho}_4]$, then the entropy condition (E1) becomes

$$2f(k) - f(\bar{\rho}_1) - f(\bar{\rho}_2) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(k) \geq f(\bar{\rho}_4)$, which is true.

Consider the case **(b)**. If $k \in [\bar{\rho}_3, \bar{\rho}_1]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) - 2f(k) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar{\rho}_3) \geq f(k)$. This implies $\bar{\rho}_1 = \bar{\rho}_3$ and so $\bar{\rho}_2 = \bar{\rho}_4$. The second statement of item (3) of the Proposition easily follows.

The proof is finished. □

Proposition 4.5. *Assume* (H1) *and that the equilibrium* $(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4)$ *for* $\mathcal{RS}$ *is composed of three bad data and one good datum.*

(1) *Assume that* $\bar{\rho}_2 \geq \sigma$*; i.e., the good datum is in an incoming arc. If the Riemann solver satisfies the entropy condition* (E1)*, then* $\bar{\rho}_1 < \sigma$, $\bar{\rho}_3 > \sigma$, $\bar{\rho}_2 \leq \bar{\rho}_4$*, and* $f(\bar{\rho}_1) \leq \max\{f(\bar{\rho}_2), f(\bar{\rho}_3)\}$. *If* $\bar{\rho}_1 < \sigma$, $\bar{\rho}_3 > \sigma$, $\bar{\rho}_2 \leq \bar{\rho}_4$*, and* $f(\bar{\rho}_1) \leq \max\{f(\bar{\rho}_2), f(\bar{\rho}_3)\}$*, then* $\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, k) \geq 0$ *for every* $k \in [0, 1]$.

(2) *Assume that* $\bar{\rho}_3 \leq \sigma$*; i.e., the good datum is in an outgoing arc. If the Riemann solver satisfies the entropy condition* (E1)*, then* $\bar{\rho}_2 < \sigma$, $\bar{\rho}_4 > \sigma$, $\bar{\rho}_3 \geq \bar{\rho}_1$*, and* $f(\bar{\rho}_4) \leq \max\{f(\bar{\rho}_2), f(\bar{\rho}_3)\}$. *If* $\bar{\rho}_2 < \sigma$, $\bar{\rho}_4 > \sigma$, $\bar{\rho}_3 \geq \bar{\rho}_1$*, and* $f(\bar{\rho}_4) \leq \max\{f(\bar{\rho}_2), f(\bar{\rho}_3)\}$*, then* $\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, k) \geq 0$ *for every* $k \in [0, 1]$.

Proof. Assume first that $\bar{\rho}_2 \geq \sigma$ and that the Riemann solver satisfies the entropy condition (E1). We easily deduce that $\bar{\rho}_1 < \sigma < \bar{\rho}_3 \leq \bar{\rho}_4$. We have the following possibilities.

(a) $\bar{\rho}_1 < \sigma \leq \bar{\rho}_2 \leq \bar{\rho}_3 \leq \bar{\rho}_4$.
(b) $\bar{\rho}_1 < \sigma < \bar{\rho}_3 \leq \bar{\rho}_2 \leq \bar{\rho}_4$.
(c) $\bar{\rho}_1 < \sigma < \bar{\rho}_3 \leq \bar{\rho}_4 \leq \bar{\rho}_2$.

Consider the case **(a)**. If $k \in [\bar{\rho}_1, \bar{\rho}_2]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar{\rho}_1)$. This implies that $f(\bar{\rho}_2) \geq f(\bar{\rho}_1)$. If $k \in [\bar{\rho}_2, \bar{\rho}_3]$, then (3.3) becomes

$$2f(k) - f(\bar{\rho}_1) - f(\bar{\rho}_2) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4) - 2f(k),$$

equivalent to $2f(k) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4)$, which is true. If $k \in [\bar{\rho}_3, \bar{\rho}_4]$, then the entropy condition (E1) reads

$$2f(k) - f(\bar{\rho}_1) - f(\bar{\rho}_2) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(k) \geq f(\bar{\rho}_4)$, which is true.

Consider the case **(b)**. If $k \in [\bar{\rho}_1, \bar{\rho}_3]$, then the entropy condition (E1) reads

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_3) + f(\bar{\rho}_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar{\rho}_1)$. This implies that $f(\bar{\rho}_3) \geq f(\bar{\rho}_1)$. If $k \in [\bar{\rho}_3, \bar{\rho}_2]$, then (3.3) becomes

$$f(\bar{\rho}_2) - f(\bar{\rho}_1) \geq f(\bar{\rho}_4) - f(\bar{\rho}_3),$$

equivalent to $f(\bar\rho_3) \geq f(\bar\rho_1)$. If $k \in [\bar\rho_2, \bar\rho_4]$, then the entropy condition (E1) reads

$$2f(k) - f(\bar\rho_1) - f(\bar\rho_2) \geq f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(k) \geq f(\bar\rho_4)$, which is true.

Consider the case **(c)**. If $k \in [\bar\rho_4, \bar\rho_2]$, then the entropy condition (E1) reads

$$f(\bar\rho_2) - f(\bar\rho_1) \geq 2f(k) - f(\bar\rho_3) - f(\bar\rho_4),$$

equivalent to $f(\bar\rho_2) \geq f(k)$. This implies that $\bar\rho_2 = \bar\rho_4$, and so we are in the case **(b)**. The second statement in item 1 of the Proposition easily follows.

Assume now that $\bar\rho_3 \leq \sigma$ and that the Riemann solver satisfies the entropy condition (E1). We easily deduce that $\bar\rho_1 \leq \bar\rho_2 < \sigma < \bar\rho_4$. We have the following possibilities.

(a) $\bar\rho_1 \leq \bar\rho_2 \leq \bar\rho_3 \leq \sigma < \bar\rho_4$.
(b) $\bar\rho_1 \leq \bar\rho_3 \leq \bar\rho_2 < \sigma < \bar\rho_4$.
(c) $\bar\rho_3 \leq \bar\rho_1 \leq \bar\rho_2 < \sigma < \bar\rho_4$.

Consider the case **(a)**. If $k \in [\bar\rho_1, \bar\rho_2]$, then the entropy condition (E1) reads

$$f(\bar\rho_2) - f(\bar\rho_1) \geq f(\bar\rho_3) + f(\bar\rho_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar\rho_1)$, which is true. If $k \in [\bar\rho_2, \bar\rho_3]$, then (3.3) becomes

$$2f(k) - f(\bar\rho_1) - f(\bar\rho_2) \geq f(\bar\rho_3) + f(\bar\rho_4) - 2f(k),$$

equivalent to $2f(k) \geq f(\bar\rho_1) + f(\bar\rho_2)$, which is true. If $k \in [\bar\rho_3, \bar\rho_4]$, then the entropy condition (E1) reads

$$2f(k) - f(\bar\rho_1) - f(\bar\rho_2) \geq f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(k) \geq f(\bar\rho_4)$. This implies that $f(\bar\rho_3) \geq f(\bar\rho_4)$.

