Advances in Differential Equations Volume 16, Numbers 5-6 (2011), 551–571

SIMULTANEOUS EXACT CONTROLLABILITY: AN ELASTODYNAMIC SYSTEM AND MAXWELL'S EQUATIONS

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(Submitted by: Jean-Michel Coron)

Abstract. A new boundary observability inequality for the Maxwell equations and the elastodynamic system is obtained. We use modified multipliers to obtain such an inequality as long as a geometric condition on the region holds and important parameters of the model are (numerically) related. This allow us to use the HUM to conclude a "simultaneous" boundary exact controllability result.

1. Introduction

We consider a pair of evolution systems modelling two distinct hyperbolic dynamics: Maxwell's equations and the elastodynamic system for elastic waves. In vector-valued variables, $\{E, H, u\}$ satisfies

$$\begin{cases} \mathcal{E}E_t - \operatorname{curl} H = 0 \\ \mu H_t + \operatorname{curl} E = 0 & \text{in } \Omega \times (0, +\infty) \\ \operatorname{div} E = 0,\ \operatorname{div} H = 0 \end{cases} \tag{1.1}$$

$$E(x,0) = E_0(x), H(x,0) = H_0(x) \text{ in } \Omega \tag{1.2}$$

$$\eta \times E = P(x,t) \text{ on } \partial\Omega \times (0, +\infty) \tag{1.3}$$

and

$$\rho u_{tt} - \lambda_2 \Delta u - (\lambda_1 + \lambda_2) \operatorname{grad}(\operatorname{div} u) = 0 \text{ in } \Omega \times (0, +\infty) \tag{1.4}$$

Accepted for publication: November 2010.
AMS Subject Classifications: 35Q99, 74F99, 35B40.

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \Omega \tag{1.5}$$

$$u = Q(x,t) \text{ on } \partial\Omega \times (0,+\infty). \tag{1.6}$$

In (1.1), $E = (E_1, E_2, E_3)$ denotes the electric field (or displacement current), $H = (H_1, H_2, H_3)$ is the magnetic field (or magnetic induction), $\mathcal{E}$ and μ are positive constants representing the permittivity and magnetic permeability respectively. In (1.3) $\eta = \eta(x)$ denotes the unit normal vector at $x \in \partial\Omega$ pointing towards the exterior of Ω and $\times$ denotes the usual vector product in $\mathbb{R}^3$. In (1.4), $u = (u_1, u_2, u_3)$ represents the displacement vector, λ_1 and λ_2 are Lame's constants (which are assumed to be positive) and ρ denotes the scalar density ($\rho > 0$). Finally, grad, curl, Δ and div denote the usual gradient, rotational, Laplace and divergence operator respectively. The vector-valued functions $E_0(x), H_0(x)$, $u_0(x)$ and $u_1(x)$ are the initial data for problems (1.1)-(1.3) and (1.4)-(1.6) respectively. P and Q in (1.3) and (1.6) are the so-called "control" functions and are the ones whose existence we would like to prove so we can have simultaneous controllability of problems (1.1)-(1.3) and (1.4)-(1.6) in the sense we will describe below.

Roughly speaking the problem of exact controllability can be stated as follows: Given a time $T > 0$ and any initial data and desired terminal data, find a corresponding control $F(x,t)$ driving the system to the terminal data at time T. One of the most useful methods to solve such problems of controllability is the Hilbert uniqueness method (HUM) introduced by J.L. Lions in the middle 80's and is based on the construction of appropriate Hilbert space structures on the space of initial data. These Hilbert structures are closely connected with uniqueness properties. Several important contributions on controllability for problem (1.1)-(1.3) were given by D.L. Russell [28], J. Lagnese [18], K. Kime [15], P. Martinez [23], M. Eller and D. Masters [3], N. Weck [29], K. Phung [26], V. Komornik [17] and B. Kapitonov [6-13] among others. Controllability results for problem (1.4)-(1.6) were treated by J. Lagnese [18,20], F. Alabau and V. Komornik [1], M. Horn [5], B. Kapitonov and G. Perla Menzala [10,14] among others.

The above mentioned authors obtained results on existence of controls P and Q of (1.1)-(1.3) or (1.4)-(1.6), which were not necessarily related.

Around 1986, D.L. Russell [28] and J.L. Lions [21] raised the question if it is possible to solve the exact controllability problem for two evolution models using only one control. They named this problem "simultaneous" exact controllability. In the absence of dissipative effects, like the case we are considering in (1.1)-(1.3) and (1.4)-(1.6), due to technical difficulties "simultaneous" exact controllability was only obtained by a number of authors for one system with two different boundary conditions. See for instance, J.L. Lions [22], B.V. Kapitonov [8,9], B. Kapitonov and M.A. Raupp [13], B.V. Kapitonov and G. Perla Menzala [14]. Let us formulate the "simultaneous" exact controllability for systems (1.1)-(1.3) and (1.4)-(1.6): Given $T > 0$, initial state (E_0, H_0, u_0, u_1), and terminal state $(\tilde{E}_0, \tilde{H}_0, \tilde{u}_0, \tilde{u}_1)$ in suitable function spaces, we ask: Is it possible to find one vector-valued function $Q(x,t)$ such that the solution $\{E, H, u\}$ of (1.1)-(1.6) satisfies at the terminal time T:

$$\big(E(\cdot,T), H(\cdot,T), u(\cdot,T), u_t(\cdot,T)\big) = (\tilde{E}_0, \tilde{H}_0, \tilde{u}_0, \tilde{u}_1),$$

$Q(x,t)$ being a control function for (1.4)-(1.6) and $P(x,t)$ (given in terms of Q) being a control function for (1.1)-(1.3)?

Our main result in this paper gives an affirmative answer for the above question and shows that P can be taken to be $\mu\lambda_2\eta \times (\eta \times Q_t)$.

Let us briefly describe all sections of this paper: In Section 2 we indicate the function spaces where the solutions of (1.1)-(1.3) and (1.4)-(1.6) will be considered. Then, we use convenient multipliers to obtain (in case $P \equiv 0$, $Q \equiv 0$) a boundary observability inequality valid for both systems (Theorem 2.2) as long as a geometric condition on the region Ω holds. The final result of Theorem 2.2 is still not suitable to apply the necessary steps of the Hilbert uniqueness method. We find another boundary observability inequality provided a (numerical) relation between important parameters of (1.1)-(1.6) hold. This new inequality is appropriate for using HUM. In Section 3 we prove the main conclusions of this article (Theorem 3.1).

Systems (1.1)-(1.3) and (1.4)-(1.6) are not directly coupled to each other. A more interesting problem will be to study the case when they are coupled say with coupling terms $-\gamma \operatorname{curl} E$ and $\gamma \operatorname{curl} u_t$ (with $\gamma > 0$) for system (1.1)-(1.3) and (1.4)-(1.6) respectively. As far as we know this remains an open problem.

We will use standard notation which can be found in G. Duvant and J.L. Lions' book [2]. For example $H^m(\Omega)$ and $H^r(\partial\Omega)$ will denote the Sobolev spaces of order m and r on Ω and $\partial\Omega$ respectively. Given a real-valued function g we denote by $\int_{\partial\Omega} g \, d\Gamma$ the surface integral of g over the surface $S = \partial\Omega$. If X is a vector space, then X^m means $X \times X \times \cdots \times X$ m-times. Thus, for example, $L^2(\Omega)^3 = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, $H^{-1/2}(\partial\Omega)^3 = H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ and so forth. For any vector $v \in \mathbb{R}^3$, $|v|$ denotes the usual norm of v in $\mathbb{R}^3$.

In order to simplify notation we will always denote by C a positive constant which may be different from line to line.

