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SIGN-CHANGING SOLUTIONS FOR AN ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION WITH DEEPENING POTENTIAL WELL

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Abstract. In this paper, by using variational methods, we obtain the existence of sign-changing solutions for an asymptotically linear Schrödinger equation with deepening potential well.

1. Introduction

In this paper we are concerned with the following nonlinear Schrödinger equation:

$$-\Delta u+\lambda g(x)u=f(u),\quad x\in\mathbb{R}^N,\ N\geq 3, \tag{P}$$

where λ is a positive parameter, the function g satisfies the condition

(G) $g\in L^\infty(\mathbb{R}^N,\mathbb{R})$ and there exists a nonempty bounded smooth domain $\Omega\subset\mathbb{R}^N$ such that

$$g(x)\equiv 0 \text{ on } \bar\Omega,\quad g(x)\in(0,1] \text{ on } \mathbb{R}^N\setminus\bar\Omega \text{ and } \lim_{|x|\to\infty} g(x)=1.$$

We note that, by the condition (G), the potential function $g(x)$ represents a potential well with bottom Ω whose depth is controlled by the parameter λ.

We will make the following assumptions on f.

(f_1) $f(t)\in C(\mathbb{R},\mathbb{R})$, $f(t)t>0$ for all $t\in\mathbb{R}$, $t\neq 0$.

(f_2) $\lim_{|t|\to 0}\frac{f(t)}{t}=0$.

(f_3) There exists $\alpha\in(0,\infty)$ such that $\lim_{|t|\to\infty}\frac{f(t)}{t}=\alpha$.

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(f_4) $f(t)/t$ is nondecreasing in $t \geq 0$, and nonincreasing in $t \leq 0$.

Usually, $f(t)$ with the property (f_3) is called asymptotically linear in t at infinity. When f is sublinear ($\lim_{|t|\to\infty} f(t)/t = 0$) or superlinear ($\lim_{|t|\to\infty} f(t)/t = \infty$), some existence results related to the problem (P) were obtained recently, see for example [1, 2, 3, 6, 9] and the references therein. When f is asymptotically linear at infinity, the existence and nonexistence of positive solutions of the problem (P) have been completely discussed by using variant methods, see for example [4, 8, 10], where, for given α, the properties of the principle eigenvalue (cf. [7]) of the following eigenvalue problem, obtained by linearizing the nonlinear term $f(u)$,

$$-\Delta u + \lambda g(x)u = \alpha u, \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

play an important role. To the best of our knowledge, the existence of sign-changing solutions of the problem (P) have not been obtained.

In order to obtain sign-changing solutions of the problem (P), we now reconsider the problem (1.1) for a fixed λ. Let $L_\lambda := -\Delta + \lambda g$. We have shown in [11] that the principle eigenvalue $\alpha_1(\lambda)$ of L_λ always exists for any $\lambda \in (\Gamma_1, +\infty)$ with

$$\Gamma_1 = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2\, dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (1-g)u^2\, dx = 1 \Big\},$$

and there exists a positive eigenfunction $u_{\alpha_1(\lambda)}$ corresponding to $\alpha_1(\lambda)$. Moreover, $\alpha_1(\lambda)$ is simple in the sense that

$$\ker(L_\lambda - \alpha_1(\lambda)I) = \operatorname{span}\{u_{\alpha_1(\lambda)}\} := V_1.$$

Define

$$\Gamma_2 = \inf_{u\in H^1(\mathbb{R}^N)\cap V_1^\perp} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\int_{\mathbb{R}^N} (1-g)u^2\, dx}.$$

Now let $\lambda \in (\Gamma_2, +\infty)$ be fixed and define

$$\alpha_2(\lambda) = \inf\{\Phi(u) : u \in H^1(\mathbb{R}^N) \cap V_1^\perp \text{ and } \|u\|_2 = 1\};$$

then $\alpha_2(\lambda)$ is the second eigenvalue of the operator L_λ (cf. [11]).

We separate the first quadrant of the (λ, α)-plane into several parts (cf. Fig. 1.1), namely, I $= \{(\lambda,\alpha) : \alpha > \alpha_2(\lambda) \text{ and } \lambda > \alpha\}$; II $= \{(\lambda,\alpha) : \alpha > \alpha_2(\lambda) \text{ and } \lambda < \alpha\}$; III $= \{(\lambda,\alpha) : \alpha_1(\lambda) < \alpha < \alpha_2(\lambda) \text{ and } \lambda > \alpha\}$; IV $= \{(\lambda,\alpha) : \lambda \leq \Gamma_1 \text{ and } \lambda < \alpha\}$ and V $= \{(\lambda,\alpha) : \lambda \geq \Gamma_1 \text{ and } \alpha \leq \alpha_1(\lambda)\}$. By using the method of invariant sets and the concentration-compactness principle, we will show that the problem (P) has at least a sign-changing solution in the regions I, II$\backslash\sigma_p(L_\lambda)$ in Figure 1.1 and the dotted diagonal line $\lambda = \alpha$ in Figure 1.1, where $\sigma_p(L_\lambda) = \{\alpha :$ there exists $u \in H^1(\mathbb{R}^N)$ such that $-\Delta u + \lambda g u = \alpha u\}$. For the region II, we need another assumption for g:

(G^*) there exist $c, m, R_0 > 0$ such that $1 - g(x) \geq \frac{c}{|x|^m}$ for $|x| \geq R_0$.

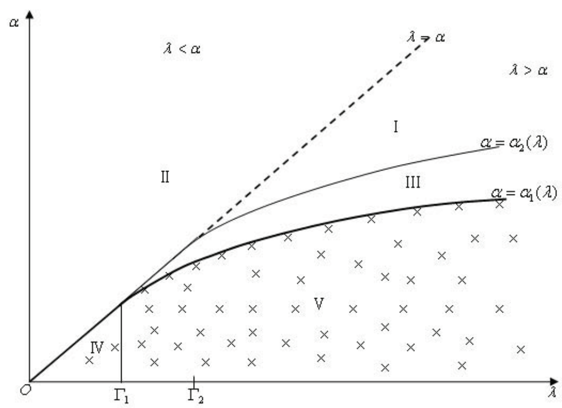

Figure 1.1

Now we give the main results of this paper.

Theorem 1.1. *Assume that* (G), (f_1)-(f_4) *hold, and also assume that* (G^*) *holds for* $\alpha > \lambda$. *If* $\alpha > \alpha_2(\lambda)$ *and* $\alpha \notin \sigma_p(L_\lambda)$, *then the problem (P) has a sign-changing solution (besides the positive one and the negative one).*

For the case of $\lambda > \alpha > \alpha_2(\lambda)$; i.e., (λ, α) is in the region I in Fig.1.1, we may obtain the same result without assuming monotonicity conditions. Actually, if f depends on x, we have proved that the problem (P) has at least a sign-changing solution in [12]. In this paper, we also consider this case. Since the conditions are a little different from [12], we give a sketch of the proof of the existence of sign-changing solutions. Moreover, for the case of $\lambda > \alpha > \alpha_1(\lambda)$; i.e., (λ, α) is in the regions I and III in Figure 1.1, by using the mountain pass lemma (see [15]) we may obtain the existence of a positive solution (and a negative solution) of the problem (P) without assuming monotonicity conditions. If f does not depend on x, in [10] the authors proved that the problem (P) has a positive solution for this case under the condition (f_4). In fact, by [8] we know that the problem (P) has no solutions in the regions IV and V.

In the rest of the section, we list some preliminaries which we will use later.

Recall that a functional I defined on a Banach space X is said to satisfy the Palais-Smale condition (the (PS) condition for short) if any sequence $\{u_n\} \subset X$ satisfying $|I(u_n)| \leq c$ and $I'(u_n) \to 0$ $(n \to \infty)$ possesses a convergent subsequence. I is said to satisfy the $(\text{Cerami})_c$ condition if any sequence $\{u_n\} \subset X$ with $I(u_n) \to c$, $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ $(n \to \infty)$ possesses a convergent subsequence.

Proposition 1.1. ([5, Theorem 3.2]) *Let X be a Hilbert space and let f be a C^1 functional defined on X. Assume that f satisfies the (PS) condition on X and $f'(u)$ has the expression $f'(u) = u - Au$ for $u \in X$. Assume that D_1 and D_2 are open convex subsets of X with the properties that $D_1 \cap D_2 \neq \varnothing$, $A(\partial D_1) \subset D_1$ and $A(\partial D_2) \subset D_2$. If there exists a path $h : [0,1] \to X$ such that*

$$h(0) \in D_1 \setminus D_2, \qquad h(1) \in D_2 \setminus D_1$$

and

$$\inf_{u \in \overline{D_1} \cap \overline{D_2}} f(u) > \sup_{t \in [0,1]} f(h(t)),$$

then f has at least four critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D_2}$, one in $D_2 \setminus \overline{D_1}$, and one in $X \setminus (\overline{D_1} \cup \overline{D_2})$.

Proposition 1.2. ([13, Lemma 2.1]) *Let $\{\rho_n\}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geq 0 \; \forall \, x \in \mathbb{R}^N, \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \rho_n \, dx = \eta > 0,$$

where η is fixed. Then there exists a subsequence of $\{\rho_n\}$, still denoted by $\{\rho_n\}$, satisfying one of the following two possibilities:

(i) (*Vanishing*) $\lim\limits_{n \to \infty} \sup\limits_{y \in \mathbb{R}^N} \int_{y + B_R} \rho_n \, dx = 0$, *for all $R > 0$.*

(ii) (*Non-vanishing*) *There exist $\nu, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that* $\varliminf\limits_{n \to \infty} \int_{y_n + B_R} \rho_n \, dx \geq \nu > 0$.

Lemma 1.1. ([14, Lemma I.1]) *Let $1 < p \leq +\infty$, $1 \leq q < +\infty$, with $q \neq \frac{Np}{N-p} \, \forall \, p < N$. Assume that $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$, $\{\nabla u_n\}$ is bounded in $L^p(\mathbb{R}^N)$, and*

$$\sup_{y \in \mathbb{R}^N} \int_{y + B_R} |u_n|^q \, dx \to 0, \;\; n \to \infty \quad \text{for some } R > 0.$$

Then $u_n \to 0$ $(n \to \infty)$ in $L^\theta(\mathbb{R}^N)$, for θ between q and $\frac{Np}{N-p}$.

