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LOCAL AND GLOBAL PROPERTIES OF SOLUTIONS OF HEAT EQUATION WITH SUPERLINEAR ABSORPTION

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Abstract. We study the limit when $k \to \infty$ of the solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0,\infty)$ with initial data $k\delta$, when f is a positive superlinear increasing function. We prove that there exist essentially three types of possible behaviour according to whether f^{-1} and $F^{-1/2}$ belong or not to $L^1(1,\infty)$, where $F(t) = \int_0^t f(s)ds$. We use these results for providing a new and more general construction of the initial trace and some uniqueness and nonuniqueness results for solutions with unbounded initial data.

1. Introduction

In this article we investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + f(u) = 0 \tag{1.1}$$

in $Q_\infty := \mathbb{R}^N \times (0,\infty)$ $(N \geq 2)$, where $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, nondecreasing, and positive on $(0,\infty)$, vanishes at 0 and tends to infinity at infinity. As a *model equation* we shall consider the following nonlinear term, with $\alpha > 0$,

$$\partial_t u - \Delta u + u \ln^\alpha(u+1) = 0, \tag{1.2}$$

which points out all the delicate features of *weakly superlinear absorption*. In contrast, for power-like absorption $f(u) = |u|^\beta u$ with $\beta > 0$ much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x,t) = t^{-1/\beta} w(x/\sqrt{t}). \tag{1.3}$$

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In this case the critical exponent $\beta_c = 2/N$ plays a fundamental role in the description of isolated singularities and the study of the initial trace. This is due to the fact that, for $0 < \beta < \beta_c$, there exists a positive self-similar solution with an isolated singularity at $(0,0)$ and vanishing on $\mathbb{R}^N \setminus \{0\} \times \{0\}$, while no such solution exists when $\beta \geq \beta_c$ and more generally, no solution with isolated singularities.

In the case of (1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any $k > 0$ there exists a unique $u = u_k \in C(\overline{Q_\infty} \setminus \{(0,0)\}) \cap C^{2,1}(Q_\infty)$ a solution of

$$\begin{cases} \partial_t u - \Delta u + u \ln^\alpha(u+1) = 0 & \text{in } Q_\infty \\ u(.,x) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \tag{1.4}$$

There are two critical values for α: $\alpha = 1$ and $\alpha = 2$, the explanation of which comes from the study of the two singular problems

$$\begin{cases} \phi' + \phi \ln^\alpha(\phi+1) & = 0 \qquad \text{in } (0,\infty) \\ \phi(0) & = \infty, \end{cases} \tag{1.5}$$

and, for any $\epsilon > 0$,

$$\begin{cases} -\Delta\psi + \psi \ln^\alpha(\psi+1) & = 0 \qquad \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim\limits_{|x|\to\epsilon} \psi(x) & = \infty, \end{cases} \tag{1.6}$$

where $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$. When it exists, the solution ϕ_∞ of (1.5) is given implicitly by

$$\int_{\phi_\infty(t)}^\infty \frac{ds}{s \ln^\alpha(s+1)} = t \qquad \forall t > 0, \tag{1.7}$$

and such a formula is valid if and only if $\alpha > 1$. For problem (1.6) an explicit expression of the solution is not valid, but this solution exists if and only if $\alpha > 2$; in this case of the Keller-Osserman condition (see (1.12) below) holds.

Having in mind this model we study (1.1) assuming the *weak singularity condition* on f:

$$\int_1^\infty s^{-2-\frac{2}{N}} f(s)ds < \infty. \tag{1.8}$$

Proposition 1.1. *Assume* (1.8) *holds. Then for any* $k > 0$*, there exists a unique solution* $u := u_k$ *to*

$$\begin{cases} \partial_t u - \Delta u + f(u) & = 0 \qquad \text{in } Q_\infty \\ u(\cdot, 0) & = k\delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \tag{1.9}$$

Furthermore, if ψ_n *is a sequence of positive integrable functions converging to* $k\delta$ *in the weak-star topology, then the sequence* u_{ψ_n} *of solutions of* (1.1) *in* Q_∞ *with initial data* ψ_n *converges to* $u_{k\delta}$*, locally uniformly.*

Another important condition on f is

$$\int_1^\infty \frac{ds}{f(s)} < \infty. \tag{1.10}$$

Under assumption (1.10) there exists a solution $\phi := \phi_\infty$ to

$$\begin{cases} \phi' + f(\phi) & = 0 \qquad \text{in } (0, \infty) \\ \phi(0) & = \infty. \end{cases} \tag{1.11}$$

The function ϕ_∞ is the maximal solution of (1.11) and it is made explicit by a formula similar to (1.7) in which $s \ln^\alpha(s+1)$ is replaced by $f(s)$.

The next important condition on f we shall encounter is the Keller-Osserman condition, i.e.,

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \tag{1.12}$$

where

$$F(s) = \int_0^s f(\sigma)d\sigma, \quad \forall s \in [1, \infty). \tag{1.13}$$

If (1.12) is satisfied, by [4, Theorem III] for any $\epsilon > 0$ there exists a maximal solution $\psi := \psi_\epsilon$ to

$$\begin{cases} -\Delta\psi + f(\psi) & = 0 \qquad \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim\limits_{|x| \to \epsilon} \psi(x) & = \infty. \end{cases} \tag{1.14}$$

Assumptions (1.10) and (1.13), which are simultaneously satisfied in the case of a power-like absorption, but not in our model case, are the Ariane shred which illuminates the structure of the set of solutions of (1.1), in particular in view of the initial trace problem.

The first question we consider is the study of the limit of u_k when $k \to \infty$. This question is natural since $k \mapsto u_k$ is increasing. In order to treat it, we need some additional conditions.

(C1)- The function $s \mapsto \frac{f(s)}{s}$ is increasing on $(0,\infty)$ and satisfies

$$\lim_{s\to 0}\frac{f(s)}{s}=0 \quad \text{and} \quad \lim_{s\to\infty}\frac{f(s)}{s}=\infty.$$

(C2)- The function f is convex on $(0,\infty)$.

(C3)- If $\liminf_{s\to\infty} f(s)/(s\ln^{\alpha} s) = 0, \forall\alpha > 2$, then there exists $\beta\in(1,2]$ such that

$$\limsup_{s\to\infty}\frac{f(s)}{s\ln^{\beta} s}<\infty.$$

In the second section, we prove the following results.

Theorem 1.2. *Assume the conditions* (C1) *and* (C3) *hold. If f satisfies*

$$\int_1^{\infty}\frac{ds}{f(s)}=\infty, \tag{1.15}$$

then the solutions u_k of (1.9) *satisfy* $\lim_{k\to\infty} u_k(x,t)=\infty$ *for every* $(x,t)\in Q_\infty$.

Theorem 1.3. *Assume the conditions* (C1)–(C3) *hold. If f satisfies* (1.10) *and*

$$\int_1^{\infty}\frac{ds}{\sqrt{F(s)}}=\infty, \tag{1.16}$$

where F is defined in (1.13)*, then the solutions u_k of* (1.9) *satisfy* $\lim_{k\to\infty} u_k(x,t) = \phi_\infty(t)$ *for every* $(x,t)\in Q_\infty$*, where ϕ_∞ is the solution of* (1.11)*.*

We denote by $\mathcal{U}_0$ the set of positive solutions u of (1.1) in Q_∞, which are continuous in $\overline{Q_\infty}\setminus\{(0,0)\}$, vanish on the set $\{(x,0): x\neq 0\}$, and satisfy

$$\lim_{t\to 0}\int_{B_\epsilon} u(x,t)dx=\infty \tag{1.17}$$

for any $\epsilon>0$.

Theorem 1.4. *Assume f satisfies* (1.8), (1.12)*, and* (C2)*. Then we have that* $\underline{U}:=\lim_{k\to\infty} u_k$ *is the minimal element of* $\mathcal{U}_0$.

In the third section we study the set of positive and locally bounded solutions of (1.1) in Q_∞. This set differs considerably according to the assumptions on f. This is due to the properties of the radial solutions of the associated stationary equation

$$-\Delta w + f(w) = 0 \qquad \text{in } \mathbb{R}^N. \tag{1.18}$$

The next result is based upon the Picard-Lipschitz fixed-point theorem and a result of Vazquez and Véron [11].

Proposition 1.5. *Assume* (1.16) *holds. For any* $a > 0$, *there exists a unique positive function* $w := w_a \in C^2([0, \infty))$ *to the problem*

$$\begin{cases} -w'' - \dfrac{N-1}{r} w' + f(w) = 0 & \text{in } \mathbb{R}_+ \\ w'(0) = 0, \quad w(0) = a. \end{cases} \tag{1.19}$$

A striking consequence of the existence of such solutions is the following nonuniqueness result.

Theorem 1.6. *Assume* f *satisfies* (1.10) *and* (1.16). *Then for any* $u_0 \in C(\mathbb{R}^N)$ *satisfying, for some* $b > a > 0$, $w_a(x) \leq u_0(x) \leq w_b(x)$ $\forall x \in \mathbb{R}^N$, *there exist two solutions* $\underline{u}, \overline{u} \in C(\overline{Q_\infty})$ *of* (1.1) *with initial value* u_0. *They satisfy respectively*

$$0 \leq \underline{u}(x,t) \leq \min\{w_b(x), \phi_\infty(t)\} \qquad \forall (x,t) \in Q_\infty, \tag{1.20}$$

thus $\lim_{t\to\infty} \underline{u}(x,t) = 0$ *uniformly with respect to* $x \in \mathbb{R}^N$; *and*

$$w_a(x) \leq \overline{u}(x,t) \leq w_b(x) \qquad \forall (x,t) \in Q_\infty, \tag{1.21}$$

thus $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$ *uniformly with respect to* $t \geq 0$.

