Advances in Differential Equations Volume 16, Numbers 1–2 (2011), 85–112

# SELF-SIMILAR BLOW UP WITH A CONTINUOUS RANGE OF VALUES OF THE AGGREGATED MASS FOR A DEGENERATE KELLER-SEGEL SYSTEM

Y. Sugiyama
Department of Mathematics, Tsuda University
2-1-1 Tsuda-machi, Kodaira-shi, Tokyo 187-8577, Japan

J.J.L. Velázquez
ICMAT (CSIC-UAM-UC3M-UCM), Facultad de Matemáticas
Universidad Complutense, Madrid 28040, Spain

(Submitted by: Yoshikazu Giga)

**Abstract.** In this paper we show that there exist radial solutions of the Keller-Segel system with porous medium like diffusion and critical exponents that blow up in a self-similar manner with a continuous range of masses. The situation is very different from the one that takes place for the usual Keller-Segel system with semilinear diffusion, that for critical nonlinearities yields blow up with a discrete set of values for the mass.

## 1. Introduction

In this paper we will study a class of self-similar solutions that develop singularities in finite time for the system of equations

$$u_t = \Delta u^m - \nabla(u\nabla v), \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$0 = \Delta v + u, \quad x \in \Omega, \quad t > 0, \tag{1.2}$$

where

$$m = 2 - \tfrac{2}{N}, \; N \geq 3. \tag{1.3}$$

We will also assume in the following that $\Omega = \mathbb{R}^N$ and that

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.4}$$

In the case of bounded domains (1.1), (1.2) must be complemented with suitable boundary conditions for $u$, $v$. The most commonly considered ones are homogeneous Neumann boundary conditions.

Accepted for publication: July 2010.
AMS Subject Classifications: 35K45, 35K57, 35K65.

This problem corresponds to the so-called critical case. In the case $N = 2$ with $m = 2 - \frac{2}{N}$ the system (1.1), (1.2) reduces to the following classical model that has been extensively studied:

$$u_t = \Delta u - \nabla(u\nabla v), \quad x \in \Omega \subset \mathbb{R}^2, \quad t > 0, \tag{1.5}$$

$$0 = \Delta v + u, \quad x \in \Omega \subset \mathbb{R}^2, \quad t > 0. \tag{1.6}$$

The mathematical properties of the system (1.5), (1.6) have been extensively studied. The review [11], [12] contains a lot of detailed information about the results available for (1.5), (1.6). We will recall here some of the results that are relevant for the problem considered in this paper.

The conditions for global existence and blow up for the system (1.5), (1.6) are by now reasonably well understood both in bounded domains and in the whole space for radial and nonradial solutions (cf. for instance [15], [16], [17], [6], [4]). It has been also proved that, under some suitable conditions, for instance radial behavior, positivity and integrability of the solutions, the function $u$ behaves asymptotically near the blow-up time $t = T$ as

$$u(x,t) \rightharpoonup 8\pi\delta(x) + f(x) \quad \text{as} \quad t \to T^- \tag{1.7}$$

with $f \in L^1(\Omega)$ (cf. [20]). Similar results exist in nonradial cases under the additional assumption that the blow-up point is isolated (cf. [18], [19]). A construction of a class of solutions with the asymptotics (1.7) with a derivation of their precise asymptotic behaviour can be found in [10]. On the other hand, it has been proved in [19] that for finite mass solutions blow up of the solutions of (1.5), (1.6) in nonradial situations takes place in a finite number of points. Similar results for (1.1), (1.2) under the integrability assumption

$$\int^T \int (u(x,t))^m dxdt < \infty$$

have been proved in [21], [22].

It is widely believed that, in the radial case, blow up can take place for (1.5), (1.6) only by means of the aggregation of the mass $8\pi$. In this paper we find that such mass selection does not occur for the system (1.1), (1.2). At first glance this is a bit surprising because the choice of the exponent in (1.3) is the critical one for this problem. More precisely, the mass of the initial data $\|u_0\|_{L^1(\Omega)}$ is critical from the point of view of dimensional analysis. Therefore, by analogy with the results obtained for the two-dimensional case, global existence for the solutions of (1.1), (1.2), (1.3), (1.4) with suitable boundary conditions and conditions on $\|u_0\|_{L^1(\Omega)}$ could be expected to be defined globally for $t > 0$ and shown in ([23]–[25]). On the contrary, blow up of solutions of this problem with large $\|u_0\|_{L^1(\Omega)}$ might be expected. Results in this direction have been obtained in ([2], [24], [25]) for the following system, that has some better mathematical properties than (1.1), (1.2):

$$u_t = \Delta u^m - \nabla(u\nabla v), \quad x \in \mathbb{R}^N, \quad t > 0$$
$$0 = \Delta v + u - \gamma v, \quad x \in \mathbb{R}^N, \quad t > 0$$

with $\gamma \geq 0$.

On the other hand, for $m > 2 - \frac{2}{N}$, a unique value of the mass aggregating cannot be expected for equations like (1.1), (1.2) that have a rescaling invariance property. Indeed, even if solutions exhibiting mass aggregation exist, the amount of mass aggregating can be changed just by rescaling the solutions. This fact has been known a long time ago for equations like (1.5), (1.6), where a whole family of solutions aggregating arbitrary values of the mass has been described in [8].

The way in which we will prove the existence of a family of solutions of (1.1), (1.2) that yields aggregation with values of the masses in an interval is the following. We will construct a family of radial self-similar solutions of this system. In this way the problem will be reduced to an ODE problem that we will study in detail using analytical and asymptotic arguments. At some point, in order to show in a convincing manner that the range of possible aggregating masses lies in an interval, we will need to show that a suitable coefficient is different from zero. We will verify this for space dimensions $N = 3$ to 10 solving numerically a differential equation free of other parameters in Subsection 3.5. We will also compute the asymptotics of such a coefficient for $N$ large using matched asymptotic expansions in Subsection 3.6.

The system (1.1)-(1.3) has been recently studied in [3]. In this paper it has been proved that for a suitable class of weak solutions, namely the free energy solutions, blow up can occur if the mass is above a critical value $M_c$ and that global existence of such solutions takes place if the mass of the initial data is smaller than the critical value $M_c$. The number $M_c$ has been characterized by means of the solution of a suitable variational problem. Information about the asymptotic behaviour near the blow-up point for solutions having exactly the amount of mass $M_c$ has been also obtained.

One of the questions that still has not been answered is the uniqueness of the weak solutions of the system (1.1)-(1.3). The main difficulty is the nonlocal chemotactic term combined with the nonlinear character of the diffusion. The difficulty due to the nonlinear character of the diffusion has been solved for the porous medium equation (cf. [5]). However, uniqueness for equations of the type (1.1), (1.2) if $m > 1$ has not been proved yet.

We finally remark that the solutions constructed in this paper satisfy the assumption

$$\int_0^T \int_{\mathbb{R}^N} (u(x,t))^m dx dt < \infty. \tag{1.8}$$

It has been proved in [21], [22] that for the solutions of (1.1), (1.2) satisfying (1.8) the blow up takes place in a finite number of points and the solutions develop Dirac function singularities at such blow-up points. The results of this paper show that the class of solutions satisfying (1.8) is not empty. Although this is still far from a general proof of the fact that all the solutions satisfy (1.8), at least it suggests that this condition is not unreasonable.

The Keller-Segel system is a classical model in mathematical biology that was introduced in [13] to describe chemotactic aggregation phenomena. The results in this paper are purely mathematically motivated and probably they do not have a direct biological application. However, they indicate that different diffusive mechanisms can yield different qualitative behaviours for the solutions.

## 2. Construction of the self-similar solutions

2.1. **Reformulating the system of self-similar solutions.** We will assume in the rest of the paper that $N \geq 3$. Taking into account the rescaling properties of (1.1), (1.2) it is natural to look for solutions of the form

$$u(x,t) = \frac{1}{(T-t)}\Phi(y), \quad v(x,t) = (T-t)^{\frac{2}{N}-1}\Psi(y), \quad y = \frac{x}{(T-t)^{\frac{1}{N}}}, \tag{2.1}$$

where $T > 0$ is the blow-up time and where the functions $\Phi$, $\Psi$ are smooth in the region where $\Phi(y) \geq 0$, $\Psi(y) > 0$ in $\mathbb{R}^N$. We will assume also that $\Phi(0) > 0$. Due to the degenerate character of the nonlinear diffusion in (1.1), (1.2) we cannot rule out the possibility of $\Phi$ being compactly supported. Actually, this will be the case for the solutions that we will obtain in this paper. For this reason we will assume that $u, v$ solve (1.1), (1.2) in weak form, as usual for equations of the porous medium type. We will make this concept precise in Subsection 2.4.

Plugging (2.1) into (1.1), (1.2) we obtain that $\Phi$, $\Psi$ satisfy

$$\Delta(\Phi^m) - \nabla \cdot (\Phi \nabla \Psi) - \frac{y \cdot \nabla \Phi}{N} - \Phi = 0, \tag{2.2}$$

$$\Delta \Psi + \Phi = 0. \tag{2.3}$$

We are interested in obtaining radial solutions of (2.2), (2.3). Notice that (2.2) can be written as

$$\nabla\Big(\nabla(\Phi^m) - \Phi\nabla\Psi - \frac{1}{N} y\cdot\Phi\Big) = 0 \tag{2.4}$$

with $m$ as in (1.3).