2. A boundary observability inequality

We consider suitable function spaces where the solution $\{E, H, u\}$ of problems (1.1)-(1.3) and (1.4)-(1.6) (with $P \equiv Q \equiv \vec{0}$) will be considered. Let Ω be a bounded region of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ and $\mathcal{E} > 0$, $\mu > 0$. We consider the Hilbert space $X = L^2(\Omega)^3 \times L^2(\Omega)^3$ with inner product

$$\langle v, w \rangle_X = \int_\Omega (\mathcal{E} v_1 \cdot w_1 + \mu v_2 \cdot w_2)\, dx$$

for any pair $v = (v_1, v_2)$, $w = (w_1, w_2)$ in X. The dot $\cdot$ means the usual inner product in $\mathbb{R}^3$. Next, we consider the Hilbert space

$$H(\mathrm{curl}; \Omega) = \{w \in L^2(\Omega)^3 \ : \ \mathrm{curl}\, w \in L^2(\Omega)^3\}$$

with inner product

$$\langle v, w \rangle_{H(\mathrm{curl};\Omega)} = \int_\Omega (v \cdot w + \mathrm{curl}\, v \cdot \mathrm{curl}\, w)\, dx$$

for any $v = (v_1, v_2)$, $w = (w_1, w_2)$ in $H(\mathrm{curl}; \Omega)$. Let $\mathcal{D}(\Omega)$ be the space of test functions and denote by

$$H_0(\mathrm{curl}; \Omega) = \text{ closure of } \mathcal{D}(\Omega)^3 \text{ in } H(\mathrm{curl}; \Omega).$$

It is well known ([2]; Chapter 7, page 338) that

$$H_0(\mathrm{curl}; \Omega) = \{v \in H(\mathrm{curl}; \Omega) : \eta \times v = 0 \text{ on } \partial\Omega\}.$$

Here $\eta = \eta(x)$ denotes (all through this article) the unit vector normal at $x \in \partial\Omega$ pointing towards the exterior of Ω. We denote by $\mathcal{D}(\overline{\Omega}) = \{\varphi|_\Omega : \varphi \in \mathcal{D}(\mathbb{R}^3)\}$. The following result is well known ([2], Chapter IX, page 204).

Theorem 2.1. *Under the above assumptions on Ω, the trace map $\gamma_\tau \colon v \mapsto \eta \times v|_{\partial\Omega}$ defined from $C_0^1(\overline{\Omega})^3$ into $C^1(\partial\Omega)^3$ extends by continuity to a continuous linear mapping from $H(\mathrm{curl}; \Omega)$ onto $H^{-1/2}(\partial\Omega)^3$. If we still denote by γ_τ this extension, then the kernel of γ_τ coincides with $H_0(\mathrm{curl}; \Omega)$.*

Now, we define the operator $A \colon \mathcal{D}(A) \subseteq X \mapsto X$ with domain

$$\mathcal{D}(A) = H_0(\mathrm{curl}; \Omega) \times H(\mathrm{curl}; \Omega).$$

The operator is given by

$$A(v, w) = (\mathcal{E}^{-1} \mathrm{curl}\, w, -\mu^{-1} \mathrm{curl}\, v)$$

for any (v, w) in $\mathcal{D}(A)$. We can verify that the operator A is skew-selfadjoint, therefore A generates a one-parameter group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on X. Observe that in order that $U(t)(E_0, H_0)$ be a solution of problem

(1.1)-(1.3) (with $P \equiv \vec{0}$) with $(E_0, H_0) \in \mathcal{D}(A)$ it remains to prove that the components of $U(t)(E_0, H_0)$ are divergence free.

Let $N = \ker A^* = \{v \in \mathcal{D}(A^*) : A^*v = 0\}$. Clearly N is not trivial, because it contains elements of the form $w = (\operatorname{grad}\varphi_1, \operatorname{grad}\varphi_2)$ with φ_1 and φ_2 in $\mathcal{D}(\overline{\Omega})$. We can verify that if we denote by $N_1 = N^\perp$ then $U(t)$ takes $N_1 \cap \mathcal{D}(A)$ into itself. In fact, let $v \in N_1 \cap \mathcal{D}(A)$ and $w \in N$; then

$$\frac{d}{dt}\langle U(t)v, w\rangle_X = \langle AU(t)v, w\rangle_X = \langle U(t)v, A^*w\rangle_X = 0$$

for any t. Thus $\langle U(t)v, w\rangle$ = constant (in t). Choosing $t = 0$ we have that constant= $\langle v, w\rangle_X = 0$. Consequently $U(t)v \in N_1 \cap \mathcal{D}(A)$ for any $t \in \mathbb{R}$. Observe that for any element $v = (v_1, v_2) \in N_1 \cap \mathcal{D}(A)$ and $w = (0, \operatorname{grad}\varphi_2)$ with $\varphi_2 \in \mathcal{D}(\overline{\Omega})$ we have

$$0 = \langle v, w\rangle_X = \int_\Omega \mu v_2 \cdot \operatorname{grad}\varphi_2 \, dx = \mu \int_{\partial\Omega} \varphi(v_2 \cdot \eta)\, d\Gamma.$$

Since φ_2 is arbitrary we conclude that

$$v_2 \cdot \eta = 0 \text{ on } \partial\Omega \tag{2.1}$$

in the distributional sense.

We remark that elements $v = (v_1, v_2) \in N_1 \cap \mathcal{D}(A)$ possess the property

$$\operatorname{div} v_1 = 0, \qquad \operatorname{div} v_2 = 0 \qquad \text{in } \Omega$$

in the sense of distributions.

Next, we consider problem (1.4)-(1.6) with $Q \equiv \vec{0}$. Let us denote by Y the Hilbert space $Y = H_0^1(\Omega)^3 \times L^2(\Omega)^3$ with inner product

$$\langle u, v\rangle_Y = \int_\Omega \left\{\rho u_2 \cdot v_2 + \lambda_2 \sum_{j=1}^{3} \frac{\partial u_1}{\partial x_j} \cdot \frac{\partial v_1}{\partial x_j} + (\lambda_1 + \lambda_2)\operatorname{div} u_1 \operatorname{div} v_1\right\} dx.$$

In Y we define the unbounded operator $\tilde{A} \colon \mathcal{D}(\tilde{A}) \subseteq Y \to Y$ with domain

$$\mathcal{D}(\tilde{A}) = \{(u, v), u \in [H^2(\Omega) \cap H_0^1(\Omega)]^3,\ v \in H_0^1(\Omega)^3\}$$

and given by

$$\tilde{A}(u, v) = (v, \rho^{-1}[\lambda_2 \Delta u + (\lambda_1 + \lambda_2)\operatorname{grad}(\operatorname{div} u)]).$$

We can easily verify that $\tilde{A}$ is skew-selfadjoint. Thus the operator $\tilde{A}$ generates a one-parameter group of unitary operators $\{S(t)\}_{t\in\mathbb{R}}$ on Y. Consequently, given $(u_0, u_1) \in \mathcal{D}(\tilde{A})$ we have a unique strong solution of (1.3)-(1.6) (with $Q \equiv 0$). We can also obtain more regularity considering smoother initial data in the standard way.

Now, let us be concerned with a boundary observability inequality. We will use multiplier techniques. They have to be conveniently modified in order to handle the extra boundary terms which will appear after integration of the identities. Let $h = h(x)\colon \overline{\Omega} \mapsto \mathbb{R}$ be an auxiliary scalar smooth function which we will choose later. For problem (1.1)-(1.3) (with $P \equiv 0$) we consider the multipliers

$$\mathcal{M}_1 = tE + \mu \operatorname{grad} h \times H$$

and

$$\mathcal{M}_2 = tH - \mathcal{E} \operatorname{grad} h \times E.$$

Since $\{E, H\}$ solves (1.1)-(1.3) we have the identity

$$\begin{aligned} 0 = 2\mathcal{M}_1 \cdot (\mathcal{E}E_t - \operatorname{curl} H) + 2\mathcal{M}_2 \cdot (\mu H_t + \operatorname{curl} E) \\ + 2\mathcal{E}(\operatorname{grad} h \cdot E) \operatorname{div} E + 2\mu(\operatorname{grad} h \cdot H) \operatorname{div} H. \end{aligned} \tag{2.2}$$

