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WELL POSEDNESS AND STABILITY IN THE PERIODIC CASE FOR THE BENNEY SYSTEM

J. Angulo
Department of Mathematics, IME-USP
Rua do Matão 1010, Cidade Universitária, CEP 05508-090
São Paulo, SP, Brazil

A.J. Corcho
Universidade Federal de Alagoas Instituto de Matemática
Campus A. C. Simões, Tabuleiro dos Martins
57072-900 Maceió, AL, Brazil

S. Hakkaev
Faculty of Mathematics and Informatics, Shumen University
9712 Shumen, Bulgaria

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Abstract. We establish local well-posedness results in weak periodic function spaces for the Cauchy problem of the Benney system. The Sobolev space $H^{1/2} \times L^2$ is the lowest regularity attained and also we cover the energy space $H^1 \times L^2$, where global well posedness follows from the conservation laws of the system. Moreover, we show the existence of a smooth explicit family of periodic travelling waves of *dnoidal* type and we prove, under certain conditions, that this family is orbitally stable in the energy space.

1. Introduction

In this paper we consider the system introduced by Benney in [10] which models the interaction between short and long waves, for example, in the

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theory of resonant water wave interaction in a nonlinear medium:

$$\begin{cases} iu_t + u_{xx} = uv + \beta|u|^2u, & (x,t) \in \mathcal{M} \times \triangle T \\ v_t = (|u|^2)_x, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function representing the envelope of short waves, and $v = v(x,t)$ is a real-valued function representing the long wave. Here β is a real parameter, $\triangle T$ is the time interval $[0,T]$ and $\mathcal{M}$ is the real line $\mathbb{R}$ or the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

We let $H^s(\mathcal{M})$ denote the classical Sobolev space with the norm

$$\|f\|_s = \Big(\int_{-\infty}^{\infty} (1+|\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi \Big)^{1/2} \quad \text{if } \mathcal{M} = \mathbb{R},$$

and

$$\|f\|_s = \Big(\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |\hat{f}(n)|^2 \Big)^{1/2} \quad \text{if } \mathcal{M} = \mathbb{T},$$

where $\hat{f}(\xi)$ and $\hat{f}(n)$ denote the Fourier transform and Fourier coefficient of f, respectively. We consider the initial data (u_0, v_0) in the space $H^r(\mathcal{M}) \times H^s(\mathcal{M})$ with the induced norm

$$\|(f,g)\|_{r\times s} := \|f\|_r + \|g\|_s.$$

The quantities

$$E_1[u(.,t)] = \int_I |u(x,t)|^2 dx, \tag{1.2}$$

$$E_2[u(.,t), v(.,t)] = \int_I \Big[v(x,t)|u(x,t)|^2 + |u_x(x,t)|^2 + \tfrac{\beta}{2}|u(x,t)|^4 \Big] dx \tag{1.3}$$

and

$$E_3[u(.,t), v(.,t)] = \int_I \Big[|v(x,t)|^2 + 2\text{Im } (u(x,t)\bar{u}_x(x,t)) \Big] dx \tag{1.4}$$

with the interval $I = (-\infty, +\infty)$ if $\mathcal{M} = \mathbb{R}$ and $I = [0,1]$ if $\mathcal{M} = \mathbb{T}$ are invariants by the flux of the system (1.1); i.e, the natural energy space for the system is $H^1(\mathcal{M}) \times L^2(\mathcal{M})$.

1.1. **Some results in the continuous case.** When $\mathcal{M} = \mathbb{R}$ the local well posedness for (1.1) for data $(u_0, v_0) \in H^{(s+1/2)}(\mathbb{R}) \times H^s(\mathbb{R})$ with indices $s \geq 0$ was established in the works [8], [16] and [24]. Furthermore, in [24] also was proved global well posedness in $H^{(s+1/2)}(\mathbb{R}) \times H^s(\mathbb{R})$ for $s = 0$ if $\beta = 0$ and for $s \in \mathbb{Z}^+$ and any real β by using the conservation laws (1.2), (1.3) and (1.4).

Recently, in [15] Corcho showed that for $\beta < 0$ (focusing case) and for data $(u_0, v_0) \in H^r(\mathbb{R}) \times H^s(\mathbb{R})$, with $0 \leq 3r + 1 < 1$ and $r(2s+3) + 1 \geq 0$, this problem is ill posed in the following sense: the data-solution mapping fails to be uniformly continuous on bounded subsets of $H^r(\mathbb{R}) \times H^s(\mathbb{R})$.

Concerning the existence and stability of solitary waves solutions for (1.1) of the general form

$$\begin{cases} u(x,t) = e^{i\omega t} e^{ic(x-ct)/2} \phi_s(x-ct), \\ v(x,t) = \psi_s(x-ct), \end{cases} \tag{1.5}$$

where $\phi_s, \psi_s : \mathbb{R} \to \mathbb{R}$ are smooth, $c > 0$, $\omega \in \mathbb{R}$, and $\phi_s(\xi), \psi_s(\xi) \to 0$ as $|\xi| \to \infty$, Laurençot in [21] studied, for $\beta = 0$, the nonlinear stability of the orbit

$$\Omega_{(\Phi,\Psi)} = \left\{ (e^{i\theta}\Phi(\cdot + x_0), \Psi(\cdot + x_0));\ (\theta, x_0) \in [0, 2\pi) \times \mathbb{R} \right\},$$

in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ by the flow generated by (1.1). Here we have that $\Phi(\xi) = e^{ic\xi/2}\phi_s(\xi)$, $\Psi(\xi) = \psi_s(\xi)$, and

$$\phi_s(\xi) = \sqrt{2c\sigma}\,\mathrm{sech}(\sqrt{\sigma}\xi), \qquad \psi_s(\xi) = -\frac{1}{c}\phi_s^2(\xi), \tag{1.6}$$

$\sigma = \omega - \frac{c^2}{4} > 0$.

1.2. **Main results in the periodic case.** In the present work we focus our attention on the case $\mathcal{M} = \mathbb{T}$ and we study the following problems:

- well posedness in Sobolev spaces with low regularity,
- existence and nonlinear stability of periodic travelling waves.

As follows we define the concepts of well posedness and the stability that will be used in this work. The space $H^r(\mathbb{T}) \times H^s(\mathbb{T})$ will be denoted by $H^r_{per} \times H^s_{per}$.

Definition 1.1 (Well Posedness and Ill Posedness). *We say that the system* (1.1) *is locally well posed, in time, in the space $H^r_{per} \times H^s_{per}$ if the following conditions hold:*

(a) *for every* (u_0, v_0) *in the space* $H^r_{per} \times H^s_{per}$ *there exists a positive time* $T = T(\|u_0\|_r, \|v_0\|_s)$ *and a distributional solution* $(u, v) : \mathbb{T} \times \triangle T \longrightarrow \mathbb{C} \times \mathbb{R}$ *which is in the space* $C\left(\triangle T;\; H^r_{per} \times H^s_{per}\right)$*;*

(b) *the data-solution mapping* $(u_0, v_0) \longmapsto (u, v)$ *is uniformly continuous from* $H^r_{per} \times H^s_{per}$ *to* $C\left(\triangle T;\; H^r_{per} \times H^s_{per}\right)$*;*

(c) *there is an additional Banach space* $\mathcal{X}$ *such that* (u, v) *is the unique solution to the Cauchy problem in* $\mathcal{X} \cap C\left(\triangle T;\; H^r_{per} \times H^s_{per}\right)$*.*

Moreover, we say that the problem is ill posed if, at least, one of the above conditions fails.

Before stating our well- and ill-posedness results we will give some useful notation. Let η be a function in $C^\infty_0(\mathbb{R})$ such that $0 \le \eta(t) \le 1$,

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \le 1, \\ 0 & \text{if } |t| \ge 2, \end{cases}$$

and $\eta_\delta(t) = \eta(\frac{t}{\delta})$. We denote by $\lambda\pm$ a number slightly larger, respectively smaller, than λ and by $\langle\cdot\rangle$, $\langle\xi\rangle = 1 + |\xi|$. The characteristic function on the set A is denoted by χ_A. Furthermore, we will work with the auxiliary periodic Bourgain space $X^{s,b}_{per}$ defined as follows: first we denote by $\mathcal{X}$ the space of functions $f : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ such that

(i) $f(x, \cdot) \in \mathcal{S}(\mathbb{R})$ for each $x \in \mathbb{T}$;

(ii) $f(\cdot, t) \in C^\infty(\mathbb{T})$ for each $t \in \mathbb{R}$.

For $s, b \in \mathbb{R}$, the spaces $H^b_t H^s_{per}$ and $X^{s,b}_{per}$ are the completion of $\mathcal{X}$ with respect to the norms

$$\|f\|_{H^b_t H^s_{per}} = \Big(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} (1 + |n|)^{2s} (1 + |\tau|)^{2b} |\widehat{f}(n, \tau)|^2 d\tau \Big)^{\frac{1}{2}} \tag{1.7}$$

and

$$\begin{aligned} \|f\|_{X^{s,b}_{per}} &= \|S(-t)f\|_{H^b_t H^s_{per}} \\ &= \Big(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} (1 + |n|)^{2s} (1 + |\tau + n^2|)^{2b} |\widehat{f}(n, \tau)|^2 d\tau \Big)^{\frac{1}{2}}, \end{aligned} \tag{1.8}$$

respectively, where $S(t) := e^{it\partial_x^2}$ is the corresponding Schrödinger generator (unitary group) associated to the linear problem

$$iu_t + u_{xx} = 0, \qquad u(x, 0) = g_(x). \tag{1.9}$$

For any $r, s \in \mathbb{R}$ and $b_1, b_2 > 1/2$, we have the embedding $X^{r,b_1}_{per} \hookrightarrow C\left(\mathbb{R}; H^r_{per}\right)$ and $H^{b_2}_t H^s_{per} \hookrightarrow C\left(\mathbb{R}; H^s_{per}\right)$. For the case $b = 1/2$ the embedding can be guaranteed by considering the following slight modifications of the Bourgain spaces:

$$\|f\|_{X^r_{per}} := \|f\|_{X^{r,1/2}_{per}} + \|\langle n\rangle^r \widehat{f}(n,\tau)\|_{\ell^2_n L^1_\tau} \tag{1.10}$$

and

$$\|f\|_{Y^s_{per}} := \|f\|_{H^{1/2}_t H^s_{per}} + \|\langle n\rangle^s \widehat{f}(n,\tau)\|_{\ell^2_n L^1_\tau} \tag{1.11}$$

Concerning local well posedness we obtain the following result.