The paper is organized as follows. In Section 2 we consider the limit problem associated to the problem (P). In Section 3 we give several preliminary results which will be used to obtain the existence of sign-changing solutions of the problem (P). In Section 4 we prove that the problem (P) has a sign-changing solution. In Section 5, for the case of $\lambda > \alpha$, we prove the existence of sign-changing solutions of the problem (P) without assuming monotonicity conditions.

Throughout this paper, $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively. B_R denotes the ball in $\mathbb{R}^N$ centered at zero with radius R. $|\cdot|_q$ denotes the standard norm in $L^q(\mathbb{R}^N)$ for $1 \le q \le +\infty$. We use $\langle u, v\rangle = \int_{\mathbb{R}^N} \big(\nabla u \nabla v + \lambda g(x) uv\big)\, dx$ as the inner product in the Hilbert space $X = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda g(x) u^2\, dx < \infty\}$ with the induced norm $\|u\|_\lambda = \sqrt{\langle u, u\rangle}$ (Clearly, $\|u\|_\lambda$ is equivalent to the norm $\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx\Big)^{1/2}$ by (G).). $c, c_0, c_1, c_2, ...$ denote (possibly different) positive constants.

2. Some results of the limit problem associated to (P)

We want to prove Theorem 1.1 by using Proposition 1.1, and encounter the difficulty caused by the lack of compactness. To overcome this difficulty we first consider the following problem:

$$-\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^N,\ N \ge 3, \tag{P_∞}$$

which is usually called the limit problem associated to (P) at infinity. Its energy functional is defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\, dx - \int_{\mathbb{R}^N} F(u)\, dx,$$

where $F(u) = \int_0^u f(t)\, dt$.

Proposition 2.1. *Assume (f_1)-(f_4) and (G) hold.*

(i) *If $\lambda < \alpha$, then the problem (P_∞) has positive solutions, and has a least energy positive solution among positive solutions. Moreover, the corresponding conclusion holds for negative solutions.*

(ii) *If $\lambda \ge \alpha$, then the problem (P_∞) has no non-trivial solutions. For the sake of convenience we will assume that the least energy is ∞.*

Proof. (i) We consider the following auxiliary functions:

$$f_+(t) = \begin{cases} f(t), & t \ge 0, \\ 0, & t \le 0, \end{cases} \quad \text{and} \quad f_-(t) = \begin{cases} f(t), & t \le 0, \\ 0, & t \ge 0, \end{cases}$$

and the associated functionals

$$J_{\pm}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda u^2)\,dx - \int_{\mathbb{R}^N} F_{\pm}(u)\,dx,$$

where $F_{\pm}(u) = \int_0^u f_{\pm}(t)\,dt$. We only consider the case of J_+. Similarly, the same results can be obtained for the case of J_-. By (f_1)-(f_3), we see that J_+ possesses a mountain pass geometry. Define

$$d_+^* = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma} J_+(u) > 0,$$

where $\Gamma = \{\gamma : \gamma \in C([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0, J_+(\gamma(1)) < 0\}$. Since the functional J_+ has a mountain pass geometry, we deduce (see [17]) the existence of a Cerami sequence at the mountain pass level d_+^*; namely, there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that

$$J_+(u_n) \to d_+^* \text{ and } (1 + \|u_n\|)\|J_+'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

We will prove that, up to a translation, $\{u_n\}$ has a subsequence weakly converging to a non-trivial critical point of J_+.

Firstly, we will show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We use an indirect argument. Suppose that $\|u_n\| \to \infty\,(n \to \infty)$. Let $w_n = u_n/\|u_n\|$. Clearly, $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By Proposition 1.2, there exists a subsequence (still denoted by $\{w_n\}$) such that one of the following cases occurs:

(1) (Vanishing) $\lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^N}\int_{y+B_R} w_n^2\,dx = 0,\ \forall\, R > 0.$

(2) (Non-vanishing) There exist $\nu, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\varliminf\limits_{n\to\infty}\int_{y_n+B_R} w_n^2\,dx \geq \nu > 0.$

Next we prove that both cases will lead to a contradiction.

Claim 1. Vanishing cannot occur. By the condition (f_3), for any fixed $p \in (2, 2^*)$, we have $\lim_{|t|\to\infty} f(t)/t^{p-1} = 0$ for all $x \in \mathbb{R}^N$; this, together with (f_2), implies that, for any $\varepsilon' > 0$, there exists $C_{\varepsilon'} > 0$ such that

$$|f_+(t)| \leq \varepsilon'|t| + C_{\varepsilon'}|t|^{p-1}. \tag{2.1}$$

Here we adopted some of the ideas in [10]. For any $L > 0$, we have

$$J_+\Big(\frac{L}{\|u_n\|}u_n\Big) = \frac{1}{2}L^2\int_{\mathbb{R}^N}(|\nabla w_n|^2 + \lambda w_n^2)\,dx - \int_{\mathbb{R}^N} F_+(Lw_n)\,dx.$$

By (2.1) and Lemma 1.1, we see that

$$\Big|\int_{\mathbb{R}^N} F_+(Lw_n)\,dx\Big| \leq \varepsilon' L^2|w_n|_2^2 + C_{\varepsilon'}L^p|w_n|_p^p \leq \varepsilon' L^2 + o(1)$$

for n large enough. Noting that ε' is arbitrary, we get

$$\varliminf_{n\to\infty} J_+\Big(\frac{L}{\|u_n\|}u_n\Big) \geq cL^2,$$

where $c > 0$ is a constant. Since $\|u_n\| \to \infty$, $L/\|u_n\| \in (0,1)$ for n large enough, and

$$\max_{t\in[0,1]} J_+(tu_n) \geq J_+\Big(\frac{L}{\|u_n\|}u_n\Big) \geq cL^2.$$

Choose L so large that $d_+^* < cL^2$; then the maximum in the above inequality cannot be attained at $t = 1$, thus, for each sufficiently large n, there exists $t_n \in (0,1)$ such that

$$J_+(t_nu_n) = \max_{t\in[0,1]} J_+(tu_n).$$

Thus, by the arbitrariness of L we get

$$J_+(t_nu_n) \to \infty, \quad \text{as } n \to \infty. \tag{2.2}$$

By the fact that $\langle J_+'(t_nu_n), t_nu_n\rangle = 0$ and (f_4),

$$\begin{aligned} J_+(t_nu_n) &= J_+(t_nu_n) - \frac{1}{2}\langle J_+'(t_nu_n), t_nu_n\rangle \\ &= \int_{\mathbb{R}^N} \big(\frac{1}{2}f_+(t_nu_n)t_nu_n - F_+(t_nu_n)\big)\,dx \\ &\leq \int_{\mathbb{R}^N} \big(\frac{1}{2}f_+(u_n)u_n - F_+(u_n)\big)\,dx \\ &= \big(J_+(u_n) - \frac{1}{2}\langle J_+'(u_n), u_n\rangle\big) \to d_+^*, \end{aligned}$$

which contradicts (2.2). So, vanishing cannot occur.

Claim 2 Non-vanishing cannot occur.

Since $\{w_n\}$ is non-vanishing, and the problem (P_∞) is translation invariant, replacing w_n by $\tilde{w}_n = w_n(\cdot + y_n)$, if $|y_n| \to \infty$, we may assume that there exists $0 \not\equiv w \in X$ such that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have $\langle J_+'(u_n), \varphi\rangle/\|u_n\| = o(1)$; that is,

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla\varphi + \lambda w_n\varphi)\,dx = \int_{\mathbb{R}^N} \frac{f_+(u_n)}{u_n} w_n\varphi\,dx + o(1). \tag{2.3}$$

We claim that

$$\int_{\mathbb{R}^N} \frac{f_+(u_n)}{u_n} w_n\varphi\,dx \to \alpha \int_{\mathbb{R}^N} w_+\varphi\,dx. \tag{2.4}$$

By the conditions (f_3) and (f_4), $f_+(t)/t$ is bounded for all $t \in \mathbb{R}$. Denote $\Omega_0 = \{x \in \mathbb{R}^N : w(x) = 0\}$; then $w_n(x) \to w(x) = 0$ for almost every $x \in \Omega_0$, and

$$\frac{f_+(u_n(x))}{u_n(x)} w_n(x) \to 0 = \alpha w(x) \quad \text{a.e. } x \in \Omega_0.$$

Denote $\Omega_1 = \{x \in \mathbb{R}^N : w(x) \neq 0\}$; then $w_n(x) \to w(x)$ for almost every $x \in \Omega_1$, and $|u_n(x)| = \|u_n\| \, |w_n(x)| \to \infty$. By (f_3) and the definition of f_+,

$$\frac{f_+(u_n(x))}{u_n(x)} w_n(x) \to \alpha w_+(x) \quad \text{a.e. } x \in \Omega_1.$$

Therefore,

$$\frac{f_+(u_n(x))}{u_n(x)} w_n(x) \to \alpha w(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

Also since the sequence $\{\frac{f_+(u_n)}{u_n} w_n\}$ is bounded in $L^2(\mathbb{R}^N)$, $\{\frac{f_+(u_n)}{u_n} w_n\} \rightharpoonup \alpha w$ in $L^2(\mathbb{R}^N)$. Hence (2.4) holds. Combining (2.3), (2.4) and the weak convergence of $\{w_n\}$ in $H^1(\mathbb{R}^N)$, we see that w satisfies

$$-\Delta w + \lambda w = \alpha w_+.$$

Multiplying by w_-, we see that $w_- = 0$. Hence $w \geq 0$ satisfies

$$-\Delta w = (\alpha - \lambda) w.$$

This is impossible for any λ, α since $\sigma_p(-\Delta, H^1(\mathbb{R}^N)) = \emptyset$ (see [18, Theorem 3.8]).

Combining Claims 1 and 2, we see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Next we show that, up to a translation, $\{u_n\}$ converges weakly to a nontrivial solution of the problem (P_∞). Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, by Proposition 1.2, there exists a subsequence (still denoted by $\{u_n\}$) such that one of the following cases occurs:

(1) (Vanishing) $\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{y+B_R} u_n^2 \, dx = 0, \ \forall \, R > 0$.