The next theorem shows that if two solutions of (1.1) have the same initial data and the same asymptotic behaviour as $|x| \to \infty$, then they coincide.

Theorem 1.7. *Assume* f *satisfies* (C1) *and* (1.16). *Let* $u, \tilde{u} \in C(\overline{Q}_\infty) \cap C^{2,1}(Q_\infty)$ *be two positive solutions of* (1.1) *with initial data* u_0. *If for any* $\epsilon > 0$,

$$u(x,t) - \tilde{u}(x,t) = o(w_\epsilon(|x|)) \text{ as } x \to \infty \tag{1.22}$$

locally uniformly with respect to $t \geq 0$, *then* $u = \tilde{u}$.

On the contrary, if the Keller-Osserman condition holds, a continuous solution is uniquely determined by the positive initial value $u_0 \in C(\mathbb{R}^N)$, and uniqueness still holds if $C(\mathbb{R}^N)$ is replaced by $\mathfrak{M}_+(\mathbb{R}^N)$.

Theorem 1.8. *Assume* f *satisfies* (1.12) *and* (C2). *Then we have the following:*

(*i*) *For any nonnegative function* $u_0 \in C(\mathbb{R}^N)$ *there exists a unique nonnegative solution* $u \in C(\overline{Q}_\infty)$ *of* (1.1) *in* Q_∞ *with initial value* u_0.

(*ii*) *For any for any nonnegative measure* $\mu \in \mathfrak{M}(\mathbb{R}^N)$, *there exists at most one nonnegative solution* $u \in C(Q_\infty)$ *of* (1.1) *in* Q_∞ *such that* $f(u) \in$

$L^1_{loc}(\overline{Q}_\infty)$ *satisfying*

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N}\zeta(x)d\mu(x) \quad \forall\zeta\in C_c(\mathbb{R}^N). \tag{1.23}$$

In the last section we use the tools studied in the previous sections to develop a *new construction of the initial trace* of locally bounded positive solutions of (1.1) in Q_∞. In contrast to the power-like case [5], where the initial trace was constructed by duality arguments based upon the Hölder inequality and a delicate choice of test functions, our new method has the advantage of being based only on the maximum principle, using either the Keller-Osserman condition, if (1.16) holds, or the asymptotics of the u_k if (1.16) does not hold. We first prove

Proposition 1.9. *Let $u\in C^{2,1}(Q_\infty)$ be a positive solution of* (1.1) *in Q_∞. The set $\mathcal{R}(u)$ of the points $z\in\mathbb{R}^N$ such that there exists an open ball $B_r(z)$ such that $f(u)\in L^1(Q_T^{B_r(z)})$ is an open subset. Furthermore, there exists a positive Radon measure $\mu:=\mu(u)$ on $\mathcal{R}(u)$ such that*

$$\lim_{t\to 0}\int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)}\zeta(x)d\mu(x) \qquad \forall\zeta\in C_c(\mathcal{R}(u)). \tag{1.24}$$

Due to Proposition 1.9, we introduce the definition of the initial trace.

Definition 1.10. *The couple $(\mathcal{S}(u),\mu)$, where $\mathcal{S}(u)=\mathbb{R}^N\setminus\mathcal{R}(u)$, is called the initial trace of u in Ω and will be denoted by $tr_{\mathbb{R}^N}(u)$. The set $\mathcal{R}(u)$ is called the regular set of the initial trace of u and the measure μ the regular part of the initial trace. The set $\mathcal{S}(u)$ is closed and is called the singular part of the initial trace of u.*

The initial trace can also be represented by a positive, outer regular Borel measure, not necessarily locally bounded. The space of these measures on $\mathbb{R}^N$ will be denoted by $\mathcal{B}^{\rm reg}_+(\mathbb{R}^N)$. If for every open subset $A\subset\mathbb{R}^N$ we denote by $\mathfrak{M}_+(A)$ the space of positive Radon measures on A, there is a one-to-one correspondence between $\mathcal{B}^{\rm reg}_+(\mathbb{R}^N)$ and the set of couples

$$CM_+(\mathbb{R}^N)=\left\{(\mathcal{S},\mu):\mathcal{S}\subset\mathbb{R}^N\text{ closed, }\mu\in\mathfrak{M}_+(\mathcal{R})\text{ with }\mathcal{R}=\mathbb{R}^N\setminus\mathcal{S}\right\}. \tag{1.25}$$

The Borel measure $\nu\in\mathcal{B}^{\rm reg}_+(\mathbb{R}^N)$ corresponding to a couple $(\mathcal{S},\mu)\in CM_+(\mathbb{R}^N)$ is given by

$$\nu(A)=\begin{cases}\infty & \text{if } A\cap\mathcal{S}\neq\emptyset\\ \mu(A) & \text{if } A\subseteq\mathcal{R},\end{cases} \qquad \forall A\subset\mathbb{R}^N,\ A\text{ Borel}. \tag{1.26}$$

If u is a solution of (1.1), we shall use the notation $tr_{\mathbb{R}^N}(u)$ (respectively $Tr_{\mathbb{R}^N}(u)$) for the trace considered as an element of $CM_+(\mathbb{R}^N)$ (respectively $\mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$).

We consider the case when the Keller-Osserman condition holds.

Theorem 1.11. *Assume f is nondecreasing and satisfies* (1.12). *If $u \in C^{2,1}(Q_\infty)$ is a positive solution of* (1.1), *it possesses an initial trace $\nu \in \mathcal{B}_+^{reg}(\mathbb{R}^N)$.*

Furthermore, the following theorem deals with the existence of the maximal solution and the minimal solution of (1.1) with a given initial trace $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$.

Theorem 1.12. *Assume f is nondecreasing and satisfies* (1.12), (1.8), *and* (C2). *Then for any $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$ there exist a maximal solution $\overline{u}_{\mathcal{S},\mu}$ and a minimal solution $\underline{u}_{\mathcal{S},\mu}$ of* (1.1) *in Q_∞, with initial trace $(\mathcal{S}, \mu)$, in the following sense:*

$$\underline{u}_{\mathcal{S},\mu} \leq v \leq \overline{u}_{\mathcal{S},\mu} \tag{1.27}$$

for every positive solution $v \in C^{2,1}(Q_\infty)$ of (1.1) *in Q_∞ such that $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$.*

If the Keller-Osserman condition does not hold, we obtain the following results, which depend upon whether $\lim_{k\to\infty} u_k$ is equal to ϕ_∞ or is infinite (we recall that u_k is the solution of (1.9)).

Theorem 1.13. *Assume* (1.8), (1.10), *and* (1.16) *are satisfied and $\lim_{k\to\infty} u_k = \phi_\infty$. If u is a positive solution of* (1.1) *in Q_∞, it possesses an initial trace which is either the Borel measure ν_∞ which satisfies $\nu_\infty(\mathcal{O}) = \infty$ for any nonempty open subset $\mathcal{O} \subset \mathbb{R}^N$, or is a positive Radon measure μ on $\mathbb{R}^N$. This result holds in particular if* (C1) *and* (C3) *hold.*

A consequence of Theorem 1.13 which is worth mentioning is the following.

Proposition 1.14. *Under the assumptions of Theorem* 1.13, *for any $b > 0$ there exists a positive solution $u \in C(Q_\infty)$ of* (1.1) *in* (1.1) *satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x,t) \leq \phi_\infty(t) + w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{1.28}$$

Consequently there exist infinitely many positive solutions of (1.1) *with initial trace ν_∞. Furthermore, ϕ_∞ is the smallest of all these solutions.*

Theorem 1.15. *Assume f satisfies* (1.8), (1.15), (1.16), *and $\lim_{k\to\infty} u_k = \infty$. If u is a positive solution of* (1.1) *in Q_∞, it possesses an initial trace which* *is a positive Radon measure μ on $\mathbb{R}^N$. This result holds in particular if* (C1) *and* (C3) *hold.*

The proofs are a combination of methods developed in [8] for elliptic equations, stability results, Theorem 1.2, and Theorem 1.3.

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2. Isolated singularities

In order to study (1.1), we start by proving Proposition 1.1.

Proof of Proposition 1.1. We denote by $E(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ the fundamental solution of the heat equation in Q_∞. Since kE ($k > 0$) is a supersolution for (1.1), it is classical to prove that if

$$I := \int_0^1 \int_{B_R} f(kE(x,t))dx\,dt < \infty \tag{2.1}$$

for any $R > 0$, then there exists a unique solution $u = u_k$ to (1.1) satisfying the initial condition $u_k(\cdot,0) = k\delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$. Furthermore, the mapping $k \mapsto u_k$ is increasing. Actually, it is proved in [6, Theorem 1.1] that if f satisfies the weak singularity assumption (1.8), then for any positive bounded Borel measure there exists a unique solution $u := u_\mu$ to (1.1) satisfying $u_\mu(\cdot,0) = \mu$. Furthermore, if $\{\mu_n\}$ is a sequence of positive bounded measures which converge to a measure μ in the weak-star topology of measures, then the sequence of corresponding solutions $\{u_{\mu_n}\}$ converges locally uniformly to u_μ, and $\{f(u_{\mu_n})\}$ converges to $f(u_\mu)$ in $L^1_{loc}(\mathbb{R}^N \times [0,\infty))$.