For radial solutions (2.4) can be written as

$$\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\Big(r^{N-1}\Big(\frac{\partial(\Phi^m)}{\partial r} - \Phi\frac{\partial\Psi}{\partial r} - \frac{r\Phi}{N}\Big)\Big) = 0, \quad r = |y|$$

whence

$$r^{N-1}\Big(\frac{\partial(\Phi^m)}{\partial r} - \Phi\frac{\partial\Psi}{\partial r} - \frac{r\Phi}{N}\Big) = C_1$$

for some suitable $C_1 \in \mathbb{R}$. Using the regularity of $\Phi,\ \Psi$ in a neighbourhood of the origin as well as the fact that $N \geq 3$ we obtain

$$\frac{\partial(\Phi^m)}{\partial r} - \Phi\frac{\partial\Psi}{\partial r} - \frac{r\Phi}{N} = 0. \tag{2.5}$$

Notice that at the points $y \in \mathbb{R}^N$ where $\Phi(y) > 0$ we can rewrite (2.5) as

$$\frac{\partial}{\partial r}\Big(\frac{m}{m-1}\Phi^{m-1} - \Psi - \frac{r^2}{2N}\Big) = 0.$$

This equation can be integrated once

$$\frac{m}{m-1}\Phi^{m-1} - \Psi - \frac{r^2}{2N} = C_2. \tag{2.6}$$

In order to determine the constant $C_2$, we will use the fact that the function $v$ in (1.1), (1.2) is determined, up to the addition of an arbitrary function $f(t)$. In a similar way, the solutions $(\Phi, \Psi)$ of (2.2), (2.3) are determined, up to the addition of an arbitrary constant to $\Psi$. In particular, we can select $\Psi$ in such a way that $\Psi(0) = \frac{m}{m-1}\Phi^{m-1}(0)$, and we can then assume that $C_2 = 0$ in (2.6). Then

$$\Phi = \Big((\frac{m-1}{m})(\Psi + \frac{r^2}{2N})\Big)^{\frac{1}{m-1}}$$

and using (1.3) it follows that

$$\Phi = c_N Q^{\frac{N}{N-2}}, \tag{2.7}$$

where

$$c_N = \Big(\frac{N-2}{2(N-1t)}\Big)^{\frac{N}{N-2}}, \tag{2.8}$$

$$Q(r) = \Psi + \frac{r^2}{2N}. \tag{2.9}$$

Using (2.9) to eliminate $\Psi$ from (2.3) and using also (2.7) we obtain

$$Q'' + \frac{N-1}{r}Q' + c_N Q^{\frac{N}{N-2}} - 1 = 0. \tag{2.10}$$

The solution of (2.10) is uniquely determined given the value

$$Q(0) = \beta > 0. \tag{2.11}$$

A large portion of our analysis will be focused on the understanding of the asymptotics of $Q = Q_\beta$, the solution of (2.10), (2.11), as $\beta \to \infty$. In order to make this analysis, it is convenient to introduce the following rescaled set of variables:

$$Q_\beta(r) = \beta \bar{Q}_\beta(\eta), \quad \eta = \sqrt{c_N}\beta^{\frac{1}{N-2}} r. \tag{2.12}$$

Taking into account (2.10), (2.11) it follows that $\bar{Q} = \bar{Q}_\beta$ satisfies

$$\bar{Q}'' + \frac{N-1}{\eta}\bar{Q}' + \bar{Q}^{\frac{N}{N-2}} - \frac{1}{c_N \beta^{\frac{N}{N-2}}} = 0, \tag{2.13}$$

$$\bar{Q}(0) = 1. \tag{2.14}$$

Notice that due to the singular character of the equation (2.13) at $r = 0$ only, the initial value $\bar{Q}(0)$ is needed to prescribe uniquely the function $\bar{Q}$ in spite of the fact that the equation is a second-order equation. The same will happen for several of the equations later.

It is natural to approximate the solutions of (2.13), (2.14) by means of the solutions of the following auxiliary problem:

$$\bar{Q}_\infty'' + \frac{N-1}{\eta}\bar{Q}_\infty' + \bar{Q}_\infty^{\frac{N}{N-2}} = 0, \tag{2.15}$$

$$\bar{Q}_\infty(0) = 1. \tag{2.16}$$

Notice that the reformulation of the problem as (2.13), (2.14) makes immediate the proof of the following approximation result for the self-similar solutions.

In order to describe the asymptotics of the functions $\bar{Q}_\beta$ we need to describe the behaviour of the function $\bar{Q}_\infty$ defined by means of (2.15), (2.16). Since (2.15) can be transformed into an autonomous second-order system by means of a change of variables, the description of it can be made by means of the analysis of a phase portrait.

2.2. **On the description of the function $\bar{Q}_\infty$.** We now study the solutions of (2.15), (2.16). The results of this section are rather standard, but we recall them here in a form suitable for our purposes.

We can transform (2.15) into an autonomous system by means of the following change of variables:

$$\bar{Q}_\infty(\eta) = \frac{1}{\eta^{N-2}} G(\xi), \tag{2.17}$$

where

$$\eta = e^{\xi}. \tag{2.18}$$

Therefore,

$$G_{\xi\xi} - (N-2)G_\xi + G^{\frac{N}{N-2}} = 0$$

and writing $W = G_\xi$ we obtain the system

$$G_\xi = W, \tag{2.19}$$

$$W_\xi = (N-2)W - G^{\frac{N}{N-2}}. \tag{2.20}$$

On the other hand, the initial condition (2.16) in this reformulation of the problem becomes

$$G(\xi) \sim e^{(N-2)\xi} \quad \text{as} \quad \xi \to -\infty. \tag{2.21}$$

Local analysis of the solutions of (2.19), (2.20) shows that there exists a unique trajectory contained in the half-plane $\{\mathrm{Re}(G) > 0\}$ such that

$$W \sim (N-2)G, \ (G, W) \to 0 \text{ as } \ \xi \to -\infty.$$

Along such a trajectory the functions $G$, $W$ satisfy

$$G(\xi) \sim Ce^{(N-2)\xi} \quad \text{as} \quad \xi \to -\infty, \ \ W(\xi) \sim (N-2)Ce^{(N-2)\xi} \tag{2.22}$$

as $\xi \to -\infty$. There exists another family of trajectories contained in $\{\mathrm{Re}(G) > 0\}$ and approaching towards the origin with the asymptotics

$$W \sim \frac{G^{\frac{N}{N-2}}}{(N-2)}, \ (G, W) \to 0 \quad \text{as} \quad \xi \to -\infty. \tag{2.23}$$

Notice that in this case $\frac{dW}{dG} \to 0$ as $\xi \to -\infty$. For these trajectories

$$G(\xi) \sim \frac{(N-2)^{N-2}}{2^{\frac{N-2}{2}}} \frac{1}{(-\xi)^{\frac{N-2}{2}}} \quad \text{as} \quad \xi \to -\infty. \tag{2.24}$$

We are not concerned in this paper with the trajectories associated to (2.19), (2.20) satisfying (2.24), but only with the trajectory satisfying (2.22) with $C = 1$ (see also (2.21)). We will now show that this specific trajectory of the phase portrait of (2.19), (2.20) enters into the region $\{W < 0,\ G > 0\}$ at some value $\xi = \xi^*$(cf. Figure 1). To this end we argue as follows. The next order in the asymptotics of $G(\xi)$ can be computed approximating $G^{\frac{N}{N-2}}$ in (2.20) as (cf. (2.21)) $G^{\frac{N}{N-2}}(\xi) \sim e^{N\xi}$ as $\xi \to -\infty$. It then follows from (2.20), (2.21) that

$$W(\xi) \sim (N-2)e^{(N-2)\xi} - \frac{e^{N\xi}}{2} \quad \text{as} \quad \xi \to -\infty \tag{2.25}$$

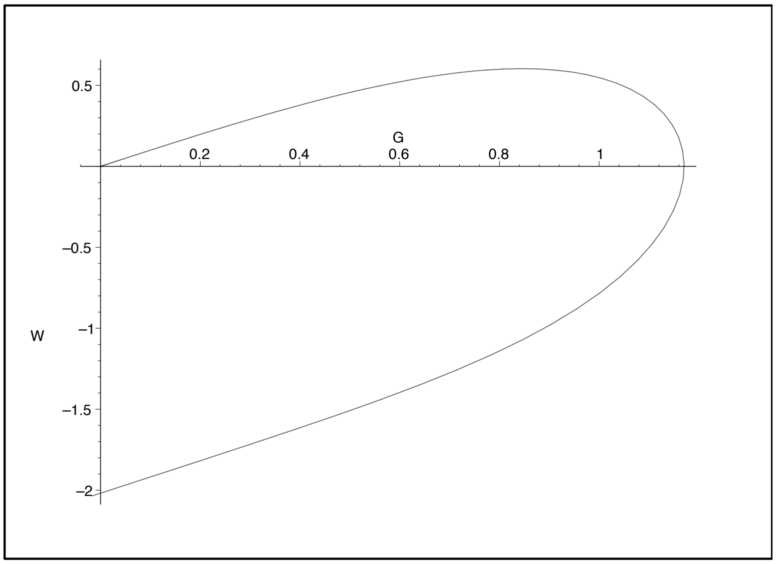
and using (2.19), (2.21) we obtain

$$G(\xi) \sim e^{(N-2)\xi} - \frac{e^{N\xi}}{2N} \quad \text{as} \quad \xi \to -\infty. \tag{2.26}$$