(2) (Non-vanishing) There exist $\eta, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\varliminf_{n\to\infty} \int_{y_n+B_R} u_n^2 \, dx \geq \eta > 0$.

Now we prove that vanishing cannot occur. Otherwise, by (2.1) and Lemma 1.1, we get

$$\int_{\mathbb{R}^N} f_+(u_n) u_n \, dx \to 0 \quad \text{as } n \to \infty.$$

This, together with the fact that $\langle J_+'(u_n), u_n \rangle \to 0$, we see that $\|u_n\| \to 0$ as $n \to \infty$. This is impossible since $J_+(u_n) \to d_+^* > 0$ as $n \to \infty$.

Thus $\{u_n\}$ is non-vanishing. Let $v_n = u_n(x+y_n)$; then $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and

$$\varliminf_{n\to\infty} \int_{B_R} v_n^2\,dx \geq \eta > 0.$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^N)$. Since $J_+'(u_n) \to 0$, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\nabla v_n \nabla\varphi + \lambda v_n\varphi)\,dx = \int_{\mathbb{R}^N} f(v_n)\varphi\,dx + o(1)\|\varphi\|. \tag{2.5}$$

By the conditions (f_1)-(f_3), we see that there exists $C > 0$ such that $|f_+(u)| \leq C|u|$. Also since $v_n \to v$ in $L^2_{\rm loc}(\mathbb{R}^N)$, by [19, Lemma A.2] we have $f_+(v_n) \to f_+(v)$ in $L^2_{\rm loc}(\mathbb{R}^N)$, thus

$$\int_{\mathbb{R}^N} f_+(v_n)\varphi\,dx \to \int_{\mathbb{R}^N} f_+(v)\varphi\,dx, \quad \text{as } n\to\infty. \tag{2.6}$$

Combining (2.5), (2.6) and the weak convergence of $\{v_n\}$ in $H^1(\mathbb{R}^N)$, we see that v is a non-trivial critical point of J_+. Thus by the strong maximum principle we have $v > 0$ in $\mathbb{R}^N$. Hence $v > 0$ is a solution of the problem (P_∞).

Finally, we will prove that there exists a least energy positive solution v of the problem (P_∞). If $u > 0$ is a solution of the problem (P_∞), then

$$0 = \langle J'(u), u\rangle = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,dx - \int_{\mathbb{R}^N} f(u)u\,dx.$$

Thus, by the condition (f_4),

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx = \int_{\mathbb{R}^N} \big(\frac{1}{2}f(u)u - F(u)\big)\,dx \geq 0.$$

Set

$$d_+ = \inf\{J(u) : u \text{ is a positive solution of } (P_\infty)\}.$$

Let $\{u_n\}$ be a sequence of positive solutions of the problem (P_∞) with $J(u_n) \to d_+$ as $n\to\infty$. The same argument just used above implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\{u_n\}$ is non-vanishing; that is, there exist $\eta, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\varliminf_{n\to\infty} \int_{y_n+B_R} u_n^2\,dx \geq \eta > 0$. Let $v_n = u_n(x+y_n)$; then $v_n \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^N)$. Similar to the above discussion, we see that v is a positive solution of the problem (P_∞), and from Fatou's lemma we get

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda v^2)\,dx - \int_{\mathbb{R}^N} F(v)\,dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \big(\frac{1}{2} f(v)v - F(v)\big)\, dx \le \lim_{n\to\infty} \int_{\mathbb{R}^N} \big(\frac{1}{2} f(v_n)v_n - F(v_n)\big)\, dx \\
&= \lim_{n\to\infty} J(v_n) = \lim_{n\to\infty} J(u_n) = d_+,
\end{aligned}
$$

which implies that v achieves d_+. Hence (i) is proved.

(ii) Let u be a solution of the problem (P_∞); by the conditions (f_3) and (f_4), we get

$$
\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\, dx = \int_{\mathbb{R}^N} f(u)u\, dx \le \alpha \int_{\mathbb{R}^N} u^2\, dx,
$$

thus by $\lambda \ge \alpha$ we see that $u \equiv 0$; that is, (P_∞) has no non-trivial solutions. For this case we suppose $d_+ = \infty$. □

Proposition 2.2. *Assume* (f_1)-(f_4) *and* (G) *hold. Then*

(i) $d_+^* = d_+$, $d_-^* = d_-$.

(ii) *Suppose that* u *is a sign-changing solution of the problem* (P_∞)*, then* $J(u) \ge d_+ + d_-$.

Proof. Let u be a least energy positive solution of the problem (P_∞), then $J(u) = d_+$, $\langle J'(u), u\rangle = 0$, and

$$
\begin{aligned}
\frac{d}{dt} J(tu) &= t \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\, dx - \int_{\mathbb{R}^N} f(tu)u\, dx \\
&= \int_{\mathbb{R}^N} \big(tf(u)u - f(tu)u\big)\, dx.
\end{aligned}
$$

By the condition (f_4), we have $\frac{d}{dt} J(tu) \ge 0$ for $0 < t \le 1$ and $\frac{d}{dt} J(tu) \le 0$ for $t \ge 1$. Thus $J(u) = \sup_{t\ge 0} J(tu)$.

On the other hand, by the conditions (f_3) and (f_4), we get

$$
\begin{aligned}
\lim_{t\to\infty} \frac{J(tu)}{t^2} &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\, dx - \lim_{t\to\infty} \int_{\mathbb{R}^N} \frac{F(tu)}{t^2} \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\, dx - \frac{1}{2}\alpha \int_{\mathbb{R}^N} u^2\, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (f(u)u - \alpha u^2)\, dx < 0.
\end{aligned}
$$

Thus $J(tu) \to -\infty$ as $t \to \infty$. By the definition of d_+^*, we have $d_+^* \le \sup_{t\ge 0} J(tu) = d_+$. By Proposition 2.1, we have found a positive solution u^* such that $J(u^*) \le d_+^*$. Hence $d_+ \le J(u^*) \le d_+^* \le d_+$ and $d_+^* = d_+$. Similarly, we can prove $d_-^* = d_-$.

(ii) Suppose that u is a sign-changing solution of the problem (P_∞) and let $u = u_+ + u_-$, where $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$; then we have $\langle J'(u_\pm), u_\pm\rangle = 0$. Moreover,

$$J(u_\pm) = \sup_{t\geq 0} J(tu_\pm) \geq d^*_\pm = d_\pm,$$

thus $J(u) = J(u_+) + J(u_-) \geq d_+ + d_-$. □

3. PROPERTIES OF SOLUTIONS OF THE PROBLEM (P)

In this section, we will prove the existence of least energy positive (negative) solutions of the problem (P). Moreover, we use the concentration-compactness principle to analysis the behavior of a Cerami sequence, which lies at the heart of the proof of the existence of sign-changing solutions of the problem (P).

The functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$, corresponding to the problem (P), is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda g(x)u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx,$$

where $F(u) = \int_0^u f(t)\,dt$. Similar to Proposition 2.1, we define

$$I_\pm(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda g(x)u^2)\,dx - \int_{\mathbb{R}^N} F_\pm(u)\,dx,$$

where $F_\pm(u) = \int_0^u f_\pm(t)\,dt$.

Proposition 3.1. *Suppose that (f_1)-(f_4) and (G) hold, and $\alpha > \alpha_1(\lambda)$, $\alpha > \lambda$.*

(i) *If $c < d_\pm$, then $I_\pm$ satisfies the (Cerami)$_c$ condition.*
(ii) *(P) has positive solutions and negative solutions.*
(iii) *(P) has a least energy positive(and negative) solution. Moreover, if $c_\pm$ is the least energy of positive(and negative) solutions, then $c_\pm < d_\pm$.*
(iv) *If u is a sign-changing solution of the problem (P), then $I(u) \geq c_+ + c_-$.*

Proof. We only consider the positive solutions.

(i) Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^N)$ such that $I_+(u_n) \to c$, $(1 + \|u_n\|)\|I'_+(u_n)\| \to 0$ as $n \to \infty$. Firstly, we will show $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Secondly, we will prove $\{u_n\}$ has a convergent subsequence.

Step 1: $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

We use an indirect argument. Suppose that $\|u_n\| \to \infty\,(n \to \infty)$. Let $w_n = u_n/\|u_n\|$. Clearly, $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By Proposition 1.2, there exists a subsequence (still denoted by $\{w_n\}$) such that one of the following cases occurs:

(1) (Vanishing) $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{y+B_R} w_n^2\,dx = 0,\ \forall\, R > 0.$

(2) (Non-vanishing) There exist $\nu, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\varliminf\limits_{n\to\infty} \int_{y_n+B_R} w_n^2\,dx \geq \nu > 0.$

Next we will prove that these two cases cannot occur.

Claim 1. Vanishing cannot occur.

The proof of Claim 1 is similar to Proposition 2.1.

Claim 2. Non-vanishing cannot occur if $\{y_n\}$ is bounded.

Since $\{w_n\}$ is non-vanishing and $\{y_n\}$ is bounded, there exists $0 \not\equiv w \in H^1(\mathbb{R}^N)$ such that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have $\langle I'_+(u_n), \varphi\rangle/\|u_n\| = o(1)$; that is,

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla\varphi + \lambda g(x) w_n \varphi)\,dx = \int_{\mathbb{R}^N} \frac{f_+(u_n)}{u_n} w_n \varphi\,dx + o(1). \tag{3.1}$$

Similar to the proof of (2.4), we have

$$\int_{\mathbb{R}^N} \frac{f_+(u_n)}{u_n} w_n \varphi\,dx \to \alpha \int_{\mathbb{R}^N} w_+ \varphi\,dx. \tag{3.2}$$

Combining (3.1), (3.2) and the weak convergence of $\{w_n\}$ in $H^1(\mathbb{R}^N)$, we have

$$-\Delta w + \lambda g(x) w = \alpha w_+.$$

Multiplying by w_-, we see that $w_- = 0$. Hence $w \geq 0$ satisfies

$$-\Delta w + \lambda g(x) w = \alpha w.$$

This is impossible since $\alpha > \alpha_1(\lambda)$.

Claim 3. Non-vanishing cannot occur if $\{y_n\}$ is unbounded.