We want to obtain identities so that (2.2) can be written in the form

$$\frac{\partial I}{\partial t} = \operatorname{div}(\vec{B}) + D. \tag{2.3}$$

In fact, straightforward calculations give us the following identities:

a) $2t[H \cdot \operatorname{curl} E - E \cdot \operatorname{curl} H] = -2t \operatorname{div}(H \times E)$
 because $\operatorname{div}(\vec{a} \times \vec{b}) = \operatorname{curl} \vec{a} \cdot \vec{b} - \vec{a} \cdot \operatorname{curl} \vec{b}$ is valid.

b) $\operatorname{grad} h \cdot \frac{\partial}{\partial t}(H \times E) = (\operatorname{grad} h \times H) \cdot E_t - (\operatorname{grad} h \times E) \cdot H_t$ where we used $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

c) $t\frac{\partial}{\partial t}[\mathcal{E}|E|^2 + \mu|H|^2] = \frac{\partial}{\partial t}[t(\mathcal{E}|E|^2 + \mu|H|^2)] - (\mathcal{E}|E|^2 + \mu|H|^2)$.

d) $\operatorname{div}[2\mathcal{E}E(E \cdot \operatorname{grad} h) + 2\mu H(H \cdot \operatorname{grad} h)] =$
 $= 2\mathcal{E}(E \cdot \operatorname{grad} h) \operatorname{div} E + 2\mathcal{E}E \cdot \operatorname{grad}(E \cdot \operatorname{grad} h) +$
 $+ 2\mu(H \cdot \operatorname{grad} h) \operatorname{div} H + 2\mu H \cdot \operatorname{grad}(H \cdot \operatorname{grad} h)$
 where we used $\operatorname{div}(f\vec{F}) = f \operatorname{div} \vec{F} + \vec{F} \cdot \operatorname{grad} f$ for f an scalar valued function and $\vec{F}$ a vector field.

e) Using the same property as in b) and equation (1.1) we deduce
 $-2\mu\mathcal{E}\frac{\partial}{\partial t}\{\operatorname{grad} h \cdot (H \times E)\} =$
 $= -2\mu(\operatorname{grad} h \times H) \cdot \operatorname{curl} H - 2\mathcal{E}(\operatorname{grad} h \times E) \cdot \operatorname{curl} E.$

f) Using d) we can verify the identity
 $2\mathcal{E}(\operatorname{grad} h \cdot E) \operatorname{div} E + 2\mu(\operatorname{grad} h \cdot H) \operatorname{div} H =$
 $= \operatorname{div}[2\mathcal{E}E(E \cdot \operatorname{grad} h) + 2\mu H(H \cdot \operatorname{grad} h)] -$
 $- 2\mathcal{E}E \cdot \operatorname{grad}(E \cdot \operatorname{grad} h) - 2\mu H \cdot \operatorname{grad}(H \cdot \operatorname{grad} h) =$
 $= \operatorname{div}[2\mathcal{E}E(E \cdot \operatorname{grad} h) + 2\mu H(H \cdot \operatorname{grad} h) -$

$$-(\mathcal{E}|E|^2+\mu|H|^2)\,\mathrm{grad}\,h]-$$
$$-2\sum_{i,j=1}^{3}(\mathcal{E}E_iE_j+\mu H_iH_j)\tfrac{\partial^2 h}{\partial x_i\partial x_j}+[\mathcal{E}|E|^2+\mu|H|^2]\Delta h.$$

Adding the above identities a) up to f) we deduce that $\frac{\partial I}{\partial t}-\mathrm{div}\,\vec{B}-D$ is equal to the right-hand side of (2.2) where

$$I=I(x,t)=t[\mathcal{E}|E|^2+\mu|H|^2]+4\mathcal{E}\mu\,\mathrm{grad}\,h\cdot(H\times E) \tag{2.4}$$

$$\begin{aligned}\vec{B}=\vec{B}(x,t)&=2t(H\times E)+[\mathcal{E}|E|^2+\mu|H|^2]\,\mathrm{grad}\,h\\&\quad-2\mathcal{E}E(E\cdot\mathrm{grad}\,h)-2\mu H(H\cdot\mathrm{grad}\,h)\end{aligned} \tag{2.5}$$

and

$$D=2\sum_{i,j=1}^{3}\frac{\partial^2 h}{\partial x_i\partial x_j}[\mathcal{E}E_iE_j+\mu H_iH_j]-(\Delta h-1)[\mathcal{E}|E|^2+\mu|H|^2] \tag{2.6}$$

where

$$|E|^2=\sum_{j=1}^{3}E_j^2,\quad |H|^2=\sum_{j=1}^{3}H_j^2.$$

Similarly, for problem (1.4)-(1.6) with $Q\equiv 0$ we consider the multiplier

$$\mathcal{M}_3=tu_t+(\mathrm{grad}\,h\cdot\mathrm{grad})u+u$$

where

$$(\mathrm{grad}\,h\cdot\mathrm{grad})u=\sum_{j=1}^{3}\frac{\partial h}{\partial x_j}\frac{\partial u}{\partial x_j}.$$

Since u solves (1.4)-(1.6) we have the identity

$$0=2\mathcal{M}_3\cdot[\rho u_{tt}-\lambda_2\Delta u-(\lambda_1+\lambda_2)\,\mathrm{grad}(\mathrm{div}\,u)]. \tag{2.7}$$

We want to obtain identities so that (2.7) can be written as

$$\frac{\partial I_1}{\partial t}=\mathrm{div}\,\vec{F}-D_1. \tag{2.8}$$

Straightforward calculations give us the identities

$$\text{a}_1)\ \ \tfrac{\partial}{\partial t}\Big[t\Big\{\rho|u_t|^2+\lambda_2\sum_{j=1}^{3}|\tfrac{\partial u}{\partial x_j}|^2+(\lambda_1+\lambda_2)(\mathrm{div}\,u)^2\Big\}$$
$$+2\rho\{u_t\cdot(\mathrm{grad}\,h\cdot\mathrm{grad})u+u_t\cdot u\}\Big]$$
$$=3\rho|u_t|^2+(\lambda_1+\lambda_2)(\mathrm{div}\,u)^2+\lambda_2\sum_{j=1}^{3}|\tfrac{\partial u}{\partial x_j}|^2$$

$$+2t\lambda_2 \sum_{j=1}^{3} \frac{\partial u_t}{\partial x_j} \cdot \frac{\partial u}{\partial x_j} + 2t(\lambda_1+\lambda_2)\operatorname{div} u \operatorname{div} u_t$$
$$+2\rho u_t \cdot \sum_{j=1}^{3} \frac{\partial h}{\partial x_j}\frac{\partial u_t}{\partial x_j} + 2\rho u_{tt} \cdot \Big\{u + \sum_{j=1}^{3} \frac{\partial h}{\partial x_j}\frac{\partial u}{\partial x_j} + tu_t\Big\}.$$

b$_1$) Let $F_k = 2[tu_t + (\operatorname{grad} h \cdot \operatorname{grad})u + u] \cdot \Big[\lambda_2 \frac{\partial u}{\partial x_k} + (\lambda_1+\lambda_2)(\operatorname{div} u)\ell_k\Big]$
$$+\frac{\partial h}{\partial x_k}\Big[\rho|u_t|^2 - \lambda_2 \sum_{j=1}^{3} |\frac{\partial u}{\partial x_j}|^2 - (\lambda_1+\lambda_2)(\operatorname{div} u)^2\Big], \quad k=1,2,3$$
$\ell_1 = (1,0,0), \quad \ell_2 = (0,1,0)$ and $\ell_3 = (0,0,1)$.