Theorem 1.2 (Local Well Posedness)**.** *For any $(u_0, v_0) \in H^r_{per} \times H^s_{per}$ with r, s satisfying the condition*

$$\max\{0,\, r-1\} \leq s \leq \min\{r,\, 2r-1\}, \tag{1.12}$$

there exist a positive time $T = T\left(\|u_0\|_r, \|v_0\|_s\right)$ and a unique solution $(u(t), v(t))$ of the initial-value problem (1.1), *satisfying*

(a) $(\eta_T(t)u, \eta_T(t)v) \in X^r_{per} \times Y^s_{per}$;

(b) $(u, v) \in C\left(\triangle T;\ H^r_{per} \times H^s_{per}\right)$.

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ is locally uniformly continuous from $H^r_{per} \times H^s_{per}$ into $C\left(\triangle T;\ H^r_{per} \times H^s_{per}\right)$.

The proof of Theorem 1.2 is based on the Banach fixed-point theorem applied on the integral formulation of the system combined with new sharp periodic bilinear estimates, in adequate mixed Bourgain spaces $X^{r,b_1}_{per} \times H^{b_2}_t H^s_{per}$, for the coupling terms uv and $\partial_x(|u|^2)$.

Also we find a region for which the Cauchy problem is not locally well posed; more precisely, we prove the following theorem.

Theorem 1.3. *Let $\beta \neq 0$. Then, for any $r < 0$ and $s \in \mathbb{R}$, the initial-value problem* (1.1) *is locally ill posed for data in $H^r_{per} \times H^s_{per}$.*

Regarding the stability of periodic travelling waves, namely, solutions for (1.1) of the form

$$\begin{cases} u(t,x) = e^{-i\omega t} e^{ic(x-ct)/2} \varphi_{\omega,c}(x-ct) \\ v(x,t) = n_{\omega,c}(x-ct), \end{cases} \tag{1.13}$$

where $\varphi_{\omega,c}$, $n_{\omega,c}$ are real, smooth, L-periodic functions, $c > 0$, and $\omega < 0$, we have the following definition.

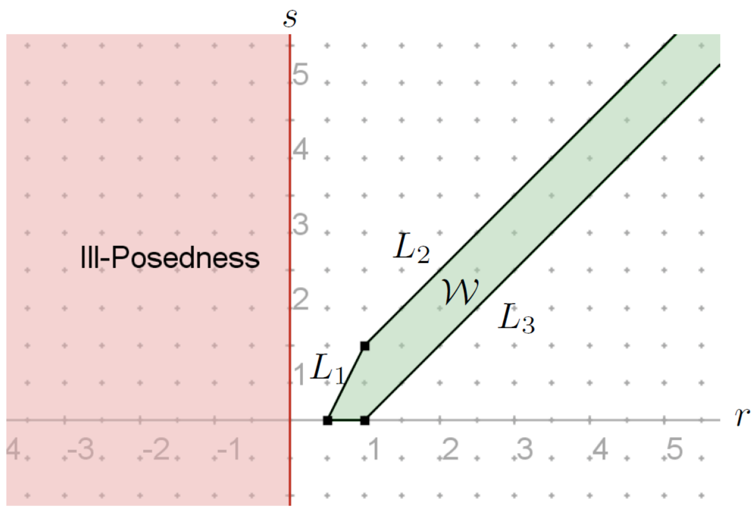

FIGURE 1. Well-posedness results for periodic Benney system. The region $\mathcal{W}$, limited by the lines $L_1 : s = 2r - 1$, $L_2 : s = r$ and $L_3 : s = r-1$, contains the indices (r, s) where the local well posedness is achieved in Theorem 1.2.

Definition 1.4 (Non-Linear Stability)**.** *The periodic travelling wave* $\Phi(\xi) = e^{ic\xi/2}\varphi_{\omega,c}(\xi)$, $\Psi(\xi) = n_{\omega,c}(\xi)$, *is orbitally stable in* $H^1_{per}([0, L]) \times L^2_{per}([0, L])$ *if, for all* $\varepsilon > 0$, *there exists* $\delta > 0$, *such that if* $||(u_0, v_0) - (\Phi, \Psi)||_{H^1_{per} \times L^2_{per}} < \delta$ *and* $(u(t), v(t))$ *is the solution of* (1.1) *with* $(u(0), v(0)) = (u_0, v_0)$, *then*

$$\inf_{s\in[0,2\pi)} \inf_{r\in\mathbb{R}} ||(u(t), v(t)) - (e^{is}\Phi(\cdot + r), \Psi(\cdot + r))||_{H^1_{per} \times L^2_{per}} < \varepsilon, \qquad t \in \mathbb{R}.$$

Otherwise (Φ, Ψ) *is called orbitally unstable.*

We will show below that there exists a smooth explicit family of profiles solutions of minimal period L,

$$(\omega, c) \in \mathcal{A}_\beta \to (\varphi_{\omega,c}, n_{\omega,c}) \in H^r_{per}([0, L]) \times H^s_{per}([0, L]),$$

where $\mathcal{A}_\beta = \{(x, y) : y > 0, 1 > \beta y, \quad and \quad x < -\frac{2\pi^2}{L^2} - \frac{y^2}{4}\}$ and which depends on the Jacobian elliptic function *dn* called *dnoidal*; more precisely,

$$\begin{cases} \varphi_{\omega,c}(\xi) = \sqrt{\frac{c}{1-\beta c}}\ \eta_1 dn\big(\frac{\eta_1}{\sqrt{2}}\xi; \kappa\big) \\ n_{\omega,c}(\xi) = -\frac{\eta_1^2}{1-\beta c} dn^2\big(\frac{\eta_1}{\sqrt{2}}\xi; \kappa\big) \end{cases} \tag{1.14}$$

with $\eta_1 = \eta_1(\omega, c)$ and $\kappa = \kappa(\omega, c)$, being smooth functions of ω and c.

So, by following Angulo [4] and Grillakis *et al.* [17], [18], we obtain the following stability theorem.

Theorem 1.5 (Stability Theory)**.** *Let* $(\omega, c) \in \mathcal{A}_\beta$ *such that for* $c > 0$ *there is* $q \in \mathbb{N}$ *satisfying* $4\pi q/c = L$*. Define* $\sigma \equiv -\omega - \frac{c^2}{4}$*. Then* $\Phi(\xi) = e^{ic\xi/2}\varphi_{\omega,c}(\xi)$, $\Psi(\xi) = n_{\omega,c}(\xi)$*, with* $\varphi_{\omega,c}, n_{\omega,c}$ *given in* (1.14)*, is orbitally stable in* $H^1_{per}([0,L]) \times L^2_{per}([0,L])$ *by the periodic flow generated by* (1.1) *:*

(a) *for* $\beta \leq 0$*,*

(b) *for* $\beta > 0$ *and* $8\beta\sigma - 3c(1-\beta c)^2 \leq 0$*.*

2. LOCAL THEORY

We prove Theorem 1.2 using the standard technique; that is, we use the Duhamel integral formulation for the system (1.1) combined with the Banach fixed-point theorem in adequate Bourgain spaces $X^r_{per} \times Y^s_{per}$ with the objective of getting the desired solution. The main difficulty is the necessity to prove two new mixed periodic bilinear estimates, which we will prove in the following sections.

2.1. **Sharp Periodic Bilinear Estimates.** We begin by recalling the following elementary inequalities, which will be used in the proof of the next main estimates.

Lemma 2.1. *Let* $\theta_1, \theta_2 > 0$ *with* $\theta_1 + \theta_2 > 1$ *and* $\lambda > 1/2$*. Then, there are positive constants* C_1 *and* C_2 *such that*

(a) $\int_{-\infty}^{+\infty} \frac{dx}{\langle x-a\rangle^{\theta_1}\langle x-b\rangle^{\theta_2}} \leq \frac{C_1}{\langle a-b\rangle^{\mu}}$*, where* $\mu := \min\{\theta_1, \theta_2, \theta_1+\theta_2-1\}$*;*

(b) $\sum\limits_{n\in\mathbb{Z}} \frac{1}{\langle n^2+an+b\rangle^{\lambda}} \leq C_2$*, with* $a, b \in \mathbb{R}$*.*

Proof. For details of the proof we refer, for instance, to the works [19] and [7]. □

Lemma 2.2. *Let* $0 < \theta < 1/4$*. Then, the estimates*

$$\|uv\|_{X^{r,-1/2}_{per}} \lesssim \|u\|_{X^{r,1/2-\theta}_{per}}\|v\|_{H^{1/2}_t H^s_{per}} + \|u\|_{X^{r,1/2}_{per}}\|v\|_{H^{1/2-\theta}_t H^s_{per}} \tag{2.1}$$

$$\left\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau+n^2\rangle} \right\|_{\ell^2_n L^1_\tau} \lesssim \|u\|_{X^{r,1/2-\theta}_{per}}\|v\|_{H^{1/2}_t H^s_{per}} + \|u\|_{X^{r,1/2}_{per}}\|v\|_{H^{1/2-\theta}_t H^s_{per}} \tag{2.2}$$

hold provided $r \geq 0$ *and* $\max\{0, r-1\} \leq s$*.*

Proof. First we prove (2.1). We define $f(n,\tau) := \langle \tau + n^2\rangle^{b_1}\langle n\rangle^r \widehat{u}(n,\tau)$ and $g(n,\tau) := \langle \tau\rangle^{b_2}\langle n\rangle^s \widehat{v}(n,\tau)$. Then, using duality arguments we obtain

$$\|uv\|_{X^{r,-1/2}_{per}} = \sup\big\{W(\varphi) : \|\varphi\|_{\ell^2_n L^2_\tau} \le 1\big\},$$

where

$$W(\varphi) = \sum_{(n,n_1)\in\mathbb{Z}^2} \int_{\mathbb{R}^2} \frac{\langle \tau + n^2\rangle^{-1/2}\langle n\rangle^r f(n_1,\tau_1) g(n-n_1,\tau-\tau_1)\varphi(n,\tau)}{\langle \tau_1 + n_1^2\rangle^{b_1}\langle \tau-\tau_1\rangle^{b_2}\langle n_1\rangle^r\langle n-n_1\rangle^s} d\tau d\tau_1. \tag{2.3}$$