If $\{y_n\}$ is unbounded, passing to a subsequence we may assume that $|y_n| \to \infty$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, let $\varphi_n(x) = \varphi(x - y_n)$; then

$$|\langle I'_+(u_n), \varphi_n\rangle| \leq \|I'_+(u_n)\|\,\|\varphi_n\| = \|I'_+(u_n)\|\,\|\varphi\| \to 0$$

as $n \to \infty$. Let $\tilde{u}_n = u_n(x + y_n)$, $\tilde{w}_n = w_n(x + y_n)$; then

$$\int_{\mathbb{R}^N} \big(\nabla \tilde{w}_n \nabla\varphi + \lambda g(x + y_n) \tilde{w}_n \varphi\big)\,dx = \int_{\mathbb{R}^N} \frac{f_+(\tilde{u}_n)}{\tilde{u}_n} \tilde{w}_n \varphi\,dx + o(1). \tag{3.3}$$

Since

$$\varliminf_{n\to\infty}\int_{B_R}\tilde{w}_n^2\,dx=\varliminf_{n\to\infty}\int_{y_n+B_R}w_n^2\,dx\geq\eta>0,$$

there exists $0\not\equiv\tilde{w}\in H^1(\mathbb{R}^N)$ such that $\tilde{w}_n\rightharpoonup\tilde{w}$ in $H^1(\mathbb{R}^N)$. Similar to the above discussion, we have $\tilde{w}\geq 0$. By the condition (G), since $g\in L^\infty(\mathbb{R}^N)$ and $g(x)\to 1$ as $|x|\to\infty$, we have

$$\int_{\mathbb{R}^N}g(x+y_n)\tilde{w}_n\varphi\,dx\to\int_{\mathbb{R}^N}\tilde{w}\varphi\,dx. \tag{3.4}$$

Similar to the argument of (2.4), we obtain

$$\int_{\mathbb{R}^N}\frac{f_+(\tilde{u}_n)}{\tilde{u}_n}\tilde{w}_n\varphi\,dx\to\alpha\int_{\mathbb{R}^N}\tilde{w}\varphi\,dx. \tag{3.5}$$

Combining (3.3)-(3.5) we see that $\tilde{w}$ satisfies

$$-\Delta\tilde{w}=(\alpha-\lambda)\tilde{w}.$$

This is impossible for any λ,α since $\sigma_p(-\Delta,H^1(\mathbb{R}^N))=\emptyset$ (see [18, Theorem 3.8]).

Combining Claims 1, 2 and 3, we see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Step 2: $\{u_n\}$ has a convergent subsequence.

By Step 1, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, thus by Proposition 1.2, there exists a subsequence (still denoted by $\{u_n\}$) such that one of the following cases occurs:

(1) (Vanishing) $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{y+B_R}u_n^2\,dx=0,\ \forall\, R>0.$

(2) (Non-vanishing) There exist $\eta,R>0$ and $\{y_n\}\subset\mathbb{R}^N$ such that $\varliminf_{n\to\infty}\int_{y_n+B_R}u_n^2\,dx\geq\eta>0.$

Similar to the argument of Proposition 2.1, we see that vanishing cannot occur. Next we will prove that non-vanishing cannot occur if $\{y_n\}$ is unbounded.

Let $v_n=u_n(x+y_n)$, then $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$; passing to a subsequence we may assume that $v_n\rightharpoonup v\neq 0$ in $H^1(\mathbb{R}^N)$. Similar to the proof of Proposition 2.1, we see that v is a positive solution of (P_∞). Thus, by Fatou's lemma,

$$\begin{aligned}d_+\leq J(v)&=\int_{\mathbb{R}^N}\big(\frac{1}{2}f(v)v-F(v)\big)\,dx\leq\varliminf_{n\to\infty}\int_{\mathbb{R}^N}\big(\frac{1}{2}f(v_n)v_n-F(v_n)\big)\,dx\\&=\lim_{n\to\infty}I(v_n)=\lim_{n\to\infty}I(u_n)=c,\end{aligned}$$

which contradicts the assumption $c<d_+$.

Hence $\{y_n\}$ is bounded if non-vanishing occurs. Similar to the argument of Proposition 2.1, we see that there exists $0 \neq u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, and u is a positive solution of the problem (P). By the concentration-compactness principle (see [14]), there exist sequences $\{v_i\}_{i=1}^p \subset H^1(\mathbb{R}^N)$, $\{y_{n_i}\}_{n=1}^\infty \subset \mathbb{R}^N (i = 1, 2, \cdots, p)$ such that $v_1, v_2, \cdots, v_p$ are non-trivial solutions of (P_∞), $|y_{n_i} - y_{n_j}| \to \infty$ as $n \to \infty$, $i \neq j$, and $|y_{n_i}| \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} I(u_n) = I(u) + \sum_{i=1}^p J(v_i),$$

$$\lim_{n\to\infty} \|u_n - u - \sum_{i=1}^p v_i(x + y_{n_i})\| = 0. \tag{3.6}$$

If $p \neq 0$, then

$$\lim_{n\to\infty} I(u_n) = I(u) + \sum_{i=1}^p J(v_i) \geq d_+,$$

which contradicts the assumption $c < d_+$. So $p = 0$. Thus by (3.6) we have $u_n \to u$ in $H^1(\mathbb{R}^N)$.

(ii) Under the assumptions of Proposition 3.1, I_+ possesses a mountain pass geometry; we define

$$c_+^* = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} I_+(u) > 0,$$

where $\Gamma = \{\gamma : \gamma \in C([0, 1], H^1(\mathbb{R}^N)), \gamma(0) = 0, I_+(\gamma(1)) < 0\}$. Let u be a least energy positive solution of the problem (P_∞); we have $d_+ = \sup_{t\geq 0} J_+(tu)$. By the definition of c_+^* and the condition (G), there exists $t^* > 0$ such that

$$c_+^* \leq \sup_{t\geq 0} I_+(tu) = I_+(t^*u) < J_+(t^*u) \leq d_+.$$

Thus by (i) we know that I_+ satisfies the Cerami condition. Hence c_+^* is a critical value of I_+; i.e., (P) has a positive solution. Similarly, a negative solution can be obtained for the case of I_-.

(iii) Define

$$c_\pm = \inf\{I_\pm(u) : u \text{ is a critical point of } I_\pm\};$$

then $c_\pm \leq c_\pm^* < d_\pm$. So, by (i), there exist $u > 0$ and $u < 0$ that achieve c_+ and c_-, respectively.

(iv) If u is a sign-changing solution of the problem (P), using the same argument as in the proof of Proposition 2.2 (ii), we have $I(u) \geq c_+ + c_-$. □

Before we give the main result of this section, let us give some definitions. Define the convex cones $P = \{u \in H^1(\mathbb{R}^N) : u \geq 0\}$ and $-P = \{u \in H^1(\mathbb{R}^N) : u \leq 0\}$.

Lemma 3.1. *Set $c^* = \min\{c_+ + d_-, c_- + d_+\}$. Suppose that $\{u_n\}$ is a Cerami sequence, and u_n is sign-changing for every $n \in \mathbb{N}$, and there exists $\varepsilon > 0$ such that $dist(u_n, P) \geq \varepsilon$ and $dist(u_n, -P) \geq \varepsilon$. If $c = \lim_{n\to\infty} I(u_n) < c^*$, then $\{u_n\}$ has a subsequence which converges to a sign-changing solution of the problem (P).*

Proof. Similar to the discussion of Proposition 3.1, we know $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We claim that

$$\text{if } v \text{ is a non-trivial solution of } (P_\infty), \\ \text{then there exists } \delta > 0 \text{ such that } \|v\| \geq \delta.$$

In fact, if v is a solution of the problem (P_∞), then

$$\langle J'(v), v\rangle = \int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda v^2)\, dx - \int_{\mathbb{R}^N} f(v)v\, dx = 0;$$

this, together with (2.1), implies that we have

$$\|v\|^2 \leq \varepsilon\|v\|^2 + c\|v\|^p,$$

thus, $v \equiv 0$ or there exists $\delta > 0$ such that $\|v\| \geq \delta$. Moreover, this claim is true for (P). According to the concentration-compactness principle(see [14]), there exist sequences $\{v_i\}_{i=1}^p \subset H^1(\mathbb{R}^N)$, $\{y_{n_i}\}_{n=1}^\infty \subset \mathbb{R}^N (i = 1, 2, \cdots, p)$ such that $v_1, v_2, \cdots, v_p$ are non-trivial solutions of the problem (P_∞), $|y_{n_i} - y_{n_j}| \to \infty$ as $n \to \infty$, $i \neq j$, and $|y_{n_i}| \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \|u_n - u - \sum_{i=1}^p v_i(x + y_{n_i})\| = 0,$$

$$\lim_{n\to\infty} I(u_n) = I(u) + \sum_{i=1}^p J(v_i),$$

$$\lim_{n\to\infty} \|u_n\|^2 = \|u\|^2 + \sum_{i=1}^p \|v_i\|^2.$$

If $u \equiv 0$, suppose that one of $\{v_i\}$, say v_1, is sign-changing, thus $c \geq J(v_1) \geq d_+ + d_- > c^*$, which contradicts the assumption $c < c^*$. Suppose that none of $\{v_i\}$ are sign-changing; then one of $\{v_i\}$, say v_1, is positive and another one of $\{v_i\}$, say v_2, is negative. Thus we have $c \geq J(v_1) + J(v_2) \geq d_+ + d_- > c^*$, which also contradicts the assumption $c < c^*$. So $u \not\equiv 0$.

If u is sign-changing, we have $I(u) \geq c_+ + c_-$. Suppose that $p \neq 0$; then there exists at least one solution v of the problem (P_∞), and we get $J(v) \geq \min\{d_+, d_-\}$. Thus

$$c \geq I(u) + J(v) \geq c_+ + c_- + \min\{d_+, d_-\} \geq \min\{c_+ + d_-, c_- + d_+\} = c^*,$$

a contradiction. So $p = 0$. Hence $u_n \to u$ in $H^1(\mathbb{R}^N)$.

If u is positive, then $I(u) \geq c_+$, and at least one of $\{v_i\}$, say v_1, is not positive, thus

$$c \geq I(u) + J(v_1) \geq c_+ + d_- \geq c^*,$$

which is a contradiction. Hence u is not positive.