Calculating the partial derivatives $-\frac{\partial F_k}{\partial x_k}$ and adding we get, for $F = (F_1, F_2, F_3)$,

$$-\operatorname{div} \vec{F} = -2\Big[tu_t + \sum_{j=1}^{3} \frac{\partial h}{\partial x_j}\frac{\partial u}{\partial x_j} + u\Big] \cdot [\lambda_2 \Delta u + (\lambda_1+\lambda_2)\operatorname{grad}(\operatorname{div} u)]$$
$$-\Delta h\Big[\rho|u_t|^2 + \lambda_2 \sum_{j=1}^{3} |\frac{\partial u}{\partial x_j}|^2 - (\lambda_1+\lambda_2)(\operatorname{div} u)^2\Big]$$
$$-\sum_{k=1}^{3} \frac{\partial h}{\partial x_k}\Big[2\rho u_t \cdot \frac{\partial u_t}{\partial x_k} - 2\lambda_2 \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} \cdot \frac{\partial^2 u}{\partial x_j \partial x_k} - 2(\lambda_1+\lambda_2)\operatorname{div} u \operatorname{div}(\frac{\partial u}{\partial x_k})\Big]$$
$$-2\sum_{k=1}^{3} \Big[t\frac{\partial u_t}{\partial x_k} + \sum_{j=1}^{3} \frac{\partial h}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{3} \frac{\partial^3 h}{\partial x_j \partial x_k}\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_k}\Big]\Big[\lambda_2 \frac{\partial u}{\partial x_k} + (\lambda_1+\lambda_2)(\operatorname{div} u)\ell_k\Big].$$

Adding a$_1$) to b$_1$) gives us that $\frac{\partial I_1}{\partial t} - \operatorname{div} \vec{F} + D_1$ is equal to the right-hand side of (2.7) where

$$I_1 = t\Big\{\rho|u_t|^2 + \lambda_2 \sum_{j=1}^{3} |\frac{\partial u}{\partial x_j}|^2 + (\lambda_1+\lambda_2)(\operatorname{div} u)^2\Big\}$$
$$+ 2\rho(u_t \cdot (\operatorname{grad} h \cdot \operatorname{grad})u + u_t \cdot u\}, \tag{2.9}$$

F_k is given as in b$_1$), and

$$D_1 = \rho(3-\Delta h)|u_t|^2 + (\Delta h - 1)\Big(\lambda_2 \sum_{j=1}^{3} |\frac{\partial u}{\partial x_j}|^2 + (\lambda_1+\lambda_2)(\operatorname{div} u)^2\Big)$$
$$- 2\sum_{i,p=1}^{3} \frac{\partial^2 h(x)}{\partial x_p \partial x_i}\Big[\lambda_2 \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_p} + (\lambda_1+\lambda_2)\frac{\partial u_i}{\partial x_p}\operatorname{div} u\Big]. \tag{2.10}$$

This proves (2.8).

Remark 1. If we choose $h(x) = \frac{1}{2}|x - x_0|^2$ for some $x_0 \in \mathbb{R}^3$, then, we can verify that D and D_1 given by (2.3) and (2.11) are identically zero.

Thus, with that choice, (2.3) and (2.11) represent conservation laws for problem (1.1)-(1.3) and (1.4)-(1.6) respectively. Those modified multipliers (related to dilation invariance) were already used for related problems by B. Kapitonov [6,7], V. Komornik [17] and F. Alabau and V. Komornik [1] among others.

Lemma 2.1. *Let $\Phi = \Phi(x)$ be a solution of the elliptic problem*

$$\begin{cases} \Delta\Phi = 1 \text{ in } \Omega \\ \dfrac{\partial\Phi}{\partial\eta} = \dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Area}(\partial\Omega)} \text{ on } \partial\Omega. \end{cases}$$

We define

$$\alpha = \alpha(\Omega) = 2 \max_{\substack{x\in\Omega \\ |\xi|=1}} \sum_{i,j=1}^{3} \frac{\partial^2\Phi(x)}{\partial x_i \partial x_j}\xi_i\xi_j, \tag{2.11}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is an arbitrary vector of $\mathbb{R}^3$ with $|\xi| = 1$. Then, $\alpha(\Omega) \geq 2/3$. In case Ω is a ball then $\alpha = 2/3$.

Proof. We take $\xi = (1,0,0), (0,1,0)$ or $(0,0,1)$, then replacing those vectors in (2.11) we get

$$\alpha(\Omega) \geq 2\frac{\partial^2\Phi}{\partial x_j^2}, \quad j = 1,2,3.$$

Consequently, adding the inequalities we obtain $3\alpha \geq 2\Delta\Phi = 2$. Thus $\alpha(\Omega) \geq 2/3$. If $\Omega = \{x \in \mathbb{R}^3 : |x - x_0| < R\}$ for some $x_0 \in \mathbb{R}^3$ and $R > 0$, then the function $\Phi(x) = \frac{1}{6}|x - x_0|^2$ satisfies $\Delta\Phi = 1$ in Ω and $\frac{\partial\Phi}{\partial\eta} = \frac{R}{3} = \frac{\mathrm{Vol}(\Omega)}{\mathrm{area}(\partial\Omega)}$ on $\partial\Omega$. Clearly

$$\frac{\partial^2\Phi(x)}{\partial x_i \partial x_j} = \begin{cases} \frac{1}{3} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Therefore, $\alpha = \frac{2}{3}\max_{|\xi|=1}\sum_{j=1}^{3}\xi_j^2$. We conclude that $\alpha(\Omega) = \frac{2}{3}$ in this case.

Theorem 2.2. *Let P and Q be identically zero (in (1.3) and (1.6) respectively). Let $(E_0, H_0) \in N_1 \cap \mathcal{D}(A)$, $(u_0, u_1) \in \mathcal{D}(\tilde{A})$ and $\{E, H, u\}$ be the solution of problem (1.1)-(1.6). Assume we can find $\delta_1 > 0$ sufficiently small such that*

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Area}(\partial\Omega)}\delta_1 + (x - x_0)\cdot\eta \geq 0 \text{ on } \partial\Omega$$

for some $x_0 \in \mathbb{R}^3$. *Then, there exists a positive constant* C *and some time* $T_0 > 0$ *such that*

$$(T-T_0)\int_\Omega \Big\{\mathcal{E}|E|^2+\mu|H|^2+\rho|u_t|^2+\lambda_2\sum_{i=1}^{3}\Big|\frac{\partial u}{\partial x_i}\Big|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\Big\}dx$$

$$\le C\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{\mu|H\times\eta|^2-\mathcal{E}(E\cdot\eta)^2+(\lambda_1+2\lambda_2)(\operatorname{div}u)^2+\lambda_2|\operatorname{curl}u|^2\}d\Gamma dt$$

for any $T > T_0$.

Proof. Integration in $\Omega\times(0,T)$ of identity (2.3) gives us

$$T\int_\Omega\{\mathcal{E}|E|^2+\mu|H|^2\}\,dx+2\mathcal{E}\mu\int_\Omega \operatorname{grad}h\cdot(H\times E)\,dx\Big|_{t=0}^{t=T}$$
$$=\int_0^T\int_{\partial\Omega}\vec{B}\cdot\eta\,d\Gamma\,dt+\int_0^T\int_\Omega D\,dxdt, \tag{2.12}$$

where

$$\vec{B}\cdot\eta=2t\eta\cdot(H\times E)+\frac{\partial h}{\partial\eta}(\mathcal{E}|E|^2+\mu|H|^2)$$
$$-2\mathcal{E}(E\cdot\eta)(E\cdot\operatorname{grad}h)-2\mu(H\cdot\eta)(H\cdot\operatorname{grad}h). \tag{2.13}$$

Using the boundary condition (1.3) and vector identities we obtain

$$\begin{aligned}\eta\cdot(H\times E)&=-(\eta\times E)\cdot H=0\\ |E|^2&=(E\cdot\eta)^2+|E\times\eta|^2=(E\cdot\eta)^2\\ E&=\eta\times(E\times\eta)+\eta(E\cdot\eta)=\eta(E\cdot\eta)\end{aligned} \tag{2.14}$$

on $\partial\Omega$. Also, since $|\eta|=1$ and (2.1) holds, we obtain

$$|H|^2=(H\cdot\eta)^2+|H\times\eta|^2=|H\times\eta|^2. \tag{2.15}$$

We can use the vector identity

$$(U\cdot W)(V\cdot Z)-(U\cdot Z)(V\cdot W)=(U\times V)\cdot(W\times Z)$$

valid for any four vectors in $\mathbb{R}^3$. In our case $U=\operatorname{grad}h$, $W=\eta$, $V=Z=E$ give us

$$\frac{\partial h}{\partial\eta}|E|^2=(E\cdot\eta)(E\cdot\operatorname{grad}h)\text{ on }\partial\Omega. \tag{2.16}$$

Therefore, from (2.14)-(2.15) we obtain that

$$\vec{B}\cdot\eta=\frac{\partial h}{\partial\eta}(\mu|H\times\eta|^2-\mathcal{E}(E\cdot\eta)^2). \tag{2.17}$$

Next, we estimate the term $\int_0^T\int_\Omega D\,dxdt$ in (2.12). Let Φ be as in Lemma 2.1.