Consider the case **(b)**. If $k \in [\bar\rho_1, \bar\rho_3]$, then the entropy condition (E1) reads

$$f(\bar\rho_2) - f(\bar\rho_1) \geq f(\bar\rho_3) + f(\bar\rho_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar\rho_1)$, which is true. If $k \in [\bar\rho_3, \bar\rho_2]$, then (3.3) becomes

$$f(\bar\rho_2) - f(\bar\rho_1) \geq f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(\bar\rho_3) \geq f(\bar\rho_1)$, which is true. If $k \in [\bar\rho_2, \bar\rho_4]$, then the entropy condition (E1) reads

$$2f(k) - f(\bar\rho_1) - f(\bar\rho_2) \geq f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(k) \geq f(\bar\rho_4)$. This implies that $f(\bar\rho_2) \geq f(\bar\rho_4)$.

Consider the case **(c)**. If $k \in [\bar\rho_3, \bar\rho_1]$, then the entropy condition (E1) reads

$$f(\bar\rho_1) + f(\bar\rho_2) - 2f(k) \geq f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(\bar\rho_3) \geq f(k)$. This implies that $\bar\rho_1 = \bar\rho_3$ and so we are in the case **(b)**. The second statement in item (2) of the Proposition easily follows.

The proof is finished. □

Proposition 4.6. *Assume* (H1) *and that the equilibrium* $(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4)$ *for* $\mathcal{RS}$ *is composed of four bad data. If the Riemann solver satisfies the entropy condition* (E1), *then* $\bar\rho_1 \leq \bar\rho_2 < \sigma < \bar\rho_3 \leq \bar\rho_4$. *Moreover, if* $\bar\rho_1 \leq \bar\rho_2 < \sigma < \bar\rho_3 \leq \bar\rho_4$, *then* $\mathcal{F}(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4, k) \geq 0$ *for every* $k \in [0, 1]$.

Proof. It is sufficient to check the entropy condition (E1). If $k \in [\bar\rho_1, \bar\rho_2]$, then the entropy condition (E1) reads

$$f(\bar\rho_2) - f(\bar\rho_1) \geq f(\bar\rho_3) + f(\bar\rho_4) - 2f(k),$$

equivalent to $f(k) \geq f(\bar\rho_1)$, which is true. If $k \in [\bar\rho_2, \bar\rho_3]$, then (3.3) becomes

$$2f(k) - f(\bar\rho_1) - f(\bar\rho_2) \geq f(\bar\rho_3) + f(\bar\rho_4) - 2f(k),$$

equivalent to $2f(k) \geq f(\bar\rho_3) + f(\bar\rho_4)$, which is true. If $k \in [\bar\rho_3, \bar\rho_4]$, then the entropy condition (E1) reads

$$2f(k) - f(\bar\rho_1) - f(\bar\rho_2) \geq f(\bar\rho_4) - f(\bar\rho_3),$$

equivalent to $f(k) \geq f(\bar\rho_4)$, which is true. This concludes the proof. □

Remark 5. Note that there exist Riemann solvers satisfying the consistency condition and the entropy condition (E1). Here we construct a Riemann solver $\mathcal{RS}$ with such properties. Consider an initial condition $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$. Denote with $(\hat\rho_1, \hat\rho_2, \hat\rho_3, \hat\rho_4)$ the image of the initial condition through $\mathcal{RS}$; i.e., $(\hat\rho_1, \hat\rho_2, \hat\rho_3, \hat\rho_4) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$. If h is the number of bad initial data, then we define $\mathcal{RS}$ according to the following possibilities.

$h = 0$. We put $\hat\rho_1 = \hat\rho_2 = \hat\rho_3 = \hat\rho_4 = \sigma$. By Proposition 4.2, this provides an entropy-admissible equilibrium. Moreover,

$$\mathcal{RS}\left(\mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})\right) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0}).$$

$h = 1$. Let $\bar l \in \{1, 2, 3, 4\}$ be such that $\rho_{\bar l,0}$ is a bad datum. We have two possibilities: $\bar l \leq 2$ or $\bar l \geq 3$.

Assume first $\bar l \leq 2$. We put $\hat\rho_{\bar l} = \rho_{\bar l,0}$ and $\hat\rho_l = \sigma$ for $l \in \{1, 2\}$, $l \neq \bar l$. Moreover, we define $\hat\rho_3 = \hat\rho_1$ and $\hat\rho_4 = \hat\rho_2$. Assume now $\bar l \geq 3$. We put $\hat\rho_{\bar l} = \rho_{\bar l,0}$ and $\hat\rho_l = \sigma$ for $l \in \{3, 4\}$, $l \neq \bar l$. Moreover, we define $\hat\rho_1 = \hat\rho_3$

Bad data	admissible configurations
0	$\bar\rho_1 = \bar\rho_2 = \bar\rho_3 = \bar\rho_4 = \sigma$
1	$\bar\rho_1 \le \bar\rho_3 \le \bar\rho_4 \le \sigma = \bar\rho_2, \qquad \bar\rho_1 < \sigma$
	$\bar\rho_3 = \sigma \le \bar\rho_1 \le \bar\rho_2 \le \bar\rho_4, \qquad \bar\rho_4 > \sigma$
2	$\bar\rho_1 \le \bar\rho_3 \le \bar\rho_4 \le \bar\rho_2 < \sigma$
	$\sigma < \bar\rho_3 \le \bar\rho_1 \le \bar\rho_2 \le \bar\rho_4$
	$\bar\rho_1 \le \bar\rho_3 \le \sigma \le \bar\rho_2 \le \bar\rho_4, \qquad \bar\rho_1 < \sigma < \bar\rho_4$
3	$\bar\rho_1 < \sigma < \bar\rho_3 \le \bar\rho_4,\ \ \sigma \le \bar\rho_2 \le \bar\rho_4,\ \ f(\bar\rho_1) \le \max\{f(\bar\rho_2), f(\bar\rho_3)\}$
	$\bar\rho_1 \le \bar\rho_2 < \sigma < \bar\rho_4,\ \ \bar\rho_1 \le \bar\rho_3 \le \sigma,\ \ f(\bar\rho_4) \le \max\{f(\bar\rho_2), f(\bar\rho_3)\}$
4	$\bar\rho_1 \le \bar\rho_2 < \sigma < \bar\rho_3 \le \bar\rho_4$

Table 1. All the possible configurations for an equilibrium $(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4)$ of a $\mathcal{RS}$ satisfying the entropy condition (E1). By symmetry, we assume that $\bar\rho_1 \le \bar\rho_2$ and $\bar\rho_3 \le \bar\rho_4$; i.e., (H1) holds.

and $\hat\rho_2 = \hat\rho_4$. By Proposition 4.3, these solutions provide entropy-admissible equilibria. Moreover,

$$\mathcal{RS}\left(\mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})\right) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0}).$$

$h = 2$. Let $l_1, l_2 \in \{1,2,3,4\}$, $l_1 \neq l_2$, be such that $\rho_{l_1,0}$ and $\rho_{l_2,0}$ are bad data. We have three different possibilities.