If u is negative, the same argument just used above implies that $c \geq c_- + d_+ \geq c^*$, which is also a contradiction. Hence u is not negative. Consequently, we have u is sign-changing. The lemma is proved. □

4. EXISTENCE OF SIGN-CHANGING SOLUTIONS

By the assumptions on g and f, the functional $I : X \to \mathbb{R}$, corresponding to the problem (P), defined by

$$I(u) = \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(u)\,dx$$

is continuously (Fréchet) differentiable, where $F(u) = \int_0^u f(t)\,dt$. We follow the arguments of [5] and write the gradient of I at u by

$$I'(u) = u - A(u), \quad A : X \to X, \quad A(u) = \big(-\Delta + \lambda g\big)^{-1} f(u).$$

Then $\langle A(u), v\rangle = \int_{\mathbb{R}^N} f(u)v\,dx$ for all $v \in X$. We consider the convex cones $P = \{u \in X : u \geq 0\}$ and $-P = \{u \in X : u \leq 0\}$. For $\varepsilon > 0$, we denote

$$D_1 = \{u \in X : \operatorname{dist}(u, P) < \varepsilon\} \text{ and } D_2 = \{u \in X : \operatorname{dist}(u, -P) < \varepsilon\}.$$

Obviously, $D_1 \cap D_2 \neq \emptyset$. Note that D_1 and D_2 are open convex subsets of X, therefore, $X \setminus (\overline{D_1} \cup \overline{D_2})$ contains only sign-changing functions. Define $K = \{u \in X : I'(u) = 0\}$. For $c \in \mathbb{R}$, we set $K_c = \{u \in X : I(u) = c,\ I'(u) = 0\}$. Denote $X_0 = X \setminus K$.

A Lipschitz continuous map $W : X_0 \to X$ is called a pseudogradient vector field for I if it satisfies

$$\langle I'(u), W(u)\rangle \geq \|I'(u)\|^2; \quad \|W(u)\| \leq 2\|I'(u)\| \quad \text{for } u \in X_0.$$

For $u \in X_0$, we consider the following initial-value problem in X_0:

$$\begin{cases} \dfrac{d}{dt}u(t) = -W(u), \ t \geq 0, \\ u(0) = u_0. \end{cases} \tag{4.1}$$

By the theory of ordinary differential equations in Banach spaces, (4.1) has a unique solution, denoted by $u(t, u_0)$, with the maximal interval of existence $[0, \tau(u_0))$. Note that $I(u(t, u_0))$ is strictly decreasing in $t \in [0, \tau)$. Define $C_X(D_1 \cap D_2) := (D_1 \cap D_2) \cup \{u_0 \in X_0 : \text{there exists } t \geq 0 \text{ such that } u(t, u_0) \in D_1 \cap D_2\}$. Let $\partial C_X(D_1 \cap D_2)$ be the boundary of $C_X(D_1 \cap D_2)$ in X. For simplicity, we denote $u(t, u_0)$ by u_t.

Let h be a path : $[0, 1] \to X$ such that

$$\sup_{t \in [0,1]} I(h(t)) < \inf_{u \in \overline{D_1} \cap \overline{D_2}} I(u). \tag{4.2}$$

Define $\gamma(s, t) : [0, 1] \times [0, 1] \to X$ such that $\gamma(0, t) = 0$, $\gamma(1, t) = h(t)$ for all $t \in [0, 1]$, and $\gamma(s, 1) \in D_1$, $\gamma(s, 0) \in D_2$ for all $s \in [0, 1]$. Then we can find a point $u^* \in \gamma(s, t)$ such that the flow $u(t, u^*)$ (denoted by u_t) generated by (4.1) satisfies $u_t \in \partial C_X(D_1 \cap D_2) \backslash (D_1 \cup D_2)$ for $t \in [0, \tau(u^*))$. Moreover, we can find a sequence t_n, $t_n \to \tau$ as $n \to \infty$, satisfying

$$\sup_{u \in \overline{D_1} \cap \overline{D_2}} I(u) \leq c = \lim_{n \to \infty} I(u_{t_n}) \leq I(u^*)$$

and

$$(1 + \|u_{t_n}\|)\|I'(u_{t_n})\| \to 0 \text{ as } n \to \infty.$$

In particular $\{u_{t_n}\}$ is sign-changing. If $I(u^*) < c^*$ (c^* is defined in Lemma 3.1), from Lemma 3.1 we see that $\{u_{t_n}\}$ has a subsequence which converges to a sign-changing solution of the problem (P). So we will construct $\gamma(s, t)$ satisfying (4.2), and by the argument in [5, Theorem 3.2],

$$\sup_{[0,1] \times [0,1]} I(\gamma(s, t)) < c^*.$$

Without loss of generality, we may assume $c_+ + d_- \leq c_- + d_+$, thus $c^* = c_+ + d_-$.

Lemma 4.1. *Assume (G), (G^*) and (f_1)-(f_3) hold. Suppose that φ is a least energy positive solution of (P); i.e., $I(\varphi) = c_+$; and ψ is a least energy negative solution of (P_∞); i.e., $J(\psi) = d_-$. Let $\mathbf{e}$ be a fixed unit vector in $\mathbb{R}^N$, define $\psi_R(\cdot) = \psi(\cdot - 2R\mathbf{e})$ and let $R \in \mathbb{R}$ be a parameter. Set*

$\gamma(s,t) = s\varphi + t\psi_R$, $s,t \geq 0$. *Then there exists* R *large enough such that*

$$\sup_{s,t\geq 0} I(\gamma(s,t)) < c_+ + d_-, \tag{4.3}$$

$$I(\gamma(s,t)) \to -\infty \text{ as } s^2+t^2 \to \infty. \tag{4.4}$$

Similarly, if φ *is a least energy negative solution of (P),* ψ *is a least energy positive solution of* (P_∞)*. Then*

$$\sup_{t,s\geq 0} I(\gamma(s,t)) < c_- + d_+, \text{ and } I(\gamma(s,t)) \to -\infty \text{ as } s^2+t^2 \to \infty.$$

To prove this lemma, we need the following regularity result.

Lemma 4.2. *Under the assumptions* (G) *and* (f_1)*-*(f_3)*, if* u *is a solution of the problem* (P_∞)*, then the following exponential decay at infinity holds:*

$$\int_{\mathbb{R}^N\setminus B_R} (u^2 + |\nabla u|^2)\,dx \leq Ce^{-\delta R},$$

for some positive constants C, $\delta > 0$. *The same conclusion holds for solutions of the problem (P).*

Proof. Let $u \in H^1(\mathbb{R}^N)$ be a solution of the problem (P_∞); similar to the argument of Lemma 5.10 of [20] (see also [21, Lemma 2.8]), we have $u(x) \to 0$ as $|x| \to \infty$. Since u is a solution of (P_∞), we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla\varphi + \lambda u\varphi)\,dx = \int_{\mathbb{R}^N} f(u)\varphi\,dx.$$

We use a hole-filling trick. Take $\varphi = u\chi$ where χ is a cut-off function such that $\chi = 0$ for $|x| \leq R$ and $\chi = 1$ for $|x| \geq R+1$, $|\nabla\chi| \leq 2$ for all $x \in \mathbb{R}^N$; then

$$\int_{\mathbb{R}^N} (|\nabla u|^2\chi + \lambda u^2\chi)\,dx = -\int_{\mathbb{R}^N} u\nabla u\nabla\chi\,dx + \int_{\mathbb{R}^N} f(u)u\chi\,dx. \tag{4.5}$$

By the conditions (f_1)-(f_3), for R_0 large enough, we have

$$|f(u)| \leq \varepsilon'|u| + C_{\varepsilon'}|u|^p \leq 2\varepsilon'|u| \quad \text{for } |x| \geq R_0.$$

This, together with (4.5), implies, for $R > R_0$,

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{R+1}} (|\nabla u|^2+u^2)dx &\leq C\int_{\mathbb{R}^N} |u\nabla u|\,|\nabla\chi|\,dx \leq C\int_{\mathbb{R}^N} (u^2+|\nabla u|^2)|\nabla\chi|\,dx \\ &\leq C\Big(\int_{\mathbb{R}^N\setminus B_R} (|\nabla u|^2+u^2)\,dx - \int_{\mathbb{R}^N\setminus B_{R+1}} (|\nabla u|^2+u^2)\,dx\Big).\end{aligned}$$

Let $R_n = R_0 + n$. Define a sequence

$$\alpha_n = \int_{\mathbb{R}^N \setminus B_{R_0+n}} (|\nabla u|^2 + u^2)\, dx,$$

then $\alpha_{n+1} \le C(\alpha_n - \alpha_{n+1})$, which implies that $\alpha_{n+1} \le \theta \alpha_n$ where $\theta = \frac{C}{C+1} < 1$. Thus there exists C (independent of n) such that $\alpha_n \le C\theta^n$. Hence the exponential decay is proved. □

Proof of Lemma 4.1. We divide the proof into three steps.