Notice that (2.25), (2.26) imply that the point $(G(\xi), W(\xi))$ is contained in the region $\mathcal{U} = \{W < (N-2)G,\ W > 0,\ G > 0\}$ for $\xi \to -\infty$. On the other hand the slope of the vector field defined by the right-hand side of (2.19), (2.20) is

$$\frac{(N-2)W - G^{\frac{N}{N-2}}}{W} = (N-2) - \frac{G^{\frac{N}{N-2}}}{W} < (N-2)$$

for $W = (N-2)G,\ G > 0$. Therefore the region $\mathcal{U}$ is invariant and the trajectory satisfying (2.21) remains in this region for any $\xi \in \mathbb{R}$ as long as it is defined. On the other hand (2.21) implies also that the point $(G(\xi), W(\xi))$ is contained in the set $\{W > \frac{G^{\frac{N}{N-2}}}{N-2}\}$ as $\xi \to -\infty$. Notice that the curves $\Gamma_1 \equiv \{W = (N-2)G\}$ and $\Gamma_2 \equiv \{W = \frac{G^{\frac{N}{N-2}}}{N-2}\}$ intersect at the points $(0,0)$ and $((N-2)^{(N-2)}, (N-2)^{(N-1)})$. On the other hand, $W$ is larger for the points in the curve $\Gamma_1$ than for the points in the curve $\Gamma_2$ for a given value of $G \in (0, (N-2)^{(N-2)})$ (see Figure 1). It then follows from the invariance of the set $\mathcal{U}$ that the trajectory $(G, W)$ reaches the curve $\Gamma_2$ at some $\xi_0 \in \mathbb{R}$ where $G(\xi_0) \in (0, (N-2)^{(N-2)})$, $W(\xi_0) \in (0, (N-2)^{(N-1)})$. We now remark that, due to (2.20), $W(\xi)$ is decreasing if $W < \frac{G^{\frac{N}{N-2}}}{N-2}$. Then, as long as the trajectory remains in the set $\mathcal{V} = \{0 < W < \frac{G^{\frac{N}{N-2}}}{N-2},\ G > 0\}$ we have $W(\xi) \leq W(\xi_0) \leq (N-2)^{(N-2)}$. On the other hand, $G(\xi)$ is increasing in this region (cf. (2.19)). We now claim that the trajectory $(G, W)$ leaves the region $\mathcal{V}$ at a finite value $\xi = \xi_*$. Indeed, let us assume the opposite. Then, due to the absence of equilibria for (2.19), (2.20) in the set $\bar{\mathcal{V}}$ it would follow that $G(\xi) \to \infty$ as $\xi \to \infty$. However, due to the boundedness of $W$ in $\mathcal{V}$, (2.20) would then imply that $W_\xi \leq -1$ for $\xi$ sufficiently large, and




therefore $W$ would become negative for a finite value of $\xi = \xi_*$, yielding a contradiction. Then, the unique trajectory satisfying (2.21) enters in the region $\mathcal{W} = \{W < 0,\ G > 0\}$ at a finite value of $\xi$.

Finally, we prove that such a trajectory arrives at the line $G = 0,\ W < 0$ in a finite time. Indeed, this is just a consequence of the fact that in the region $\mathcal{W}$ we have $W_\xi < 0$, thus, $W \le -\kappa < 0$ for $\xi \ge \xi_1$ for some $\xi_1 < \infty$, as long as the trajectory remains in $\mathcal{W}$. It then follows from (2.19) that $G$ vanishes at some $\xi^* < \infty$, whence the result follows.

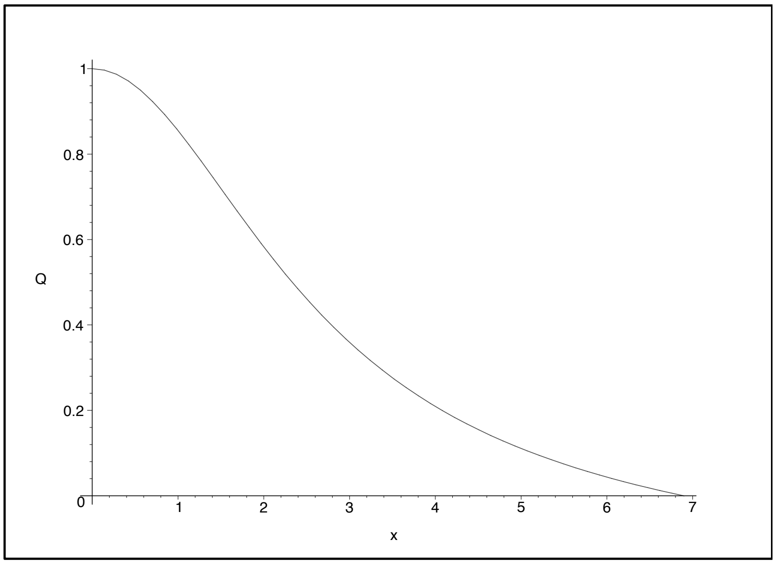
We summarize the previous results in the following lemma.

**Lemma 1.** *There exists* $\xi^*$ *such that for the unique solution of* (2.19), (2.20) *satisfying* (2.21) *we have*

$$\begin{aligned} &G(\xi^*) = 0,\ \ W(\xi^*) < 0,\\ &G(\xi) > 0 \ \ \textit{for} \ \ -\infty < \xi < \xi^*,\\ &W(\xi) < (N-2)G(\xi) \ \ \textit{for} \ \ -\infty < \xi < \xi^*. \end{aligned}$$

Using (2.17) we can reformulate the results of the lemma in terms of the function $\bar{Q}_\infty$. The only part of the following lemma that is not immediate, namely, the derivative estimate, follows from the fact that

$$\bar{Q}'_\infty(\eta) = \frac{1}{\eta^{N-1}}\Big[ -(N-2)G(\log(\eta)) + G_\xi(\log(\eta))\Big]$$




$$= \frac{1}{\eta^{N-1}}(W - (N-2)G)(\log(\eta)) < 0 \quad \text{for } -\infty < \xi < \xi^*.$$

**Proposition 2.** *There exists a unique solution of* (2.15), (2.16) *defined in an interval* $[0, \eta^*]$ *with* $\eta^* = e^{\xi^*}$. *Moreover,*

$$\bar{Q}_\infty(\eta^*) = 0, \;\; \bar{Q}'_\infty(\eta^*) < 0$$
$$\bar{Q}'_\infty(\eta) \leq 0 \;\; \textit{for} \;\; \eta \in [0, \eta^*].$$

**Remark 3.** Notice that $\eta^*$ depends on the space dimension $N$. We will use indistinctly the notations $\eta^*$ and $\eta^*(N)$ for this number. In particular we will use the second one if the explicit dependence of $\eta^*$ on $N$ plays a crucial role in the argument.

Figure 2 shows the aspect of the function $Q = \bar{Q}_\infty$.

2.3. **Construction of the functions $\bar{Q}_\beta$ for $\beta$ sufficiently large.** We can analyze now the solutions $\bar{Q}_\beta$ of (2.13), (2.14) for large $\beta$ using standard ODE arguments.

**Theorem 4.** *There exists* $\beta_0 > 0$ *sufficiently large such that, for any* $\beta > \beta_0$, *there exist* $\eta_\beta > 0$ *and a unique solution* $\bar{Q}_\beta$ *of* (2.13), (2.14) *defined for* $\eta \in [0, \eta_\beta]$. *Moreover,*

$$\bar{Q}_\beta(\eta_\beta) = 0 \tag{2.27}$$

*and*

$$\lim_{\beta\to\infty} \bar{Q}_\beta(\eta) = \bar{Q}_\infty(\eta) \tag{2.28}$$

*uniformly in compact subsets of* $I = \{\eta \in [0,\infty) : \bar{Q}_\infty(\eta) > 0\}$. $\lim_{\beta\to\infty} \eta_\beta = \eta^*$ *with* $\bar{Q}_\infty$, $\eta^*$ *as in Proposition* 2.

**Remark 5.** We will use also the notation $\eta_\beta(N)$ for the number $\eta_\beta$ if the space dimension plays a role in the argument (cf. Remark 3).

**Proof of Theorem 4.** This theorem just follows from standard continuous dependence results for ODEs. The only nontrivial part is due to the fact that the equation (2.13) is singular at $\eta = 0$. However, this difficulty can be overcome using the change of variables $\bar{Q}_\beta(\eta) = G_\beta(\log(\eta)) = G_\beta(\xi)$ (cf. (2.17), (2.18)) that transforms (2.13), (2.14) into

$$G_{\beta,\xi\xi} - (N-2)G_{\beta,\xi} + G_\beta^{\frac{N}{N-2}} = \frac{e^{N\xi}}{c_N \beta^{\frac{N}{N-2}}}, \quad G_\beta(\xi) \sim e^{(N-2)\xi} \text{ as } \xi \to -\infty.$$

The existence, uniqueness and continuous dependence on $\beta$ of $G_\beta$ then follows from standard continuous dependence results for stable manifolds of ODEs (cf. [7]). □

2.4. **Compactly supported self-similar solutions of the system (1.1), (1.2).** The previous analysis can be thought of as the construction of a one-parameter family of self-similar solutions for the system (1.1), (1.2). Due to the presence of the interfaces we need to use a suitable concept of weak solutions. We will use the following standard definition for equations of the porous medium type (cf. [23]).