Assume first that $l_1, l_2 \in \{1,2\}$. In this case we put $\hat\rho_{l_1} = \rho_{l_1,0}$, $\hat\rho_{l_2} = \rho_{l_2,0}$, $\hat\rho_3 = \hat\rho_1$, and $\hat\rho_4 = \hat\rho_2$. Assume now that $l_1, l_2 \in \{3,4\}$. In this case we put $\hat\rho_{l_1} = \rho_{l_1,0}$, $\hat\rho_{l_2} = \rho_{l_2,0}$, $\hat\rho_1 = \hat\rho_3$, and $\hat\rho_2 = \hat\rho_4$. Consider finally the last case. For simplicity suppose that $l_1 = 1$ and $l_2 = 4$. We define $\hat\rho_{l_1} = \rho_{l_1,0}$, $\hat\rho_{l_2} = \rho_{l_2,0}$, $\hat\rho_2 = \hat\rho_{l_2}$, and $\hat\rho_3 = \hat\rho_{l_1}$. By Proposition 4.4, these solutions provide entropy-admissible equilibria. Moreover

$$\mathcal{RS}\left(\mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})\right) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0}).$$

$h = 3$. Let $\bar l \in \{1,2,3,4\}$ be such that $\rho_{\bar l,0}$ is a good datum. We have two possibilities: $\bar l \le 2$ or $\bar l \ge 3$.

Assume first $\bar l \le 2$; say $\bar l = 2$ for simplicity. If $f(\rho_{3,0}) + f(\rho_{4,0}) - f(\rho_{1,0}) \in [\min\{f(\rho_{3,0}), f(\rho_{4,0})\}, f(\sigma)]$, then we put $\hat\rho_l = \rho_{l,0}$ for every $l \in \{1,2,3,4\}$, $l \neq \bar l$ and $\hat\rho_{\bar l} \in [\sigma, 1]$ such that $f(\hat\rho_2) = f(\rho_{3,0}) + f(\rho_{4,0}) - f(\rho_{1,0})$. If $f(\rho_{3,0}) + f(\rho_{4,0}) - f(\rho_{1,0}) > f(\sigma)$ and $f(\rho_{3,0}) \ge f(\rho_{4,0})$, then $\hat\rho_1 = \hat\rho_3 = \rho_{1,0}$ and $\hat\rho_2 = \hat\rho_4 = \rho_{4,0}$. If $f(\rho_{3,0}) + f(\rho_{4,0}) - f(\rho_{1,0}) > f(\sigma)$ and $f(\rho_{3,0}) < f(\rho_{4,0})$, then $\hat{\rho}_2 = \hat{\rho}_3 = \rho_{3,0}$ and $\hat{\rho}_1 = \hat{\rho}_4 = \rho_{1,0}$. If $f(\rho_{3,0}) + f(\rho_{4,0}) - f(\rho_{1,0}) < \min\{f(\rho_{3,0}), f(\rho_{4,0})\}$, then $\hat{\rho}_2 = \hat{\rho}_4 = \rho_{3,0}$ and $\hat{\rho}_1 = \hat{\rho}_3 = \rho_{4,0}$.

Assume now $\bar{l} \geq 3$; say $\bar{l} = 3$ for simplicity.

If $f(\rho_{1,0}) + f(\rho_{2,0}) - f(\rho_{4,0}) \in [\min\{f(\rho_{1,0}), f(\rho_{2,0})\}, f(\sigma)]$, then we put $\hat{\rho}_l = \rho_{l,0}$ for every $l \in \{1,2,3,4\}$, $l \neq \bar{l}$ and $\hat{\rho}_{\bar{l}} \in [0,\sigma]$ such that $f(\hat{\rho}_3) = f(\rho_{1,0}) + f(\rho_{2,0}) - f(\rho_{4,0})$. If $f(\rho_{1,0}) + f(\rho_{2,0}) - f(\rho_{4,0}) > f(\sigma)$ and $f(\rho_{1,0}) \geq f(\rho_{2,0})$, then $\hat{\rho}_1 = \hat{\rho}_4 = \rho_{4,0}$ and $\hat{\rho}_2 = \hat{\rho}_3 = \rho_{2,0}$. If $f(\rho_{1,0}) + f(\rho_{2,0}) - f(\rho_{4,0}) > f(\sigma)$ and $f(\rho_{2,0}) > f(\rho_{1,0})$, then $\hat{\rho}_2 = \hat{\rho}_4 = \rho_{4,0}$ and $\hat{\rho}_1 = \hat{\rho}_3 = \rho_{1,0}$. If $f(\rho_{1,0}) + f(\rho_{2,0}) - f(\rho_{4,0}) < \min\{f(\rho_{1,0}), f(\rho_{2,0})\}$, then $\hat{\rho}_2 = \hat{\rho}_4 = \rho_{2,0}$ and $\hat{\rho}_1 = \hat{\rho}_3 = \rho_{1,0}$.

By Propositions 4.4 and 4.5, these solutions provide entropy-admissible equilibria. Moreover,

$$\mathcal{RS}\left(\mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})\right) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0}).$$

$h = 4$. We have some different cases.

Assume first that $f(\rho_{1,0}) + f(\rho_{2,0}) = f(\rho_{3,0}) + f(\rho_{4,0})$. We put $\hat{\rho}_1 = \rho_{1,0}$, $\hat{\rho}_2 = \rho_{2,0}$, $\hat{\rho}_3 = \rho_{3,0}$, and $\hat{\rho}_4 = \rho_{4,0}$. Assume now that $f(\rho_{1,0}) + f(\rho_{2,0}) < f(\rho_{3,0}) + f(\rho_{4,0})$. For simplicity suppose that $f(\rho_{1,0}) \leq f(\rho_{2,0})$ and $f(\rho_{3,0}) \geq f(\rho_{4,0})$. If $f(\rho_{4,0}) > f(\rho_{2,0})$, then we put $\hat{\rho}_1 = \hat{\rho}_3 = \rho_{1,0}$ and $\hat{\rho}_2 = \hat{\rho}_4 = \rho_{2,0}$. If $f(\rho_{4,0}) \leq f(\rho_{2,0})$, then we put $\hat{\rho}_1 = \rho_{1,0}$, $\hat{\rho}_2 = \rho_{2,0}$, $\hat{\rho}_4 = \rho_{4,0}$ and $\hat{\rho}_3 \in [0,\sigma]$ such that $f(\hat{\rho}_3) = f(\hat{\rho}_1) + f(\hat{\rho}_2) - f(\hat{\rho}_4)$.

Assume finally that $f(\rho_{1,0}) + f(\rho_{2,0}) > f(\rho_{3,0}) + f(\rho_{4,0})$. For simplicity suppose that $f(\rho_{1,0}) \leq f(\rho_{2,0})$ and $f(\rho_{3,0}) \geq f(\rho_{4,0})$. If $f(\rho_{1,0}) > f(\rho_{3,0})$, then we put $\hat{\rho}_1 = \hat{\rho}_3 = \rho_{3,0}$ and $\hat{\rho}_2 = \hat{\rho}_4 = \rho_{4,0}$. If $f(\rho_{1,0}) \leq f(\rho_{3,0})$, then we put $\hat{\rho}_1 = \rho_{1,0}$, $\hat{\rho}_3 = \rho_{3,0}$, $\hat{\rho}_4 = \rho_{4,0}$, and $\hat{\rho}_2 \in [\sigma, 1]$ such that $f(\hat{\rho}_2) = f(\hat{\rho}_3) + f(\hat{\rho}_4) - f(\hat{\rho}_1)$. By Propositions 4.4, 4.5, and 4.6, these solutions provide entropy-admissible equilibria. Moreover,

$$\mathcal{RS}\left(\mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})\right) = \mathcal{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0}).$$

Remark 6. Another example of a Riemann solver satisfying the entropy condition (E1) for a node with two incoming and two outgoing arcs is a particular case of the Riemann solver $\mathcal{RS}_2$, defined in Section 5.2; see Proposition 5.5.