(i) We estimate $I(s\varphi + t\psi_R) - I(s\varphi) - I(t\psi_R)$. A straightforward computation gives us

$$I(s\varphi + t\psi_R) - I(s\varphi) - I(t\psi_R) = \int_{\mathbb{R}^N} st \nabla\varphi \nabla\psi_R\, dx + \lambda \int_{\mathbb{R}^N} stg(x)\varphi\psi_R\, dx - \int_{\mathbb{R}^N} \big(F(s\varphi + t\psi_R) - F(s\varphi) - F(t\psi_R)\big)\, dx.$$

Now we estimate the three terms on the right:

$$\begin{aligned} st\Big|\int_{\mathbb{R}^N} \nabla\varphi\nabla\psi_R\, dx\Big| &\le st\Big(\int_{B_R} |\nabla\varphi|\,|\nabla\psi_R|\, dx + \int_{\mathbb{R}^N\setminus B_R} |\nabla\varphi|\,|\nabla\psi_R|\, dx\Big) \\ &\le cst\Big[\Big(\int_{B_R} |\nabla\psi_R|^2\, dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^N\setminus B_R} |\nabla\varphi|^2\, dx\Big)^{1/2}\Big] \\ &\le cst\Big[\Big(\int_{\mathbb{R}^N\setminus B_R} |\nabla\psi|^2\, dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^N\setminus B_R} |\nabla\varphi|^2\, dx\Big)^{1/2}\Big] \\ &\le cste^{-\delta R} \le c(s^2+t^2)e^{-\delta R}. \end{aligned} \tag{4.6}$$

By the condition (G), we have

$$\begin{aligned} \lambda st\Big|\int_{\mathbb{R}^N} g(x)\varphi\psi_R\, dx\Big| &\le cst\Big[\Big(\int_{B_R} |\psi_R|^2\, dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^N\setminus B_R} |\varphi|^2\, dx\Big)^{1/2}\Big] \\ &\le cst\Big[\Big(\int_{\mathbb{R}^N\setminus B_R} |\psi|^2\, dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^N\setminus B_R} |\varphi|^2\, dx\Big)^{1/2}\Big] \\ &\le cste^{-\delta R} \le c(s^2+t^2)e^{-\delta R}. \end{aligned} \tag{4.7}$$

We now turn to estimating the last term:

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N} \big(F(s\varphi + t\psi_R) - F(s\varphi) - F(t\psi_R)\big)\, dx\Big| \\ &\le \int_{B_R} |F(s\varphi + t\psi_R) - F(s\varphi)|\, dx + \int_{B_R} |F(t\psi_R)|\, dx \end{aligned}$$

$$+\int_{\mathbb{R}^N\setminus B_R}|F(s\varphi+t\psi_R)-F(t\psi_R)|\,dx+\int_{\mathbb{R}^N\setminus B_R}|F(s\varphi)|\,dx.$$

By the conditions (f_3) and (f_4), we obtain

$$\int_{B_R}|F(t\psi_R)|\,dx\le c\int_{B_R}t^2\psi_R^2\,dx\le ct^2e^{-\delta R},$$

and by the mean value theorem,

$$\begin{aligned}&\int_{B_R}|F(s\varphi+t\psi_R)-F(s\varphi)|\,dx=\int_{B_R}|f(s\varphi+\theta t\psi_R)t\psi_R|\,dx\\&\le c\int_{B_R}(s\varphi+t|\psi_R|)t|\psi_R|\,dx\le c(s^2+t^2)e^{-\delta R},\end{aligned}$$

where $\theta\in(0,1)$. In the same way, we also have

$$\int_{\mathbb{R}^N\setminus B_R}|F(s\varphi+t\psi_R)-F(t\psi_R)|\,dx\le c(s^2+t^2)e^{-\delta R}$$

and

$$\int_{\mathbb{R}^N\setminus B_R}|F(s\varphi)|\,dx\le cs^2e^{-\delta R}.$$

Thus,

$$\Big|\int_{\mathbb{R}^N}\big(F(s\varphi+t\psi_R)-F(s\varphi)-F(t\psi_R)\big)\,dx\Big|\le c(s^2+t^2)e^{-\delta R}. \tag{4.8}$$

Here c denotes possibly different positive constants. Combining (4.6)-(4.8), we obtain

$$|I(s\varphi+t\psi_R)-I(s\varphi)-I(t\psi_R)|\le c(s^2+t^2)e^{-\delta R}. \tag{4.9}$$

(ii) We shall show that $I(s\varphi+t\psi_R)\to-\infty$ as $s^2+t^2\to\infty$. Since φ is a solution of (P), we have

$$\begin{aligned}I(s\varphi)&=\frac{1}{2}s^2\int_{\mathbb{R}^N}(|\nabla\varphi|^2+\lambda g\varphi^2)\,dx-\int_{\mathbb{R}^N}F(s\varphi)\,dx\\&=\frac{1}{2}s^2\int_{\mathbb{R}^N}f(\varphi)\varphi\,dx-\int_{\mathbb{R}^N}F(s\varphi)\,dx.\end{aligned}$$

Thus, by the conditions (f_3) and (f_4),

$$\begin{aligned}\lim_{s\to\infty}\frac{I(s\varphi)}{s^2}&=\frac{1}{2}\int_{\mathbb{R}^N}f(\varphi)\varphi\,dx-\lim_{s\to\infty}\int_{\mathbb{R}^N}\frac{F(s\varphi)}{s^2}\,dx\\&=\frac{1}{2}\int_{\mathbb{R}^N}(f(\varphi)\varphi-\alpha\varphi^2)\,dx<0.\end{aligned} \tag{4.10}$$

Also, since ψ is a solution of (P_∞), we get

$$\begin{aligned}\lim_{t\to\infty}\frac{I(t\psi_R)}{t^2} &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\psi_R|^2+\lambda g\psi_R^2)\,dx-\lim_{t\to\infty}\int_{\mathbb{R}^N}\frac{F(t\psi_R)}{t^2}\,dx\\ &\le \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\psi_R|^2+\lambda\psi_R^2-\alpha\psi_R^2)\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N}(f(\psi)\psi-\alpha\psi^2)\,dx<0.\end{aligned} \tag{4.11}$$

From (4.10) and (4.11), there exist constants $c_1, c_2 > 0$ (independent of s, t, R) such that

$$I(s\varphi)+I(t\psi_R)\le -c_1(s^2+t^2)+c_2;$$

this, together with (4.9), implies that

$$I(s\varphi+t\psi_R)\le -c_1(s^2+t^2)+c_2+c(s^2+t^2)e^{-\delta R}.$$

Taking R_0 so large that

$$I(s\varphi+t\psi_R)\le -c_3(s^2+t^2)+c_4,\quad \text{for all } R\ge R_0,$$

where c_3, c_4 are constants, we have $I(s\varphi+t\psi_R)\to-\infty$ as $s^2+t^2\to\infty$.

(iii) We will give the complete estimate of $\sup_{s,t\ge0} I(\gamma(s,t))$. By step (ii), we see that there exists $M>0$ such that

$$I(s\varphi+t\psi_R)\le 0\quad\text{for all } s^2+t^2\ge M^2.$$

Now suppose that $s^2+t^2\le M^2$,

$$I(s\varphi)\le I(\varphi)=c_+, \tag{4.12}$$

and there exists $t_1>0$ such that, for $t\le t_1$,

$$I(t\psi_R)\le \tfrac{1}{2}d_-. \tag{4.13}$$

For $t_1\le t\le M$, by the condition (G^*), we have

$$\begin{aligned}I(t\psi_R) &= J(t\psi_R)-\frac{1}{2}\lambda t^2\int_{\mathbb{R}^N}(1-g(x))\psi_R^2\,dx\\ &\le J(\psi_R)-\frac{1}{2}\lambda t_1^2\int_{\mathbb{R}^N}(1-g(x))\psi_R^2\,dx\\ &= d_- -\frac{1}{2}\lambda t_1^2\int_{B_R}(1-g(x+2R\mathbf{e}))\psi^2\,dx\\ &\le d_- -\frac{1}{2}\lambda t_1^2\frac{c}{R^m}\int_{B_1}\psi^2\,dx=d_- -\frac{c_0}{R^m}.\end{aligned} \tag{4.14}$$

From step (i), combining (4.12)-(4.14), we obtain

$$\sup_{s,t\geq 0} I(s\varphi + t\psi_R) \leq c_+ + d_- - \frac{c_0}{R^m} + O(e^{-\delta R}) < c_+ + d_-,$$

provided R is large enough.

Lemma 4.3. *Assume* (f_1)-(f_3) *and* (G) *hold; then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon, 0 < \varepsilon \leq \varepsilon_0$, $A(\partial D_1) \subset D_1$ *and* $A(\partial D_2) \subset D_2$. *Moreover, if* $u \in D_1$ *or* $u \in D_2$ *is a solution of the problem* (P), *then* $u \in P$ *or* $u \in -P$, *respectively.*

Proof. Indeed, if $u \in X$ and $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, then

$$\operatorname{dist}(A(u), P) = \min_{w\in P} \|A(u) - w\|_\lambda = \min_{w\in P} \|A(u)^+ + A(u)^- - w\|_\lambda \leq \|A(u)^-\|_\lambda.$$

Let $s \in [2, 2^*)$; then by the continuity of the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$, there exists $C_s > 0$ such that

$$|u^{\mp}|_s = \min_{w\in\pm P} |u - w|_s \leq C_s \min_{w\in\pm P} \|u - w\|_\lambda = C_s \operatorname{dist}(u, \pm P). \tag{4.15}$$

We claim that $A(u) \in D_1$ for any $u \in \partial D_1$. It follows from (2.1), (4.15), (f_1) and the Hölder inequality that

$$\operatorname{dist}(A(u), P)\|A(u)^-\|_\lambda \leq \|A(u)^-\|_\lambda^2 = \langle A(u), A(u)^-\rangle = \int_{\mathbb{R}^N} f(u)A(u)^-\,dx$$
$$\leq \int_{\{u\geq 0\}} f(u)A(u)^-\,dx + \int_{\{u\leq 0\}} f(u)A(u)^-\,dx\,dx \leq \int_{\{u\leq 0\}} f(u^-)A(u)^-\,dx$$
$$\leq \int_{\{u\leq 0\}} (\varepsilon'|u^-| + C_{\varepsilon'}|u^-|^{p-1})A(u)^-\,dx$$
$$\leq \varepsilon'\tilde{c}\operatorname{dist}(u, P)\|A(u)^-\|_\lambda + \tilde{C}_{\varepsilon',p}\operatorname{dist}(u, P)^{p-1}\|A(u)^-\|_\lambda.$$

Taking $\varepsilon' = \frac{1}{2\tilde{c}}$, we then get

$$\operatorname{dist}(A(u), P) \leq \tfrac{1}{2}\operatorname{dist}(u, P) + \tilde{C}\operatorname{dist}(u, P)^{p-1},$$

where $\tilde{C} > 0$ is a constant. Let $\varepsilon_0 = \left(\frac{1}{4\tilde{C}}\right)^{\frac{1}{p-2}}$; then, for all ε with $0 < \varepsilon \leq \varepsilon_0$, we have

$$\operatorname{dist}(A(u), P) \leq \tfrac{3}{4}\operatorname{dist}(u, P) \tag{4.16}$$

for all $u \in D_1$. Clearly, $\operatorname{dist}(A(u), P) \leq \frac{3}{4}\varepsilon < \varepsilon$ for every $u \in \partial D_1$; that is, $A(u) \in D_1$ for all $u \in \partial D_1$. Hence $A(\partial D_1) \subset D_1$. In a similar way, $A(\partial D_2) \subset D_2$. If $u \in D_1$ is a solution of the problem (P), then $I'(u) = u - A(u) = 0$; i.e., $u = A(u)$, by (4.16), thus $u \in P$. Similarly, if $u \in D_2$, then $u \in -P$. □