**Definition 6.** *A pair* $(u, v)$ *of nonnegative functions defined in* $\mathbb{R}^N \times (0,T)$ *is a weak solution of the system* (1.1)-(1.3) *if*

(*i*) $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(0,T'; L^\infty(\mathbb{R}^N))$ *for any* $T' \in (0,T)$.
(*ii*) $\nabla(u^m) \in L^2(0,T'; L^2(\mathbb{R}^N))$ *for any* $T' \in (0,T)$.
(*iii*) $v \in L^\infty(0,T'; H^1(\mathbb{R}^N))$ *for any* $T' \in (0,T)$.
(*iv*) $(u,v)$ *satisfies the following:*

$$\int_0^T \int_{\mathbb{R}^N} (\nabla(u^m)\cdot\nabla\varphi - u\nabla v\cdot\nabla\varphi - u\varphi_t)dxdt = 0, \tag{2.29}$$

$$\int_{\mathbb{R}^N} (\nabla v\cdot\nabla\psi - u\psi)dx = 0 \;\; a.a\; t \in (0,T) \tag{2.30}$$

*for all* $\varphi,\ \psi \in C_0^\infty((0,T)\times\mathbb{R}^N)$.

The following theorem holds.

**Theorem 7.** *Suppose that* $\beta > \beta_0$ *with* $\beta_0$ *as in Theorem* 4. *Let us define* $\Phi_\beta$, $\Psi_\beta$ *as*

$$\Phi_\beta(r) = c_N \beta^{\frac{N}{N-2}} \left(\bar{Q}_\beta\left(\sqrt{c_N}\beta^{\frac{1}{N-2}} r\right)\right)^{\frac{N}{N-2}}, \quad 0 \le r \le \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}, \tag{2.31}$$

$$\Psi_\beta(r) = \beta\bar{Q}_\beta\left(\sqrt{c_N}\beta^{\frac{1}{N-2}} r\right) - \frac{r^2}{2N}, \quad 0 \le r \le \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}, \tag{2.32}$$

$$r = |y|, \tag{2.33}$$

*where* $\bar{Q}$ *is as in Theorem* 4 *and* $c_N$ *is as in* (2.8).

$$\Phi_\beta(r) = 0, \quad r \ge \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}, \tag{2.34}$$

$$\Psi_\beta(r) = a_\beta + \frac{b_\beta}{r^{N-1}}, \quad r \ge \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}} \tag{2.35}$$

*with*

$$a_\beta = -\frac{1}{2N}\Big(\frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}\Big)^2 - \frac{b_\beta}{(\frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}})^{N-1}}, \tag{2.36}$$

$$b_\beta = \frac{1}{(N-1)}\Big(\frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}\Big)^N\Big[\frac{\eta_\beta}{N\sqrt{c_N}\beta^{\frac{1}{N-2}}} - \sqrt{c_N}\beta^{\frac{N-1}{N-2}}\bar{Q}'_\beta(\eta_\beta)\Big]. \tag{2.37}$$

*Then the pair* $(u_\beta, v_\beta)$ *defined by means of (cf.* (2.1)*)*

$$u_\beta(x,t) = \frac{1}{(T-t)}\Phi_\beta(y),\ v_\beta(x,t) = (T-t)^{\frac{2}{N}-1}\Psi_\beta(y),\ y = \frac{x}{(T-t)^{\frac{1}{N}}} \tag{2.38}$$

*is a weak solution of the system* (2.2), (2.3) *in the sense of Definition* 6.

**Remark 8.** Notice that the family of solutions obtained in Theorem 7 is a one-parameter family of solutions depending on the parameter $\beta > \beta_0$. The existence of such a continuum of forward self-similar blowing-up solutions is unusual for parabolic or parabolic-elliptic problems, and to our knowledge it is the first example of this class of behaviour in this class of problems. In typical parabolic problems, the families of backward blow-up solutions are a discrete set (cf. for instance [1], [9], [14]). On the other hand, as we will check later, the family of solutions can be parameterized for large values of $\beta$ using as parameter the total mass of the solution

$$m_\beta = \int_{\mathbb{R}^N} u_\beta(x,t)dx = \int_{\mathbb{R}^N} u_\beta(x,0)dx \tag{2.39}$$

instead of $\beta$. Therefore, assuming that the solutions described in Theorem 7 are stable, they would be selected by the total amount of mass aggregating at a given point. Notice that this number is not necessarily the same as the total mass of the initial data $u_0(x)$ because due to the finite speed of propagation for the support of the solutions of (1.1), (1.2) one can easily construct different solutions with different masses and the same blow-up mechanism, just putting together noninteracting compactly supported pieces like the ones described in Theorem 7.

**Proof.** Equation (2.30) is satisfied due to the fact that we have chosen the values of $a_\beta,\ b_\beta$ in (2.36), (2.37) to obtain continuity for $\Psi_\beta$ and $\Psi'_\beta$ at the point $r = \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}$. Since $\Phi_\beta,\ \Psi_\beta$ solve (2.2), (2.3) classically for $r < \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}$ and $r > \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}$, we only need to check the possible contributions in (2.29) due to the terms in the set $r = \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}$. In particular, the possible contribution of the discontinuity of the derivative of $(\Phi_\beta)^{m-1} = (\Phi_\beta)^{\frac{N-2}{N}}$ at $r = \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}$ must be examined. Notice, however, that since

$$(\Phi_\beta(r))^{\frac{N-2}{N}} = (c_N)^{\frac{N-2}{N}}\beta\bar{Q}_\beta(\sqrt{c_N}\beta^{\frac{1}{N-2}}r)$$

and $\bar{Q}'_\beta(\eta_\beta^-) < 0$, this discontinuity is similar to the one in the Barenblatt solution and the fact that $u_\beta,\ v_\beta$ are weak solutions of (1.1), (1.2) can be checked with a similar argument to the one used for such a solution. □

In the rest of the paper we will study the asymptotic behaviour of the solutions $(u_\beta, v_\beta)$ as $\beta \to \infty$. In particular this will allow us to show that these solutions can be parameterized by their mass $m_\beta$ as defined in (2.39).

## 3. Asymptotics of $m_\beta$ as $\beta \to \infty$

3.1. **First-order expansion of $\bar{Q}_\beta$ in powers of $\frac{1}{\beta^{\frac{N}{N-2}}}$ as $\beta \to \infty$.** In order to compute the first-order asymptotics of $m_\beta$ for large values of $\beta$ we need to compute first-order asymptotic formulas for $\bar{Q}_\beta$, the solution of (2.13), (2.14). Such asymptotics are a consequence of standard analytic perturbative theory for ODEs (cf. [7]). Therefore, we just indicate the formal argument to derive the expression of the first-order corrective terms.

We write $\bar{Q}_\beta$ as the following formal series:

$$\bar{Q}_\beta(\eta) = \bar{Q}_\infty(\eta) + \frac{1}{\beta^{\frac{N}{N-2}}} H(\eta) + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big). \tag{3.1}$$

Plugging (3.1) into (2.13), (2.14) and comparing similar powers of $\beta$, we obtain that $H(\eta)$ satisfies

$$H'' + \frac{N-1}{\eta} H' + \frac{N(\bar{Q}_\infty)^{\frac{2}{N-2}}}{N-2} H - \frac{1}{c_N} = 0, \tag{3.2}$$

$$H(0) = 0. \tag{3.3}$$

Notice that given $\bar{Q}_\infty$, the solution of (2.15), (2.16), we can obtain a family of solutions satisfying (2.15) (but not (2.16)), just by rescaling

$$q_\infty(\eta;\theta) = \theta \bar{Q}_\infty(\theta^{\frac{1}{N-2}} \eta). \tag{3.4}$$

The solution of (3.2), (3.3) can be computed using the fact that the function

$$\psi(\eta) \equiv \frac{\partial q_\infty(\eta;1)}{\partial \theta} = \bar{Q}_\infty(\eta) + \frac{1}{N-2} \eta \bar{Q}'_\infty(\eta) \tag{3.5}$$

satisfies the homogeneous equation associated to (3.2)

$$\psi'' + \frac{N-1}{\eta}\psi' + \frac{N(\bar{Q}_\infty)^{\frac{2}{N-2}}}{N-2}\psi = 0. \tag{3.6}$$

The solution of (3.2), (3.3) can then be obtained in the form $H(\eta) = a(\eta)\psi(\eta)$, where $a(\eta)$ satisfies

$$a'' + \frac{N-1}{\eta} a' + 2\frac{\psi'}{\psi} a' = \frac{1}{c_N}\frac{1}{\psi}, \qquad a(0) = 0.$$

This equation can be explicitly solved:

$$a(\eta) = \frac{1}{c_N} \int_0^\eta \frac{d\xi}{\xi^{N-1}(\psi(\xi))^2} \int_0^\xi s^{N-1}\psi(s)ds.$$

Therefore,

$$H(\eta) = \frac{\psi(\eta)}{c_N} \int_0^\eta \frac{d\xi}{\xi^{N-1}(\psi(\xi))^2} \int_0^\xi s^{N-1}\psi(s)ds. \tag{3.7}$$

The following result is by now standard.

**Proposition 9.** *Let $\bar{Q}_\beta$ be the unique solution of* (2.13), (2.14) *with* $\beta > 0$. *Then the asymptotics* (3.1) *hold, with $\psi$ as in* (3.5) *and $H$ as in* (3.7). *The asymptotics of $\bar{Q}'_\beta$ as $\beta \to \infty$ can be obtained by formally differentiating in both sides of* (3.1).