Clearly $\Phi \in C^2(\Omega)\cap C^1(\overline{\Omega})$. Let $\delta>0$ and $x_0\in\mathbb{R}^3$. We define

$$h(x)=\delta\Phi(x)+\frac{1}{2}|x-x_0|^2. \tag{2.18}$$

Substitution of $h(x)$ given by (2.18) into (2.6) gives us

$$D=2\delta\sum_{i,j=1}^{3}\frac{\partial^2\Phi}{\partial x_i\partial x_j}(\mathcal{E}E_iE_j+\mu H_iH_j)-\delta(\mathcal{E}|E|^2+\mu|H|^2). \tag{2.19}$$

Let $\alpha=\alpha(\Omega)$ be given by (2.11). From (2.19) we obtain

$$|D(x,t)|\le\delta(6\alpha-1)[\mathcal{E}|E|^2+\mu|H|^2]. \tag{2.20}$$

Therefore,

$$\Big|\int_0^T\int_\Omega D\,dxdt\Big|\le\delta(6\alpha-1)T\int_\Omega\{\mathcal{E}|E|^2+\mu|H|^2\}\,dx. \tag{2.21}$$

Finally, let us estimate the term $2\mathcal{E}\mu\int_\Omega\operatorname{grad}h\cdot(H\times E)\,dx$ in (2.12). Let $\beta=\max_{x\in\overline{\Omega}}\{|\operatorname{grad}\Phi(x)|+|x-x_0|\}$. Since $|H\times E|\le 2|H||E|$ and $h(x)$ is given by (2.18) we deduce

$$\Big|2\int_\Omega\mathcal{E}\mu\operatorname{grad}h\cdot(H\times E)\,dx\Big|\le 4(1+\delta)\beta\sqrt{\mathcal{E}\mu}\int_\Omega[\mathcal{E}|E|^2+\mu|H|^2]\,dx \tag{2.22}$$

for any $0\le t\le T$. Using (2.17), (2.20) and (2.22) together with identity (2.12) we deduce the estimate

$$\begin{aligned}(T-T_1)&\int_\Omega\{\mathcal{E}|E|^2+\mu|H|^2\}\,dx\\ &\le C\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{\mu|H\times\eta|^2-\mathcal{E}(E\cdot\eta)^2\}\,d\Gamma dt\end{aligned} \tag{2.23}$$

for any $T>T_1$, where $T_1=\frac{8(1+\delta)\beta\sqrt{\mathcal{E}\mu}}{1-\delta(6\alpha-1)}>0$, $C=[1-\delta(6\alpha-1)]^{-1}$, and $0<\delta<(6\alpha-1)^{-1}$. Similarly, integration over $\Omega\times(0,T)$ of identity (2.8) gives us

$$\begin{aligned}T&\int_\Omega\Big\{\rho|u_t|^2+\lambda_1\sum_{i=1}^{3}\Big|\frac{\partial u}{\partial x_i}\Big|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\Big\}\,dx\\ &=-2\rho\int_\Omega[u_t\cdot(\operatorname{grad}h\cdot\operatorname{grad})u+u_t\cdot u]\,dx\Big|_{t=0}^{t=T}\end{aligned}$$

$$+\int_0^T\int_{\partial\Omega}\sum_{j=1}^3 F_j\eta_j\,d\Gamma dt+\int_0^T\int_\Omega D_1\,dxdt \tag{2.24}$$

where F_j and D_1 are given as in b_1) and (2.10) respectively. Due to our choice (2.18) it follows that $\Delta h=\delta+3$ in Ω and

$$\frac{\partial^2 h}{\partial x_i\partial x_j}=\delta\frac{\partial^2\Phi}{\partial x_i\partial x_j}+\delta_{ij}$$

where

$$\delta_{ij}=\begin{cases}1 & \text{if} \quad i=j\\ 0 & \text{if} \quad i\neq j.\end{cases}$$

Thus from (2.10) we obtain

$$D_1=\delta\rho|u_t|^2+\delta\Big[\lambda_2\sum_{j=1}^3\Big|\frac{\partial u}{\partial x_j}\Big|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\Big]$$
$$-2\delta\sum_{i,j=1}^3\frac{\partial^2\Phi(x)}{\partial x_i\partial x_j}\Big[\lambda_2\frac{\partial u}{\partial x_i}\cdot\frac{\partial u}{\partial x_j}+(\lambda_1+\lambda_2)\frac{\partial u_i}{\partial x_j}\operatorname{div}u\Big].$$

We can easily prove that D_1 satisfies

$$D_1\le\delta C\Big\{\rho|u_t|^2+\lambda_2\sum_{j=1}^3\Big|\frac{\partial u}{\partial x_j}\Big|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\Big\}$$

where C is a positive constant. Thus,

$$\int_0^T\int_\Omega D_1\,dxdt\le\delta CT\int_\Omega\{\rho|u_t|^2+\lambda_2\sum_{j=1}^3\Big|\frac{\partial u}{\partial x_j}\Big|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\}\,dx. \tag{2.25}$$

To get a bound for the term

$$\Big|-2\rho\int_\Omega[u_t\cdot(\operatorname{grad}h\cdot\operatorname{grad})u+u_t\cdot u]\Big|_{t=0}^{t=T}\Big|$$

we can use the Cauchy-Schwarz, Hölder's and Poincaré's inequality to conclude that the above expression is bounded by

$$C\int_\Omega\Big\{\rho|u_t|^2+\lambda_2\sum_{j=1}^3\Big|\frac{\partial u}{\partial x_j}\Big|^2\Big\}\,dx\Big|_{t=0}^{t=T} \tag{2.26}$$

for some positive constant C. Finally, to get an estimate for the term $\sum_{j=1}^{3}\int_0^T\int_{\partial\Omega} F_j\eta_j\, d\Gamma dt$ in (2.24) we can proceed using similar steps as the ones given in [1] or [11] using the boundary condition $u=0$ on $\partial\Omega\times(0,T)$ (and $u\in H^2(\Omega)^3$) which gives us the identities

$$\frac{\partial u_i}{\partial x_j}=\eta_j\frac{\partial u_i}{\partial\eta},\quad \operatorname{div}u=\eta\cdot\frac{\partial u}{\partial\eta},\quad \operatorname{curl}u=\eta\times\frac{\partial u}{\partial\eta}$$

$$|\operatorname{grad}u|^2=|\frac{\partial u}{\partial\eta}|^2=\left|\eta\cdot\frac{\partial u}{\partial\eta}\right|^2+\left|\eta\times\frac{\partial u}{\partial\eta}\right|^2=(\operatorname{div}u)^2+|\operatorname{curl}u|^2$$

on $\partial\Omega\times(0,T)$. The estimate we obtain is

$$\sum_{j=1}^{3}\int_0^T\int_{\partial\Omega}F_j\eta_j\,d\Gamma dt \tag{2.1}$$

$$\leq C\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\left\{(\lambda_1+2\lambda_2)(\operatorname{div}u)^2+\lambda_2|\operatorname{curl}u|^2\right\}\,d\Gamma dt \tag{2.27}$$

for some positive constant C.