We will divide the space $\mathbb{Z}^2\times\mathbb{R}^2$ into three regions, namely $\mathbb{Z}^2\times\mathbb{R}^2 = A_0\cup A_1\cup A_2$ and we separate the integral W as follows:

$$W(\varphi) = W_0(\varphi) + W_1(\varphi) + W_2(\varphi), \tag{2.4}$$

where

$$W_j(\varphi) = \sum\int\sum\int_{(n,n_1,\tau,\tau_1)\in A_j} \frac{\langle \tau + n^2\rangle^{-1/2}\langle n\rangle^r f(n_1,\tau_1) g(n-n_1,\tau-\tau_1)\varphi(n,\tau)}{\langle \tau_1 + n_1^2\rangle^{b_1}\langle \tau-\tau_1\rangle^{b_2}\langle n_1\rangle^r\langle n-n_1\rangle^s},$$

for $j=0,1,2$. It is easy to see that to obtain (2.1) is suffices to prove that whenever $r,\ s\ge 0$ and $r-s\le 1$ the estimate

$$W_j(\varphi) \lesssim \|f\|_{\ell^2_n L^2_\tau}\|g\|_{\ell^2_n L^2_\tau}\|\varphi\|_{\ell^2_n L^2_\tau} = \|u\|_{X^{b_1,r}_{per}}\|v\|_{H^{b_2}_t H^s_{per}}\|\varphi\|_{\ell^2_n L^2_\tau} \tag{2.5}$$

holds with $b_1 = 1/2-\theta$ and $b_2 = 1/2$ or with $b_1 = 1/2$ and $b_2 = 1/2-\theta$. Indeed, next we will prove the following estimates:

$$\begin{aligned} W_j(\varphi) &\lesssim \|u\|_{X^{1/2}_{per}}\|v\|_{H^{1/2-\theta}_t H^s_{per}}\|\varphi\|_{\ell^2_n L^2_\tau}, \text{ for } j=0,1,\\ W_2(\varphi) &\lesssim \|u\|_{X^{1/2-\theta,r}_{per}}\|v\|_{H^{1/2}_t H^s_{per}}\|\varphi\|_{\ell^2_n L^2_\tau}. \end{aligned} \tag{2.6}$$

For this purpose, in region A_0 we integrate first over (n_1,τ_1), in region A_1 we integrate first over (n,τ) and in region A_2 we integrate first over $(n_2,\tau_2) = (n-n_1,\tau-\tau_1)$; then using the Cauchy-Schwarz inequality we easily see that it remains only to bound uniformly the following three expressions:

$$\widetilde{W}_0 := \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau + n^2\rangle} \sum_{n_1}\int_{A_0} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle\langle \tau_2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2s}} \tag{2.7}$$

$$\widetilde{W}_1 := \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_{n}\int_{A_1} \frac{\langle n\rangle^{2r} d\tau}{\langle \tau + n^2\rangle\langle \tau_2\rangle^{1-2\theta}\langle n_2\rangle^{2s}} \tag{2.8}$$

$$\widetilde{W}_2 := \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2s}\langle\tau_2\rangle} \sum_n \int_{A_2} \frac{\langle n\rangle^{2r} d\tau}{\langle\tau+n^2\rangle\langle\tau_1+n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}}. \tag{2.9}$$

Now we define the regions A_0, A_1 and A_2. We use the notation

$$\mathcal{L} := \max\Big\{\big|\tau+n^2\big|, \big|\tau_1+n_1^2\big|, |\tau_2|\Big\}. \tag{2.10}$$

and first we introduce the subsets

$$\begin{aligned} A_{0,1} &:= \big\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|\le 100\big\}, \\ A_{0,2} &:= \big\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|>100 \text{ and } |n|\le 2|n_1|\big\}, \\ A_{0,3} &:= \Big\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|>100, |n_1|<|n|/2 \text{ and } \mathcal{L}=\big|\tau+n^2\big|\Big\}. \end{aligned} \tag{2.11}$$

Then, we put

$$A_0 := A_{0,1}\cup A_{0,2}\cup A_{0,3}, \tag{2.12}$$

$$A_1 := \Big\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|>100, |n_1|<|n|/2 \text{ and } \mathcal{L}=\big|\tau_1+n_1^2\big|\Big\},$$

$$A_2 := \Big\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|>100, |n_1|<|n|/2 \text{ and } \mathcal{L}=|\tau_2|\Big\}.$$

For later use, we recall that the dispersive relation of this bilinear estimate is

$$\tau+n^2-(\tau_1+n_1^2)-\tau_2 = n^2-n_1^2, \tag{2.13}$$

where $\tau-\tau_1=\tau_2$ and $n-n_1=n_2$.

We begin with the analysis of (2.7). In the region $A_{0,1}$, using the facts that $|n|\lesssim 1$ and $r,\ s\ge 0$ we have

$$\begin{aligned} \widetilde{W}_{0,1} &:= \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle\tau+n^2\rangle} \sum_{n_1} \int_{A_{0,1}} \frac{d\tau_1}{\langle\tau_1+n_1^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{n,\tau} \frac{1}{\langle\tau+n^2\rangle} \sum_{n_1} \int_{-\infty}^{+\infty} \frac{d\tau_1}{\langle\tau_1+n_1^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{\langle\tau+n_1^2\rangle^{1-2\theta}} \lesssim 1, \end{aligned} \tag{2.14}$$

where in the last inequality we have used the fact that $0<\theta<1/4$ combined with Lemma 2.1.

In the region $A_{0,2}$, we have $\langle n\rangle^{2r} \lesssim \langle n_1\rangle^{2r}$. Thus, similarly to the previous case, we get

$$\widetilde{W}_{0,2} := \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau+n^2\rangle} \sum_{n_1} \int\limits_{A_{0,2}} \frac{d\tau_1}{\langle \tau_1+n_1^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2s}} \tag{2.15}$$

$$\lesssim \sup_{n,\tau} \frac{1}{\langle \tau+n^2\rangle} \sum_{n_1} \int\limits_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1+n_1^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_2\rangle^{2s}}$$

$$\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{\langle \tau+n_1^2\rangle^{1-2\theta}} \lesssim 1.$$

In the region $A_{0,3}$ we have $|n_1| < |n|/2$ and $|n| > 100$, which imply that $|n-n_1| \sim |n+n_1| \sim |n|$. Moreover, the dispersive relation (2.13) says that

$$\mathcal{L} = |\tau+n^2| \gtrsim |n^2-n_1^2| = |n-n_1||n+n_1| \sim |n|^2.$$

Therefore,

$$\widetilde{W}_{0,3} := \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau+n^2\rangle} \sum_{n_1} \int\limits_{A_{0,3}} \frac{d\tau_1}{\langle \tau_1+n_1^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2s}} \tag{2.16}$$

$$\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s}}{\langle \tau+n^2\rangle} \sum_{n_1} \int\limits_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1+n_1^2\rangle\langle\tau_2\rangle^{1-2\theta}}$$

$$\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s}}{\langle n\rangle^2} \sum_{n_1} \frac{1}{\langle \tau+n_1^2\rangle^{1-2\theta}} \lesssim 1,$$

since $r \geq 0$, $r-s \leq 1$ and $0 < \theta < 1/4$.

Putting together the estimates (2.14), (2.15) and (2.16) we conclude that

$$|\widetilde{W}_0| \leq |\widetilde{W}_{0,1}| + |\widetilde{W}_{0,2}| + |\widetilde{W}_{0,3}| \lesssim 1,$$

obtaining the desired bounds for (2.7).

Next we estimate the contribution of (2.8). In the region A_1, we know that $|n_1| < |n|/2$, $|n| > 100$ and $\mathcal{L} = |\tau_1+n_1^2|$. So, $|n_2| \sim |n|$ and the dispersive relation (2.13) implies that $|\tau_1+n_1^2| \gtrsim n^2$. Thus,

$$\widetilde{W}_1 = \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1+n_1^2\rangle} \sum_{n} \int\limits_{A_1} \frac{\langle n\rangle^{2r}}{\langle \tau+n^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_2\rangle^{2s}} d\tau$$

$$\lesssim \sup_{\tau_1} \sum_n \int_{-\infty}^{+\infty} \frac{\langle n\rangle^{2r-2s-2}}{\langle \tau+n^2\rangle \langle \tau_2\rangle^{1-2\theta}} d\tau \lesssim \sup_{\tau_1} \sum_n \frac{1}{\langle \tau_1+n^2\rangle^{1-2\theta}} \lesssim 1,$$

since $r \geq 0$, $r-s \leq 1$ and $0<\theta<1/4$.

Finally, we bound (2.9) by noting that, in the region A_2, $|n|>100$, $|n_1|<|n|/2$ and $\mathcal{L}=|\tau_2|$. Then, $|n_2|\sim |n|$ and the dispersive relation (2.13) yields $|\tau_2| \gtrsim n^2$. Using these conditions and the fact that $r\geq 0$, $r-s\leq 1$ we obtain

$$\widetilde{W}_2 = \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2s}\langle \tau_2\rangle} \sum_n \int_{A_2} \frac{\langle n\rangle^{2r}}{\langle \tau+n^2\rangle \langle \tau_1+n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}} d\tau$$

$$\lesssim \sup_{n_2,\tau_2} \sum_n \int_{-\infty}^{+\infty} \frac{\langle n\rangle^{2r-2s-2}}{\langle \tau+n^2\rangle \langle \tau_1+n_1^2\rangle^{1-2\theta}} d\tau \lesssim \sup_{n_2,\tau_2} \sum_n \frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}$$

$$= \sup_{n_2,\tau_2} \Big\{ \sum_{n\in H_1} \frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}} + \sum_{n\in H_2} \frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}} \Big\},$$

where $H_1 := \big\{ n\in\mathbb{Z} : \big|n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\big| < 2\big\}$ and $H_2 := \big\{ n\in\mathbb{Z} : \big|n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\big| \geq 2\big\}$. Now we note that $\#H_1 \leq 4$ and for any $n\in H_2$ we have

$$\langle 2n_2(n+\tfrac{\tau_2}{2n_2}-\tfrac{n_2}{2})\rangle \gtrsim \langle n\rangle \langle n+\tfrac{\tau_2}{2n_2}-\tfrac{n_2}{2}\rangle,$$

since $|n_2|\sim|n|$. Then, by Hölder's inequality,

$$\sum_{n\in H_1} \frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}} + \sum_{n\in H_2} \frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}$$

$$\leq 4 + \sum_{n\in H_2} \frac{1}{\langle n\rangle^{1-2\theta}\langle n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\rangle^{1-2\theta}}$$

$$4 + \Big(\sum_n \frac{1}{\langle n\rangle^{2(1-2\theta)}}\Big)^{1/2} \Big(\sum_n \frac{1}{\langle n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\rangle^{2(1-2\theta)}}\Big)^{1/2} \lesssim 1,$$

since $0<\theta<1/4$. This completes the proof of (2.1).