This existence result and the next proposition lead to the conclusion of Proposition 1.1. □

Proposition 2.1. *If f satisfies* (1.8) *and* (C1), *then* (2.1) *is fulfilled.*

Proof. We set

$$h(r) = \frac{f(r)}{r} \qquad r \in (0,\infty). \tag{2.2}$$

I is rewritten as

$$I = kC^* \int_0^1 \int_{B_R} t^{-N/2} e^{-|x|^2/4t} h(kC^* t^{-N/2} e^{-|x|^2/4t})dx\,dt,$$

where $C^* = (4\pi)^{-N/2}$. By a change of variables,

$$I = kC^* \int_0^1 \int_0^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4})\rho^{N-1} d\rho\,dt.$$

We set

$$I_1 := kC^* \int_0^1 \int_0^1 e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho \, dt,$$

$$I_2 := kC^* \int_0^1 \int_1^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho \, dt.$$

Since $e^{-\rho^2/4}\rho^{N-1}$ is bounded in $[0,\infty)$, then there exists a constant c_1 depending only on k such that

$$I_1 < c_1 \int_0^1 \int_0^1 h(kC^* t^{-N/2}) d\rho \, dt = c_1 \int_0^1 h(kC^* t^{-N/2}) dt < \infty.$$

Next we show that under the condition (1.8), $I_2 < \infty$. In order to do that we introduce the variable τ such that $t^{-N/2} e^{-\rho^2/4} = \tau^{-N/2}$. Then $t = \tau e^{-\frac{\rho^2}{2N}}$ and $dt = e^{-\frac{\rho^2}{2N}} d\tau$. Therefore

$$I_2 \leq kC^* \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} \left(\int_0^{e^{\rho^2/2N}} h(kC^* \tau^{-N/2}) d\tau \right) d\rho. \tag{2.3}$$

Since f satisfies (1.8), there exists $\epsilon > 0$ (depending only on k) such that

$$\int_0^\epsilon h(kC^* \tau^{-N/2}) d\tau$$

takes a finite value, denoted by c_2. Hence

$$\int_0^{e^{\rho^2/2N}} h(kC^* \tau^{-N/2}) d\tau \leq c_2 + h(kC^* \epsilon^{-N/2})(e^{\frac{\rho^2}{2N}} - \epsilon). \tag{2.4}$$

Inserting (2.4) into the right-hand side of (2.3), we obtain

$$I_2 \leq c_3 \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} d\rho + c_4 \int_1^\infty e^{-\frac{\rho^2}{4}} \rho^{N-1} d\rho < \infty,$$

where $c_3 = kC^* c_2$ and $c_4 = kC^* h(kC^* \epsilon^{-N/2})$. Thus $I = I_1 + I_2 < \infty$. □

The functions which satisfy the following ODE are particular solutions of (1.1):

$$\phi' + f(\phi) = 0 \qquad \text{in } (0,\infty). \tag{2.5}$$

For $a > 0$, we denote by ϕ_a the solution of (2.5) with initial data $\phi(0) = a$. If (1.15) holds then $\lim_{a\to\infty} \phi_a(t) = \infty$ for any $t \in (0,\infty)$, while if (1.10) holds there exists a maximal solution ϕ_∞ given explicitly by

$$t = \int_{\phi_\infty(t)}^\infty \frac{ds}{f(s)} < \infty.$$

The proof of the next result is performed by standard contradiction arguments and is easy to verify.

Lemma 2.2. *If* (1.15) *holds, then*

$$\liminf_{r\to\infty}\frac{f(r)}{r\ln^{\alpha} r}=0,\quad \forall\alpha>1.$$

If (1.10) *holds, then*

$$\limsup_{r\to\infty}\frac{f(r)}{r\ln^{\alpha} r}=\infty,\quad \forall 0<\alpha\leq 1.$$

Proof of Theorem 1.2. Since (1.15) holds, by Lemma 2.2 and the definition (2.2) of h,

$$\liminf_{r\to\infty}\frac{h(r)}{\ln^{\alpha} r}=0\quad \forall\alpha>1.$$

Thus

$$\liminf_{r\to\infty}\frac{h(r)}{\ln^{\alpha} r}=0\quad \forall\alpha>2.$$

By (C3), there exists $\beta\in(1,2]$ such that $\limsup_{r\to\infty} h(r)/\ln^{\beta} r<\infty$. Hence there exist $M>0$ and $r_0>0$ such that

$$h(r)<M\ln^{\beta} r\qquad \forall r\in(r_0,\infty). \tag{2.6}$$

Step 1. Let $k>0$; we claim that

$$\theta_k(t)<2^{\beta-1}Mt(\ln k)^{\beta}+\frac{MN^{\beta}}{2}\int_0^1(\ln(\tau^{-1}))^{\beta}d\tau\qquad \forall t\in(0,1), \tag{2.7}$$

where

$$\theta_k(t)=\int_0^t h(kC^*\tau^{-N/2})d\tau$$

with $C^*=(4\pi)^{-N/2}$. Set $r=kC^*\tau^{-N/2}$; then (2.6) becomes

$$h(kC^*\tau^{-N/2})<M[\ln(kC^*)+\frac{N}{2}\ln(\tau^{-1})]^{\beta}\qquad \forall\tau\in(0,\tau_0),$$

where $\tau_0=(kC^*)^{2/N}r_0^{-2/N}$. We put $a_1=\ln k$ and $a_2=\frac{N}{2}\ln(\tau^{-1})$, and apply the following inequality,

$$(a_1+a_2)^{\beta}\leq 2^{\beta-1}(a_1^{\beta}+a_2^{\beta}),$$

in order to obtain

$$\begin{aligned}h(kC^*\tau^{-N/2})&<M[\ln(k)+\tfrac{N}{2}\ln(\tau^{-1})]^{\beta}\\&\leq 2^{\beta-1}M[(\ln k)^{\beta}+(\tfrac{N}{2})^{\beta}\ln^{\beta}(\tau^{-1})]\qquad \forall\tau\in(0,\tau_0)\end{aligned} \tag{2.8}$$

(notice that $C^* = (4\pi)^{-N/2} < 1$). Integrating over $[0,t]$ yields (2.7).

Step 2. It follows from (2.8) that (1.8) is fulfilled; hence, by Proposition 1.1 there exists a unique solution u_k of (1.1) in Q_∞ with initial data $k\delta_0$. By the maximum principle, $u_k(x,t) \le kE(x,t)$ for every $(x,t) \in Q_\infty$, which implies $u_k(x,t) \le kC^* t^{-N/2}$ for every $(x,t) \in Q_\infty$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^* t^{-N/2}) \ge 0.$$

If we set $v_k(x,t) = e^{\theta_k(t)} u_k(x,t)$, we obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)}[\partial_t u_k - \Delta u_k + u_k h(kC^* t^{-N/2})] \ge 0$$

and $v_k(\cdot,0) = u_k(\cdot,0) = k\delta_0$. By the maximum principle, there holds

$$v_k(x,t) \ge kC^* t^{-N/2} e^{-|x|^2/4t} \Longleftrightarrow u_k(x,t) \ge kC^* t^{-N/2} e^{-\theta_k(t) - |x|^2/4t}. \quad (2.9)$$

By Step 1,

$$e^{-\theta_k(t)} \ge c_5 e^{-M_\beta t (\ln k)^\beta} \qquad \forall t \in (0,1), \tag{2.10}$$

where

$$c_5 = \exp\Big(- \frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau \Big)$$

and $M_\beta = M 2^{\beta-1}$. Inserting (2.10) into the right-hand side of (2.9), we get

$$u_k(x,t) \ge c_5 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t} \qquad \forall (x,t) \in Q_1 := \mathbb{R}^N \times (0,1).$$

If $\lim_{k\to\infty} u_k(x,t) < \infty$ for all $(x,t) \in Q_\infty$, we put $\underline{U} := \lim_{k\to\infty} u_k$; then

$$\underline{U}(x,t) \ge c_5 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t} \quad \forall (x,t) \in Q_1, \quad \forall k > 0.$$

Let $\{t_n\} \subset (0,1]$ be a sequence converging to 0. We choose

$$k_n = \exp\left((2M_\beta t_n)^{\frac{1}{1-\beta}} \right);$$

then $\ln k_n - M_\beta t_n (\ln k_n)^\beta = \frac{1}{2}\ln k_n$. Next we restrict x in order to have

$$\ln k_n - M_\beta t_n (\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2}\ln k_n - \frac{|x|^2}{4t_n} \ge 0$$
$$\Longleftrightarrow |x| \le 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}} t_n^{\frac{\beta-2}{2(\beta-1)}}.$$

Therefore, since $1 < \beta \le 2$,

$$\lim_{n\to\infty} \underline{U}(x,t_n) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-1/2}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t\to 0} \underline{U}(x,t) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} if $\beta = 2$.