The Riemann solver $\mathcal{RS}$, constructed in Remark 5, differs from the Riemann solver $\mathcal{RS}_2$. The key difference is that a permutation of initial data in incoming (respectively outgoing) arcs influences the solution in outgoing (respectively incoming) arcs in the case of $\mathcal{RS}$, but not in the case of $\mathcal{RS}_2$. Consider the following example. Let $f(\rho) = 4\rho(1-\rho)$ be the flux. Assume that $\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right)$ and $\left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ are two initial conditions. In both cases, we have only one bad datum and so, using the notation of Remark 5, $h = 1$. Hence we deduce

$$\mathcal{RS}\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) = \left(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}\right) \quad \text{and} \quad \mathcal{RS}\left(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\right),$$

while $\mathcal{RS}_2\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) = \mathcal{RS}_2\left(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$; see Section 5.2.

5. EXAMPLES

This section deals with some examples of Riemann solvers, introduced in literature for describing car and data traffic. For each of them, we analyze the entropy conditions (E1) and (E2). First we need some notation.

Consider the set

$$\mathcal{A} := \left\{ A = \{a_{ji}\}_{\substack{i=1,\dots,n \\ j=n+1,\dots,n+m}} : \begin{array}{c} 0 < a_{ji} < 1 \ \forall i, j, \\ \sum_{j=n+1}^{n+m} a_{ji} = 1 \ \forall i \end{array} \right\}. \tag{5.1}$$

Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. For every $i = 1, \dots, n$, we denote $H_i = \{e_i\}^\perp$. If $A \in \mathcal{A}$, then we write, for every $j = n+1, \dots, n+m$, $a_j = (a_{j1}, \dots, a_{jn}) \in \mathbb{R}^n$ and $H_j = \{a_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $\mathbf{k} = (k_1, ..., k_\ell)$, $1 \le \ell \le n-1$, such that $0 \le k_1 < k_2 < \cdots < k_\ell \le n+m$ and for every $\mathbf{k} \in \mathcal{K}$ define $H_{\mathbf{k}} = \bigcap_{h=1}^{\ell} H_{k_h}$. Writing $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and following [6] we define the set

$$\mathfrak{N} := \left\{ A \in \mathcal{A} : \mathbf{1} \notin H_{\mathbf{k}}^\perp \text{ for every } \mathbf{k} \in \mathcal{K} \right\}. \tag{5.2}$$

Notice that, if $n > m$, then $\mathfrak{N} = \emptyset$. The matrices of $\mathfrak{N}$ will give rise of a unique solution to the Riemann problem at J.

For later use, define the set

$$\Theta = \left\{ \boldsymbol{\theta} = (\theta_1, \dots, \theta_{n+m}) \in \mathbb{R}^{n+m} : \begin{array}{c} \theta_1 > 0, \cdots, \theta_{n+m} > 0, \\ \sum_{i=1}^{n} \theta_i = \sum_{j=n+1}^{n+m} \theta_j = 1 \end{array} \right\}. \tag{5.3}$$

5.1. **Riemann Solver $\mathcal{RS}_1$.** In this subsection, we consider the Riemann solver introduced for car traffic in [6]. The construction can be done in the following way.

(1) Fix a matrix $A \in \mathfrak{N}$ and consider the closed, convex, and non-empty set

$$\Omega = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}. \tag{5.4}$$

(2) Find the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \Omega$ which maximizes the function

$$E(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \cdots + \gamma_n, \tag{5.5}$$

and define $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})^T := A \cdot (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)^T$. Since $A \in \mathfrak{N}$, then $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ is unique.

(3) For every $i \in \{1, \ldots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Finally, define $\mathcal{RS}_1 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_1(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}). \tag{5.6}$$

The following result holds.

Lemma 5.1. *The function defined in* (5.6) *satisfies the consistency condition, in the sense of Definition* 3.3.

For a proof, see [6, 17]. We show that this Riemann solver satisfies neither entropy condition (E1) nor (E2).

Proposition 5.1. *The Riemann solver $\mathcal{RS}_1$ does not satisfy the entropy condition* (*E*2) *in the sense of Definition* 3.6 *and, consequently, does not satisfy the entropy condition* (*E*1) *in the sense of Definition* 3.5.

Proof. Consider a node with 2 incoming and 2 outgoing arcs, the flux function $f(\rho) = 4\rho(1-\rho)$, a matrix

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$$

and the initial conditions $\rho_{1,0} = \frac{3}{4}$, $\rho_{2,0} = \frac{1}{8}$, $\rho_{3,0} = \frac{8+\sqrt{34}}{16}$, and $\rho_{4,0} = \frac{1}{10}$. In this case the set Ω in (5.4) is

$$\left\{ (\gamma_1, \gamma_2) \in [0,1] \times \left[0, \tfrac{7}{16}\right] : 0 \leq \frac{\gamma_1}{3} + \frac{\gamma_2}{2} \leq \frac{15}{32},\, 0 \leq \frac{2\gamma_1}{3} + \frac{\gamma_2}{2} \leq 1 \right\};$$

see Figure 2. Therefore, we deduce that $\bar{\gamma}_1 = 1$, $\bar{\gamma}_2 = \frac{13}{48}$, $\bar{\gamma}_3 = \frac{15}{32}$, $\bar{\gamma}_4 = \frac{77}{96}$, $\bar{\rho}_1 = \sigma$, $\bar{\rho}_2 > \sigma$, $\bar{\rho}_3 = \rho_{3,0}$, and $\bar{\rho}_4 < \sigma$. The entropy condition (3.4) in this case becomes

$$f(\bar{\rho}_2) - f(\sigma) \geq f(\bar{\rho}_3) - f(\sigma) + f(\sigma) - f(\bar{\rho}_4),$$

which is equivalent to

$$0 \leq f(\bar{\rho}_2) - f(\sigma) - f(\bar{\rho}_3) + f(\bar{\rho}_4) = \frac{13}{48} - 1 - \frac{15}{32} + \frac{77}{96} = -\frac{19}{48}.$$

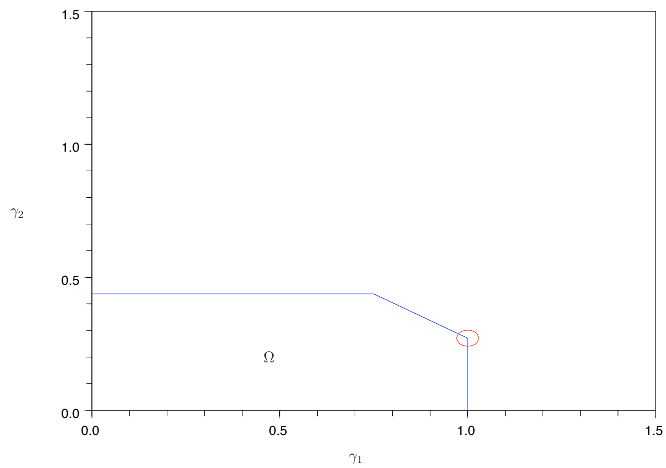

FIGURE 2. The set Ω of Proposition 5.1.