Next, we prove (2.2). We let $a\in(1/2,\ 3/4-\theta)$. By using the Cauchy-Schwarz inequality, we have

$$\Big\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau+n^2\rangle} \Big\|^2_{\ell^2_n L^1_\tau} \leq \sum_n \langle n\rangle^{2r} \Big\{ \int_{-\infty}^{+\infty} \frac{|\widehat{uv}(n,\tau)|^2}{\langle \tau+n^2\rangle^{2(1-a)}} d\tau \int_{-\infty}^{+\infty} \frac{d\tau}{\langle \tau+n^2\rangle^{2a}} \Big\}. \tag{2.17}$$

Now, we separate $\mathbb{Z}^2 \times \mathbb{R}^2$ in the same regions used to estimate (2.1) and we note that, except in the region $A_{0,3}$, the right-hand side of (2.17) can be estimated in the same way as (2.1). To see this, we observe that the integral $\int_{-\infty}^{+\infty} \frac{d\tau}{\langle \tau+n^2\rangle^{2a}}$ is convergent and we replace the term $\langle \tau + n^2\rangle$ by $\langle \tau + n^2\rangle^{2(1-a)}$ in (2.7), (2.8) and (2.9), then we follow the same steps to bound the corresponding expressions in each region, using the fact that the condition $2(1-a)+(1-2\theta)-1 > 1/2$ holds for $a \in (1/2,\ 3/4-\theta)$.

Now we proceed with the estimate of the right-hand side of (2.17) in $A_{0,3}$. Here, by using the fact that $|\tau + n^2| \gtrsim |n|^2$ we have that

$$\int\limits_{A_{0,3}} \frac{d\tau}{\langle \tau+n^2\rangle^{2a}} \lesssim \langle n\rangle^{2(1-2a)}. \tag{2.18}$$

Then, using (2.18), we have

$$\left\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau+n^2\rangle} \chi_{A_{0,3}} \right\|^2_{\ell^2_n L^1_\tau} \lesssim \widetilde{W}_{0,3} \|u\|^2_{X^{r,1/2}_{per}} \|v\|^2_{H^{1/2-\theta}_t H^s_{per}}, \tag{2.19}$$

where

$$\widetilde{W}_{0,3} = \sup_{n,\tau} \frac{\langle n\rangle^{2r} n^{2(1-2a)}}{\langle \tau+n^2\rangle^{2(1-a)}} \sum_{n_1} \int\limits_{A_{0,3}} \frac{d\tau_1}{\langle \tau_1+n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}}. \tag{2.20}$$

Similarly to the estimate made in (2.16) we obtain

$$\begin{aligned} \widetilde{W}_{0,3} &\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s+2-4a}}{\langle \tau+n^2\rangle^{2(1-a)}} \sum_{n_1} \int\limits_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1+n_1^2\rangle \langle \tau_2\rangle^{1-2\theta}} \qquad (2.21)\\ &\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s+2-4a}}{\langle n\rangle^{4(1-a)}} \sum_{n_1} \frac{1}{\langle \tau+n_1^2\rangle^{1-2\theta}} \lesssim 1, \end{aligned}$$

since $0<\theta<1/4$ and $r-s\le 1$. Finally, combining (2.18) and (2.21) we get

$$\left\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau+n^2\rangle} \chi_{A_{0,3}} \right\|_{\ell^2_n L^1_\tau} \lesssim \|u\|_{X^{r,1/2}_{per}} \|v\|_{H^{1/2-\theta}_t H^s_{per}},$$

as we desired. Then, we have finished the proof of Lemma 2.2. □

The next result shows that the conditions obtained above for indices r and s are necessary.

Proposition 2.3. *For any real numbers b_1 and b_2, the veracity of the inequality*

$$\|uv\|_{X^{r,-1/2}} \lesssim \|u\|_{X^{r,b_1}} \|v\|_{H^{b_2}_t H^s_x}$$

implies that $\max\{0, r-1\} \le s$.

Proof. Firstly, we fix $N \gg 1$ a large integer and define the sequences

$$\alpha_1(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_1(n) = \begin{cases} 1 & \text{if } n = -2N, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u_{1_N}(x,t)$ and $v_{1_N}(x,t)$ be given by $\widehat{u}_{1_N}(n,\tau) = \alpha_1(n)\chi_{[-1,1]}(\tau+n^2)$ and $\widehat{v}_{1_N}(n,\tau) = \beta_1(n)\chi_{[-1,1]}(\tau)$. Taking into account the dispersive relation

$$\tau + n^2 - (\tau_1 + n_1^2) - \tau_2 = n^2 - n_1^2,$$

we can easily compute that

$$\|u_{1_N} v_{1_N}\|_{X^{r,-1/2}} \sim N^r, \quad \|u_{1_N}\|_{X^{r,b_1}} \sim N^r \quad \text{and} \quad \|v_{1_N}\|_{H_t^{b_2} H_x^s} \sim N^s.$$

Hence, from the bound $\|u_{1_N} v_{1_N}\|_{X^{r,-1/2}} \lesssim \|u_{1_N}\|_{X^{r,b_1}} \|v_{1_N}\|_{H_t^{b_2} H_x^s}$ we must have $N^r \lesssim N^{r+s}$ for $N \gg 1$, which implies that $s \ge 0$.

Secondly, we define the sequences

$$\alpha_2(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_2(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widehat{u}_{2_N}(n,\tau) = \alpha_2(n)\chi_{[-1,1]}(\tau+n^2)$ and $\widehat{v}_{2_N}(n,\tau) = \beta_2(n)\chi_{[-1,1]}(\tau)$. Again, it is easy to see that

$$\|u_{2_N} v_{2_N}\|_{X^{r,-1/2}} \sim N^{r-1}, \quad \|u_{2_N}\|_{X^{r,b_1}} \sim 1 \quad \text{and} \quad \|v_{2_N}\|_{H_t^{b_2} H_x^s} \sim N^s.$$

Hence, the bound $\|u_{2_N} v_{2_N}\|_{X^{r,-1/2}} \lesssim \|u_{2_N}\|_{X^{r,b_1}} \|v_{2_N}\|_{H_t^{b_2} H_x^s}$ implies $N^{r-1} \lesssim N^s$ for $N \gg 1$, so we must have $r - 1 \le s$. □

Lemma 2.4. *Let* $0 < \theta < 1/4$. *Then, the estimates*

$$\|\partial_x(u\bar{w})\|_{H_t^{-1/2} H_{per}^s} \lesssim \|u\|_{X_{per}^{r,1/2-\theta}} \|w\|_{X_{per}^{r,1/2}} + \|u\|_{X_{per}^{r,1/2}} \|w\|_{X_{per}^{r,1/2-\theta}} \tag{2.22}$$

$$\left\| \langle n \rangle^s \frac{\widehat{\partial_x(u\bar{w})}(n,\tau)}{\langle \tau \rangle} \right\|_{\ell_n^2 L_\tau^1} \lesssim \|u\|_{X_{per}^{r,1/2-\theta}} \|w\|_{X_{per}^{r,1/2}} + \|u\|_{X_{per}^{r,1/2}} \|w\|_{X_{per}^{r,1/2-\theta}} \tag{2.23}$$

hold provided $0 \le s \le \min\{2r-1,\ r\}$.

Proof. The proof is similar to Lemma (2.2). Here, the relevant dispersive relation is given by

$$(\tau_1 + n_1^2) + (\tau_2 - n_2^2) - \tau = n_1^2 - n_2^2, \tag{2.24}$$

where $\tau_2 = \tau - \tau_1$ and $n_2 = n - n_1$.

To prove (2.22), by duality arguments, it suffices to bound uniformly the following expressions:

$$Z_0 = \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_n \int\limits_{C_0} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau, \tag{2.25}$$

$$Z_1 = \sup_{n,\tau} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle} \sum_{n_1} \int\limits_{C_1} \frac{d\tau_1}{\langle\tau_1 + n_1^2\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2r}}, \tag{2.26}$$

$$Z_2 = \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2r}\langle \tau_2 - n_2^2\rangle} \sum_n \int\limits_{C_2} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_1 + n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}} d\tau, \tag{2.27}$$

where C_0, C_1 and C_2 are defined as follows. We denote by

$$\mathcal{L} := \max\big\{|\tau|, |\tau_1 + n_1^2|, |\tau_2 - n_2^2|\big\}$$

and then we define de following sets:

$C_{0,1} := \{(n,\tau,n_1,\tau_1) : |n| \le 100\}$,

$C_{0,2} := t\{(n,\tau,n_1,\tau_1) : |n| > 100, \frac{|n_2|}{2} \le |n_1| \le 2|n_2|\}$,

$C_{0,3} := \big\{(n,\tau,n_1,\tau_1) : |n| > 100, |n_1| < \frac{|n_2|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau_1 + n_1^2|\big\}$.

Now we put

$C_0 := C_{0,1} \cup C_{0,2} \cup C_{0,3}$,

$C_1 := \{(n,\tau,n_1,\tau_1) : |n| > 100, |n_1| < \frac{|n_2|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau|\}$,

$C_2 := \{(n,\tau,n_1,\tau_1) : |n| > 100, |n_1| < \frac{|n_2|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau_2 - n_2^2|\}$.

Now, we bound (2.25). In the region $C_{0,1}$, $|n| \le 100$. Hence,

$$Z_{0,1} := \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_n \int\limits_{C_{0,1}} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau$$

$$\lesssim \sup_{n_1,\tau_1} \sum_{|n|\le 100} \int\limits_{-\infty}^{+\infty} \frac{d\tau}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}} \lesssim \sup_{n_1,\tau_1} \sum_{|n|\le 100} \frac{1}{\langle\tau_1 + (n-n_1)^2\rangle^{1-2\theta}} \lesssim 1,$$

since $r \ge 0$ and $1 - 2\theta > 0$.

In the region $C_{0,2}$, we have $|n_1| \sim |n_2|$. Hence,

$$Z_{0,2} := \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_n \int\limits_{C_{0,2}} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau$$

$$\lesssim \sup_{n_1,\tau_1} \frac{\langle n_1\rangle^{2s-4r+2}}{\langle \tau_1+n_1^2\rangle} \sum_n \int_{-\infty}^{+\infty} \frac{d\tau}{\langle\tau\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}}$$
$$\lesssim \sup_{n_1,\tau_1} \sum_n \frac{1}{\langle\tau_1+(n-n_1)^2\rangle^{1-2\theta}} \lesssim 1,$$

for $0\leq s\leq 2r-1$ and $0<\theta<1/4$.

In the region $C_{0,3}$, the dispersion relation (2.24)and the assumptions $|n_1| \not\sim |n_2|$, $|n|\geq 100$ and $\mathcal{L}=|\tau_1+n_1^2|$ imply that $|\tau_1+n_1^2| \gtrsim (\max\{|n_1|,|n_2|\})^2$. Then,

$$Z_{0,3} := \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle\tau_1+n_1^2\rangle} \sum_n \int_{C_{0,3}} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau$$
$$\lesssim \sup_{n_1,\tau_1} \sum_n \int_{-\infty}^{+\infty} \frac{\big\langle \max\{|n_1|,|n_2|\}\big\rangle^{2s-2r}}{\langle\tau\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}} d\tau$$
$$\lesssim \sup_{n_1,\tau_1} \sum_n \frac{1}{\langle\tau_1+(n-n_1)^2\rangle^{1-2\theta}} \lesssim 1,$$

for $0\leq s\leq r$ and $0<\theta<1/4$. Then, the inequality $|Z_0|\leq |Z_{0,1}|+|Z_{0,2}|+|Z_{0,3}|\lesssim 1$ yields the desired estimate for Z_0.