Using estimates (2.25), (2.26) and (2.27) we obtain

$$(T-T_2)\int_\Omega\left\{\rho|u_t|^2+\lambda_2\sum_{j=1}^{3}\left|\frac{\partial u}{\partial x_j}\right|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\right\}dx$$

$$\leq C\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{(\lambda_1+2\lambda_2)(\operatorname{div}u)^2+\lambda_2|\operatorname{curl}u|^2\}\,d\Gamma dt \tag{2.28}$$

for any $T>T_2$ where T_2 is positive and depends only on the above constants and $\delta>0$ is sufficiently small. Adding (2.23) to (2.28), choosing $T_0=\max\{T_1,T_2\}$, $\delta>0$ small enough and $\delta_1=\delta$ we have

$$\frac{\partial h}{\partial\eta}=\delta_1\frac{\operatorname{Vol}(\Omega)}{\operatorname{Area}(\partial\Omega)}+(x-x_0)\cdot\eta\geq 0 \text{ on } \partial\Omega$$

which proves Theorem 2.2. □

Remark 2. The basic assumption in Theorem 2.2,

$$\frac{\operatorname{Vol}(\Omega)}{\operatorname{area}(\partial\Omega)}\delta_1+(x-x_0)\cdot\eta(x)\geq 0 \quad \text{for all} \quad x\in\partial\Omega,$$

is called a "substar-shaped" condition. If $\delta_1=0$, it then reduces to the well-known "star-shaped" geometric condition. It seems to be hard to characterize a large class of regions where the above condition holds (without being star-shaped). In [6], B. Kapitonov suggested that due to the "continuity" of $\alpha = \alpha(\Omega)$ (given in (2.11)), there exist such regions satisfying the above geometric condition without being star-shaped.

Unfortunately, the conclusion of Theorem 2.2 is not yet suitable to use the techniques needed to apply the Hilbert uniqueness method. We present a sufficient condition in the following theorem.

Theorem 2.3. *Under the assumptions of Theorem* 2.2 *if the* (*numerical*) *relation* $\mathcal{E}\mu = \rho\lambda_2^{-1}$ *holds, then, there exist positive constant* C, $\tilde{\tilde{C}}$ *and* T_3 *such that*

$$(T - T_3)\int_\Omega \Big\{\mathcal{E}|E|^2 + \mu|H|^2 + \rho|u_t|^2 + \lambda_2\sum_{j=1}^{3}\Big|\frac{\partial u}{\partial x_j}\Big|^2 + (\lambda_1+\lambda_2)(\operatorname{div} u)^2\Big\}dx$$
$$\leq C\int_0^T\int_{\partial\Omega}\Big\{\tilde{\tilde{C}}\Big|\frac{\partial u}{\partial \eta} + \mu H\Big|^2 - \frac{\partial h}{\partial \eta}\mathcal{E}|E\cdot\eta|^2\Big\}\,d\Gamma dt$$

for any $T > T_3$.

Proof. In (2.18) we choose $\delta = \delta_1$ as in Theorem 2.2. The following identity can be verified by direct calculations:

$$\begin{aligned}
&\mu H\cdot[\rho u_{tt} - \lambda_2\Delta u - (\lambda_1+\lambda_2)\operatorname{grad}(\operatorname{div} u)]\\
&\quad + \rho\mathcal{E}^{-1}\operatorname{curl} u\cdot(\mathcal{E}E_t - \operatorname{curl} H)\\
&\quad + \rho u_t\cdot(\mu H_t + \operatorname{curl} E) - (\lambda_1 + 2\lambda_2)\mu\operatorname{div} u\operatorname{div} H\\
&\quad + \lambda_2\mathcal{E}^{-1}(\rho\lambda_2^{-1} - \mathcal{E}\mu)(\operatorname{curl} u\cdot\operatorname{curl} H)\\
&= \frac{\partial I_2}{\partial t} - \operatorname{div}\vec{B}_2,
\end{aligned} \tag{2.29}$$

where

$$I_2 = \rho\mu u_t\cdot H + \rho E\cdot\operatorname{curl} u$$

and

$$\vec{B}_2 = \rho u_t\times E + \lambda_2\mu H\times\operatorname{curl} u + (\lambda_1+2\lambda_2)\mu H\operatorname{div} u.$$

Since $\{E, H, u\}$ solves problem (1.1)-(1.6) and $\mathcal{E}\mu = \rho\lambda_2^{-1}$ we have from (2.29)

$$\frac{\partial}{\partial t}I_2 = \operatorname{div}\vec{B}_2 \quad \text{ in } \Omega\times(0,T).$$

Integration of the above identity over $\Omega\times(0,T)$ gives us

$$\int_\Omega\{\rho\mu u_t\cdot H + \rho E\cdot\operatorname{curl} u\}\,dx\Big|_{t=0}^{t=T}$$

$$= \int_0^T \int_{\partial\Omega} \lambda_2 \mu \operatorname{curl} u \cdot (\eta \times H)\, d\Gamma dt, \tag{2.30}$$

where we used the boundary conditions and (2.1). The identity

$$\begin{aligned} 2\lambda_2\mu \operatorname{curl} u \cdot (\eta \times H) &= \lambda_2 |\operatorname{curl} u + \mu(\eta \times H)|^2 \\ &- \lambda_2 |\operatorname{curl} u|^2 - \lambda_2 \mu^2 |\eta \times H|^2 \end{aligned}$$

can be used in (2.30) to obtain

$$\begin{aligned} &\int_\Omega \{\rho\mu u_t \cdot H + \rho E \cdot \operatorname{curl} u\}\, dx \Big|_{t=0}^{t=T} \qquad (2.31) \\ &= \frac{\lambda_2}{2} \int_0^T \int_{\partial\Omega} \Big\{ |\operatorname{curl} u + \mu(\eta \times H)|^2 - \Big| \frac{\partial u}{\partial \eta} \times \eta \Big|^2 - \mu^2 |H \times \eta|^2 \Big\}\, d\Gamma dt, \end{aligned}$$

where we used the condition $u = 0$ on $\partial\Omega \times (0, T)$ (and $u \in [H^2(\Omega) \cap H_0^1(\Omega)]^3$) to assure that $\frac{\partial u_i}{\partial x_j} = \eta_j \frac{\partial u_i}{\partial \eta}$ on $\partial\Omega \times (0, T)$. Thus $\operatorname{curl} u = \eta \times \frac{\partial u}{\partial \eta}$ on $\partial\Omega \times (0, T)$. Clearly,

$$|\operatorname{curl} u|^2 \le 2 \sum_{j=1}^3 \Big| \frac{\partial u}{\partial x_j} \Big|^2. \tag{2.32}$$

Using (2.32) we can estimate the left-hand side of (2.31) to obtain

$$\begin{aligned} &\Big| \int_\Omega \{\rho\mu u_t \cdot H + \rho E \cdot \operatorname{curl} u\}\, dx \Big| \\ &\le C \int_\Omega \Big\{ \rho |u_t|^2 + \mathcal{E}|E|^2 + \mu |H|^2 + \lambda_2 \sum_{j=1}^3 \Big| \frac{\partial u}{\partial x_j} \Big|^2 \Big\}\, dx, \end{aligned} \tag{2.33}$$

where $C = \max\{\frac{\sqrt{\rho\mu}}{2}, \frac{\rho}{2\sqrt{\mathcal{E}\lambda_2}}\}$. We multiply identity (2.31) by $\beta \max\{1, \lambda_2^{-1}\mu^{-1}\}$, where $\beta = \max_{x \in \overline{\Omega}} \{|\operatorname{grad} \Phi(x)| + |x - x_0|\}$, add the identity thus obtained to the inequality in the conclusion of Theorem 2.2 and use (2.33) to obtain the inequality

$$\begin{aligned} &(T - T_3) \int_\Omega \Big\{ \mathcal{E}|E|^2 + \mu|H|^2 + \rho|u_t|^2 + \lambda_2 \sum_{j=1}^3 \Big| \frac{\partial u}{\partial x_j} \Big| + (\lambda_1 + \lambda_2)(\operatorname{div} u)^2 \Big\} dx \\ &\le C \int_0^T \int_{\partial\Omega} \Big\{ \tilde{C}\lambda_2 |\operatorname{curl} u + \mu(\eta \times H)|^2 \\ &\qquad + \frac{\partial h}{\partial \eta}(\lambda_1 + 2\lambda_2)(\operatorname{div} u)^2 - \frac{\partial h}{\partial \eta} \mathcal{E}(E \cdot \eta)^2 \Big\} d\Gamma dt \end{aligned} \tag{2.34}$$

for some positive constants C and $\tilde{C}$.