3.2. **First-order asymptotics of $\eta_\beta$ as $\beta \to \infty$.** We now derive the asymptotics of $\eta_\beta$ as $\beta \to \infty$. This is just a consequence of the implicit function theorem combined with (3.1). We recall (cf. (2.27)) that $\bar{Q}_\beta(\eta_\beta) = 0$, whence, using (3.1),

$$\bar{Q}_\infty(\eta^*) + \bar{Q}'_\infty(\eta^*)(\eta_\beta - \eta^*) + \frac{1}{\beta^{\frac{N}{N-2}}} H(\eta^*) + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big) + O((\eta_\beta - \eta^*)^2) = 0$$

as $\beta \to \infty$, where $\eta^*$ is the same as in Proposition 2.

Since $\bar{Q}_\infty(\eta^*) = 0$ and $\bar{Q}'_\infty(\eta^*) < 0$ by Proposition 2, it then follows that

$$\eta_\beta - \eta^* = -\frac{H(\eta^*)}{\bar{Q}'_\infty(\eta^*)} \frac{1}{\beta^{\frac{N}{N-2}}} + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big) \quad \text{as} \quad \beta \to \infty. \tag{3.8}$$

Summarizing, we have the following.

**Proposition 10.** *Suppose that $\bar{Q}_\beta$ is the unique solution of* (2.13), (2.14) *with $\beta > 0$ sufficiently large. Then the function $\eta_\beta$ defined by means of* (2.27) *satisfies the asymptotics* (3.8).

3.3. **A more convenient way of writing $m_\beta$.** We now rewrite $m_\beta$ in (2.39) in a more convenient way. Using (2.31), (2.34), (2.39) and (2.38) we obtain, after some elementary changes of variables,

$$\begin{aligned} m_\beta &= c_N \int_{\{|y| \le \frac{\eta_\beta}{\sqrt{c_N}\beta^{\frac{1}{N-2}}}\}} \beta^{\frac{N}{N-2}} \left(\bar{Q}_\beta\left(\sqrt{c_N}\beta^{\frac{1}{N-2}}|y|\right)\right)^{\frac{N}{N-2}} dy \\ &= (c_N)^{1-\frac{N}{2}} \, |\partial B_1(0)| \int_0^{\eta_\beta} \left(\bar{Q}_\beta(\eta)\right)^{\frac{N}{N-2}} \eta^{N-1} d\eta, \end{aligned} \tag{3.9}$$

where $|\partial B_1(0)|$ denotes the $(N-1)-$dimensional surface measure of the unit sphere in $\mathbb{R}^N$.

We now use equation (2.13) to derive a more convenient representation formula for $m_\beta$. Notice that (2.13) can be rewritten as

$$\frac{d}{d\eta}\Big(\eta^{N-1}\frac{d\bar{Q}}{d\eta}\Big) + \eta^{N-1}\bar{Q}^{\frac{N}{N-2}} - \frac{\eta^{N-1}}{c_N\beta^{\frac{N}{N-2}}} = 0.$$

Integrating this equation in $[0, \eta_\beta]$ we obtain

$$\int_0^{\eta_\beta} (\bar{Q}_\beta(\eta))^{\frac{N}{N-2}} \eta^{N-1} d\eta = \frac{(\eta_\beta)^N}{N c_N \beta^{\frac{N}{N-2}}} - (\eta_\beta)^{N-1} \bar{Q}'_\beta(\eta_\beta),$$

whence

$$m_\beta = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta_\beta)^{N-1}\Big[\frac{\eta_\beta}{Nc_N\beta^{\frac{N}{N-2}}} - \bar{Q}'_\beta(\eta_\beta)\Big]. \tag{3.10}$$

3.4. **Asymptotic expansion of $m_\beta$.** We now combine Theorem 4, Propositions 9 and 10 and (3.10) to derive an asymptotic formula for $m_\beta$ as $\beta \to \infty$. Using Proposition 9 and (3.10) we obtain

$$m_\beta = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta_\beta)^{N-1}\Big[\frac{\eta_\beta}{Nc_N\beta^{\frac{N}{N-2}}} - \bar{Q}'_\infty(\eta_\beta) - \frac{1}{\beta^{\frac{N}{N-2}}}H'(\eta_\beta)\Big] + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big). \tag{3.11}$$

On the other hand, Proposition 10 implies that $\eta_\beta \to \eta^*$ as $\beta \to \infty$. Thus

$$m_\beta = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta^*)^{N-1}\Big(1 + (N-1)\frac{\eta_\beta - \eta^*}{\eta^*}\Big)\times \tag{3.12}$$
$$\Big[-\bar{Q}'_\infty(\eta^*) + \frac{\eta^*}{Nc_N\beta^{\frac{N}{N-2}}} - \bar{Q}''_\infty(\eta^*)(\eta_\beta - \eta^*) - \frac{1}{\beta^{\frac{N}{N-2}}}H'(\eta^*)\Big] + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big).$$

Using (3.8) we obtain

$$m_\beta = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta^*)^{N-1}\Big(1 - \frac{(N-1)}{\eta^*}\frac{H(\eta^*)}{\bar{Q}'_\infty(\eta^*)}\frac{1}{\beta^{\frac{N}{N-2}}}\Big)\cdot$$
$$\Big[-\bar{Q}'_\infty(\eta^*) + \frac{\eta^*}{Nc_N\beta^{\frac{N}{N-2}}} + \bar{Q}''_\infty(\eta^*)\frac{H(\eta^*)}{\bar{Q}'_\infty(\eta^*)}\frac{1}{\beta^{\frac{N}{N-2}}} - \frac{1}{\beta^{\frac{N}{N-2}}}H'(\eta^*)\Big]$$
$$+ o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big),$$

and, after some computations,

$$m_\beta = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta^*)^{N-1}\cdot$$
$$\Big[-\bar{Q}'_\infty(\eta^*) + \frac{\eta^*}{Nc_N\beta^{\frac{N}{N-2}}} + \Big(\frac{\bar{Q}''_\infty(\eta^*)}{\bar{Q}'_\infty(\eta^*)} + \frac{(N-1)}{\eta^*}\Big)\frac{H(\eta^*)}{\beta^{\frac{N}{N-2}}} - \frac{1}{\beta^{\frac{N}{N-2}}}H'(\eta^*)\Big]$$
$$+ o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big).$$

Using (2.15) we obtain

$$\Big(\frac{\bar{Q}''_\infty(\eta^*)}{\bar{Q}'_\infty(\eta^*)} + \frac{(N-1)}{\eta^*}\Big) = -\frac{(\bar{Q}_\infty(\eta^*))^{\frac{N}{N-2}}}{\bar{Q}'_\infty(\eta^*)} = 0.$$

Thus

$$m_\beta = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta^*)^{N-1} \times \Big[ -\bar{Q}'_\infty(\eta^*) + \frac{\eta^*}{Nc_N\beta^{\frac{N}{N-2}}} - \frac{1}{\beta^{\frac{N}{N-2}}} H'(\eta^*)\Big] + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big). \tag{3.13}$$

Taking into account (3.7), we obtain

$$H'(\eta^*) = \frac{\psi'(\eta^*)}{c_N} \int_0^{\eta^*} \frac{d\xi}{\xi^{N-1}(\psi(\xi))^2} \int_0^\xi s^{N-1}\psi(s)ds + \frac{1}{c_N(\eta^*)^{N-1}\psi(\eta^*)} \int_0^{\eta^*} s^{N-1}\psi(s)ds. \tag{3.14}$$

Using (3.5) we obtain

$$\int_0^{\eta^*} s^{N-1}\psi(s)ds = \frac{1}{N-2}\int_0^{\eta^*} s^2 \frac{d}{ds}(s^{N-2}\bar{Q}_\infty(s))ds = -\frac{2}{N-2}\int_0^{\eta^*} s^{N-1}\bar{Q}_\infty(s)ds, \tag{3.15}$$

$$\psi(\eta^*) = \frac{1}{N-2}\eta^*\bar{Q}'_\infty(\eta^*), \tag{3.16}$$

$$\psi'(\eta^*) = \Big(1+\frac{1}{N-2}\Big)\bar{Q}'_\infty(\eta^*) + \frac{1}{N-2}\eta^*\bar{Q}''_\infty(\eta^*),$$

and using (2.15) as well as the fact that $\bar{Q}_\infty(\eta^*) = 0$,

$$\psi'(\eta^*) = \Big(1+\frac{1}{N-2}\Big)\bar{Q}'_\infty(\eta^*) - \frac{N-1}{N-2}\bar{Q}'_\infty(\eta^*) = 0. \tag{3.17}$$

Plugging (3.15), (3.17) into (3.14) we obtain

$$H'(\eta^*) = -\frac{2}{c_N(N-2)(\eta^*)^{N-1}\psi(\eta^*)}\int_0^{\eta^*} s^{N-1}\bar{Q}_\infty(s)ds,$$

and using (3.16) it then follows that

$$H'(\eta^*) = -\frac{2}{c_N(\eta^*)^N\bar{Q}'_\infty(\eta^*)}\int_0^{\eta^*} s^{N-1}\bar{Q}_\infty(s)ds.$$

Then, using (3.13), we obtain

$$m_\beta = m_\infty + \frac{A}{\beta^{\frac{N}{N-2}}} + o\Big(\frac{1}{\beta^{\frac{N}{N-2}}}\Big) \quad \text{as} \quad \beta\to\infty, \tag{3.18}$$

where, using the fact that $\bar{Q}'_\infty(\eta^*) < 0$,

$$m_\infty = |\partial B_1(0)|(c_N)^{1-\frac{N}{2}}(\eta^*)^{N-1}|\bar{Q}'_\infty(\eta^*)|, \tag{3.19}$$

$$A = \frac{|\partial B_1(0)|(c_N)^{-\frac{N}{2}}(\eta^*)^N}{N}\Big[1 - \frac{2N}{|\bar{Q}'_\infty(\eta^*)|}\frac{\int_0^{\eta^*} s^{N-1}\bar{Q}_\infty(s)ds}{(\eta^*)^{N+1}}\Big]. \tag{3.20}$$

3.5. **Numerical values.** In this subsection we collect the values of $m_\infty$, $A$, $\eta^*$ that we have computed numerically for dimensions $3 \le N \le 10$.

| $N$ | $\frac{m_\infty}{(\eta^*)^N}$ | $\eta^*$ | $B = \frac{A}{(\eta^*)^N}$ |
|---|---|---|---|
| 3 | 0.618 | 6.897 | 0.0377 |
| 4 | 1.229 | 6.346 | 0.0621 |
| 5 | 1.543 | 6.784 | 0.0681 |
| 6 | 1.652 | 7.402 | 0.0681 |
| 7 | 1.607 | 8.077 | 0.0660 |
| 8 | 1.458 | 8.773 | 0.0631 |
| 9 | 1.250 | 9.478 | 0.0601 |
| 10 | 1.020 | 10.186 | 0.0572 |

Table 1 : Values of $\dfrac{m_\infty}{(\eta^*)^N}$, $\eta^*$, $B$

Notice that $A > 0$ and therefore (3.18) implies $m_\beta \searrow m_\infty$ as $\beta \to \infty$.