This concludes the proof. □

The maximization of the function E over Ω, which defines the Riemann solver $\mathcal{RS}_1$, is, however, in connection with the maximization of the entropy $\mathcal{F}$. In order to explain this fact, let us introduce some notation.

Given Ω in (5.4), define

$$\Phi = \left\{ (\rho_1, \dots, \rho_{n+m}) \in \prod_{l=1}^{n+m} \Phi_l : \begin{array}{c} (f(\rho_1), \dots, f(\rho_n)) \in \Omega, \\ \begin{pmatrix} f(\rho_{n+1}) \\ \vdots \\ f(\rho_{n+m}) \end{pmatrix} = A \cdot \begin{pmatrix} f(\rho_1) \\ \vdots \\ f(\rho_n) \end{pmatrix} \end{array} \right\} \tag{5.7}$$

and the functional

$$\begin{array}{rcl} \mathcal{G} : \Phi & \longrightarrow & \mathbb{R} \\ (\rho_1, \dots, \rho_{n+m}) & \longmapsto & \mathcal{F}(\rho_1, \dots, \rho_{n+m}, \sigma), \end{array} \tag{5.8}$$

which is the restriction of $\mathcal{F}$ on $\Phi \times \{\sigma\}$. Note that the set Φ consists of all the possible solutions at J satisfying Definition 3.1 and the distribution rule, determined by the matrix $A \in \mathfrak{N}$. It is easy to see that there exists a one-to-one correspondence between Ω and Φ.

For every $\mathcal{H} \subseteq \{1, \dots, n+m\}$ of cardinality h, with $0 \le h \le n-1$, define

$$\Omega_{\mathcal{H}} = \left\{ (\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : \begin{array}{l} (\gamma_{n+1}, \dots, \gamma_{n+m})^T = A \cdot (\gamma_1, \dots, \gamma_n)^T, \\ (\gamma_{n+1}, \dots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} \Omega_j, \\ \gamma_l = \max \Omega_l \quad \text{if} \quad l \in \mathcal{H}, \\ \gamma_l < \max \Omega_l \quad \text{if} \quad l \notin \mathcal{H}, \end{array} \right\} \tag{5.9}$$

and

$$\Phi_{\mathcal{H}} = \left\{ (\rho_1, \dots, \rho_{n+m}) \in \prod_{l=1}^{n+m} \Phi_l : \begin{array}{c} (f(\rho_1), \dots, f(\rho_n)) \in \Omega_{\mathcal{H}}, \\ \begin{pmatrix} f(\rho_{n+1}) \\ \vdots \\ f(\rho_{n+m}) \end{pmatrix} = A \cdot \begin{pmatrix} f(\rho_1) \\ \vdots \\ f(\rho_n) \end{pmatrix} \end{array} \right\}. \tag{5.10}$$

Notice that $\Omega_{\mathcal{H}}$ and $\Phi_{\mathcal{H}}$ depend on the initial condition $(\rho_{1,0}, \dots, \rho_{n+m,0})$ and on the matrix $A \in \mathfrak{N}$. There is a one-to-one correspondence between $\Omega_{\mathcal{H}}$ and $\Phi_{\mathcal{H}}$, given by the one-to-one function

$$\begin{array}{ccc} \Phi_{\mathcal{H}} & \longrightarrow & \Omega_{\mathcal{H}} \\ (\rho_1, \dots, \rho_{n+m}) & \longmapsto & (f(\rho_1), \dots, f(\rho_n)). \end{array}$$

Moreover, if $\Omega_{\mathcal{H}} \neq \emptyset$, then $\Omega_{\mathcal{H}}$ has, at most, topological dimension $n-h$.

The following proposition holds.

Proposition 5.2. *Let $\mathcal{H} \subseteq \{1, \dots, n+m\}$ be a set of cardinality h, with $0 \le h \le n-1$, and suppose that $\Omega_{\mathcal{H}} \neq \emptyset$. The functional $\mathcal{G}$, restricted to $\Phi_{\mathcal{H}}$, is given by*

$$\mathcal{G}(\rho_1, \dots, \rho_{n+m}) = \sum_{l \in \{1, \dots, n+m\} \setminus \mathcal{H}} [f(\rho_l) - f(\sigma)] + \sum_{l \in \mathcal{H}} [f(\sigma) - f(\rho_l)] . \tag{5.11}$$

Proof. Fix $(\rho_1 \dots, \rho_{n+m}) \in \Phi_{\mathcal{H}}$ and $l \in \{1, \dots, n+m\}$. We have some different possibilities.

(1) $l \le n$ and $l \in \mathcal{H}$. In this case the term $\operatorname{sgn}(\rho_l - \sigma)(f(\rho_l) - f(\sigma))$ becomes $f(\sigma) - f(\rho_l)$.
(2) $l \le n$ and $l \notin \mathcal{H}$. In this case the term $\operatorname{sgn}(\rho_l - \sigma)(f(\rho_l) - f(\sigma))$ becomes $f(\rho_l) - f(\sigma)$.
(3) $l \ge n+1$ and $l \in \mathcal{H}$. In this case the term $-\operatorname{sgn}(\rho_l - \sigma)(f(\rho_l) - f(\sigma))$ becomes $f(\sigma) - f(\rho_l)$.
(4) $l \ge n+1$ and $l \notin \mathcal{H}$. In this case the term $-\operatorname{sgn}(\rho_l - \sigma)(f(\rho_l) - f(\sigma))$ becomes $f(\rho_l) - f(\sigma)$.

Therefore the proof is finished. □

Corollary 5.1. *Let $\mathcal{H} \subseteq \{1, \ldots, n+m\}$ be a set of cardinality h, with $0 \le h \le n-1$ and suppose that $\Omega_{\mathcal{H}} \neq \emptyset$. The problem of maximizing $\mathcal{G}$ on the set $\Phi_{\mathcal{H}}$ is equivalent to the problem of maximizing the function E, defined in* (5.5)*, on the set $\Omega_{\mathcal{H}}$.*

Proof. Notice that, by Proposition 5.2, the function $\mathcal{G}$ on the set $\Phi_{\mathcal{H}}$ coincides with

$$\sum_{l \in \{1,\ldots,n+m\} \setminus \mathcal{H}} f(\rho_l) + C,$$

where C is a constant, depending on $\mathcal{H}$ and on the initial conditions. Indeed, if $l \in \mathcal{H}$, then ρ_l is completely determined by the initial condition $\rho_{l,0}$. More precisely, ρ_l is equal to $\rho_{l,0}$ when $\rho_{l,0}$ is a bad datum, while ρ_l is equal to σ in the other case. Therefore, if $(\rho_1, \ldots, \rho_{n+m}) \in \Phi_{\mathcal{H}}$, then we deduce that

$$\begin{aligned}
\mathcal{G}(\rho_1, \ldots, \rho_{n+m}) &= \sum_{i \in \{1,\ldots,n\} \setminus \mathcal{H}} f(\rho_i) + \sum_{j \in \{n+1,\ldots,n+m\} \setminus \mathcal{H}} f(\rho_j) + C \\
&= \sum_{i \in \{1,\ldots,n\} \setminus \mathcal{H}} f(\rho_i) + \sum_{j \in \{n+1,\ldots,n+m\}} f(\rho_j) + C_1 \\
&= \sum_{i \in \{1,\ldots,n\} \setminus \mathcal{H}} f(\rho_i) + \sum_{i \in \{1,\ldots,n\}} f(\rho_j) + C_1 \\
&= 2 \sum_{i \in \{1,\ldots,n\} \setminus \mathcal{H}} f(\rho_i) + C_2,
\end{aligned}$$

where C_1 and C_2 are constants. Finally, note that the function E, restricted on $\Omega_{\mathcal{H}}$, is given by

$$E(\gamma_1, \ldots, \gamma_n) = \sum_{i \in \{1,\ldots,n\} \setminus \mathcal{H}} \gamma_i + C_2 - C_1.$$