We pick some point x_0 in $\mathbb{R}^N$ (respectively B_{r_2}) if $1 < \beta < 2$ (respectively $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, it follows that

$$\underline{U}(x,t) \geq u_{k\delta_{x_0}}(x,t) = u_k(x - x_0, t),$$

by the comparison principle. Letting $k \to \infty$ yields $\underline{U}(x,t) \geq \underline{U}(x - x_0, t)$. Interchanging the role of 0 and x_0 yields to $\underline{U}(x,t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x,t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x,t)$ is independent of x and therefore it is a solution of (1.11). By (1.15), $\underline{U}(x,t) = \infty$ for any $(x,t) \in Q_\infty$, which is a contradiction, and the conclusion follows. □

Proposition 2.3. *Assume* (1.10) *is satisfied. For any* $k > 0$*, there holds*

$$u_k(x,t) \leq \phi_\infty(t) \qquad \forall (x,t) \in Q_\infty.$$

Proof. For any small $\epsilon > 0$, we set $\phi_{\infty\epsilon}(t) = \phi_\infty(t - \epsilon)$, $t \in [\epsilon, \infty)$; then $\phi_{\infty\epsilon}$ is a solution of (1.1) in (ϵ, ∞), which dominates u_k on $\mathbb{R}^N \times \{\epsilon\}$ for any $k > 0$. By the comparison principle, $u_k(x,t) \leq \phi_{\infty\epsilon}(t)$ for every $(x,t) \in \mathbb{R}^N \times [\epsilon, \infty)$. Letting $\epsilon \to 0$ yields the claim. □

A necessary and sufficient condition for the existence of a maximal solution to the stationary equation

$$-\Delta w + f(w) = 0$$

in a bounded domain Ω is the Keller-Osserman condition (1.12) ([4], [9]). If f is convex and (1.12) holds, then (1.10) is fulfilled. The Keller-Osserman condition can be replaced by another condition, which is due to the following result.

Lemma 2.4. *Assume* f *is convex on* $(0, \infty)$*. Set*

$$L := \int_1^\infty \frac{ds}{\sqrt{sf(s)}}.$$

Then (1.12) *holds if and only if* $L < \infty$.

Proof. In order to obtain the assertion, it is sufficient to show that

$$s\,f(\frac{s}{2}) \leq F(s) \leq s\,f(s) \qquad \forall s \geq 1. \tag{2.11}$$

The right-hand estimate in (2.11) follows from the monotone property of f. The assumption of convexity of f in $(0,\infty)$ implies

$$f(s) \geq f(\frac{s}{2}) + \frac{s}{2}f'(\frac{s}{2}) \qquad \forall s > 0.$$

Define $\varphi(s) = \int_0^s f(\sigma)d\sigma - sf(\frac{s}{2})$; then $\varphi'(s) = f(s) - f(\frac{s}{2}) - \frac{s}{2}f'(\frac{s}{2}) \geq 0$. Hence $\varphi(s) > \varphi(0) = 0$, which leads to the left-hand side estimate in (2.11). □

By using the same argument by contradiction as in Lemma 2.2 and thanks to Lemma 2.4, we obtain the following result.

Lemma 2.5. *If* (1.16) *holds, then*

$$\liminf_{r\to\infty} \frac{f(r)}{r\ln^{\alpha}(r)} = 0 \quad \forall \alpha > 2.$$

If (1.12) *holds, then*

$$\limsup_{r\to\infty} \frac{f(r)}{r\ln^{\alpha}(r)} = \infty \quad \forall 0 < \alpha \leq 2.$$

Proof of Theorem 1.3. The proof follows the same lines as that of Theorem 1.2. The only difference is that the final function $\underline{U}$, which is also independent of x by the same argument as in the proof Theorem 1.2, is now identified with ϕ_∞, since, by the assumptions we have made, it is the only solution independent of x which tends to ∞ when $t \to 0$. □

Proposition 2.6. *Assume* (1.12) *and* (1.8) *are satisfied. Then for any* $k > 0$ *there holds*

$$u_k(x,t) \leq \Phi(|x|) \quad \forall (x,t) \in Q_\infty,$$

where Φ *is a solution to the problem*

$$\begin{cases} -\Phi'' + f(\Phi) &= 0 \qquad \text{in } (0,\infty) \\ \lim\limits_{s\to 0}\Phi(s) &= \infty. \end{cases}$$

Proof. Step 1: Upper estimate. Since f satisfies (1.12), by [4] for any $R > 0$, there exists a solution w_R to the problem

$$\begin{cases} -\Delta w_R + f(w_R) &= 0 \qquad \text{in } B_R, \\ \lim\limits_{|x|\to R} w_R(x) &= \infty, \end{cases} \tag{2.12}$$

and w_R is nonnegative since $f(0)=0$. Notice also that $R\mapsto w_R$ is decreasing, since f is nondecreasing; moreover, $\lim_{R\to\infty} w_R=0$, since $f(0)=0$ and f is positive on $(0,\infty)$. Let $x_0\neq 0$ be arbitrary in $\mathbb{R}^N$. Set $\mathbb{E}=\{\vec{e}:|\vec{e}|=1\}$ and take $\vec{e}\in\mathbb{E}$. Put $x_{\vec{e}}=|x_0|\,\vec{e}$ and for $n>|x_0|$ put $a_n=n\vec{e}$. Denote by $\mathbb{H}_{\vec{e}}$ the open half-space generated by $\vec{e}$ and its orthogonal hyperplane at the origin; then $x_{\vec{e}},a_n\in\mathbb{H}_{\vec{e}}$. Take R such that $n-|x_0|<R<n$. We set $W_{\vec{e},n,R}(x)=w_R(x-a_n)$; then $W_{\vec{e},n,R}$ is a solution of (1.1) in $B_R(a_n)$ and blows-up on the boundary $\lim\limits_{|x-a_n|\to R} W_{\vec{e},n,R}(x)=\infty$. By the maximum principle,

$$u_k(x,t)\leq W_{\vec{e},n,R}(x)\quad \forall (x,t)\in B_R(a_n)\times(0,\infty). \tag{2.13}$$

The sequence $\{W_{\vec{e},n,R}\}$ is decreasing with respect to R and is bounded from below by u_k; then there exists $W_{\vec{e},n}:=\lim\limits_{R\to n} W_{\vec{e},n,R}$ satisfying

$$u_k(x,t)\leq W_{\vec{e},n}(x)\quad \forall (x,t)\in B_n(a_n)\times(0,\infty). \tag{2.14}$$

The sequence $\{W_{\vec{e},n}\}$ is also decreasing with respect to n and is bounded from below by u_k; then there exists $W_{\vec{e},\infty}:=\lim\limits_{n\to\infty} W_{\vec{e},n}$. Letting $n\to\infty$ in (2.14) yields

$$u_k(x,t)\leq W_{\vec{e},\infty}(x)\quad \forall (x,t)\in \mathbb{H}_{\vec{e}}\times(0,\infty). \tag{2.15}$$

In particular,

$$u_k(x_{\vec{e}},t)\leq W_{\vec{e},\infty}(x_{\vec{e}}). \tag{2.16}$$

Since u_k is radial, it follows that

$$u_k(x_0,t)=u_k(x_{\vec{e}},t)\leq W_{\vec{e},\infty}(x_{\vec{e}}).$$

For any $r>0$, $n>r$, $n-r<R<n$, and $\vec{e},\vec{e'}\in\mathbb{E}$, since w_R is radial,

$$w_R(r\vec{e}-n\vec{e})=w_R(r\vec{e'}-n\vec{e'}).$$

Letting successively $R\to n$ and $n\to\infty$ yields

$$W_{\vec{e},\infty}(r\vec{e})=W_{\vec{e'},\infty}(r\vec{e'}).$$

Define $\tilde{\Phi}(r):=W_{\vec{e},\infty}(r\vec{e})$, $\forall r\in(0,\infty)$; then it satisfies

$$\begin{cases} -\tilde{\Phi}''-\dfrac{N-1}{r}\tilde{\Phi}'+f(\tilde{\Phi}) &=0 \qquad \text{in } (0,\infty)\\ \lim\limits_{r\to 0}\tilde{\Phi}(r) &=\infty, \end{cases} \tag{2.17}$$

and

$$u_k(x,t)\leq\tilde{\Phi}(|x|)\qquad \forall (x,t)\in Q_\infty. \tag{2.18}$$

Step 2: End of the proof. We claim that

$$\tilde{\Phi}(r) \leq \Phi(r) \qquad \forall r \in (0, \infty). \tag{2.19}$$

For any $\epsilon > 0$, we set $\Phi_\epsilon(r) = \Phi(r - \epsilon)$, $r > \epsilon$; then Φ_ϵ is a solution of

$$-\Phi_\epsilon'' + f(\Phi_\epsilon) = 0 \qquad \text{in } (\epsilon, \infty) \tag{2.20}$$

satisfying $\lim_{r \to \epsilon} \Phi_\epsilon(r) = \infty$. Since $\Phi_\epsilon' \leq 0$, Φ_ϵ is a supersolution of the equation in (2.17) in (ϵ, ∞), which dominates $\tilde{\Phi}$ at $r = \epsilon$. By the maximum principle, $\tilde{\Phi} \leq \Phi_\epsilon$ in (ϵ, ∞). Letting $\epsilon \to 0$ yields (2.19). Combining (2.18) and (2.19) leads to the conclusion. □

Remark. Combining Proposition 2.3 and Proposition 2.6 yields

$$u_k(x,t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty,\ \forall k > 0. \tag{2.21}$$

Proof of Theorem 1.4. Since f is convex, (1.12) implies (1.10). Actually, only $\liminf_{s \to \infty} \frac{f(s)}{s} > 0$ is needed for this implication. The sequence $\{u_k\}$ is increasing with respect to k and is bounded from above by (2.21); then there exists $\underline{U} := \lim_{k \to \infty} u_k$ satisfying

$$\underline{U}(x,t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty,\ \forall k > 0. \tag{2.22}$$

Moreover, $\underline{U} \in \mathcal{U}_0$ because $\underline{U}$ has the following properties:
(i) It is positive in Q_∞, belongs to $C(\overline{Q} \setminus \{(0,0)\})$, and vanishes on $\mathbb{R}^N \setminus \{0\} \times \{0\}$.
(ii) It satisfies (1.1) and

$$\lim_{t \to 0} \int_{B_\sigma} \underline{U}(x,t)dx = \infty, \quad \forall \sigma > 0. \tag{2.23}$$

In the sense of initial trace in Definition 4.3, $\underline{U}$ has initial trace $tr_{\mathbb{R}^N}(\underline{U}) = (\{0\}, 0)$ (here $\{0\}$ is the singular part and the Radon measure on $\mathbb{R}^N \setminus \{0\}$ is the zero measure) and the conclusion follows from Proposition 4.5. □

By a simple adaptation of the proof of Proposition 2.3 and Proposition 2.6 it is possible to extend (2.22) to any positive solution vanishing on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$.