3.6. **Asymptotics as $N \to \infty$.** In this subsection we compute, using formal asymptotic methods, the function $\bar{Q}_\infty$ and some related quantities, like its mass and the size of its support, as $N \to \infty$. Notice that the problem (2.15), (2.16) formally reduces in the limit $N \to \infty$ to

$$\frac{N-1}{\eta}\bar{Q}'_\infty + \bar{Q}_\infty = 0, \tag{3.21}$$

$$\bar{Q}_\infty(0) = 1. \tag{3.22}$$

This approximation remains valid as long as the derivatives $\bar{Q}''_\infty$ are of order one and $\bar{Q}_\infty$ remains bounded and uniformly away from zero. Using (3.21), (3.22) we obtain the following approximation:

$$\bar{Q}_\infty(\eta) = \exp\Big(-\frac{\eta^2}{2(N-1)}\Big). \tag{3.23}$$

This approximation breaks down when $\eta$ is sufficiently large and $(\bar{Q}_\infty)^{\frac{2}{N-1}}$ differs from one in a meaningful way. In order to describe how this happens, we rewrite (2.15), (2.16) as

$$\bar{Q}_\infty'' + \frac{N-1}{\eta}\bar{Q}_\infty' + \bar{Q}_\infty \exp\left(\frac{2}{N-2}\log(\bar{Q}_\infty)\right) = 0 \tag{3.24}$$

and we make the change of variables

$$S = \frac{\log(\bar{Q}_\infty)}{(N-2)}, \quad \eta = (N-1)\zeta \tag{3.25}$$

that transforms (3.24) into

$$\frac{1}{(N-2)}S_{\zeta\zeta} + (S_\zeta)^2 + \frac{(N-1)}{(N-2)}\frac{S_\zeta}{\zeta} + \left(\frac{N-1}{N-2}\right)^2 e^{2S} = 0. \tag{3.26}$$

In the limit $N \to \infty$, (3.26) becomes to the leading order

$$(S_\zeta)^2 + \frac{S_\zeta}{\zeta} + e^{2S} = 0, \tag{3.27}$$

and taking into account (3.23), we must solve (3.27) with the matching condition

$$S(\zeta) \sim -\frac{\zeta^2}{2} \quad \text{as} \quad \zeta \to 0. \tag{3.28}$$

Notice that the equation (3.27) provides two roots for $S_\zeta$ for each value of $\zeta$, $S$. However, the only one that is compatible with the asymptotics (3.28) for small $\zeta$ is

$$S_\zeta = \frac{1}{2\zeta}\Big[-1 + \sqrt{1 - 4\zeta^2 e^{2S}}\Big]. \tag{3.29}$$

This equation can be solved explicitly using the change of variables

$$H = \zeta^2 e^{2S} \tag{3.30}$$

that transforms (3.28), (3.29) into

$$\zeta H_\zeta = H\big[-1 + \sqrt{1-4H}\big] + 2H = H\big[1 + \sqrt{1-4H}\big], \tag{3.31}$$

$$H(\zeta) \sim \zeta^2 \quad \text{as} \quad \zeta \to 0. \tag{3.32}$$

The unique solution of (3.31), (3.32) can be computed explicitly:

$$-\frac{1}{4H} + \frac{\sqrt{1-4H}}{4H} - \operatorname{arctanh}\sqrt{1-4H} = \log(\zeta) + C.$$

Using the asymptotics

$$\operatorname{arctanh}\sqrt{1-4H} \sim -\frac{1}{2}\log(H) + O(H) \quad \text{as} \quad H \to 0,$$

as well as (3.30), we obtain $C = -\frac{1}{2}$, whence

$$-\frac{1}{4H} + \frac{\sqrt{1-4H}}{4H} - \operatorname{arctanh}\sqrt{1-4H} = \log(\zeta) - \frac{1}{2}. \tag{3.33}$$

Equation (3.31) implies that $H$ is increasing, and the value $H = \frac{1}{4}$ is reached at a finite value of $\zeta = \zeta_0$. For values of $\zeta \geq \zeta_0$ the function $H$ cannot be continued as a solution of (3.31). Using the fact that $H(\zeta_0) = \frac{1}{4}$ it follows from (3.33) that

$$\zeta_0 = \frac{1}{\sqrt{e}} \tag{3.34}$$

and (3.30) implies that

$$S(\zeta_0) = \frac{1}{2}\log\left(\frac{1}{4\zeta_0^2}\right) = \frac{1}{2}\log(\frac{e}{4}). \tag{3.35}$$

In a neighbourhood of $\zeta_0$, (3.26) cannot be approximated as (3.27) due to the fact that $S_{\zeta\zeta}$ diverges as $\zeta \to \zeta_0^-$. This follows from (3.29) that yields the following outer asymptotics:

$$S_{\zeta\zeta} \sim -\frac{1}{4\zeta_0^{3/2}}\frac{1}{\sqrt{\zeta_0-\zeta}} \quad \text{as} \quad \zeta \to \zeta_0^-, \tag{3.36}$$

where we have used the fact that

$$1 - 4\zeta^2 e^{2S} \sim \frac{1}{\zeta_0}(\zeta_0 - \zeta) \quad \text{as} \quad \zeta \to \zeta_0^-.$$

In order to describe the transition taking place as $\zeta$ approaches $\zeta_0$, we introduce new variables $\sigma$, $z$ as follows:

$$S(\zeta) = S(\zeta_0) - \frac{1}{2\zeta_0}(\zeta - \zeta_0) + \frac{1}{N}\sigma(z), \tag{3.37}$$

$$z = N^{2/3}(\zeta - \zeta_0), \tag{3.38}$$

where this specific rescaling has been obtained assuming that the term $\frac{1}{(N-2)}S_{\zeta\zeta}$ in (3.26) becomes comparable to the other terms from that equation as $\zeta \to \zeta_0^-$. Plugging (3.37), (3.38) into (3.26) and keeping the leading-order terms as $N \to \infty$ we obtain, using (3.35),

$$\sigma_{zz} + (\sigma_z)^2 + \frac{z}{4\zeta_0^3} = 0. \tag{3.39}$$

On the other hand, (3.36) yields the following matching condition:

$$\sigma_z \sim \frac{1}{2\zeta_0^{3/2}}\sqrt{-z} \quad \text{as} \quad z \to -\infty. \tag{3.40}$$

The solution of (3.39), (3.40) is uniquely defined and it develops a singularity at a finite value $z^*$ that can be computed explicitly using Airy's functions but whose precise value is not particularly important.

Near the singularity at $z = z^*$ we have the following asymptotics for the solution of (3.39):

$$\sigma_z \sim \frac{1}{(z-z^*)} - \frac{z^*}{12\zeta_0^3}(z-z^*) + .... \text{ as } z \to (z^*)^-, \tag{3.41}$$

$$\sigma \sim \log(|z-z^*|) + b - \frac{z^*}{24\zeta_0^3}(z-z^*)^2 + .... \text{ as } z \to (z^*)^-, \tag{3.42}$$

where $b$ is a numerical constant that can be explicitly computed in terms of Airy functions.