The contribution of (2.26) can be estimated as follows. In the region C_1, we have $|n|\sim \max\{|n_1|,|n_2|\}$ and $|\tau|\geq (\max\{|n_1|,|n_2|\})^2$. Thus,

$$Z_1 \leq \sup_{n,\tau} \frac{\langle n\rangle^{2s+2}}{\langle\tau\rangle} \sum_{n_1} \int_{C_1} \frac{d\tau_1}{\langle\tau_1+n_1^2\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2r}}$$
$$\lesssim \sup_{n,\tau} \sum_{n_1} \int_{-\infty}^{\infty} \frac{\big\langle \max\{|n_1|,|n_2|\}\big\rangle^{2s-2r}}{\langle\tau_1+n_1^2\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}} d\tau_1 \lesssim \sup_{n,\tau} \sum_{n_1} \frac{1}{\langle\tau+n_1^2-n_2^2\rangle^{1-2\theta}}$$
$$\lesssim \sup_{n,\tau} \sum_{n_1} \frac{1}{\langle 2nn_1+\tau-n^2\rangle^{1-2\theta}} \lesssim 1,$$

for $0\leq s\leq r$ and $0<\theta<1/4$, using the same arguments to estimate $\widetilde{W}_2$ in Lemma 2.2.

On the other hand, the expression (2.27) can be controlled by using the fact that, in the region C_2, $|n|\sim\max\{|n_1|,|n_2|\}$ and $|\tau_2-n_2^2| \gtrsim$

$(\max\{|n_1|, |n_2|\})^2$. Then,

$$Z_2 = \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2r}\langle \tau_2 - n_2^2\rangle} \sum_n \int_{C_2} \frac{|n|^2\langle n\rangle^{2s}}{\langle \tau\rangle\langle \tau_1 + n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}} d\tau$$
$$\lesssim \sup_{n_2,\tau_2} \sum_n \int_{-\infty}^{+\infty} \frac{\big\langle \max\{|n_1|, |n_2|\}\big\rangle^{2s-2r}}{\langle \tau\rangle\langle \tau_1 + n_1^2\rangle^{1-2\theta}} d\tau$$
$$\lesssim \sup_{n_2,\tau_2} \sum_n \frac{1}{\langle (n+n_2)^2 - \tau_2\rangle^{1-2\theta}} \lesssim 1,$$

for $s \le r$ and $0 < \theta < 1/4$. Collecting all the estimates above we obtain the claimed estimate (2.22).

The prove of (2.23) follows from a similar way to the proof of (2.2). □

Now we exhibit examples showing the necessity of the conditions for r and s used in Lemma 2.4.

Proposition 2.5. *For any real numbers b_1 and b_2 the veracity of the inequality*

$$\|\partial_x(u\bar{w})\|_{H_t^{-1/2}H_{per}^s} \lesssim \|u\|_{X^{r,b_1}}\|w\|_{X^{r,b_2}}$$

implies that $s \le \min\{2r-1,\ r\}$.

Proof. For a fixed large integer $N \gg 1$, we define the following sequences:

$$\alpha_1(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_1(n) = \begin{cases} 1 & \text{if } n = -N, \\ 0 & \text{otherwise.} \end{cases}$$

Putting

$$\widehat{u}_{1_N}(n,\tau) = \alpha_1(n)\chi_{[-1,1]}(\tau + n^2), \quad \widehat{w}_{1_N}(n,\tau) = \beta_1(n)\chi_{[-1,1]}(\tau + n^2),$$

a simple calculation using the dispersive relation (2.24) gives

$$\|(u_1\bar{w}_1)_x\|_{H_t^{-1/2}H_{per}^s} \sim N^{s+1} \quad \text{and} \quad \|u_1\|_{X^{r,b_1}} \sim N^r \sim \|w_1\|_{X^{r,b_2}}.$$

Hence, the inequality $\|(u_1\bar{w}_1)_x\|_{H_t^{-1/2}H_{per}^s} \lesssim \|u_1\|_{X^{r,b_1}}\|w_1\|_{X^{r,b_2}}$ implies

$$N^{s+1} \le N^{2r}, \text{ for } N \gg 1 \iff s \le 2r-1.$$

Finally, we define

$$\alpha_2(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_2(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise,} \end{cases}$$

and we put

$$\widehat{u}_{2_N}(n,\tau)=\alpha_2(n)\chi_{[-1,1]}(\tau+n^2),\quad \widehat{w}_{2_N}(n,\tau)=\beta_2(n)\chi_{[-1,1]}(\tau+n^2).$$

Then, by similar calculations as in the previous case we obtain

$$\|(u_2\bar{w}_2)_x\|_{H_t^{-1/2}H^s_{per}}\sim N^s,\quad \|u_2\|_{X^{r,b_1}}\sim 1\quad\text{and}\sim\|w_2\|_{X^{r,b_2}}\sim N^r.$$

Again, the inequality $\|(u_2\bar{w}_2)_x\|_{H_t^{-1/2}H^s_{per}}\lesssim\|u_2\|_{X^{r,b_1}}\|w_2\|_{X^{r,b_2}}$ implies

$$N^s\le N^r,\ \text{for } N\gg 1\Longleftrightarrow s\le r.$$

Thus, we have finished the proof. □

2.2. **Proof of Local Theorem.** The next lemmas will be useful in the proof of Theorem 1.2.

Lemma 2.6. *For any $s\in\mathbb{R}$, $\delta\in(0,1]$, $0<\mu<1/2$ and $-1/2<b_1\le b_2<1/2$ we have*

(a) $\|\eta_\delta(\cdot)F\|_{X^{s,1/2}_{per}}\le C\delta^{-\mu}\|F\|_{X^{s,1/2}_{per}}$ *and*
$\|\eta_\delta(\cdot)F\|_{H_t^{1/2}H^s_{per}}\le C\delta^{-\mu}\|F\|_{H_t^{1/2}H^s_{per}}$;

(b) $\|\eta_\delta(\cdot)F\|_{X^{s,b_1}_{per}}\le C\delta^{b_2-b_1}\|F\|_{X^{s,b_2}_{per}}$ *and*
$\|\eta_\delta(\cdot)F\|_{H_t^{b_1}H^s_{per}}\le C\delta^{b_2-b_1}\|F\|_{H_t^{b_2}H^s_{per}}$.

Proof. The proof of this result can be found, for instance, in [23] and [7]. □

Lemma 2.7 (Trilinear Estimate)**.** *For any $s\ge 0$, we have*

$$\|uv\bar{w}\|_{X^r_{per}}\lesssim\|u\|_{X^{s,3/8}_{per}}\|v\|_{X^{s,3/8}_{per}}\|w\|_{X^{s,3/8}_{per}}.$$

Proof. See [13] and [7]. □

Now we give the sketch of the proof of the local theorem. First, we let $(u_0,v_0)\in H^r_{per}\times H^s_{per}$ where r and s satisfy

$$\max\{0,\,r-1\}\le s\le\min\{r,\,2r-1\}$$

and we consider the operator $\Phi=(\Phi_1,\,\Phi_2)$, with

$$\begin{aligned}\Phi_1(u,v)&=\eta(t)u_0-i\eta(t)\int_0^t e^{i(t-t')\partial_x^2}\left((\eta_\delta u\eta_\delta v)(t')+\eta_\delta u|\eta_\delta u|^2(t')\right)dt',\\ \Phi_2(u,v)&=\eta(t)v_0+\eta(t)\int_0^t\partial_x(|\eta_\delta u|^2)(t')dt',\end{aligned}\tag{2.28}$$

defined on the ball

$$\mathcal{B}[a,b]=\left\{(u,v)\in X^r_{per}\times Y^s_{per}:\ \|u\|_{X^r_{per}}\le a\text{ and }\|v\|_{Y^s_{per}}\le b\right\}.$$

Then, by Lemmas 2.2, 2.4, 2.6 and 2.7 we have

$$\begin{aligned}\|\Phi_1(u,v)\|_{X^r_{per}} &\le C_0\|u_0\|_{H^r_{per}} + C\Big(\|\eta_\delta u\|_{X^{r,1/2-\theta}_{per}}\|\eta_\delta v\|_{H^{1/2}_t H^s_{per}} + \\ &\qquad + \|\eta_\delta u\|_{X^{r,1/2}_{per}}\|\eta_\delta v\|_{H^{1/2-\theta}_t H^s_{per}} + \|\eta_\delta u\|^3_{X^{r,3/8}_{per}}\Big) \\ &\le C_0\|u_0\|_{H^r_{per}} + C\delta^\epsilon(ab+a^3)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\|\Phi_2(u,v)\|_{Y^s_{per}} &\le C_0\|v_0\|_{H^s_{per}} + C\left(\|\eta_\delta u\|_{X^{r,1/2-\theta}_{per}}\|\eta_\delta u\|_{X^{r,1/2}_{per}}\right) \\ &\le C_0\|v_0\|_{H^s_{per}} + C\delta^\epsilon a^2,\end{aligned} \tag{2.30}$$

with ϵ enough small.

Now we put $a = 2C_0\|u_0\|_{H^r_{per}}$ and $b = 2C_0\|v_0\|_{H^s_{per}}$ and then we choose δ such that $\delta^\epsilon \le \min\left\{\frac{1}{2C(ab+a^3)}, \frac{1}{2Ca^2}\right\}$. Thus, we have that $\Phi(\mathcal{B}[a,b]) \subset \mathcal{B}[a,b]$. The contraction condition

$$\|\Phi(u,v) - \Phi(\tilde{u},\tilde{v})\|^{r\times s}_{per} \le C(a,b)\delta^\theta\|(u-\tilde{u}, v-\tilde{v})\|^{r\times s}_{per},$$

where $\|(f,g)\|^{r\times s}_{per} := \|f\|_{X^r_{per}} + \|g\|_{Y^s_{per}}$ and $C(a,b)$ is a positive constant depending only on a and b, follows similarly. This shows that the map Φ is a contraction on $\mathcal{B}[a,b]$. Thus we obtain a unique fixed point which solves the system for $T < \delta$ and we finish the proof.

Remark 2.8. We note that global well posedness in $H^1_{per} \times L^2_{per}$ follows directly from the local theorem for $(r,s) = (1,0)$ combined with the conservation laws (1.2), (1.3) and (1.4).

3. Ill posedness

In this section we will show that the solution of (1.1) cannot depend uniformly continuously on its initial data for $r < 0$ and $s \in \mathbb{R}$. We will use the same argument given in [14].