This completes the proof. □

Remark 7. Note that the set Φ is, in general, disconnected, while the set Ω is convex and so connected. The function $\mathcal{G}$ defined in (5.8), i.e., the entropy function restricted on $\Phi \times \{\sigma\}$, is continuous, since it does not have jumps in each connected component of Φ. Since there is a bijection between the sets Ω and Φ, we can consider the entropy function on Ω. More precisely, define the function

$$\begin{array}{rcl}
\Upsilon : \Omega & \longrightarrow & \Phi \\
(\gamma_1, \ldots, \gamma_n) & \longmapsto & (\rho_1, \ldots, \rho_{n+m}),
\end{array}$$

satisfying $f(\rho_i) = \gamma_i$ for every $i \in \{1, \dots, n\}$, and consider the map $\mathcal{G} \circ \Upsilon : \Omega \to \mathbb{R}$. This map, in general, is discontinuous, since it can have jumps at every point $(\gamma_1, \dots, \gamma_n) \in \overline{\Omega}_{\mathcal{H}_1} \cap \overline{\Omega}_{\mathcal{H}_2}$ with $\mathcal{H}_1 \neq \mathcal{H}_2$ different subsets of $\{1, \dots, n+m\}$ of cardinalities less than or equal to $n-1$.

5.2. **Riemann Solver $\mathcal{RS}_2$.** In this subsection, we consider the Riemann solver, introduced in [14] for data networks; see also [17]. The construction can be done in the following way.

(1) Fix $\boldsymbol{\theta} \in \Theta$ and define

$$\Gamma_{inc} = \sum_{i=1}^{n} \sup \Omega_i, \quad \Gamma_{out} = \sum_{j=n+1}^{n+m} \sup \Omega_j;$$

then the maximal possible through-flow at the crossing is

$$\Gamma = \min \{\Gamma_{inc}, \Gamma_{out}\} .$$

(2) Introduce the closed, convex, and non-empty sets

$$\begin{aligned} I &= \left\{ (\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i \colon \sum_{i=1}^{n} \gamma_i = \Gamma \right\} \\ J &= \left\{ (\gamma_{n+1}, \dots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} \Omega_j \colon \sum_{j=n+1}^{n+m} \gamma_j = \Gamma \right\} . \end{aligned}$$

(3) Denote with $(\bar{\gamma}_1, \dots, \bar{\gamma}_n)$ the orthogonal projection on the convex set I of the point $(\Gamma\theta_1, \dots, \Gamma\theta_n)$ and with $(\bar{\gamma}_{n+1}, \dots, \bar{\gamma}_{n+m})$ the orthogonal projection on the convex set J of the point $(\Gamma\theta_{n+1}, \dots, \Gamma\theta_{n+m})$.

(4) For every $i \in \{1, \dots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \dots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Finally, define $\mathcal{RS}_2 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_2(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{\rho}_{n+1}, \dots, \bar{\rho}_{n+m}) . \tag{5.12}$$

The following result holds.

Lemma 5.2. *The function defined in* (5.12) *satisfies the consistency condition*

$$\mathcal{RS}_2(\mathcal{RS}_2(\rho_{1,0}, \dots, \rho_{n+m,0})) = \mathcal{RS}_2(\rho_{1,0}, \dots, \rho_{n+m,0}) \tag{5.13}$$

for every $(\rho_{1,0}, \dots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

For a proof, see [18]. We prove now that the Riemann solver $\mathcal{RS}_2$ satisfies the entropy condition (E2).

Proposition 5.3. *Assume $n = m$ and consider a node J with n incoming roads and m outgoing roads. The Riemann solver $\mathcal{RS}_2$ satisfies the entropy condition (E2) in the sense of Definition* 3.6.

Proof. Fix an initial condition $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and define $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) = \mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0})$. We have two different cases.

$\Gamma_{inc} \le \Gamma_{out}$. In this situation, we deduce that $\bar{\rho}_i \le \sigma$ for every $i \in \{1, \ldots, n\}$. Thus the entropy reads

$$\mathcal{F}(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}, \sigma) = nf(\sigma) - \sum_{i=1}^{n} f(\bar{\rho}_i) - \sum_{j=n+1}^{n+m} \operatorname{sgn}(\bar{\rho}_j - \sigma)\left(f(\bar{\rho}_j) - f(\sigma)\right).$$

For every $j \in \{n+1, \ldots, n+m\}$, the term $-\operatorname{sgn}(\bar{\rho}_j - \sigma)\left(f(\bar{\rho}_j) - f(\sigma)\right)$ can be minorized by $f(\bar{\rho}_j) - f(\sigma)$ and so

$$\begin{aligned}\mathcal{F}(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}, \sigma) &\ge nf(\sigma) - \sum_{i=1}^{n} f(\bar{\rho}_i) + \sum_{j=n+1}^{n+m} \left(f(\bar{\rho}_j) - f(\sigma)\right) \\ &= (n-m)f(\sigma) = 0.\end{aligned}$$

$\Gamma_{inc} > \Gamma_{out}$. In this situation, we deduce that $\bar{\rho}_j \ge \sigma$ for every $j \in \{n+1, \ldots, n+m\}$. Thus the entropy reads

$$\mathcal{F}(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}, \sigma) = \sum_{i=1}^{n} \operatorname{sgn}(\bar{\rho}_i - \sigma)\left(f(\bar{\rho}_i) - f(\sigma)\right) + mf(\sigma) - \sum_{j=n+1}^{n+m} f(\bar{\rho}_j).$$

For every $i \in \{1, \ldots, n\}$, the term $\operatorname{sgn}(\bar{\rho}_i - \sigma)\left(f(\bar{\rho}_i) - f(\sigma)\right)$ can be minorized by $f(\bar{\rho}_i) - f(\sigma)$ and so

$$\begin{aligned}\mathcal{F}(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}, \sigma) &\ge \sum_{i=1}^{n} \left(f(\bar{\rho}_i) - f(\sigma)\right) + mf(\sigma) - \sum_{j=n+1}^{n+m} f(\bar{\rho}_j) \\ &= (m-n)f(\sigma) = 0.\end{aligned}$$

The proof is finished. □

In general, the Riemann solver $\mathcal{RS}_2$ does not satisfy the entropy condition (E1) even in the case $n = m$, as the next Proposition shows.

Proposition 5.4. *The Riemann solver $\mathcal{RS}_2$ does not satisfy the entropy condition* (E1) *in the sense of Definition* 3.5.