Proposition 2.7. *Assume* (1.12) *and* (C2) *are satisfied. Then any positive solution* $u \in C^{2,1}(Q_\infty)$ *of* (1.1) *satisfies*

$$u(x,t) \leq \phi_\infty(t) \qquad \forall (x,t) \in Q_\infty. \tag{2.24}$$

If we assume moreover that $u \in C(\overline{Q} \setminus \{(0,0)\})$ *vanishes on* $\mathbb{R}^N \setminus \{0\} \times \{0\}$, *there holds*

$$u(x,t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty. \tag{2.25}$$

Proof. Since $f(0) = 0$ and due to the convexity of f, the following inequality holds:

$$f(a+b) \geq f(a) + f(b) \quad \forall a, b > 0, \tag{2.26}$$

which implies that for any $R, \tau > 0$, $(x,t) \mapsto \phi_\infty(t-\tau) + w_R(x)$ is a super-solution of (1.1) in $B_R \times (\tau, \infty)$. This function dominates u on the parabolic boundary, and thus in the domain itself by the comparison principle. Since $f(r) > 0$ if $r > 0$, $\lim_{R\to\infty} w_R = 0$ in $\mathbb{R}^N$. Therefore,

$$u(x,t) \leq \phi_\infty(t) = \lim_{\tau\to 0} \lim_{R\to\infty} (\phi_\infty(t-\tau) + w_R(x)) \qquad \forall (x,t) \in Q_\infty.$$

For the second estimate we notice that (2.13) is valid with u_k replaced by u (and without assumption (1.8) since existence is assumed). The remainder of the proof of Proposition 2.6 is similar and yields

$$u(x,t) \leq \Phi(|x|) \qquad \forall (x,t) \in Q_\infty.$$

This implies (2.25). □

It is also possible to construct a maximal element of $\mathcal{U}_0$ ($\mathcal{U}_0$ is defined in Theorem 1.4). For $\ell > 0$ and $\epsilon > 0$, let $u := U_{\epsilon,\ell}$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) &= 0 \qquad \text{in } Q_\infty \\ u(x,0) &= \ell \chi_{B_\epsilon} \qquad \text{in } \mathbb{R}^N. \end{cases}$$

Lemma 2.8. *For any* $\tau > 0$ *and* $\epsilon > 0$, *there exist* $\ell > 0$ *and* $m(\tau,\epsilon) > 0$ *such that any positive solution* u *of* (1.1) *which satisfies* (i) *in the proof of Theorem* 1.4 *satisfies*

$$u(x,t) \leq U_{\epsilon,\ell}(x, t-\tau) + m(\tau,\epsilon) \qquad \forall (x,t) \in Q_\infty,\ t \geq \tau. \tag{2.27}$$

Furthermore,

$$\lim_{\tau\to 0} m(\tau,\epsilon) = 0 \qquad \forall \epsilon > 0. \tag{2.28}$$

Finally,

$$\overline{U}(x,t) = \lim_{\tau\to 0} \lim_{\epsilon\to 0} \lim_{\ell\to\infty} (U_{\epsilon,\ell}(x,t-\tau) + m(\tau,\epsilon)) \tag{2.29}$$

is the maximal element of $\mathcal{U}_0$.

Proof. We set $\ell = \phi_\infty(\tau)$; then $u(x,\tau) \le \ell$ for any $x \in \mathbb{R}^N$. Let $W := W_{\epsilon/2}$ be the solution of the following Cauchy-Dirichlet problem,

$$\begin{cases} \partial_t W - \Delta W + f(W) &= 0 \qquad \text{in } B^c_{\epsilon/2} \times (0,\infty) \\ W(x,0) &= 0 \qquad \text{in } B^c_{\epsilon/2} \\ W(x,t) &= \phi_\infty(t) \qquad \text{in } \partial B^c_{\epsilon/2} \times (0,\infty), \end{cases} \tag{2.30}$$

and put $m(\tau,\epsilon) := \max\{W_{\epsilon/2}(x,\delta) : |x| > \epsilon,\ 0 < \delta \le \tau\}$. It is clear to see that

$$\lim_{\tau\to 0} m(\tau,\epsilon) = W_{\epsilon/2}(x,0) = 0. \tag{2.31}$$

From the fact that $u(x,0) = 0$ in $B^c_{\epsilon/2}$ and $u(x,t) \le \phi_\infty(t)$ in $\partial B^c_{\epsilon/2} \times (0,\infty)$, along with the maximum principle, it follows that $u(x,t) \le W_{\epsilon/2}(x,t)$ in $B^c_{\epsilon/2} \times (0,\infty)$.

Next, we compare $U_{\epsilon,\ell}(\cdot,\cdot-\tau) + m(\tau,\epsilon)$ with u in $\mathbb{R}^N \times (\tau,\infty)$. The function $U_{\epsilon,\ell}(\cdot,\cdot-\tau) + m(\tau,\epsilon)$ is a supersolution of (1.1) in $\mathbb{R}^N \times (\tau,\infty)$. If $x \in B_\epsilon$, $U_{\epsilon,\ell}(x,0) = \ell \ge u(x,\tau)$, which implies $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$. If $x \in B^c_\epsilon$, $m(\tau,\epsilon) \ge W_{\epsilon/2}(x,\tau) \ge u(x,\tau)$; hence, $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$. So we always have $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$ for any $x \in \mathbb{R}^N$. Applying the maximum principle yields $U_{\epsilon,\ell}(\cdot,\cdot-\tau) + m(\tau,\epsilon) \ge u$ in $\mathbb{R}^N \times (\tau,\infty)$. Finally, the function $\overline{U}$ defined by (2.29) is the maximal solution because $U_{\epsilon,\ell}(x,t-\tau) \to U_{\epsilon,\ell}(x,t)$ as $\tau \to 0$ and $U_{\epsilon,\ell} \uparrow U_{\epsilon,\infty}$ when $\ell \to \infty$ and $U_{\epsilon,\infty} \downarrow \overline{U}$ when $\epsilon \to 0$. □

3. About uniqueness

We prove first the existence of global radial solutions of (1.18) under the Keller-Osserman condition.

Proof of Proposition 1.5. A solution of (1.19) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N} \int_0^s t^{N-1} f(w)\,dt\,ds. \tag{3.1}$$

Existence follows from the Picard-Lipschitz fixed-point theorem. The function is increasing and defined on a maximal interval $[0,r_a)$. By a result of Vazquez and Veron [11] $r_a = \infty$; thus, the solution is global. Uniqueness on $[0,\infty)$ follows always from local uniqueness. The function $r \mapsto w(r)$ is increasing and

$$w'(r) \ge \frac{f(a)}{N} r, \qquad w(r) \ge a + \frac{f(a)}{2N} r^2$$

for all $r > 0$. □

Proposition 3.1. *Assume* (1.16) *holds. For any* $u_0 \in C(\mathbb{R}^N)$ *which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{3.2}$$

for some $0 < a < b$, *there exists a positive function* $\overline{u} \in C(\overline{Q}_\infty) \cap C^{2,1}(Q_\infty)$ *a solution of* (1.1) *in* Q_∞ *and satisfying* $\overline{u}(\cdot, 0) = u_0$ *in* $\mathbb{R}^N$. *Furthermore,*

$$w_a(|x|) \leq u(x,t) \leq w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{3.3}$$

Proof. Clearly w_a and w_b are ordered solutions of (1.1). We denote by u_n the solution of the initial-boundary problem

$$\left\{\begin{array}{lll} \partial_t u_n - \Delta u_n + f(u_n) & = 0 & \text{in } Q_n = B_n \times (0,\infty) \\ u_n(x,t) & = (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0,\infty) \\ u_n(x,0) & = u_0(x) & \text{in } B_n. \end{array}\right. \tag{3.4}$$

By the maximum principle, u_n satisfies (3.3) in Q_n. Using locally parabolic equations regularity theory, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q}_\infty$. Using a diagonal sequence, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q}_\infty$ to some weak solution $\overline{u} \in C(\overline{Q}_\infty)$ which satisfies $\overline{u}(\cdot,0) = u_0$ in $\mathbb{R}^N$. By a standard method, $\overline{u}$ is a strong solution (at least $C^{2,1}(Q_\infty)$). □