We can obtain a bound of the term on the right-hand side of (2.34) as follows:

$$\begin{aligned}
&\tilde{C}\lambda_2|\operatorname{curl} u+\mu(\eta\times H)|^2+\frac{\partial h}{\partial\eta}(\lambda_1+2\lambda_2)(\operatorname{div} u)^2\\
&\le \tilde{C}\lambda_2|\operatorname{curl} u+\mu(\eta\times H)|^2+\beta(\lambda_1+2\lambda_2)(\operatorname{div} u)^2\\
&\le \max\{\tilde{C}\lambda_2,\beta(\lambda_1+2\lambda_2)\}\left[\left|\eta\times\left(\frac{\partial u}{\partial\eta}+\mu H\right)\right|^2+\left[\eta\cdot(\frac{\partial u}{\partial\eta}+\mu H)\right]^2\right]\\
&= \max\{\tilde{C}\lambda_2,\beta(\lambda_1+2\lambda_2)\}\left|\frac{\partial u}{\partial\eta}+\mu H\right|^2 \qquad (2.35)
\end{aligned}$$

because $u=0$ and $\eta\cdot H=0$ on $\partial\Omega$, thus $\operatorname{curl} u=\eta\times\frac{\partial u}{\partial\eta}$ and $\eta\cdot\frac{\partial u}{\partial\eta}=\operatorname{div} u$ on $\partial\Omega$. Using (2.35), the proof of Theorem 2.3 is now complete. □

Remark 3. Unfortunately we could not find a precise physical meaning of the condition $\mathcal{E}\mu=\rho/\lambda_2$. All we know is that $\mathcal{E}\mu$ in the isotropic case is very small, like $1/c^2$ where c^2 is the speed of light. We belief that the above numerical relation between the parameters is sufficient but not necessary.

Corollary 2.1. *Under the assumptions of Theorem* 2.2 *and* $\mathcal{E}\mu=\rho\lambda_2^{-1}$. *If the solution* $\{E,H,u\}$ *of problem* (1.1)-(1.6) *with zero boundary conditions satisfies the condition*

$$\frac{\partial u}{\partial\eta}+\mu H=0 \qquad \text{on } \partial\Omega\times(0,T),$$

then, for any $T>T_3$ *we will have*

$$E(x,t)\equiv H(x,t)\equiv u(x,t)\equiv 0 \qquad \text{in } \Omega\times(0,T).$$

3. SIMULTANEOUS EXACT CONTROLLABILITY

Inequalities (2.28) and (2.23) are the so-called "inverse inequalities" or "boundary observability inequalities." In general, for an evolution model those are the harder ones to obtain. In our case the "direct inequalities" for models (1.1)–(1.3) and (1.4)–(1.6) (with $P\equiv 0$ and $Q\equiv 0$) are also known. There exist positive constants $C_1(T)$ and $C_2(T)$ such that

$$\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{(\lambda_1+2\lambda_2)(vidu)^2+\lambda_2|\operatorname{curl} u|^2\}d\Gamma\,dt$$

$$\leq C_1(T)\int_\Omega\{\rho|u_t|^2+\lambda\sum_{j=1}^{3}\{\frac{\partial u}{\partial x_j}|^2+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\}dx \tag{3.1}$$

and

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{u|H\times\eta|^2-\mathcal{E}(E\cdot\eta)^2\}\,d\Gamma\,dt\\&\leq C_2(T)\int_\Omega\{\mathcal{E}|E|^2+\mu|H|^2\}\,dx\end{aligned} \tag{3.2}$$

(see, for instance [6]). Adding (3.1) and (3.2), using the condition $\mathcal{E}\mu=\rho\lambda_2^{-1}$ as in Theorem 2.3 we can deduce the existence of a positive constant $C_3(T)$ such that

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}|\frac{\partial u}{\partial\eta}+\mu H|^2d\Gamma dt\\&\leq C_3(T)\int_\Omega\{\mathcal{E}|E|^2+\mu|H|^2+\rho|u_t|^2+\lambda_2\sum_{j=1}^{3}|\frac{\partial u}{\partial x_j}|^2\\&\quad+(\lambda_1+\lambda_2)(\operatorname{div}u)^2\}\,dx.\end{aligned} \tag{3.3}$$

Let $\{E,H,u\}$ be the strong solution of problem (1.1)-(1.6) with zero boundary conditions. The function space of initial data is the following: the Hilbert space $\mathcal{F}$ obtained by completing $N_1\cap\mathcal{D}(A)\times\mathcal{D}(\tilde{A})$ (see Section 2) with respect to the norm

$$||\{f,g\}||_{\mathcal{F}}\equiv\Big(\int_0^T\int_{\partial\Omega}\Big|\frac{\partial u}{\partial\eta}+\mu H\Big|^2d\Gamma dt\Big)^{1/2}$$

where $f=(f_1,f_2)$ and $g=(g_1,g_2)$ are the initial data of problems (1.1)-(1.3) and (1.4)-(1.6) respectively and $T>T_3$. Let us denote by $||\cdot||_{\mathcal{K}}$ the energy norm; then we have

$$||\{f,g\}||_{\mathcal{K}}^2\equiv||f||_{\mathcal{K}}^2+||g||_{\mathcal{K}}^2\leq C||\{f,g\}||_{\mathcal{F}}^2$$

for some positive constant C. Also we have $\mathcal{F}\subseteq\mathcal{K}\equiv X\times Y$.

The dual space of $\mathcal{F}$ with respect to $\mathcal{K}$ will be denoted by $\mathcal{F}'$.

Next, we will define the solutions of systems (1.1)–(1.3) and (1.4)–(1.6) with nonhomogeneous boundary conditions using the well-known transposition method.

Definition 1. Given $R=R(x,t)\in L^2(\partial\Omega\times(0,T))^3$ and $(f_1,f_2,g_1,g_2)\in\mathcal{F}'$ we say that $\{E,H,u,u_t\}$ is a solution of the Maxwell/elastodynamic system if

a) $\{E, H\}$ solves (1.1)-(1.2) with boundary conditions

$$\eta \times E = \mu\lambda_2 \eta \times (\eta \times R) \quad \text{ on } \partial\Omega \times (0, T)$$

b) $\{u, u_t\}$ solves (1.4)-(1.5) with boundary conditions

$$u_t = R \quad \text{ on } \partial\Omega \times (0, T)$$

c) $\{E(\cdot, t), H(\cdot, t), u(\cdot, t), u_t(\cdot, t)\} \in L^\infty(0, T; \mathcal{F}')$ and

d) $\langle (E(\cdot, t), H(\cdot, t), u(\cdot, t), u_t(\cdot, t)), (\tilde{E}(\cdot, t), \tilde{H}(\cdot, t), \tilde{u}(\cdot, t), \tilde{u}_t(\cdot, t)) \rangle_{\mathcal{K}}$

$$= \langle (f_1, f_2, g_1, g_2), (\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2) \rangle_{\mathcal{K}}$$

$$+ \int_0^t \int_{\partial\Omega} R \cdot \Big[\lambda_2 \Big(\frac{\partial \tilde{u}}{\partial \eta} + \mu \tilde{H} \Big) + (\lambda_1 + \lambda_2) \eta \Big(\frac{\partial \tilde{u}}{\partial \eta} + \mu \tilde{H} \Big) \cdot \eta \Big] \, d\Gamma d \tag{3.4}$$

holds for all $\{\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2\} = \{\tilde{f}, \tilde{g}\} \in \mathcal{F}$ and $0 < t < T$. In (3.4)

$$\langle \{f_1, f_2, g_1, g_2\}, \{\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2\} \rangle_{\mathcal{K}} = \langle \{f_1, f_2\}, \{\tilde{f}_1, \tilde{f}_2\} \rangle_X + \langle \{g_1, g_2\}, \{\tilde{g}_1, \tilde{g}_2\} \rangle_Y$$

$$\{\tilde{E}, \tilde{H}\} = U(t)(\tilde{f}_1, \tilde{f}_2), \{\tilde{u}, \tilde{u}_t\} = S(t)\{\tilde{g}_1, \tilde{g}_2\}$$

and $\{\tilde{E}, \tilde{H}, \tilde{u}, \tilde{u}_t\}$ is the solution of (1.1)-(1.6) with zero boundary conditions.