Using (3.25) we obtain $\bar{Q}_\infty = e^{(N-2)S}$. Differentiating this formula, and taking into account (3.37), we obtain

$$\bar{Q}'_\infty(\eta^*) \sim \frac{N^{\frac{2}{3}}}{(N-1)} e^{(N-2)[S(\zeta_0) - \frac{1}{2\zeta_0}\frac{z^*}{N^{2/3}}]} \lim_{z\to(z^*)^-}(e^{\sigma(z)}\sigma_z(z)) \text{ as } N \to \infty$$

and using (3.41), (3.42) we obtain

$$\lim_{z\to(z^*)^-}(e^{\sigma(z)}\sigma_z(z)) = -e^b,$$

whence

$$\bar{Q}'_\infty(\eta^*) \sim -\frac{N^{\frac{2}{3}}e^b}{(N-1)} e^{(N-2)[S(\zeta_0) - \frac{1}{2\zeta_0}\frac{z^*}{N^{2/3}}]} \text{ as } N \to \infty. \tag{3.43}$$

In order to approximate $A$ in (3.20) it only remains to approximate

$$\frac{\int_0^{\eta^*} s^{N-1}\bar{Q}_\infty(s)ds}{(\eta^*)^{N+1}}.$$

Let us define $\zeta^* = \frac{\eta^*}{(N-1)}$. Using the change of variables $\zeta = \frac{\eta}{(N-1)}$ we obtain

$$\begin{aligned}\frac{\int_0^{\eta^*} \eta^{N-1}\bar{Q}_\infty(\eta)d\eta}{(\eta^*)^{N+1}} &= \frac{(N-1)^N\int_0^{\zeta^*} \zeta^{N-1}\bar{Q}_\infty((N-1)\zeta)d\zeta}{(N-1)^{N+1}(\zeta^*)^{N+1}} \\ &\sim \frac{1}{N(\zeta^*)^{N+1}}\int_0^{\zeta^*} \zeta^{N-1}\bar{Q}_\infty((N-1)\zeta)d\zeta\end{aligned} \tag{3.44}$$

as $N \to \infty$. Using (3.25) we can write

$$\int_0^{\zeta^*} \zeta^{N-1}\bar{Q}_\infty((N-1)\zeta)d\zeta = \int_0^{\zeta^*} \zeta^{N-1}e^{(N-2)S(\zeta)}d\zeta$$

$$= \int_0^{\zeta^*} \zeta e^{(N-2)[\log\zeta + S(\zeta)]} d\zeta. \tag{3.45}$$

The main contribution to the last integral is due to the region around the maximum of $[\log\zeta + S(\zeta)]$ in the interval $[0, \zeta^*]$. Using (3.29) it follows that $\frac{d}{d\zeta}(\log\zeta + S(\zeta)) = \frac{1}{2\zeta} + \frac{1}{2\zeta}\sqrt{1 - 4\zeta^2 e^{2S}} > 0$, at least if $\zeta$ is not very close to $\zeta^*$. It then follows that the main contribution to the integral in (3.45) is due to the values of $\zeta$ satisfying $\zeta \approx \zeta^*$. In that region we can use the approximation (3.37). Then

$$\int_0^{\zeta^*} \zeta e^{(N-2)[\log\zeta + S(\zeta)]} d\zeta \sim e^{(N-2)S(\zeta_0)} \int_{\zeta^*-\delta}^{\zeta^*} \zeta e^{(N-2)[\log\zeta - \frac{1}{2\zeta_0}(\zeta - \zeta_0)]} e^{\sigma(z)} d\zeta$$

as $N \to \infty$. Using the classical Laplace method for computing the asymptotics of the integral on the right-hand side we obtain

$$\int_{\zeta^*-\delta}^{\zeta^*} \zeta e^{(N-2)[\log\zeta - \frac{1}{2\zeta_0}(\zeta - \zeta_0)]} e^{\sigma(z)} d\zeta \sim \frac{2\sqrt{\pi}}{\sqrt{3}} \frac{(\zeta_0)^N}{N^{2/3}} N^{\frac{1}{6}} e^{\beta N^{1/9}} \quad \text{as} \quad N \to \infty,$$

$$\beta = \frac{1}{2(\zeta_0)^{4/3}} \left[ \frac{1}{(16)^{1/6}} - \frac{1}{(16)^{2/3}} \right] > 0.$$

It then follows from (3.44), (3.45) that

$$\frac{\int_0^{\eta^*} \eta^{N-1} \bar{Q}_\infty(\eta) d\eta}{(\eta^*)^{N+1}} \sim \frac{2\sqrt{\pi}}{\sqrt{3}} \frac{(\zeta_0)^N}{(\zeta^*)^{N+1}} \frac{e^{(N-2)S(\zeta_0)}}{N^{3/2}} e^{\beta N^{1/9}} \quad \text{as} \quad N \to \infty. \tag{3.46}$$

Using (3.38) we obtain

$$\frac{(\zeta_0)^N}{(\zeta^*)^N} \sim \frac{(\zeta_0)^N}{(\zeta_0 + \frac{z^*}{N^{2/3}})^N} \sim e^{-\frac{z^*}{\zeta_0} N^{1/3}} \quad \text{as} \quad N \to \infty$$

and combining this with (3.46) we obtain

$$\frac{\int_0^{\eta^*} \eta^{N-1} \bar{Q}_\infty(\eta) d\eta}{(\eta^*)^{N+1}} \sim \frac{2\sqrt{\pi}}{\sqrt{3}\zeta_0} \frac{e^{(N-2)S(\zeta_0)}}{N^{3/2}} e^{-\frac{z^*}{\zeta_0} N^{1/3}} e^{\beta N^{1/9}} \quad \text{as} \quad N \to \infty. \tag{3.47}$$

Using now (3.43) and (3.47) we obtain

$$\frac{2N}{|\bar{Q}'_\infty(\eta^*)|} \frac{\int_0^{\eta^*} s^{N-1} \bar{Q}_\infty(s)\, ds}{(\eta^*)^{N+1}} \sim \frac{4e^{-b}\sqrt{\pi}}{\sqrt{3}\zeta_0} \frac{1}{N^{\frac{1}{6}}} e^{-\frac{z^*}{2\zeta_0} N^{1/3} + \beta N^{1/9}} \tag{3.48}$$

as $N \to \infty$. Notice that the right-hand side of (3.48) approaches zero as $N \to \infty$. Therefore, using (3.20), it follows that, as $N \to \infty$,

$$m_\infty = \frac{|\partial B_1(0)|\, e^b\, (c_N)^{1-\frac{N}{2}}\, (\eta^*)^{N-1}}{N^{1/3}} \left(\frac{e}{4}\right)^{\frac{N-2}{2}} e^{-\frac{N^{1/3}}{2\zeta_0} z^*} \to \infty, \tag{3.49}$$

$$A \sim \frac{|\partial B_1(0)|(c_N)^{-\frac{N}{2}}(\eta^*)^N}{N} > 0 \quad \text{as} \quad N \to \infty, \tag{3.50}$$

where

$$|\partial B_1(0)| = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}.$$

Thus we find that $A > 0$ for $N$ large and $m_\beta > m_\infty$ in (3.18).

3.7. **Comparison between numerics and asymptotics.** We now compare the values obtained for $\eta^*$ computing numerically the solution of (2.15), (2.16) with the asymptotic formula

$$\eta^* \sim \eta^*_{asympt} = \frac{N}{\sqrt{e}} \quad \text{as} \quad N \to \infty. \tag{3.51}$$

In the enclosed table we compare this asymptotic formula with the value of $\eta$ numerically computed. It can be seen that the agreement between these quantities is not obtained until $N$ reaches rather high values.

| $N$ | $\eta^*$ | $\eta^*_{asympt}$ |
|---|---|---|
| 3 | 6.897 | 1.820 |
| 6 | 7.402 | 3.639 |
| 10 | 10.186 | 6. 065 |
| 25 | 20.581 | 15.163 |
| 50 | 37.177 | 30.327 |
| 75 | 50.881 | 45.490 |
| 100 | 63.212 | 60.653 |

Table 2 : Comparison between $\eta^*$ and $\eta^*_{asympt}$.

## 4. General comments: critical mass and type I blow up

4.1. **On the critical mass for blow up.** The system (1.1)-(1.3) has been studied in much detail in [3]. In that paper, the existence of a critical mass $M_c$ such that blow up is not possible for initial data $u_0(x)$ with smaller mass than $M_c$ has been established, among other results. It has also been shown in [3] that there exist solutions of (1.1 )-(1.3) with $\int_{\mathbb{R}^N} u_0(x)dx > M_c$ that blow up in finite time. Previous, nonoptimal bounds for the value of the critical mass have been obtained in [25] using optimal Sobolev constants. The system that has been considered in [25] is slightly different from the one in (1.1), (1.2), since the second equation has been replaced by the more general one

$$-\Delta v + \gamma v = u.$$

It has been shown in [25] that global existence always takes place if

$$\int_{\mathbb{R}^N} u_0(x)dx < \alpha(N)$$

and $\gamma \geq 0$. On the other had, it has also been proved that there are solutions exhibiting blow up with mass $\int_{\mathbb{R}^N} u_0(x)dx > \beta(N)$ if $\gamma > 0$. However, a careful examination of the argument in [25] shows that the same result applies also for $\gamma = 0$. The values of $\alpha(N)$, $\beta(N)$ are

$$\alpha(N) = \frac{\left(2\pi N^2\right)^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}, \qquad \beta(N) = \frac{\left(2\pi N^2\right)^{\frac{N}{2}} 2^{\frac{N(N-1)}{2}}}{N\Gamma\left(\frac{N}{2}\right)}.$$

These numbers agree and they take the value $8\pi$ for $N = 2$, but they satisfy $\alpha(N) < \beta(N)$ for $N \geq 3$.