Proposition 3.2. *Assume* (1.16) *and* (1.10) *hold. Then for any* $u_0 \in C(\mathbb{R}^N)$ *which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{3.5}$$

for some $0 < a < b$, *there exists a positive function* $\underline{u} \in C(\overline{Q_\infty})$ *solution of* (1.1) *in* Q_∞ *satisfying* $\underline{u}(\cdot,0) = u_0$ *in* $\mathbb{R}^N$ *and*

$$\underline{u}(x,t) \leq \min\{\phi_\infty(t), w_b(|x|)\} \qquad \forall (x,t) \in Q_\infty. \tag{3.6}$$

Proof. For any $R > 0$, let u_R be the solution of

$$\left\{\begin{array}{ll} \partial_t u_R - \Delta u_R + f(u_R) = 0 & \text{in } Q_\infty \\ u_R(x,0) = u_0(x)\chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{array}\right. \tag{3.7}$$

The solution which is constructed is dominated by the solution of the heat equation with the same initial data. Thus

$$u_R(x,t) \leq (4\pi t)^{-N/2} \int_{B_R} e^{-|x-y|^2/4t} u_0(x)dy \qquad \forall (x,t) \in Q_\infty \tag{3.8}$$

and $\lim_{|x|\to\infty} u_R(x,t) = 0$ uniformly with respect to t. The functions ϕ_∞ and w_b are solutions of (1.1) in Q_∞, which dominate u_R at $t = 0$. By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x,t) \quad \forall (x,t) \in Q_\infty. \tag{3.9}$$

The fact that the mapping $R \mapsto u_R$ is increasing and (3.9) imply that there exists $\underline{u} := \lim_{R\to\infty} u_R$ which satisfies $\underline{u}(\cdot,0) = u_0$ in $\mathbb{R}^N$. Letting $R \to \infty$ in (3.9) yields (3.6). □

Proof of Theorem 1.6. Combining Proposition 3.1 and Proposition 3.2 we see that there exist two solutions $\underline{u}$ and $\overline{u}$ with the same initial data u_0 which are ordered and different since $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$ and $\lim_{|x|\to\infty} \underline{u}(x,t) \leq \phi_\infty(t) < \infty$ for all $t > 0$. □

Proof of Theorem 1.7. There always holds

$$(ah(a) - bh(b))\,\mathrm{sign}(a-b) \geq |a-b|\, h(|a-b|) \qquad \forall a,b > 0, \tag{3.10}$$

where h is defined in (2.2) and

$$\mathrm{sign}(z) = \begin{cases} 1 & \text{if } z > 0, \\ -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \leq [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})]\,\mathrm{sign}(u - \tilde{u});$$

therefore, by (3.10),

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + |u - \tilde{u}|\, h(|u - \tilde{u}|) \leq 0. \tag{3.11}$$

Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any $R \geq R_\epsilon$,

$$0 \leq |u - \tilde{u}|\,(x,t) \leq w_\epsilon(|x|) \quad \forall (x,t) \in B_R^c \times [0,1]. \tag{3.12}$$

Since w_ϵ is a positive solution of (1.1) which dominates $|u - \tilde{u}|$ on $\partial B_R \times [0,1]$ and at $t = 0$, it follows that $|u - \tilde{u}| \leq w_\epsilon$ in $B_R \times [0,1]$. Letting $R \to \infty$ yields $|u - \tilde{u}| \leq w_\epsilon$ in $\mathbb{R}^N \times [0,1]$. Letting $\epsilon \to 0$ and since $\lim_{\epsilon\to 0} w_\epsilon(|x|) = 0$ for any $x \in \mathbb{R}^N$, we derive $|u - \tilde{u}| = 0$; thus, $u = \tilde{u}$ in $\mathbb{R}^N \times [0,1]$. Iterating yields that equality holds in Q_∞. □

Remark. If we replace condition (C1) by condition (C2), the conclusion of Theorem 1.7 remains valid. Indeed, it follows by the convexity of f that

$$(f(a) - f(b))\,\mathrm{sign}(a-b) \geq f(|a-b|) \quad \forall a,b > 0.$$

Then we proceed as in the proof of Theorem 1.7 to get the desired conclusion.

Proof of Theorem 1.8.

Proof of statement (i). The solution $\underline{u}$ which is constructed in Proposition 3.2 is a minimal solution of (1.1) in Q_∞ with the initial value u_0. Indeed, if $u \in C^{2,1}(Q_\infty)$ is a nonnegative solution of (1.1) in Q_∞ which satisfies $u(\cdot,0) = u_0$ in $\mathbb{R}^N$, then by the maximum principle $u_R \leq u$ in Q_∞, where u_R is the solution of (3.7). Letting $R \to \infty$ yields $\underline{u} \leq u$ in Q_∞. Next we construct the maximal solution. We recall that w_R is the solution of (2.12). Since f is convex, f' is nondecreasing and for w_R there holds $f'(u_R) \leq f'(w_R + u_R)$, thus there holds $f(w_R) + f(u_R) \leq f(w_R + u_R)$. Consequently $w_R + u_R$ is a supersolution in $B_R \times (0,\infty)$. If $u \in C(\overline{Q_\infty})$ is a solution of (1.1) in Q_∞ with initial data u_0, it is dominated by $w_R + u_R$ on $\partial B_R \times (0,\infty)$. Thus $u \leq w_R + u_R$, which dominates any solution $u \in C(\overline{Q_\infty})$ of (1.1) in $B_R \times (0,\infty)$. Since

$$u_R \leq u \leq w_R + u_R,$$

$w_R \to 0$ when $R \to \infty$ (by Proposition 2.6, Step 1), and $u_R \to \underline{u}$, we derive that $u = \underline{u}$.

Step 1: Proof of statement (i), construction of the minimal solution. Assume there exists at least one positive solution u of (1.1) satisfying (1.23) and $f(u) \in L^1_{loc}(\overline{Q_\infty})$; equivalently [7],

$$\int_0^\infty \int_{\mathbb{R}^N} (-u(\partial_t \eta + \Delta\eta) + f(u)\eta)\, dx\, dt = \int_{\mathbb{R}^N} \eta(x,0) d\mu(x) \tag{3.13}$$

for all $\eta \in C_c^{2,1}(\overline{Q_\infty})$. We construct first a minimal solution in the following way: let $n \in \mathbb{N}$ and $R > 0$ and let $v = v_{R,n}$ be the solution of

$$\begin{cases} \partial v - \Delta v + f(v) = 0 & \text{in } B_R \times (0,\infty) \\ v = 0 & \text{in } \partial B_R \times (0,\infty) \\ v(\cdot,0) = u(\cdot, 2^{-n}) & \text{in } B_R. \end{cases} \tag{3.14}$$

By the maximum principle, $v_{R,n}(\cdot,t) \leq u(\cdot, t + 2^{-n})$. Furthermore,

$$v_{R,n}(x, 2^{-n}) \leq u(\cdot, 2^{-n+1}) = v_{R,n-1}(x,0);$$

therefore,

$$v_{R,n}(x, t + 2^{-n}) \leq v_{R,n-1}(x,t) \qquad \text{in } B_R \times (0,\infty). \tag{3.15}$$

Using the formulation (3.13) with $v_{R,\epsilon}$, we obtain, for any $\eta \in C_c^{2,1}(\overline{Q_\infty^{B_R}})$,

$$\int_0^\infty \int_{\mathbb{R}^N} (-v_{R,n}(\partial_t \eta + \Delta \eta) + f(v_{R,n})\eta)\, dx\, dt = \int_{\mathbb{R}^N} \eta(x,0) u(x, 2^{-n}) dx. \tag{3.16}$$

The right-hand side of (3.16) converges to $\int_{\mathbb{R}^N} \eta(x,0) d\mu(x)$. Concerning the left-hand side, there holds $f(v_{R,n}(x,t)) \leq f(u(x,t+2^{-n}))$. Since $f(u) \in L^1_{loc}(\overline{Q_\infty})$, $f(v_{R,n})$ is bounded in $L^1_{loc}(\overline{Q_\infty^{B_R}})$. By the L^1 regularity theory for parabolic equations (see [6] and the references therein), the set of functions $\{v_{R,n}\}$ is locally compact in $L^1_{loc}(Q_\infty)$ and there exists a subsequence $\{n_k\}$ and a function $\underline{u}_R$ such that $v_{R,n_k} \to \underline{u}_R$, almost everywhere in $Q_\infty^{B_R}$, and $\underline{u}_R \leq u$. Noticing that the sets of functions $\{f(u(\cdot, \cdot + 2^{-n}))\}$ and $\{u(\cdot, \cdot + 2^{-n})\}$ are uniformly integrable, we obtain that the two sets $\{f(v_{R,n})\}$ and $\{v_{R,n}\}$ are also uniformly integrable in $B_R \times (0,T)$. It follows from Vitali's convergence theorem that, up to a subsequence still denoted by $\{n_k\}$, $v_{R,n_k} \to \underline{u}_R$ and $f(v_{R,n_k}) \to f(\underline{u}_R)$ in $L^1(B_R \times (0,T))$. Letting $n = n_k \to \infty$ in (3.16) we derive

$$\int_0^\infty \int_{\mathbb{R}^N} (-\underline{u}_R(\partial_t \eta + \Delta \eta) + f(\underline{u}_R)\eta)\, dx\, dt = \int_{\mathbb{R}^N} \eta(x,0) d\mu(x). \tag{3.17}$$