Proof. Consider a node with 2 incoming and 2 outgoing arcs, the flux function $f(\rho) = 4\rho(1-\rho)$, $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{12}, \frac{7}{12}\right)$ and the equilibrium configuration $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2} - \frac{\sqrt{3}}{4\sqrt{2}}, \frac{1}{2} - \frac{1}{4\sqrt{2}}\right)$. In this case equation (3.3) becomes

$$2\mathrm{sgn}\left(\tfrac{1}{4} - k\right)\left(\tfrac{3}{4} - f(k)\right) - \mathrm{sgn}\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{4\sqrt{2}} - k\right)\left(\tfrac{5}{8} - f(k)\right)$$
$$-\mathrm{sgn}\left(\tfrac{1}{2} - \tfrac{1}{4\sqrt{2}} - k\right)\left(\tfrac{7}{8} - f(k)\right) \geq 0$$

for every $k \in [0,1]$. If $k = \frac{1}{4}$, then the previous inequality becomes

$$\left(\tfrac{5}{8} - \tfrac{3}{4}\right) - \left(\tfrac{7}{8} - \tfrac{3}{4}\right) \geq 0,$$

which is clearly false. $\square$

Indeed, in some special situations, namely for nodes with 2 incoming and 2 outgoing arcs and $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, the Riemann solver $\mathcal{RS}_2$ satisfies the entropy condition (E1).

Proposition 5.5. *Fix a node J with two incoming and two outgoing arcs. If $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, then the Riemann solver $\mathcal{RS}_2$ satisfies the entropy condition* (E1), *in the sense of Definition* 3.5.

Proof. Consider an equilibrium $(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4)$ for the Riemann solver $\mathcal{RS}_2$ and denote by g the number of good data. We have the following possibilities.

$g = 4$. In this case we deduce that $(\bar\rho_1, \bar\rho_2, \bar\rho_3, \bar\rho_4) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and so the entropy condition (E1) is satisfied.

$g = 3$. Consider only the case $\Gamma = \Gamma_{inc}$, since the other case $\Gamma = \Gamma_{out}$ is completely symmetric. Thus the bad datum is in an incoming arc and so we may assume that $\bar\rho_1 < \sigma$, $\bar\rho_2 \geq \sigma$ and $\bar\rho_3 \leq \bar\rho_4 \leq \sigma$. Since $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\bar\rho_2 = \sigma$ and $\bar\rho_3 = \bar\rho_4 < \sigma$. Moreover, the fact that $f(\bar\rho_1) + f(\bar\rho_2) = f(\bar\rho_3) + f(\bar\rho_4)$ implies that

$$\bar\rho_1 < \bar\rho_3 = \bar\rho_4 < \bar\rho_2 = \sigma.$$

By item (1) of Proposition 4.3, the entropy condition (E1) holds.

$g = 2$. Consider only the case $\Gamma = \Gamma_{inc}$, since the other case $\Gamma = \Gamma_{out}$ is completely symmetric. We have two possibilities: either the bad data are in the incoming arcs or one bad datum is in an incoming arc and the other bad datum is in an outgoing arc. Assume first that the bad data are in the incoming arcs. Without loss of generality we may assume that $\bar\rho_1 \leq \bar\rho_2 < \sigma$ and $\bar\rho_3 \leq \bar\rho_4 \leq \sigma$. Since $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, then $\bar\rho_3 = \bar\rho_4$, and so the fact that $f(\bar\rho_1) + f(\bar\rho_2) = f(\bar\rho_3) + f(\bar\rho_4)$ implies that

$$\bar\rho_1 \leq \bar\rho_3 = \bar\rho_4 \leq \bar\rho_2 < \sigma.$$

By item (1) of Proposition 4.4, the entropy condition (E1) is satisfied.

Assume now that one bad datum is in an incoming arc and the other bad datum is in an outgoing arc. Without loss of generality we may assume that $\bar\rho_1 < \sigma < \bar\rho_4$ and $\bar\rho_3 \le \sigma \le \bar\rho_2$. Since $\Gamma = \Gamma_{inc}$, we deduce that $\bar\rho_2 = \sigma$. Moreover $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ implies that $f(\bar\rho_3) \ge f(\bar\rho_4)$ and so $f(\bar\rho_1) \le f(\bar\rho_4)$, since $f(\bar\rho_1) + f(\bar\rho_2) = f(\bar\rho_3) + f(\bar\rho_4)$. Therefore

$$\bar\rho_1 \le \bar\rho_3 \le \bar\rho_2 = \sigma < \bar\rho_4 \quad \text{and} \quad \bar\rho_1 < \bar\rho_2.$$

By item (3) of Proposition 4.4, the entropy condition (E1) is satisfied.

$g = 1$. Consider only the case $\Gamma = \Gamma_{inc}$, since the other case $\Gamma = \Gamma_{out}$ is completely symmetric. We have two possibilities: the good datum is in an incoming arc or in an outgoing arc. Assume first that the good datum is in an incoming arc. Without loss of generality, we may consider that $\bar\rho_1 < \sigma \le \bar\rho_2$ and $\sigma < \bar\rho_3 \le \bar\rho_4$. Since $\Gamma = \Gamma_{inc}$, then $\bar\rho_2 = \sigma$. Moreover $f(\bar\rho_1) + f(\bar\rho_2) = f(\bar\rho_3) + f(\bar\rho_4)$ implies that $f(\bar\rho_4) \ge f(\bar\rho_1)$. By item (1) of Proposition 4.5, the entropy condition (E1) is satisfied.

Assume now that the good datum is in an outgoing arc. Without loss of generality, suppose that $\bar\rho_1 \le \bar\rho_2 < \sigma$ and $\bar\rho_3 \le \sigma < \bar\rho_4$. Since $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, then $f(\bar\rho_3) \ge f(\bar\rho_4)$ and so $f(\bar\rho_4) \le f(\bar\rho_2)$ and $\bar\rho_3 \ge \bar\rho_1$, since $f(\bar\rho_1) + f(\bar\rho_2) = f(\bar\rho_3) + f(\bar\rho_4)$. By item (2) of Proposition 4.5, the entropy condition (E1) is satisfied.

$g = 0$. In this case we have that $\Gamma = \Gamma_{inc} = \Gamma_{out}$. Without loss of generality, suppose that $\bar\rho_1 \le \bar\rho_2 < \sigma < \bar\rho_3 \le \bar\rho_4$, and we conclude by Proposition 4.6.

The proof is finished. □

5.3. **Riemann Solver $\mathcal{RS}_3$.** In this subsection, we consider the Riemann solver, introduced in [26] for crossing nodes. Consider a node J with n incoming and $m = n$ outgoing arcs and fix a positive coefficient Γ_J, which is the maximum capacity of the node. The construction can be done in the following way.

(1) Fix $\boldsymbol{\theta} \in \Theta$. For every $i \in \{1, \dots, n\}$, define $\Gamma_i = \min\{\sup\Omega_i, \sup\Omega_{i+n}\}$. Then the maximal possible through-flow at J is

$$\Gamma = \sum_{i=1}^{n} \Gamma_i.$$

(2) Introduce the closed, convex, and non-empty set

$$I = \left\{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^{n} [0, \Gamma_i] \colon \sum_{i=1}^{n} \gamma_i = \min\{\Gamma, \Gamma_J\} \right\}.$$

(3) Denote with $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ the orthogonal projection on the convex set I of the point $(\min\{\Gamma, \Gamma_J\}\theta_1, \ldots, \min\{\Gamma, \Gamma_J\}\theta_n)$ and set $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{2n}) = (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$.

(4) For every $i \in \{1, \ldots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Finally, define $\mathcal{RS}_3 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_3(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}). \tag{5.14}$$

The following result holds.