Lemma 4.4. *Let u_t be the flow associated to* (4.1) *with the maximal interval of existence* $[0, \tau(u^*))$. *If* $\lim_{t\to\tau} I(u_t) = c > -\infty$, *then there exists a sequence* t_n, $t_n \to \tau$ *as* $n \to \infty$, *satisfying*

$$c = \lim_{n\to\infty} I(u_{t_n}) \le I(u^*) \text{ and } (1 + \|u_{t_n}\|)\|I'(u_{t_n})\| \to 0 \text{ as } n \to \infty.$$

Proof. We use an indirect argument. Otherwise, there exists $\delta > 0$ such that

$$(1 + \|u_t\|)\|I'(u_t)\| \ge \delta, \quad \forall\, t \in [0, \tau). \tag{4.17}$$

If $\|u_t\|$ is unbounded, say $\|u_{t_n}\| \to \infty$ as $t_n \to \tau$, then

$$\begin{aligned} I(u_0) - c &= -\int_0^{t_n} \frac{d}{dt} I(u_t)\, dt \ge \frac{1}{2}\int_0^{t_n} \|I'(u_t)\| \Big\|\frac{du}{dt}\Big\|\, dt \\ &\ge \frac{\delta}{2}\int_0^{t_n} \frac{\big\|\frac{du}{dt}\big\|\, dt}{1 + \|u_t\|} \ge \frac{\delta}{2}\int_0^{t_n} \frac{d\|u_t\|}{1 + \|u_t\|} = \frac{\delta}{2} \ln \frac{1 + \|u_{t_n}\|}{1 + \|u_0\|} \to \infty, \end{aligned}$$

which is a contradiction. So $\|u_t\|$ is bounded, and $\|I'(u_t)\| \ge \delta > 0$.

If $\tau = \infty$, by the definition of W,

$$I(u_0) - c = \int_0^\infty \langle I'(u_t), W(u_t)\rangle\, dt \ge \int_0^\infty \|I'(u_t)\|^2\, dt,$$

thus there exists $\{t_n\}$ with $t_n \to \infty$ such that $I'(u_{t_n}) \to 0$ as $n \to \infty$.

If $\tau < \infty$, by (4.1), for $0 \le t_1 < t_2$,

$$\begin{aligned} \|u_{t_2} - u_{t_1}\| &= \Big\| \int_{t_1}^{t_2} -W(u_t)\, dt \Big\| \le 2\int_{t_1}^{t_2} \|I'(u_t)\|\, dt \\ &\le 2\Big(\int_{t_1}^{t_2} \|I'(u_t)\|^2\, dt\Big)^{1/2} (t_2 - t_1)^{1/2} \\ &\le 2\Big(-\int_{t_1}^{t_2} \frac{d}{dt} I(u_t)\, dt\Big)^{1/2} (t_2 - t_1)^{1/2} \\ &\le 2\big(I(u_{t_1}) - I(u_{t_2})\big)^{1/2} (t_2 - t_1)^{1/2} \to 0 \quad \text{as } t_1, t_2 \to \tau. \end{aligned}$$

Thus there exists u^* such that

$$\lim_{t\to\tau} \|u_t - u^*\| = 0,$$

and u^* must be a critical point, otherwise the flow can be extended beyond $\tau(u^*)$. Thus $I'(u_t) \to I'(u^*) = 0$.

In both cases of $\tau = \infty$ and $\tau < \infty$, we arrived at a contradiction. Hence (4.17) does not hold; the lemma is proved. ◻

Proof of Theorem 1.1: We first consider the case of $\alpha > \lambda$. It follows from (2.1) that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(u)\,dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\varepsilon' \int_{\mathbb{R}^N} u^2\,dx - C_{\varepsilon',p}\int_{\mathbb{R}^N} |u|^p\,dx \\ &\geq -\frac{1}{2}\varepsilon'|u|_2^2 - C_{\varepsilon',p}|u|_p^p. \end{aligned}$$

By (4.15) we have $|u^{\pm}|_s \leq C_s\mathrm{dist}(u, \mp P) \leq C_s\varepsilon_0$ for every $u \in D_1 \cap D_2$. So there exists $c_0 > -\infty$ such that

$$\inf_{u\in\overline{D_1}\cap\overline{D_2}} I(u) = c_0. \tag{4.18}$$

From (4.4), we have that there exists $R_0 > 0$ such that for all $R \geq R_0$ and for all $|s| + |t|$ large, $I(\gamma(1,t)) < c_0 - 1$. Hence (4.2) holds. Using (4.2) as in the proof of [5, Theorem 3.1], we can find $u^* \in \gamma(s,t)$ such that

$$\{u_t : 0 \leq t < \tau(u^*)\} \subset \partial C_X(D_1 \cap D_2)\backslash(D_1 \cup D_2).$$

Furthermore, the argument of Lemma 4.4 shows that there exists a sequence t_n, $t_n \to \tau$ as $n \to \infty$, satisfying

$$c = \lim_{n\to\infty} I(u_{t_n}) \leq I(u^*) \text{ and } (1 + \|u_{t_n}\|)\|I'(u_{t_n})\| \to 0 \text{ as } n \to \infty.$$

By Lemma 4.1, we have $I(u^*) < c^*$. Also, since $\{u_{t_n}\} \subset X\backslash(D_1 \cup D_2)$, by Lemma 3.1 we see that $\{u_{t_n}\}$ has a subsequence which converges to a sign-changing solution of the problem (P).

For the case of $\lambda \geq \alpha$, since $\alpha > \alpha_2(\lambda)$, and $\alpha > \alpha_2(\lambda)$ if $\alpha = \lambda$, we use the eigenfunctions corresponding to the eigenvalues of (1.1) to construct the path such that (4.2) holds; the details can be found in Lemma 5.2. Moreover, for all $c < \infty$, Lemma 3.1 holds. Hence, similar to the above discussion, we can prove (P) has a sign-changing solution. □

5. Existence of sign-changing solutions in the region I

For the case of $\lambda > \alpha$, we obtain the existence of a sign-changing solution of the problem (P) without assuming monotonicity conditions. We allow f to depend on x.

We make the following assumptions on f.

(f_1') $f(x,t) \in C(\mathbb{R}^N, \mathbb{R})$, $f(x,t)t \geq 0$ for almost every $x \in \mathbb{R}^N$, for all $t \in \mathbb{R}$, $t \neq 0$.

(f_2') $\lim\limits_{|t|\to 0} \frac{f(x,t)}{t} = 0$ uniformly with respect to $x \in \mathbb{R}^N$.

(f_3') There exists $\alpha \in (0, \infty)$ such that $\lim\limits_{|t|\to\infty} \frac{f(x,t)}{t} = \alpha$ for almost every $x \in \mathbb{R}^N$, and $\frac{f(x,t)}{t}$ is bounded on $\mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$.

(f_4') $\overline{\lim\limits_{|x|\to\infty}} \sup\limits_{t\neq 0} \frac{f(x,t)}{t} = \beta < \lambda$.

In fact, by the conditions (f_3') and (f_4'), we have

$$\alpha = \lim_{|t|\to\infty} \frac{f(x,t)}{t} \le \sup_{|t|\neq 0} \frac{f(x,t)}{t};$$

letting $|x| \to \infty$, we get $\alpha \le \beta < \lambda$.

This section will be devoted to proving the following theorem.

Theorem 5.1. *Assume that* (G) *and* (f_1')-(f_4') *hold. If* $\alpha > \alpha_2(\lambda)$ *and* $\alpha \notin \sigma_p(L_\lambda)$*, then the problem* (P) *has a sign-changing solution* (*besides the positive one and the negative one*).

To prove Theorem 5.1, several preliminary results are in order. Define the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda g(x) u^2)\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx,$$

where $F(x,u) = \int_0^u f(x,t)\, dt$.

Lemma 5.1. *Assume* (f_1')-(f_4') *and* (G) *hold. If* $\alpha \notin \sigma_p(L_\lambda)$*, then* I *satisfies the* (PS) *condition; that is, any sequence* $\{u_n\} \subset X$ *with* $I'(u_n) \to 0$*,* $I(u_n) \to c$*, possesses a convergent subsequence.*

Proof. Let $\{u_n\}$ be a sequence in X such that $I(u_n) \to c$, $I'(u_n) \to 0$ as $n \to \infty$. We first prove that $\{u_n\}$ is bounded in X. In fact, otherwise, we may suppose that $\|u_n\|_\lambda \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|_\lambda}$. Obviously, $\{w_n\}$ is bounded in X. Passing to a subsequence, still denoted by $\{w_n\}$, we may assume that there exists $w \in X$ such that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{in}\, X, \\ w_n &\to w \quad \text{in}\, L^s_{\text{loc}}(\mathbb{R}^N) \text{ for } 2 \le s < 2^*, \\ w_n(x) &\to w(x) \quad \text{a.e.\,in}\, \mathbb{R}^N. \end{aligned}$$

We claim that $w \not\equiv 0$ in $\mathbb{R}^N$. If $w = 0$ then $w_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. By ($G$), we get

$$\int_{\mathbb{R}^N} (g(x) - 1) w_n^2\, dx \to 0$$

as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla w_n|^2 + \lambda w_n^2\right) dx = \lim_{n\to\infty} \|w_n\|_\lambda^2 = 1$$

and thus

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} w_n^2 \, dx \le \frac{1}{\lambda}. \tag{5.1}$$

Since $I'(u_n) \to 0$ as $n \to \infty$,

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + \lambda g(x) u_n \varphi) \, dx = \int_{\mathbb{R}^N} f(x, u_n)\varphi \, dx + o(1)\|\varphi\|_\lambda \quad \text{for all } \varphi \in X. \tag{5.2}$$

Dividing (5.2) by $\|u_n\|_\lambda$ we have

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla \varphi + \lambda g(x) w_n \varphi) \, dx = \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} w_n \varphi \, dx + o(1). \tag{5.3}$$

Taking $\varphi = w_n$ in (5.3), we get

$$\int_{\mathbb{R}^N} (|\nabla w_n|^2 + \lambda g(x) w_n^2) \, dx = \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} w_n^2 \, dx + o(1). \tag{5.4}$$

By the conditions (f_3'), (f_4') and (5.1),

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} w_n^2 \, dx \le \beta \lim_{n\to\infty} \int_{\mathbb{R}^N} w_n^2 \, dx \le \frac{\beta}{\lambda};$$

this, together with (5.4) and the fact that $\|w_n\|_\lambda = 1$, implies $\lambda \le \beta$, which contradicts the assumption (f_4'), so that we have shown $w \not\equiv 0$. Similar to the discussion of Claim 2 of (i) in Proposition 2.1, we get

$$\int_{\mathbb{R}^N} \left(\nabla w \nabla \varphi + \lambda g(x) w \varphi\right) dx = \alpha \int_{\mathbb{R}^N} w\varphi \, dx. \tag{5.5}$$

Then from (5.5) we see that $0 \not\equiv w \in X$ satisfies

$$-\Delta w + \lambda g(x) w = \alpha w,$$

which is impossible since $\alpha \notin \sigma_p(L_\lambda)$. Therefore $\{u_n\}$ is bounded in X.