This means that $\underline{u}_R$ satisfies $\underline{u}_R \leq u$ and

$$\begin{cases} \partial \underline{u}_R - \Delta \underline{u}_R + f(\underline{u}_R) = 0 & \text{in } B_R \times (0,\infty) \\ \underline{u}_R = 0 & \text{in } \partial B_R \times (0,\infty) \\ \underline{u}_R(\cdot, 0) = \chi_{B_R} \mu & \text{in } B_R. \end{cases} \tag{3.18}$$

If $\tilde{u}$ is any other nonnegative solution of (1.1) in Q_∞ with initial data μ, the same construction of $\tilde{v}_{R,n}$, a solution of (3.14) with initial data $\tilde{u}(\cdot, 2^{-n})$ instead of $u(\cdot, 2^{-n})$, converges up to a subsequence to some $\underline{\tilde{u}}_R$ which satisfies $\underline{\tilde{u}}_R \leq \tilde{u}$ and is a solution of problem (3.15). We know from [5] and [6] that this problem admits at most one solution. Therefore $\underline{\tilde{u}}_R = \underline{u}_R$, which implies that $\underline{u}_R \leq \tilde{u}$ in $Q_\infty^{B_R}$. Furthermore, in the above construction, we have used only the fact that $\tilde{u}$ is defined in a domain larger than $Q_\infty^{B_R}$ and is nonnegative. Consequently, the same comparison applies if we compare $\underline{u}_R$ and $\underline{u}_{R'}$ for $R' > R$ and we obtain

$$\underline{u}_R \leq \underline{u}_{R'} \qquad \text{in } Q_\infty^{B_R}.$$

Put $\underline{u} = \lim_{R\to\infty} \underline{u}_R$. Using the monotone convergence theorem and a test function $\eta \in C_c^{2,1}(\overline{Q_\infty})$ with compact support in $Q_\infty^{B_R}$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (-\underline{u}(\partial_t\eta + \Delta\eta) + f(\underline{u})\eta)\, dx\, dt = \int_{\mathbb{R}^N} \eta(x,0)d\mu(x) \tag{3.19}$$

from (3.17). Thus $\underline{u}$ satisfies (1.23) and $f(\underline{u}) \in L^1_{loc}(\overline{Q_\infty})$. By construction $\underline{u}$ is smaller than any other nonnegative solution.

Step 2: End of proof of statement (ii). As in the proof of statement (i), we see that, for any $n \in \mathbb{N}^*$, there holds $u \le w_R + v_{R,n}$ in $Q_\infty^{B_R}$. Consequently $u \le w_R + \underline{u}_R$, and letting $R \to \infty$, $u \le \underline{u}$. Thus, $u = \underline{u}$. □

4. Initial trace

If Ω is an open domain in $\mathbb{R}^N$, we denote by $\mathfrak{M}(\Omega)$ (respectively $\mathfrak{M}^b(\Omega)$) the set of Radon measures in Ω (respectively bounded Radon measures), and by $\mathfrak{M}_+(\Omega)$ (respectively $\mathfrak{M}_+^b(\Omega)$) its positive cone. For $T > 0$, we set $Q_T^\Omega = \Omega \times (0,T)$ and $Q_T = \mathbb{R}^N \times (0,T)$.

4.1. The regular part of the initial trace.

In this section we assume only that f is a continuous nonnegative function defined on $\mathbb{R}_+$ and that u is a $C^{2,1}$ positive solution of (1.1) in Q_T.

Lemma 4.1. *Assume G is a bounded C^2 domain in $\mathbb{R}^N$, $Q_T^{\overline{G}} := \overline{G} \times (0,T]$ and let $u \in C^{2,1}(Q_T^{\overline{G}})$ be a positive solution of (1.1) in Q_T^G such that $f(u) \in L^1(Q_T^G)$. Then $u \in L^\infty(0,T; L^1(G'))$ for any domain $G' \subset \overline{G'} \subset G$ and there exists a positive Radon measure μ_G on G such that*

$$\lim_{t\to 0} \int_G u(x,t)\zeta(x)dx = \int_G \zeta(x)d\mu_G(x) \qquad \forall \zeta \in C_c(G). \tag{4.1}$$

Proof. Let $\phi := \phi_G$ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ with corresponding eigenvalue λ_G. We assume $\phi > 0$ in G. Then

$$\frac{d}{dt}\int_G u\phi\, dx + \lambda_G \int_G u\phi\, dx + \int_G f(u)\phi\, dx + \int_{\partial G} u\phi_{\mathbf{n}}\, dS = 0,$$

where $\phi_{\mathbf{n}}$ denotes the outward normal derivative of ϕ. Since $\phi_{\mathbf{n}} < 0$, the function

$$t \mapsto e^{\lambda_G t}\int_G u(x,t)\phi(x)dx - \int_t^T \int_G e^{\lambda_G s} f(u)\phi dx\, ds$$

is increasing and

$$\int_G u(x,t)\phi(x)dx \le e^{\lambda_G(T-t)}\int_G u(x,T)\phi(x)dx + e^{-\lambda_G t}\int_t^T\int_G e^{\lambda_G s}f(u)\phi\,dx\,ds$$

for $0 < t \le T$. Thus $u \in L^\infty(0,T;L^1(G'))$ for any strict domain G' of G. If $\zeta \in C_c(G)$, there holds

$$\frac{d}{dt}\left(\int_G u(x,t)\zeta(x)dx - \int_t^T\int_G (f(u)\zeta - u\Delta\zeta)\,dx\,ds\right) = 0. \tag{4.2}$$

Consequently,

$$\lim_{t\to 0}\int_G u(x,t)\zeta(x)dx = \int_G u(x,T)\zeta(x)dx + \int_0^T\int_G (f(u)\zeta - u\Delta\zeta)\,dx\,ds. \tag{4.3}$$

This implies that $u(\cdot,t)$ admits a limit in $\mathcal{D}'(G)$, and this limit is a positive distribution. Therefore there exists a positive Radon measure μ_G on G that satisfies (4.1). □

Proof of Proposition 1.9. It is clear that $\mathcal{R}(u)$ is an open subset. If G is a strict bounded subdomain of $\mathcal{R}(u)$, i.e., $\overline{G} \subset \mathcal{R}(u)$, there exists a finite number of points z_j $(j = 1,\dots,k)$ and $r'_j > r_j > 0$ such that $u, f(u) \in L^1(Q_T^{B_{r'_j}(z_j)})$ and $\overline{G} \subset \cup_{j=1}^k B_{r_j}(z_j)$. Let $\mu_j = \mu_{B_{r_j}(z_j)}$ be the measure defined in Lemma 4.1. If $\zeta \in C_c(G)$ there exists a partition of unity $\{\eta_j\}_{j=1}^k$ relative to the cover $\{B_{r_j}(z_j)\}_{j=1}^k$ such that $\eta_j \in C_0^\infty(G)$, $\operatorname{supp}(\eta_j) \subset B_{r_j}(z_j))$ and $\zeta = \sum_{j=1}^k \eta_j\zeta$. Since

$$\lim_{t\to 0}\int_{B_{r_j}(z_j)} u(x,t)(\eta_j\zeta)(x)dx = \int_{B_{r_j}(z_j)}(\eta_j\zeta)(x)d\mu_j(x) \qquad \forall j = 1,\dots,k,$$

there exists a positive Radon measure μ on $\mathcal{R}(u)$ satisfying (1.24). Notice also that $u \in L^\infty(0,T;L^1(G))$ for any $G \subset \overline{G} \subset \mathcal{R}(u)$. □

The main problem is to analyze the behaviour of u on the singular set $\mathcal{S}(u)$.

4.2. **The Keller-Osserman condition holds.** If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (1.1) is based upon a dichotomy in the behaviour of those solutions near $t = 0$.

Lemma 4.2. *Assume u is a positive solution of* (1.1) *in Q_T and $z \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds.*
(i) There exists an open neighborhood G of z such that $u \in L^1(Q_T^G)$.
(ii) f is nondecreasing and (1.12) *holds.*
Then, for every open relative neighborhood G' of z,

$$\lim_{t\to 0}\int_{G'} u(x,t)dx = \infty. \tag{4.4}$$

Proof. First, we assume that (i) holds and let $\zeta \in C_c^2(G)$, $\zeta \geq 0$. Since $z \in \mathcal{S}(u)$, then for every open relative neighborhood G' of z, there holds

$$\int_0^T\int_{G'} f(u)dx\,dt = \infty. \tag{4.5}$$

Since there exists

$$\lim_{t\to 0}\int_t^T\int_{G'} u\Delta\zeta dx\,dt = L \in \mathbb{R},$$

it follows from (4.3) that

$$\int_{G'} u(x,t)\zeta(x)dx = \int_t^T\int_{G'} f(u)\zeta\,dx\,ds + O(1), \tag{4.6}$$

which implies (4.4).