3.1. **Proof of theorem 1.3.** It is easy to check that

$$\begin{aligned}u_{N,a}(t,x) &= a\exp(iNx)\exp(-it(N^2 + (\gamma+\beta)a^2)) \\ v_{N,a}(t,x) &= \gamma a^2,\end{aligned} \tag{3.1}$$

where $a \in \mathbb{R}$ and N is any positive integer, solves (1.1) with initial data $u_0(x) = a\exp(iNx)$ and $v_0(x) = \gamma a^2$. Moreover, for $a = \alpha(1+N^2)^{\frac{r}{2}}$, where α is a real constant, and $|\gamma| = (1+N^2)^r$ we have

$$||u_0(x)||^2_{H^r} \le c\alpha^2, \text{ and } ||v_0(x)||^2_{H^s} \le c\alpha^4$$

where c is a constant. Let $a_1 = \alpha_1(1+N^2)^{\frac{r}{2}}$ and $a_2 = \alpha_2(1+N^2)^{\frac{r}{2}}$. For the Sobolev norm of the difference of two initial data, we have

$$||u_{N,a_1}(0) - u_{N,a_2}(0)||^2_{H^r} = c|\alpha_1 - \alpha_2|^2 \to 0, \text{as } \alpha_1 \to \alpha_2$$

and

$$||v_{a_1}(0) - v_{a_2}(0)||^2_{H^s} = |\gamma|^2|\alpha_1^2 - \alpha_2^2|^2(1+N^2)^{-2r} = |\alpha_1^2 - \alpha_2^2|^2, \text{ as } \alpha_1 \to \alpha_2.$$

On the other hand we have

$$\begin{aligned} ||u_{N,a_1}(t,x) - \quad & u_{N,a_2}(t,x)||^2_{H^r} = \sum_{n=-\infty}^{+\infty} (1+|n|^2)^r |\hat{u}_{N,\alpha_1}(n) - \hat{u}_{N,\alpha_2}(n)|^2 \\ & = (1+N^2)^r |a_1 e^{-it(N^2+(\gamma+\beta)a_1^2)} - a_2 e^{-it(N^2+(\gamma+\beta)a_2^2)}|^2 \\ & = |\alpha_1 - \alpha_2 e^{it(\gamma+\beta)(\alpha_1^2-\alpha_2^2)(1+N^2)^{-r}}|^2. \end{aligned}$$

Let $r < 0$, and α_1 and α_2 be such that $\beta(\alpha_1^2 - \alpha_2^2)(1+N^2)^{-r} = \delta(1+N^2)^{\nu}$, where $\nu > 0$, and $\nu + r < 0$. Then for $t = \frac{\pi}{2}(\delta^{-1}(1+N^2)^{-\nu})$ we have

$$||u_{N,a_1}(t,x) - u_{N,a_2}(t,x)||^2_{H^r} \geq c(\alpha_1^2 + \alpha_2^2).$$

Note that t can made arbitrarily small, by choosing N sufficiently large.

4. Existence of periodic travelling wave solutions

We are interesting in this section in finding explicit solutions for (1.1) in the form

$$\begin{cases} u(t,x) = e^{-i\omega t} e^{i\frac{c}{2}(x-ct)} \varphi_{\omega,c}(x-ct), \\ v(t,x) = n_{\omega,c}(x-ct), \end{cases} \tag{4.1}$$

where $\varphi_{\omega,c}$ and $n_{\omega,c}$ are smooth and L-periodic functions, $c > 0$, $\omega \in \mathbb{R}$, and we suppose that there is a $q \in \mathbb{N}$ such that $\frac{4\pi q}{c} = L$. So, putting (1.13) into (1.1) we obtain

$$\begin{cases} \varphi''_{\omega,c} + (\omega + \frac{c^2}{4})\varphi_{\omega,c} = \varphi_{\omega,c} n_{\omega,c} + \beta\varphi^3_{\omega,c}, \\ -cn'_{\omega,c} = 2\varphi_{\omega,c}\varphi'_{\omega,c}. \end{cases} \tag{4.2}$$

If $n_{\omega,c} = \gamma\varphi^2_{\omega,c}$, then from the second equation in (4.2) we have $\gamma = -\frac{1}{c}$. Substituting $n_{\omega,c}$ in the first equation in (4.2), it follows that $\varphi_{\omega,c}$ satisfies

$$\varphi''_{\omega,c} + \left(\omega + \frac{c^2}{4}\right)\varphi_{\omega,c} = \left(\beta - \frac{1}{c}\right)\varphi^3_{\omega,c}. \tag{4.3}$$

If $1-\beta c>0$ and $\varphi_{\omega,c}=(\frac{c}{1-\beta c})^{\frac{1}{2}}\phi_{\omega,c}$, then $\phi_{\omega,c}$ satisfies the equation

$$\phi''_{\omega,c}-\sigma\phi_{\omega,c}+\phi^3_{\omega,c}=0, \tag{4.4}$$

where $\sigma=-\omega-\frac{c^2}{4}$. So, by following Angulo's arguments in ([3], [4]) we have from (4.4) that $\phi_{\omega,c}$ satisfies the first-order equation

$$[\phi'_{\omega,c}]^2=\tfrac{1}{2}P_\phi(\phi), \tag{4.5}$$

where $P_\phi(t)=-t^4+2\sigma t^2+2B_\phi$ and B_ϕ is an integration constant. Let $-\eta_1<-\eta_2<\eta_2<\eta_1$ be the zeros of the polynomial $P_\phi(t)$. Then

$$[\phi'_{\omega,c}]^2=\tfrac{1}{2}(\eta_1^2-\phi^2_{\omega,c})(\phi^2_{\omega,c}-\eta_2^2). \tag{4.6}$$

The solution of (4.6) is

$$\phi_{\omega,c}=\eta_1 dn\Big(\frac{\eta_1}{\sqrt{2}}\xi;\kappa\Big), \tag{4.7}$$

where

$$\eta_1^2+\eta_2^2=2\sigma,\ \ \kappa^2=\frac{\eta_1^2-\eta_2^2}{\eta_1^2},\ \ 0<\eta_2<\eta_1. \tag{4.8}$$

Define the function in the variable κ, $0<\kappa<1$,

$$K=K(\kappa)=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}}$$

called the complete elliptic integral of the first kind. Since dn has fundamental period $2K(\kappa)$, it follows that $\phi_{\omega,c}$ has fundamental period

$$T_{\phi_{\omega,c}}=\frac{2\sqrt{2}}{\eta_1}K(\kappa).$$

Analogously to [3] we obtain the following result.

Theorem 4.1. *Let L be a fixed but arbitrary positive constant, $1-\beta c>0$, and $-\omega-\frac{c^2}{4}>0$. Let $\sigma_0>\frac{2\pi^2}{L^2}$ and $\eta_{2,0}=\eta_2(\sigma_0)\in(0,\sqrt{\frac{\sigma_0}{3}})$ be the unique number such that $T_\phi=L$. Then the following assertions hold:*

(1) *There exists an interval $I(\sigma_0)$ around σ_0, an interval $B(\eta_{2,0})$ around $\eta_{2,0}$, and a unique smooth function $\Lambda: I(\sigma_0)\to B(\eta_{2,0})$, such that*

$$\Lambda(\sigma_0)=\eta_{2,0}\ \text{ and }\ \frac{2\sqrt{2}}{\sqrt{2\sigma-\eta_2^2}}\mathrm{K}(\kappa)=\mathrm{L},$$

where $\sigma\in I(\sigma_0),\eta_2=\Lambda(\sigma)$.

(2) *Solutions* $(\varphi_{\omega,c}, n_{\omega,c})$ *of* (4.2) *given by*

$$\begin{cases} \varphi_{\omega,c} = \sqrt{\frac{c}{1-\beta c}}\eta_1 dn\left(\frac{\eta_1}{\sqrt{2}}\xi;\kappa\right), \\ n_{\omega,c} = -\frac{\eta_1^2}{1-\beta c} dn^2\left(\frac{\eta_1}{\sqrt{2}}\xi;\kappa\right), \end{cases} \tag{4.9}$$

with $\eta_1 = \eta_1(\sigma)$, $\eta_2 = \eta_2(\sigma)$, $\eta_1^2 + \eta_2^2 = 2\sigma$, *have the fundamental period* L *and satisfy* (4.2). *Moreover, the mapping* $\sigma \in I(\sigma_0) \to (\varphi_{\omega,c}, n_{\omega,c})$ *is a smooth function.*

(3) $I(\sigma_0)$ *can be chosen as* $(\frac{2\pi^2}{L^2}, +\infty)$.

(4) *The mapping* $\sigma \to \kappa(\sigma)$ *is a strictly increasing function.*

5. Stability of travelling waves

In this section we consider the stability of the orbit

$$\Omega_{(\Phi,\Psi)} = \{(e^{i\theta}\Phi(\cdot + x_0), \Psi(\cdot + x_0)) : (\theta, x_0) \in [0, 2\pi) \times \mathbb{R}\},$$

in $H^1_{per}([0,L]) \times L^2_{per}([0,L])$ by the periodic flow generated by (1.1), where we have $\Phi(\xi) = e^{ic\xi/2}\varphi_{\omega,c}(\xi)$, $\Psi(\xi) = n_{\omega,c}(\xi)$, with $\varphi_{\omega,c}, n_{\omega,c}$ given in (4.9). Let X be the space $X = H^1_{complex}([0,L]) \times L^2_{real}([0,L])$, with real inner product

$$(\vec{u_1}, \vec{u_2}) = \Re \int_0^L (\varepsilon_1\overline{\eta}_1 + \varepsilon_{1x}\overline{\eta}_{1x} + \varepsilon_2\overline{\eta_2})dx.$$

Let T_1, T_2 be one-parameter groups of unitary operators on X defined by

$$T_1(s)\vec{u}(\cdot) = \vec{u}(\cdot + s), \quad T_2(r)\vec{u}(\cdot) = (e^{-ir}\varepsilon(\cdot), n(\cdot))$$

for $\vec{u} \in X$, s, $r \in \mathbb{R}$. Obviously

$$T_1^{'}(0) = \begin{pmatrix} -\partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}, \quad T_2^{'}(0) = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that the equation (1.1) is invariant under T_1 and T_2. If

$$\Phi_{\omega,c}(x) = (\varepsilon_{\omega,c}(x), n_{\omega,c}(x)),$$

where $\varepsilon_{\omega,c}(x) = e^{i\frac{c}{2}x}\varphi_{\omega,c}(x)$, then from Theorem 4.1 we obtain that

$$T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)$$

is a travelling wave solution of (4.2) with $\varphi_{\omega,c}(x), n_{\omega,c}(x)$ defined by (4.9).