Similarly we define the following.

Definition 2. Given $R = R(x, t) \in L^2(\partial\Omega \times (0, T))^3$ we say that $\{E, H, u, u_t\}$ is a solution of the Maxwell elastodynamic system with zero initial data at time $t = T$ if

a′) $\{E, H\}$ solves (1.1) with boundary conditions

$$\eta \times E = \mu\lambda_2 \eta \times (\eta \times R) \quad \text{ on } \partial\Omega \times (0, T)$$

b′) $\{u, u_t\}$ solves (1.4) with boundary conditions

$$u_t = R \quad \text{ on } \partial\Omega \times (0, T)$$

d′) $\langle (E(\cdot, t), H(\cdot, t), u(\cdot, t), u_t(\cdot, t)), (\tilde{E}(\cdot, t), \tilde{H}(\cdot, t), \tilde{u}(\cdot, t), \tilde{u}_t(\cdot, t)) \rangle_{\mathcal{K}}$

$$= - \int_t^T \int_{\partial\Omega} R \cdot \Big[\lambda_2 \Big(\frac{\partial \tilde{u}}{\partial \eta} + \mu \tilde{H} \Big) + (\lambda_1 + \lambda_2) \eta \Big(\frac{\partial \tilde{u}}{\partial \eta} + \mu \tilde{H} \Big) \cdot \eta \Big] \, d\Gamma d$$

holds for any $\{\tilde{f}, \tilde{g}\} \in \mathcal{F}$ and $t \in (0, T)$.

Due to linearity and reversibility of systems (1.1)-(1.3) and (1.4)-(1.6) it is clear that in order to solve the problem of exact controllability it is sufficient to prove that for any initial data $(f, g) \in \mathcal{F}'$ the corresponding solutions can be driven to the equilibrium state at time T.

Theorem 3.1. *Assume all hypothesis of Theorem* 2.3 *and consider* $T > T_3$. *Then, for any initial data* $(f_1, f_2, g_1, g_2) \in \mathcal{F}'$ *of problems* (1.1)-(1.2) *and*

(1.4)-(1.5) *there exists a control* $Q \in H^1(0,T;L^2(\partial\Omega)^3)$ *such that* $u = Q$ *on* $\partial\Omega \times (0,T)$ *and the corresponding solution satisfies*

$$(u, u_t)|_{t=T} = (0,0)$$

while the vector-valued function $P = \mu\lambda_2\eta \times (\eta \times Q_t)$ *drives system* (1.1)-(1.2) *satisfying*

$$\eta \times E = P \qquad \text{on } \partial\Omega \times (0,T)$$

to the state of rest at the same time:

$$(E, H)|_{t=T} = (0,0).$$

Proof. We use our previous discussion to apply the Hilbert uniqueness method (HUM). Let (h_1, h_2, q_1, q_2) be an (arbitrary) element of $\mathcal{F}$ and (φ, ψ, v, v_t) the solution of (1.1)-(1.6) with zero boundary conditions and initial data

$$(\varphi, \psi, v, v_t)|_{t=0} = (h_1, h_2, q_1, q_2). \tag{3.5}$$

Next, we consider $\{E, H, u, u_t\}$ the solution of the Maxwell/elastodynamic system with zero initial data at time $t = T > T_3$ and boundary function

$$R(x,t) = -\frac{1}{\lambda_2}\Big(\frac{\partial v}{\partial \eta} + \mu\psi\Big) + \frac{\lambda_1+\lambda_2}{\lambda_2(\lambda_1+2\lambda_2)}\Big[\Big(\frac{\partial v}{\partial \eta} + \mu\psi\Big)\cdot\eta\Big]\eta. \tag{3.6}$$

We consider the map $\lambda\colon \mathcal{F} \mapsto \mathcal{F}'$ given by

$$\Lambda\{h, q\} = \Lambda\{h_1, h_2, q_1, q_2\} = \{E, H, u, u_t\}\Big|_{t=0}$$

where $h = (h_1, h_2)$, $q = (q_1, q_2)$.

From d′) in Definition 2 it follows that

$$\begin{aligned}\langle \Lambda\{h,q\}, \{\tilde f, \tilde g\}\rangle_{\mathcal{K}} &= \int_0^T \int_{\partial\Omega} \Big(\frac{\partial v}{\partial \eta} + \mu\psi\Big)\cdot\Big(\frac{\partial \tilde u}{\partial \eta} + \mu\tilde H\Big)\, d\Gamma\, dt \\ &= \langle \{h,q\}, \{\tilde f, \tilde g\}\rangle_{\mathcal{F}} \end{aligned}\tag{3.7}$$

for any $\{\tilde f, \tilde g\} \in \mathcal{F}$. This implies that Λ is an isomorphism from $\mathcal{F}$ onto $\mathcal{F}'$. Now, we return to problems (1.1), (1.2), $\eta \times E = \mu\lambda_2\eta \times (\eta \times R)$ on $\partial\Omega \times (0,T)$ and (1.4), (1.5) and $u_t = R$ on $\partial\Omega \times (0,T)$. Suppose the initial data $\{f_1, f_2, g_1, g_2\}$ belongs to $\mathcal{F}'$. We set

$$\{h_1, h_2, q_1, q_2\} = \{h, q\} = \Lambda^{-1}\{f_1, f_2, g_1, g_2\} = \Lambda^{-1}\{f, g\}$$

and $R = R(x,t)$ where (φ, ψ, v, v_t) is the solution of (1.1)-(1.6) with zero boundary conditions and initial data as in (3.5).

From Definition 1 with $t = T > T_3$ we deduce

$$\langle (E(\cdot,T), H(\cdot,T), u(\cdot,T), u_t(\cdot,T)), (\tilde{E}(\cdot,T), \tilde{H}(\cdot,T), \tilde{u}(\cdot,T), \tilde{u}_t(\cdot,T))\rangle_{\mathcal{K}}$$
$$= \langle \Lambda\{h,q\}, \{\tilde{f},\tilde{g}\}\rangle_{\mathcal{K}} - \langle \{h,q\}, \{\tilde{f},\tilde{g}\}\rangle_{\mathcal{F}}. \tag{3.8}$$

Using (3.7) we conclude that the right-hand side of (3.8) is equal to zero. This means that $(E(\cdot,T), H(\cdot,T), u(\cdot,T), u_t(\cdot,T))$ generate the zero functional on $\mathcal{F}$. The conclusion of Theorem 3.1 is a consequence of the above construction: Let $R = R(x,t)$ as in (3.6) and define

$$Q(x,t) = \int_0^t R(x,t)ds + g_1(x).$$

Obviously $u = Q$ and $\eta \times E = \mu\lambda_2\eta \times (\eta \times Q_t)$ on $\partial\Omega \times (0,T)$ by construction.

Acknowledgements. Both authors were partially supported by Research Grants from the Brazilian Government (MCT, Brasil), the first author by a Grant of CNPq Project 303981-03-2 and the second author (GPM) by a Grant of CNPq (Project 301134/2009-0) and Project Universal (Proc. 474296/2008-3). G.P.M. would like to thank Prof. J.-M. Coron for the kind invitation to visit the Laboratory J.L. Lions at the University Pierre et Marie Curie during a short period in September 2009.

Both authors express their acknowledgements to the referee of this journal who suggested we include several proofs and correct our previous version.

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