Next we prove that there exists $u \in X$ such that $\|u_n\|_\lambda \to \|u\|_\lambda$ as $n \to \infty$. By the boundedness of $\{u_n\}$, passing to a subsequence, we may assume for some $u \in X$ that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$. We claim that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Indeed, we need to prove that, for any $\mu > 0$, there exists $R > 0$ such that

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N \setminus B_R} u_n^2 \, dx \le \mu. \tag{5.6}$$

Otherwise, there exists $\mu > 0$, for all $R > 0$, such that

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N \setminus B_R} u_n^2 \, dx \geq \mu. \tag{5.7}$$

We take a cut-off function $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$, $\psi = 0$ for $|x| \leq R$, $\psi = 1$ for $|x| \geq 2R$, $0 \leq \psi(x) \leq 1$ for $R \leq |x| \leq 2R$, such that there exists $c_0 > 0$ (independent of R) such that $|\nabla \psi| \leq c_0/R$ for all $x \in \mathbb{R}^N$. Then, for any $u \in X$ and for $R \geq 1$, there exists c_1 (independent of R) such that $\|\psi u\|_\lambda \leq c_1 \|u\|_\lambda$. Since $\{u_n\}$ is bounded in X, by (5.2), letting $\varphi = u_n \psi$, we get

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda g(x) u_n^2) \psi \, dx = \int_{\mathbb{R}^N} f(x, u_n) u_n \psi \, dx - \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \psi \, dx + o(1). \tag{5.8}$$

Since $\lambda > \beta$, choosing $0 < \varepsilon' < \frac{\lambda - \beta}{2}$, by (G) and (5.7),

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda g(x) u_n^2) \psi \, dx \geq \overline{\lim_{n\to\infty}} \, \lambda \int_{\mathbb{R}^N \setminus B_{2R}} g(x) u_n^2 \, dx \geq (\lambda - \varepsilon') \mu. \tag{5.9}$$

By the condition (f_4') and (5.7),

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} f(x, u_n) u_n \psi \, dx \leq \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N \setminus B_R} f(x, u_n) u_n \, dx \leq (\beta + \varepsilon') \mu. \tag{5.10}$$

Then, by (5.8)-(5.10), we conclude that

$$\begin{aligned} (\lambda - \beta - 2\varepsilon') \mu &\leq - \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \psi \, dx = - \int_{\mathbb{R}^N} u \nabla u \nabla \psi \, dx \\ &\leq \frac{c_0}{R} \int_{B_{2R} \setminus B_R} |u| \, |\nabla u| \, dx \to 0 \end{aligned}$$

as $R \to \infty$, which is a contradiction. Therefore (5.6) is proved. Since also $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, it follows that $u_n \to u$ in $L^2(\mathbb{R}^N)$. By (5.2), letting $\varphi = u_n - u$, we get

$$\begin{aligned} 0 &\leq \overline{\lim_{n\to\infty}} (\|u_n\|_\lambda^2 - \|u\|_\lambda^2) = \overline{\lim_{n\to\infty}} \langle u_n, u_n - u \rangle \\ &= \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} f(x, u_n)(u_n - u) \, dx. \end{aligned} \tag{5.11}$$

Under the condition (f_3'), there exists a positive constant c such that $\frac{f(x,t)}{t} \leq c$. By the Hölder inequality and the fact that $u_n \to u$ in $L^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(x,u_n)(u_n - u)\, dx \leq c|u_n|_2\, |u_n - u|_2 \to 0$$

as $n \to \infty$. From this and (5.11) we know that $\|u_n\|_\lambda \to \|u\|_\lambda$. Finally,

$$\|u_n - u\|_\lambda^2 = \|u_n\|_\lambda^2 - 2\langle u_n, u\rangle + \|u\|_\lambda^2 \to \|u\|_\lambda^2 - 2\langle u, u\rangle + \|u\|_\lambda^2 = 0,$$

which implies that $u_n \to u$ in X as $n \to \infty$. The lemma is proved. □

Lemma 5.2. *Assume (f_1')-(f_3') and (G) hold. If $\alpha > \alpha_2(\lambda)$, then there exists a path $h : [0,1] \to X$ such that $h(0) \in D_1 \backslash D_2$, $h(1) \in D_2 \backslash D_1$, and*

$$\inf_{u \in \overline{D_1} \cap \overline{D_2}} I(u) > \sup_{t \in [0,1]} I(h(t)). \tag{5.12}$$

Proof. It follows from (f_1')-(f_3') that, for any $\varepsilon' > 0$, there exists $C_{\varepsilon'} > 0$ such that

$$|f(x,t)| \leq \varepsilon'|t| + C_{\varepsilon'}|t|^{p-1} \quad \text{for } 2 < p < 2^*.$$

Thus

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,u)\, dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\varepsilon' \int_{\mathbb{R}^N} u^2\, dx - C_{\varepsilon',p} \int_{\mathbb{R}^N} |u|^p\, dx \\ &\geq -\frac{1}{2}\varepsilon'|u|_2^2 - C_{\varepsilon',p}|u|_p^p\,. \end{aligned}$$

By (4.15) we have $|u^\pm|_s \leq C_s \mathrm{dist}(u, \mp P) \leq C_s\varepsilon_0$ for every $u \in D_1 \cap D_2$. So there exists $c_0 > -\infty$ such that

$$\inf_{u \in \overline{D_1} \cap \overline{D_2}} I(u) = c_0. \tag{5.13}$$

Let $u_{\alpha_i(\lambda)} (i = 1,2)$ be an eigenfunction corresponding to the eigenvalue $\alpha_i(\lambda) (i = 1,2)$, and set $W := \big\{u \in X : \|u\|_\lambda = 1, u \in \mathrm{span}\{u_{\alpha_1(\lambda)}, u_{\alpha_2(\lambda)}\}\big\}$. Define a path $\gamma : [0,1] \to W$

$$\gamma(t) = u_{\alpha_1(\lambda)} \cos(\pi t) + u_{\alpha_2(\lambda)} \sin(\pi t)$$

connecting $\gamma(0) = u_{\alpha_1(\lambda)}$ and $\gamma(1) = -u_{\alpha_1(\lambda)}$. Now let $h_R(t) = R\gamma(t)$. By the condition (f_3'), $0 \leq \frac{F(x,t)}{t^2} \leq C$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, and

$$\lim_{R \to +\infty} \frac{F(x, R\gamma(t))}{R^2\gamma(t)^2} = \frac{\alpha}{2} \quad \text{a.e. } x \in \mathbb{R}^N,$$

thus it follows by Lebesgue's dominated convergence theorem that

$$\begin{aligned}\lim_{R\to+\infty}\frac{I(h_R(t))}{R^2} &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\gamma(t)|^2+\lambda g(x)\gamma(t)^2)\,dx\\ &\quad-\lim_{R\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,R\gamma(t))}{R^2\gamma(t)^2}\gamma(t)^2\,dx\\ &=\frac{1}{2}-\frac{1}{2}\alpha\int_{\mathbb{R}^N}\big(u^2_{\alpha_1(\lambda)}\cos^2(\pi t)+u^2_{\alpha_2(\lambda)}\sin^2(\pi t)\big)\,dx\\ &=\frac{1}{2}-\frac{1}{2}\alpha\Big(\frac{\cos^2(\pi t)}{\alpha_1(\lambda)}+\frac{\sin^2(\pi t)}{\alpha_2(\lambda)}\Big)<\frac{1}{2}\Big(1-\frac{\alpha}{\alpha_2(\lambda)}\Big)<0.\end{aligned}$$

So, this yields the fact that there exists R such that $I(h_R(t))<c_0-1$. This, together with (4.18), implies that (5.12) holds. Obviously $h_R(0)\in D_1\backslash D_2$, $h_R(1)\in D_2\backslash D_1$. □

Proof of Theorem 5.1. Under the assumptions of Theorem 5.1, we see that the results of Lemmas 4.3 hold, thus by Lemma 5.1, Lemma 5.2 and Proposition 1.1, we can find a critical point in $X\backslash(\overline{D_1}\cup\overline{D_2})$, which is a sign-changing solution of the problem (P). Also we have a critical point in $D_1\backslash\overline{D_2}$ and a critical point in $D_2\backslash\overline{D_1}$, which correspond to a positive solution and a negative solution of the problem (P). □

Remark 5.1. Under the assumptions (f_1')-(f_4') and (G), for the case of $\alpha>\alpha_1(\lambda)$, $\lambda>\alpha$, the proof of the (PS) condition is similar to Lemma 5.1 above, and we easily show that I has a mountain pass geometry, then by using the mountain pass lemma (see [15]) and the strong maximum principle, we may obtain the existence of a positive solution (and a negative solution) of the problem (P).

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