Now, it is easy to verify that $E_2(\vec{u})$ is invariant under T_1 and T_2

$$E(T_1(s)T_2(r)\vec{u}) = E(\vec{u}). \tag{5.1}$$

We also have

$$E(\vec{u}(t)) = E(\vec{u}_0), \tag{5.2}$$

and that equation (1.1) can be written as the Hamiltonian system

$$\frac{d\vec{u}}{dt} = JE'(\vec{u}) \tag{5.3}$$

where $\vec{u} = (u, v)$ and J is a skew-symmetric linear operator defined by

$$J = \begin{pmatrix} -i & 0 \\ 0 & 2\partial_x \end{pmatrix}$$

and

$$E'(u, v) = \begin{pmatrix} -u_{xx} + uv + \beta|u|^2 u \\ \frac{1}{2}|u|^2 \end{pmatrix}$$

is the Frechet derivative of E. Define B_1 and B_2 such that $T_1'(0) = JB_1$, $T_2'(0) = JB_2$, then

$$Q_1(\vec{u}) = \frac{1}{2}\langle B_1\vec{u}, \vec{u}\rangle = -\frac{1}{4}\int_0^L v^2 dx + \frac{1}{2} Im \int_0^L u_x \overline{u} dx$$

$$Q_2(\vec{u}) = \frac{1}{2}\langle B_2\vec{u}, \vec{u}\rangle = \frac{1}{2}\int_0^L |u|^2 dx.$$

It is easy to verify that

$$Q_1(T_1(s)T_2(r)\vec{u}) = Q_1(\vec{u}), \;\; Q_2(T_1(s)T_2(r)\vec{u}) = Q_2(\vec{u}) \tag{5.4}$$

$$Q_1(\vec{u}(t)) = Q_1(\vec{u}(0)), \;\; Q_2(\vec{u}(t)) = Q_2(\vec{u}(0)) \tag{5.5}$$

and

$$Q_1'(u, v) = \begin{pmatrix} -iu_x \\ -\frac{1}{2}v \end{pmatrix}, \quad Q_2{}'(u, v) = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

From (4.2) we have

$$E'(\Phi_{\omega,c}) - cQ_1'(\Phi_{\omega,c}) - \omega Q_2'(\Phi_{\omega,c}) = 0. \tag{5.6}$$

Define an operator from X to X^*

$$H_{\omega,c} = E''(\Phi_{\omega,c}) - cQ_1''(\Phi_{\omega,c}) - \omega Q_2''(\Phi_{\omega,c}) \tag{5.7}$$

and the function $d(\omega, c) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(\omega, c) = E(\Phi_{\omega,c}) - cQ_1(\Phi_{\omega,c}) - \omega Q_2(\Phi_{\omega,c}). \tag{5.8}$$

The operator $H_{\omega,c}$ is self-adjoint. The spectrum of $H_{\omega,c}$ consists of the real numbers λ such that $H_{\omega,c} - \lambda I$ is not invertible. From (4.2) we have

$$T_1'(0)\Phi_{\omega,c} \in Ker H_{\omega,c}, \;\; T_2'(0)\Phi_{\omega,c} \in Ker H_{\omega,c}. \tag{5.9}$$

Let $Z = \{k_1 T_1'(0)\Phi_{\omega,c} + k_2 T_2'(0)\Phi_{\omega,c} : \;\; k_1, k_2 \in \mathbb{R}\}$. By (5.9), Z is in the kernel of $H_{\omega,c}$.

Assumption. (*Spectral decomposition of $H_{\omega,c}$*): The space X is decomposed as a direct sum $X = N \oplus Z \oplus P$, where Z is defined above, N is a finite-dimensional subspace such that $\langle H_{\omega,c}\vec{u},\vec{u}\rangle < 0$ for ũ ∈ N and P is a closed subspace such that $\langle H_{\omega,c}\vec{u},\vec{u}\rangle \geq \delta||\vec{u}||^2_X$, for $\vec{u} \in P$ with some constant $\delta > 0$ independent of $\vec{u}$.

Our stability results are based on the following general theorem in [18].

Theorem 5.1. (*Abstract Stability Theorem*) *Assume that there exist three functionals E, Q_1, Q_2 satisfying* (5.1)-(5.5). *Let $n(H_{\omega,c})$ be the number of negative eigenvalues of $H_{\omega,c}$. Assume $d(\omega, c)$ is non-degenerate at (ω, c) and let $p(d'')$ be the number of positive eigenvalues of d''. If $p(d'') = n(H_{\omega,c})$, then the periodic travelling wave $\Phi_{\omega,c}(x)$ is orbitally stable.*

The idea of the proof of Theorem 1.5 is to apply the general Theorem 5.1. Initially we identify the quadratic form associated to $H_{\omega,c}$. Let $\vec{z} = (e^{i\frac{c}{2}x}z_1, z_2)$, with $z_1 = y_1 + iy_2$, $y_1 = Rez_1, y_2 = Imz_1$. By direct computation, we get

$$\langle H_{\omega,c}\vec{z_1},\vec{z_1}\rangle = \langle L_1y_1, y_1\rangle + \langle L_2y_2, y_2\rangle + \tfrac{c}{2}\int_0^L \left(z_2 + \tfrac{2}{c}\varphi_{\omega,c}y_1\right)^2 dx$$

where $L_1 = -\partial_x^2 - (\frac{c^2}{4}+\omega) + 3(\beta - \frac{1}{c})\varphi^2_{\omega,c}$, $L_2 = -\partial_x^2 - (\frac{c^2}{4}+\omega) + (\beta - \frac{1}{c})\varphi^2_{\omega,c}$. From (4.2) we also have $L_1(\partial_x\varphi_{\omega,c}) = 0$ and $L_2\varphi_{\omega,c} = 0$. Consider the following periodic eigenvalue problems for $i = 1, 2$:

$$L_if = \lambda f, \quad f(0) = f(L), \;\; f'(0) = f'(L). \tag{5.10}$$

The problem (5.10) determines a countable infinite set of eigenvalues $\{\lambda^i_n\}$ with $\lambda^i_n \to \infty$, so from the oscillation theorem [22] we have that they are distributed in the specific form $\lambda^i_0 < \lambda^i_1 \leq \lambda^i_2 < \lambda^i_3 \leq \lambda^i_4, ...$

Theorem 5.2. *Let $\sigma \in [\frac{2\pi^2}{L^2}, +\infty)$ and $(\varphi_{\omega,c}, n_{\omega,c})$ be the travelling wave solutions of* (4.9). *Then the first three eigenvalues of the operator L_1 are simple and* 0 *is the second eigenvalue of L_1 with eigenfunction $\partial_x\varphi_{\omega,c}$. The first eigenvalue of the operator L_2 is* 0*, which is simple.*

Proof. Since $L_2\varphi_{\omega,c} = 0$ and $\varphi_{\omega,c}$ has no zeros on $[0, L]$, it follows that zero is the first eigenvalue of L_2. Now since $L_1\partial_x\varphi_{\omega,c} = 0$ and $\partial\varphi_{\omega,c}$ has two zeros on $[0, L)$, it follows that the eigenvalue zero of L_1 is either $\lambda_1 = \lambda^1_1$ or $\lambda_2 = \lambda^1_2$. Let $\psi = f(\theta x)$, where $\theta^2 = \frac{2}{\eta_1^2}$. From the equality $\kappa^2 sn^2(x) + dn^2(x) = 1$ and (5.10), we obtain that ψ satisfies the equation

$$\psi'' + (\rho - 6\kappa^2 sn^2(x))\psi = 0, \tag{5.11}$$

where

$$\rho = 6 - \frac{2}{\eta_1^2}(\sigma - \lambda). \tag{5.12}$$

From Floquet theory, it follows that $(-\infty, \rho_0)$ and (ρ_1, ρ_2) are instability intervals associated to Lame's equation. Therefore the eigenvalues ρ_0, ρ_1 and ρ_2 of (5.12) are simple and the rest of the eigenvalues $\rho_3 \leq \rho_4, ...$ satisfy $\rho_3 = \rho_4, \rho_5 = \rho_6, ...$ The eigenvalues ρ_0, ρ_1, ρ_2 and their corresponding eigenfunctions ψ_0, ψ_1, ψ_2 are

$$\rho_0 = 2(1+\kappa^2 - \sqrt{1-\kappa^2+\kappa^4}), \quad \psi_0 = 1 - (1+\kappa^2 - \sqrt{1-\kappa^2+\kappa^4})sn^2(x),$$

$$\rho_1 = 4+\kappa^2, \qquad \psi_2 = sn(x)cn(x),$$

$$\rho_2 = 2(1+\kappa^2 + \sqrt{1-\kappa^2+\kappa^4}), \quad \psi_2 = 1 - (1+\kappa^2 + \sqrt{1-\kappa^2+\kappa^4})sn^2(x).$$

Since $\rho_0 < \rho_1$ for every $\kappa^2 \in (0,1)$, from (5.12) we have

$$3\lambda_0 = \frac{\eta_1^2}{2}(\kappa^2 - 2 - 2\sqrt{1-\kappa^2+\kappa^4}) < 0.$$

Therefore, $\lambda_0 = \lambda_0^1$ is a negative eigenvalue of L_1 with eigenfunction $\chi_0(x) = \psi_0(\frac{x}{\theta})$. Similarly

$$3\lambda_2 = \frac{\eta_1^2}{2}(\kappa^2 - 2 + 2\sqrt{1-\kappa^2+\kappa^4}) > 0$$

and λ_2 is the positive eigenvalue of L_1 with eigenfunction $\chi_2(x) = \psi_2(\frac{x}{\theta})$. Thus

$$\lambda_1 = \frac{\eta_1^2(\rho_1 - 6) + 2\sigma}{6} = \frac{\eta_1^2}{6}(4+\kappa^2 - 6 + 2 - \kappa^2) = 0$$

is the second eigenvalue of L_1. This complete the proof of the theorem. □

Remark 5.3. The main properties of the spectrum of L_1, namely, there is exactly a negative eigenvalue and zero is simple, can also be obtained via positive properties of the Fourier transform of the solution $\varphi_{\omega,c}$ (see Angulo&Natali [6] and Angulo [5]).

Now, from Theorem 5.2 we obtain immediately the following two results.