Lemma 5.3. *The function defined in* (5.14) *satisfies the consistency condition*

$$\mathcal{RS}_3(\mathcal{RS}_3(\rho_{1,0}, \ldots, \rho_{n+m,0})) = \mathcal{RS}_3(\rho_{1,0}, \ldots, \rho_{n+m,0}) \tag{5.15}$$

for every $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

For a proof, see Proposition 2.4 of [26].

Example 1. Consider a node J with 2 incoming arcs and 2 outgoing ones, $\boldsymbol{\theta} = \left(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}\right)$, and $\Gamma_J = \frac{64}{75}$. Moreover, assume that $f(\rho) = 4\rho(1-\rho)$. We easily see that

$$(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4) = \left(\frac{1}{5}, \frac{1}{2} + \frac{1}{10}\sqrt{\frac{59}{3}}, \frac{4}{5}, \frac{1}{2} - \frac{1}{10}\sqrt{\frac{59}{3}}\right)$$

is an equilibrium for $\mathcal{RS}_3$. Thus we have

$$\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, \sigma) =$$
$$(f(\sigma) - f(\bar{\rho}_1)) + (f(\bar{\rho}_2) - f(\sigma)) - (f(\bar{\rho}_3) - f(\sigma)) - (f(\sigma) - f(\bar{\rho}_4)) = -\frac{64}{75}.$$

Example 2. Consider a node J with 2 incoming arcs and 2 outgoing ones, $\boldsymbol{\theta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, and $\Gamma_J = \frac{7}{6}$. Moreover, assume that $f(\rho) = 4\rho(1-\rho)$. We easily see that

$$(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{3}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{3}}\right)$$

is an equilibrium for $\mathcal{RS}_3$. Thus we have

$$\mathcal{F}(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, \sigma) \quad = \quad (f(\bar{\rho}_1) - f(\sigma)) + (f(\bar{\rho}_2) - f(\sigma))$$

$$
\begin{aligned}
& - \left(f(\bar{\rho}_3) - f(\sigma)\right) - \left(f(\sigma) - f(\bar{\rho}_4)\right) \\
= \; & 2\left(f(\bar{\rho}_2) - f(\sigma)\right) = -\tfrac{2}{3}.
\end{aligned}
$$

The following result follows by the previous examples.

Proposition 5.6. *The Riemann solver $\mathcal{RS}_3$ satisfies neither the entropy condition $(E1)$ nor the entropy condition $(E2)$.*

REFERENCES

[1] M. K. Banda and M. Herty, *Multiscale modeling for gas flow in pipe networks,* Math. Methods Appl. Sci., 31 (2008), 915–936.

[2] M. K. Banda, M. Herty, and A. Klar, *Coupling conditions for gas networks governed by the isothermal Euler equations,* Netw. Heterog. Media, 1 (2006), 295–314 (electronic).

[3] M. K. Banda, M. Herty, and A. Klar, *Gas flow in pipeline networks,* Netw. Heterog. Media, 1 (2006), 41–56 (electronic).

[4] G. Bastin, B. Haut, J.-M. Coron, and B. D'andréa-Novel, *Lyapunov stability analysis of networks of scalar conservation laws,* Netw. Heterog. Media, 2 (2007), 751–759 (electronic).

[5] G. Bretti, C. D'Apice, R. Manzo, and B. Piccoli, *A continuum-discrete model for supply chains dynamics,* Netw. Heterog. Media, 2 (2007), 661–694 (electronic).

[6] G. M. Coclite, M. Garavello, and B. Piccoli, *Traffic flow on a road network,* SIAM J. Math. Anal., 36 (2005), 1862–1886 (electronic).

[7] R. M. Colombo and M. Garavello, *A well posed Riemann problem for the p-system at a junction,* Netw. Heterog. Media, 1 (2006), 495–511 (electronic).

[8] R. M. Colombo and M. Garavello, *On the p-system at a junction,* In "Control Methods in PDE-Dynamical Systems," volume 426 of Contemp. Math., pages 193–217, Amer. Math. Soc., Providence, RI, 2007.

[9] R. M. Colombo and M. Garavello, *On the Cauchy problem for the p-system at a junction,* SIAM J. Math. Anal., 39 (2008), 1456–1471.

[10] R. M. Colombo, G. Guerra, M. Herty, and V. Schleper, *Optimal control in networks of pipes and canals,* SIAM J. Control Optim., 48 (2009), 2032–2050

[11] R. M. Colombo, M. Herty, and V. Sachers, *On 2×2 conservation laws at a junction,* SIAM J. Math. Anal., 40 (2008), 605–622.

[12] R. M. Colombo and F. Marcellini, "Smooth and Discontinuous Junctions in the p-System," J. Math. Anal. Appl., 361 (2010), 440–456.

[13] R. M. Colombo and C. Mauri, *Euler system for compressible fluids at a junction,* J. Hyperbolic Differ. Equ., 5 (2008), 547–568.

[14] C. D'apice, R. Manzo, and B. Piccoli, *Packet flow on telecommunication networks,* SIAM J. Math. Anal., 38 (2006), 717–740 (electronic).

[15] M. Garavello, R. Natalini, B. Piccoli, and A. Terracina, *Conservation laws with discontinuous flux,* Netw. Heterog. Media, 2 (2007), 159–179 (electronic).

[16] M. Garavello and B. Piccoli, *Traffic flow on a road network using the Aw-Rascle model,* Comm. Partial Differential Equations, 31 (2006), 243–275.

[17] M. Garavello and B. Piccoli, "Traffic Flow on Networks," volume 1 of AIMS Series on Applied Mathematics, AIMS, 2006.

[18] M. Garavello and B. Piccoli, *Conservation laws on complex networks,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), 1925–1951,

[19] M. Gugat, M. Herty, A. Klar, and G. Leugering, *Optimal control for traffic flow networks,* J. Optim. Theory Appl., 126 (2005), 589–616.

[20] M. Gugat, G. Leugering, and E. J. P. Georg Schmidt, *Global controllability between steady supercritical flows in channel networks,* Math. Methods Appl. Sci., 27 (2004), 781–802.

[21] M. Herty, A. Klar, and B. Piccoli, *Existence of solutions for supply chain models based on partial differential equations,* SIAM J. Math. Anal., 39 (2007), 160–173.

[22] H. Holden and N. H. Risebro, *A mathematical model of traffic flow on a network of unidirectional roads,* SIAM J. Math. Anal., 26 (1995), 999–1017.

[23] S. N. Kružkov, *First order quasilinear equations with several independent variables,* Mat. Sb. (N.S.), 81 (1970), 228–255.

[24] G. Leugering and E. J. P. G. Schmidt, *On the modelling and stabilization of flows in networks of open canals,* SIAM J. Control Optim., 41 (2002), 164–180 (electronic).

[25] M. J. Lighthill and G. B. Whitham, *On kinematic waves. II. A theory of traffic flow on long crowded roads,* Proc. Roy. Soc. London. Ser. A., 229 (1955), 317–345.

[26] A. Marigo and B. Piccoli, *A fluid dynamic model for T-junctions,* SIAM J. Math. Anal., 39 (2008), 2016–2032.

[27] P. I. Richards, *Shock waves on the highway,* Operations Res., 4:42–51, 1956.