The value of $M_c$ in [3] is the same as the value of $m_\infty$ in this paper. This value is just the mass of the function $\Phi$ in (2.7) with $Q = Q_\infty(r) = \beta\bar{Q}_\infty(\sqrt{c_N}\beta^{\frac{1}{N-2}})$ with $Q$ as in (2.15), (2.16) and an arbitrary value of $\beta$. An equivalent characterization in terms of a variational problem has been given in [3]. A comparison between the values of $m_\infty$ and the estimates $\alpha(N)$, $\beta(N)$ is given in the following table:

| $N$ | $m_\infty$ | $\alpha(N)$ | $\beta(N)$ |
|---|---|---|---|
| 3 | 202.905 | 188.429 | 1279.550 |
| 4 | 1992.455 | 1684.412 | 161703.599 |
| 5 | 22168.221 | 17128.710 | $4.764 \cdot 10^7$ |
| 6 | 271644.644 | 192883.846 | $3.160 \cdot 10^{10}$ |
| 7 | $3.605 \cdot 10^6$ | $2.363 \cdot 10^6$ | $4.616 \cdot 10^{13}$ |
| 8 | $5.118 \cdot 10^7$ | $3.113 \cdot 10^7$ | $1.462 \cdot 10^{17}$ |
| 9 | $7.713 \cdot 10^8$ | $4.366 \cdot 10^8$ | $9.935 \cdot 10^{20}$ |
| 10 | $1.226 \cdot 10^{10}$ | $6.477 \cdot 10^9$ | $1.436 \cdot 10^{25}$ |

Table 3: Comparison of $m_\infty$, $\alpha(N)$, $\beta(N)$.

Notice that the definitions of $\alpha(N)$, $\beta(N)$, $m_\infty$ imply the inequality

$$\alpha(N) < m_\infty < \beta(N).$$

Table 3 shows that this inequality holds for $3 \leq N \leq 10$.

4.2. **On the possibility of parameterizing the self-similar solutions by their mass for large $\beta$.** We remark that the asymptotics (3.18) as well as the fact that the constant $A$ is positive for $N = 3, ..., 10$ (cf. Table 1) as well as for $N \to \infty$ (cf (3.50)) imply that it is possible to parameterize the family of functions obtained in this paper by means of their mass $m_\beta$ for large values of $\beta$ and those values of $N$.

Notice that, since the results in [3] imply that it is not possible to have blow up for (1.1)-(1.3) with a mass smaller than $m_\infty$, the mass of the solutions constructed in this paper is necessarily larger than $m_\infty$. Therefore $A \geq 0$. We have not attempted to prove in an analytic manner that $A > 0$, but we have checked this to be true numerically for a range of "not too large" values of $N$. We have computed also asymptotically the behaviour of $N$ as $N \to \infty$.

4.3. **Type I blow up.** The most relevant result in this paper is the existence of solutions of (1.1)-(1.3) that blow up in finite time, but whose rate of growth can be bounded by a self-similar rate. This behaviour is usually termed Type I blow up. On the contrary, if the rate of growth of some of the variables associated to the problem increases faster than the self-similar rate, the blow up is termed Type II blow up. The possibility of having Type I blow up behaviour for (1.1)-(1.3) contrasts in a striking way with the case $N = 2,\ m = 1$, since, in this case, blow up is always of Type II in radial situations (cf. [20]), and this is expected to happen also in the nonradially symmetric case.

Another remarkable fact, that is also rather different from the situation in $N = 2$, is the existence of a continuum of values for the amount of mass aggregating. In the case $N = 2$ it is known that there exist solutions where the amount of mass $8\pi$ aggregates, and it is widely expected that the only possible values of the mass aggregating are multiples of $8\pi$. In other words, there would be a quantization of the aggregating values of the mass. On the contrary, the asymptotics (3.18) as well as the positivity of $A$ show the existence of a continuum of values for the amount of mass aggregating in the solutions constructed in this paper.

4.4. **On the boundedness of $\int_0^T \int_{R^N} (u(x,t))^m \, dxdt$.** We remark that in [22] it has been proved that the condition $\int^T \int_{R^N} (u(x,t))^m \, dxdt < \infty$ would imply that blow-up points would be isolated and that the solutions would converge to Dirac masses near the blow-up points. We have not considered this result in this paper. However, we remark that the solutions constructed in this paper satisfy such an assumption. Indeed

$$\int_0^T \int_{R^N} \big(u(x,t)\big)^{\frac{2(N-1)}{N}} dxdt$$
$$= \int_0^T \frac{dt}{(T-t)^{\frac{2(N-1)}{N}}} \int_{R^N} \Big(\Phi(\frac{x}{(T-t)^{\frac{1}{N}}})\Big)^{\frac{2(N-1)}{N}} dx$$
$$= \int_0^T \frac{dt}{(T-t)^{\frac{2(N-1)}{N}-1}} \int_{R^N} (\Phi(y))^{\frac{2(N-1)}{N}} dy = C\int_0^T \frac{dt}{(T-t)^{\frac{N-2}{N}}} < \infty.$$

## 5. CONCLUDING REMARKS

In this paper we have found a class of self-similar solutions of the Keller-Segel model with nonlinear diffusion with critical exponents (cf. (1.1)-(1.3)) and $N \geq 3$. These solutions exhibit Type I blow up, in contrast with the situation for the most studied case $N = 2$ with ordinary diffusion, where blow up, at least in the radial case, can only be of Type II (cf. [20]). We have checked with a combination of asymptotic methods and the numerical computation of some coefficients that the self-similar solutions obtained in this paper can take a continuum of values of the mass. This is also a striking difference from the case $N = 2$ in which the only solutions obtained so far have the mass $8\pi$. We have computed also asymptotic formulas for the values of the critical mass for large dimensions.

There are several questions posed in this paper that could be worth studying in the future. It would be interesting to find purely analytical proofs of the results obtained in this paper, that do not relay on the help of numerical computations. On the other hand, it could be worth finding necessary and sufficient conditions for the type of asymptotic behaviour described in these pages. Finally, it would be relevant to ascertain if Type II blow up can take place also for (1.1)-(1.3), and the precise form of this type of blow up if the answer is an affirmative one.

**Acknowledgements.** The authors acknowledge the hospitality of the Max Planck Institute in Leipzig where this work was partially done. JJLV is partially supported by Research Grant MTM2007-61755. He also acknowledges the hospitality of Universidad Complutense.

## REFERENCES

[1] J. Bebernes, A. Bressan, and D. Eberly, *A description of blow-up for the solid ignition problem,* Indiana Univ. Math. J., 36 (1987), 131–136.

[2] P. Biler, T. Nadzieja, and R. Stanczy, *Nonisothermal systems of self-attracting Fermi-Dirac particles,* Banach Center Publications, 66 (2004), 61–78.
[3] A. Blanchet, J. A. Carrillo, and P. Laurencot, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimension,* Calc. Var. Partial Differential Equations, 35 (2009), 133–168.
[4] A. Blanchet, J. A. Carrillo, and Mousmadi, *Infinite time of aggregation for the critical PKS model in* $\mathbb{R}^2$, Comm. Pure Appl. Math., 61 (2008), 1449–1481.
[5] H. Brezis and M. G. Crandall, *Uniqueness of solutions of the initial value problem for* $u_t - \Delta\phi(u) = 0$, J. Math. Pures Appl., 58 (1979), 153–163.
[6] J. Dolbeault and B. Perthame, *Optimal critical mass in the two-dimensional Keller-Segel model in* $\mathbb{R}^2$, C. R. Math. Acad. Sci. Paris, 339 (2004), 611–616.
[7] J. Guckenheimer and P. Holmes, "Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields," Springer-Verlag, 1983.
[8] M. A. Herrero, E. Medina, and J. J. L. Velázquez, *Finite time aggregation into a single point in a reaction diffusion system,* Nonlinearity, 10 (1997), 1739–1754.
[9] M. A. Herrero, E. Medina, and J. J. L. Velázquez, *Self-similar blow-up for a reaction-diffusion system,* Journal of Computational and Applied Mathematics, 97 (1998), 99–119.
[10] M. A. Herrero and J. J. L. Velázquez, *Singularity patterns in a chemotaxis model,* Math. Ann., 306 (1996), 583–623.
[11] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I,* Jahreber. Deutsch. Math.-Verein, 105 (2003), 103–165.
[12] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences II,* Jahreber. Deutsch. Math.-Verein, 106 (2004), 51–69.
[13] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability,* J. Theor. Biol., 26 (1970), 399–415.
[14] L. A. Lepin, *Countable spectrum of eigenfunctions of a nonlinear heat conduction equation with distributed parameters,* Diff. Urav., 24 (1988), 1226–1234.
[15] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system,* Adv. in Math. Sci. Appl., 5 (1995), 581–601.
[16] T. Nagai, T. Senba, and K. Yoshida, *Application of the Moser-Trudinger inequality to a parabolic system of chemotaxis,* Funkc. Ekvacioj, 40 (1997), 411–433.
[17] T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains,* J. Inequal. Appl., 6 (2001), 37–55.
[18] T. Nagai, T. Senba, and T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology,* Hiroshima Math. J., 30 (2000), 463–497.
[19] T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology,* Adv. Diff. Equ., 6 (2001), 21–50.
[20] T. Senba, *Type II blow-up of solutions to a simplified Keller-Segel system in two dimensional domains,* Nonlinear Analysis, 66 (2007), 1817–1839.
[21] Y. Sugiyama, *Partial regularity and blow-up asymptotics of weak solutions to degenerate parabolic systems of porous medium type,* submitted.
[22] Y. Sugiyama, *On* $\varepsilon-$*regularity theorem and asymptotic behaviours of solutions for Keller-Segel systems,* SIAM J. Math. Anal., 41 (2009), 1664–1692.

[23] Y. Sugiyama and H. Kunii, *Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term,* J. Differential Equations, 227 (2006), 333–364.

[24] Y. Sugiyama, *Global existence in the sub-critical cases and finite time blow-up in the super-critical cases to degenerate Keller-Segel systems,* Differential and Integral Equations, 19 (2006), 841–876.

[25] Y. Sugiyama, *Application of the best constant of the Sobolev inequality to degenerate Keller-Segel models,* Adv. Differential Equations, 12 (2007), 121–144.