Lemma 5.4. *For any real function $y_1 \in H^1$ satisfying $\langle y_1, \chi_0\rangle = \langle y_1, \partial_x \varphi_{\omega,c}\rangle = 0$ there exists a positive constant $\delta_1 > 0$ such that $\langle L_1 y_1, y_1\rangle \geq \delta_1 ||y_1||_{H^1}^2$.*

Lemma 5.5. *For any real function $y_2 \in H^1$ satisfying $\langle y_2, \varphi_{\omega,c}\rangle = 0$ there exists a positive constant δ_2 such that $\langle L_2 y_2, y_2\rangle \geq \delta_2 ||y_2||_{H^1}^2$.*

Proof of Theorem 1.5. Choose $y_1^- = \chi_0, y_2^- = 0, z_2^- = -\frac{2}{c}\varphi_{\omega,c}\chi_0$ and $\Psi^- = (y_1^-, y_2^-, z_2^-)$, then $\langle H_{\omega,c}\Psi^-, \Psi^-\rangle = \lambda_0\langle\chi_0,\chi_0\rangle < 0$. So $H_{\omega,c}$ has a negative eigenvalue. Note that the vectors

$$\Psi_{0,1} = (\partial_x\varphi_{\omega,c}, 0, -\frac{2}{c}\varphi_{\omega,c}\partial_x\varphi_{\omega,c}), \quad \Psi_{0,2} = (0, \varphi_{\omega,c}, 0)$$

are in the kernel of the operator $H_{\omega,c}$. Define the following subspaces associated to $H_{\omega,c}$: $Z = \{k_1\Psi_{0,1} + k_1\Psi_{0,2} : k_1, k_2 \in \mathbb{R}\}$, $N = \{k\Psi^- : k \in \mathbb{R}\}$, and

$$P = \{\vec{p} \in X : \vec{p} = (p_1, p_2, p_3), \langle p_1, \chi_1\rangle = \langle p_1, \partial_x\varphi_{\omega,c}\rangle = \langle p_2, \varphi_{\omega,c}\rangle = 0\}.$$

For any $\vec{u} \in X, \vec{u} = (y_1, y_2, y_2)$ choose

$$a = \frac{\langle y_1, \chi_0\rangle}{\langle\chi_0,\chi_0\rangle},\ b_1 = \frac{\langle\partial_x\varphi_{\omega,c}, y_1\rangle}{\langle\partial_x\varphi_{\omega,c}, \partial_x\varphi_{\omega,c}\rangle},\ b_2 = \frac{\langle\varphi_{\omega,c}, y_2\rangle}{\langle\varphi_{\omega,c},\varphi_{\omega,c}\rangle},$$

then $\vec{u}$ can be uniquely represented by $\vec{u} = a\Psi^- + b_1\Psi_{0,1} + b_2\Psi_{0,2} + \vec{p}$, where $\vec{p} \in P$. For any $\vec{p} \in P$, by Lemmas 5.4 and 5.5, we have

$$\langle H_{\omega,c}\vec{p},\vec{p}\rangle \geq \delta_1||p_1||_{H^1}^2 + \delta_1||p_2||_{H^1}^2 + \frac{c}{2}\int_0^L \left(p_3 + \tfrac{2}{c}\varphi p_1\right)^2 dx.$$

Next we consider the following two cases:

(1) If $||p_3||_{L^2} \geq \frac{8||\varphi_{\omega,c}||_{L^\infty}}{c}||p_1||_{L^2}$, then

$$\frac{c}{2}\int_0^L \left(p_3 + \tfrac{2}{c}\varphi_{\omega,c}p_1\right)^2 dx \geq \frac{c}{2}\left[||p_3||_{L^2}^2 - \tfrac{4}{c}||\varphi_{\omega,c}||_{L^\infty}||p_1||_{L^2}||p_3||_{L^2}\right] = \tfrac{c}{4}||p_3||_{L^2}^2.$$

(2) If $||p_3||_{L^2} \leq \frac{8||\varphi_{\omega,c}||_{L^\infty}}{c}||p_1||_{L^2}$, then

$$\delta_1||p_1||_{H^1}^2 \geq \frac{\delta_1}{2}||p_1||_{H^1}^2 + \frac{\delta_1}{2}\frac{c}{8||\varphi_{\omega,c}||_{L^\infty}}||p_3||_{L^2}^2.$$

Thus, for any $\vec{p} \in P$, it follows that

$$\langle H_{\omega,c}\vec{p},\vec{p}\rangle \geq \delta_3||p_3||_{L^2}^2 + \frac{\delta_1}{2}||p_1||_{H^1}^2 + \delta_2||p_2||_{H^1}^2,$$

where $\delta_3 = \min\{\frac{\delta_1 c}{16||\varphi_{\omega,c}||_{L^\infty}}, \frac{c}{4}\}$. Finally, we have

$$\langle H_{\omega,c}\vec{p},\vec{p}\rangle \geq \delta||\vec{p}||_X^2,$$

where $\delta > 0$ is independent of $\vec{p}$. This proves that Assumption above holds, and $n(H_{\omega,c}) = 1$.

Now we shall verify that $p(d'')=1$. We have

$$d_c(\omega,c) = -Q_1(\Phi_{\omega,c}) = \frac{1}{4(1-\beta c)^2}\int_0^L \varphi_{\omega,c}^4 dx - \frac{c^2}{4(1-\beta c)}\int_0^L \varphi_{\omega,c} dx$$

$$d_\omega(\omega,c) = -Q_2(\Phi_{\omega,c}) = -\frac{c}{2(1-\beta c)}\int_0^L \varphi_{\omega,c}^2 dx.$$

From the equalities

$$\int_0^L \varphi_{\omega,c}^2 dx = \frac{8KE}{L}, \quad \int_0^L \varphi_{\omega,c}^4 dx = \frac{64}{L^3}V(\kappa)$$

where $E = E(\kappa) = \int_0^1 \sqrt{\frac{1-\kappa^2 t^2}{1-t^2}}dt$ is the complete elliptic integral of the second kind and $V(\kappa) = \frac{\kappa^2-1}{3}K^4 + \frac{2}{L}(2-\kappa^2)K^2E$, we obtain

$$\begin{aligned} d_{\omega\omega} &= \frac{4c}{L(1-\beta c)}(K'(\kappa)E(\kappa) + K(\kappa)E'(\kappa))\kappa'(\sigma), \qquad (5.13)\\ d_{\omega c} &= -\frac{4}{L(1-\beta c)^2}K(\kappa)E(\kappa) + \frac{c}{2}d_{\omega\omega},\\ d_{c\omega} &= -\frac{16}{L^3(1-\beta c)^2}V'(\kappa)\kappa'(\sigma) + \frac{c}{2}d_{\omega\omega}\\ d_{cc} &= \frac{32\beta}{L^3(1-\beta c)^3}V(\kappa) - \frac{8c}{L^3(1-\beta c)^2}V'(\kappa)\kappa'(\sigma)\\ &\quad - \frac{2c(2-\beta c)}{L(1-\beta c)^2}K(\kappa)E(\kappa) + \frac{c^2}{4}d_{\omega\omega}. \end{aligned}$$

Thus,

$$\begin{aligned} d_{cc}d_{\omega\omega} - d_{c\omega}d_{\omega c} = \quad & -\tfrac{64}{L^4(1-\beta c)^4}V'(\kappa)\kappa'(\sigma)K(\kappa)E(\kappa) +\\ & \tfrac{1}{L(1-\beta c)}\left[\tfrac{32\alpha}{L^2(1-\beta c)^2}V(\kappa) - 2cK(\kappa)E(\kappa)\right]d_{\omega\omega}. \end{aligned}$$

We have

$$V'(\kappa) = \frac{2K^2E}{\kappa(1-\kappa^2)}\left[(2-\kappa^2)E - (1-\kappa^2)K\right], \quad \frac{V}{L^2} = \frac{\sigma(\kappa^2-1)}{12(2-\kappa^2)}K^2 + \frac{\sigma}{6}KE.$$

Using the above estimates, we obtain

$$\begin{aligned} d_{cc}d_{\omega\omega} - d_{c\omega}d_{\omega c} &= \frac{4\kappa'}{L^2(1-\beta c)^2}\left\{-32\frac{K^2}{L^2}KE^2\left[(2-\kappa^2)E - 2(1-\kappa^2)K\right]\right.\\ &+ \left.\frac{c}{3}\left[\frac{8\beta\sigma(\kappa^2-1)}{2-\kappa^2}K^2 + (16\beta\sigma - 6c(1-\beta c)^2)KE\right]\left[\frac{E^2}{\kappa(1-\kappa^2)} - \frac{K^2}{\kappa}\right]\right\} \end{aligned}$$

$$= \frac{4K\kappa'}{L^2(1-\beta c)^2}\left\{-\frac{8\sigma}{2-\kappa^2}\left[(2-\kappa^2)E^3 - 2(1-\kappa^2)KE^2\right]\right\}$$
$$+ \left[\frac{8\beta\sigma c(\kappa^2-1)}{3(2-\kappa^2)}K + \frac{c(16\beta\sigma - 6c(1-\beta c)^2)}{3}E\right]\left[\frac{E^2}{\kappa(1-\kappa^2)} - \frac{K^2}{\kappa}\right]\Bigg\}.$$

From Theorem 4.1-(4), we have $\kappa' > 0$. Therefore the sign of $det(d'') = d_{cc}d_{\omega\omega} - d_{c\omega}d_{\omega c}$ depends on the sign of

$$B(c,\omega,\kappa,\beta) = \left\{-\tfrac{8\sigma}{2-\kappa^2}\left[(2-\kappa^2)E^3 - 2(1-\kappa^2)KE^2\right]\right.$$
$$\left. + \left[\tfrac{8\beta\sigma c(\kappa^2-1)}{3(2-\kappa^2)}K + \tfrac{c(16\beta\sigma-6c(1-\beta c)^2)}{3}E\right]\left[\tfrac{E^2}{\kappa(1-\kappa^2)} - \tfrac{K^2}{\kappa}\right]\right\}.$$

From the relation

$$0 < \frac{(1-\kappa^2)K}{(2-\kappa^2)E} < \frac{1}{2} \tag{5.14}$$

we get that the first term of $B(c,\omega,\kappa,\beta)$ is negative. Now we consider three cases for β.

(1) Obviously, if $\beta = 0$, then $det(d'') < 0$.

(2) For $\beta < 0$, using (5.14), we get

$$\frac{8\beta\sigma(\kappa^2-1)K}{3(2-\kappa^2)E} + \frac{c(16\beta\sigma - 6c(1-\beta c)^2)}{3}E$$
$$= -\frac{8c\beta\sigma E}{3}\left[\frac{(1-\kappa^2)K}{(2-\kappa^2)E} - 2 + \frac{6c(1-\beta c)^2}{8c\beta}\right] < 0$$

and $det(d'') < 0$.

(3) If $\beta > 0$ and $8\beta\sigma - 3c(1-\beta c)^2 \leq 0$, then all terms of $B(c,\omega,\kappa,\beta)$ are negatives and $det(d'') < 0$.

Thus, under the above three conditions, $d''(\omega, c)$ has exactly one positive and one negative eigenvalue and $p(d'') = 1$. This finishes the proof of the theorem. □

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