Next we assume that (1.12) holds and $u \notin L^1(Q_T^G)$ for every relative neighborhood G of z. If there exists an open neighborhood $G \subset \Omega$ of z such that (4.4) does not hold, there exists a sequence $\{t_n\}$ decreasing to 0 and $0 \leq M < \infty$ such that

$$\sup_{t_n}\int_G u(x,t_n)dx = M. \tag{4.7}$$

Furthermore, we can always replace G by an open ball $B_R(z) \subset G$. Thus (4.7) holds with G replaced by $B_R(z)$. Let $w := w_R$ be the maximal solution of

$$\begin{cases} -\Delta w + f(w) &= 0 \qquad \text{in } B_R(z) \\ \lim\limits_{|x-z|\to R} w(x) &= \infty. \end{cases} \tag{4.8}$$

Let $v := v_n$ be the solution of

$$\begin{cases} \partial_t v - \Delta v &= 0 & \text{in } B_R(z)\times(t_n,\infty) \\ v &= 0 & \text{in } \partial B_R(z)\times(t_n,\infty) \\ v(\cdot,t_n) &= u(\cdot,t_n) & \text{in } B_R(z). \end{cases} \tag{4.9}$$

Since $v_n \geq 0$, $f(w_R + v_n) \geq f(w_R)$ and $w_R + v_n$ is a supersolution of (1.1) in $B_R(z) \times (t_n, T)$. It dominates u on $\partial B_R(z) \times (t_n, T)$ and at $t = t_n$; thus, $u \leq w_R + v_n$ in $B_R(z) \times (t_n, T)$. We can assume that $u(\cdot, t_n) \to \nu$ for some positive and bounded measure ν on $B_R(z)$. Therefore,

$$u(x,t) \leq v(x,t) + w_R(x) \qquad \text{in } Q_T^{B_R(z)}, \tag{4.10}$$

where v is the solution of

$$\begin{cases} \partial_t v - \Delta v &= 0 & \text{in } Q_\infty^{B_R(z)} \\ v &= 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) &= \nu & \text{in } \mathcal{D}'(B_R(z)). \end{cases} \tag{4.11}$$

Since $v \in L^1(Q_T^{B_R(z)})$ and w_R is uniformly bounded in any ball $B_{R'}(z)$ for $0 < R' < R$, we conclude that $u \in L^1(Q_T^{B_{R'}(z)})$, which is a contradiction. □

Definition 4.3. *Assume f is nondecreasing and satisfies* (1.12). *Let $u \in C^{2,1}(Q_T)$ be a positive solution of* (1.1) *in Q_T. We say that u possesses an initial trace with regular part $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ and singular part $\mathcal{S}(u) = \mathbb{R}^N \setminus \mathcal{R}(u)$ if*

(i) For any $\zeta \in C_c(\mathcal{R}(u))$,

$$\lim_{t \to 0} \int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta(x)d\mu(x). \tag{4.12}$$

(ii) For any open set $G \subset \mathbb{R}^N$ such that $G \cap \mathcal{S}(u) \neq \emptyset$

$$\lim_{t \to 0} \int_G u(x,t)dx = \infty. \tag{4.13}$$

Proof of Theorem 1.11. The set $\mathcal{R}(u)$ and the measure $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ are defined by Definition 1.10 thanks to Proposition 1.9. Because (1.12) holds, $\mathcal{S}(u) = \mathbb{R}^N \setminus \mathcal{R}(u)$ inherits the property *(ii)* in Definition 4.3 because of Lemma 4.2 *(ii)*. □

If Ω is a bounded domain with a C^2 boundary and $\mu \in \mathfrak{M}_+^b(\Omega)$, we denote by u_μ the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = \mu & \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{4.14}$$

We recall the following stability result proved in [6, Theorem 1.1].

Lemma 4.4. *Let Ω be a bounded domain with a C^2 boundary. Assume f is nondecreasing and satisfies* (1.8)*. Then for any $\mu \in \mathfrak{M}^b(\Omega)$ problem* (4.14) *admits a unique solution u_μ. Moreover, if $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$ converges weakly to $\mu \in \mathfrak{M}^b(\Omega)$, then $u_{\mu_n} \to u_\mu$ locally uniformly in $\overline{\Omega} \times (0,\infty)$ and in $L^1(Q_T^\Omega)$, and $f(u_{\mu_n}) \to f(u_\mu)$ in $L^1(Q_T^\Omega)$, for every $T > 0$.*

Remark. The result remains true if Ω is unbounded, with a C^2 compact (possibly empty) boundary and the μ_n have their support in a fixed compact set. In such a case $u_{\mu_n}(x,t) \to 0$ when $|x| \to \infty$, uniformly with respect to n and t since

$$|u_{\mu_n}(x,t)| \leq \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/4t} d\,|\mu_n|\,(y) \quad \forall (x,t) \in Q_\infty. \tag{4.15}$$

By Lemma 4.4 and the remark hereafter, for every $y \in \Omega$ and $k > 0$, there exists a unique solution $v_{y,k,\Omega} := v$ to (4.14) with $\mu = k\delta_0$. By the comparison principle (see [6, Proposition 1.2]) $v_{y,k,\Omega}$ is positive, increases as k increases, and depends continuously on y. Note that if $\Omega = \mathbb{R}^N$, $v_{y,k,\mathbb{R}^N}(x,t) := v_{y,k}(x,t) = u_k(|x-y|\,,t)$; furthermore, if f satisfies (1.12), we recall that $\underline{U} = \lim_{k\to\infty} u_k$ is the minimal solution of (1.1) in Q_∞ with initial trace $(\{0\},0)$.

Proposition 4.5. *Assume f is nondecreasing and satisfies* (1.8) *and* (1.12)*. Let $u \in C^{2,1}(Q_\infty)$ be a positive solution of* (1.1) *in Q_∞ with initial trace $(\mathcal{S},\mu)$. Then for every $y \in \mathcal{S}$,*

$$\underline{U}_y(x,t) := \underline{U}(x-y,t) \leq u(x,t) \quad \textit{in } Q_\infty. \tag{4.16}$$

Proof. By translation we may suppose that $y = 0$. Since $0 \in \mathcal{S}(u)$, for any $\eta > 0$ small enough

$$\lim_{t\to 0} \int_{B_\eta} u(x,t)dx = \infty.$$

For $\epsilon > 0$, denote $M_{\epsilon,\eta} = \int_{B_\eta} u(x,\epsilon)dx$. For any $m > m_\eta = \inf_{\sigma>0} M_{\sigma,\eta}$ there exists $\epsilon = \epsilon(m,\eta)$ such that $m = M_{\epsilon,\eta}$ and $\lim_{\eta\to 0} \epsilon(m,\eta) = 0$. Let v_η be the solution of the problem

$$\begin{cases} \partial_t v_\eta - \Delta v_\eta + f(v_\eta) = 0 & \text{in } Q_\infty \\ \qquad\qquad v_\eta(x,0) = u(x,\epsilon)\chi_{B_\eta} & \text{in } \mathbb{R}^N, \end{cases}$$

where χ_{B_η} is the characteristic function of B_η. By the maximum principle $v_\eta \leq u$ in $\mathbb{R}^N \times (\epsilon,\infty)$. By Lemma 4.4 and the remark after, v_η converges to $v_{0,m}$ when η goes to zero. Letting m go to infinity yields (4.16). $\square$

Corollary 4.6. *Under the assumption of Proposition* 4.5, *there exists a minimal positive solution* $\underline{U}_{\mathcal{S}}$ *of* (1.1) *in* Q_∞ *with initial trace* $(\mathcal{S}, 0)$ *in the sense that*

$$\underline{U}_{\mathcal{S}}(x,t) \leq u(x,t) \qquad \forall (x,t) \in Q_\infty, \tag{4.17}$$

for all positive solution $u \in C^{2,1}(Q_\infty)$ *of* (1.1) *with initial trace* $(\mathcal{S}(u), \mu)$.

Proof. If we set $\tilde{\underline{U}}_{\mathcal{S}} = \sup\{U_y : y \in \mathcal{S}\}$, then $\tilde{\underline{U}}_{\mathcal{S}}$ is a subsolution of (1.1). If u is a positive solution of (1.1) with initial trace $(\mathcal{S}, \mu)$, then $u \geq \tilde{\underline{U}}_{\mathcal{S}}$ by Proposition 4.5. Therefore u is larger than the smallest solution of (1.1) in Q_∞ which is above $\tilde{\underline{U}}_{\mathcal{S}}$. We denote this minimal solution by $\underline{U}_{\mathcal{S}}$. □

If $\mathcal{S}$ contains some ball B_R we have a more precise result.

Proposition 4.7. *Let* u *be a positive solution of* (1.1) *in* Q_∞ *with initial trace* $(\mathcal{S}, \mu)$. *We assume that* $\mathcal{S}$ *has a nonempty interior, and for* $R > 0$, *we denote by* $int_R(\mathcal{S})$ *the set of* $y \in \mathcal{S}$ *such that* $\overline{B}_R(y) \subset int_R(\mathcal{S})$. *Then for any* $R' \in (0, R)$ *there holds*

$$\lim_{t \to 0} \frac{u(x,t)}{\phi_\infty(t)} = 1 \tag{4.18}$$

uniformly for $x \in \overline{B}_{R'}(y)$ *and* $y \in int_R(\mathcal{S})$.

Proof. Let $y \in int_R(\mathcal{S})$ and $w(x,t) = u(x,t) + w_R(x-y)$, when w_R is the maximal solution of (2.12). Then w is a supersolution of (1.1) in $Q_\infty^{B_R(y)}$ and $\lim_{t\to 0} w(x,t) = \infty$, uniformly with respect to $x \in B_R(y)$, by (4.16). Then, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $w(x,t) \geq \phi_\infty(\epsilon)$ in $Q_{t_\epsilon}^{B_R(y)}$. Since $\phi_\infty(t+\epsilon)$ remains bounded on $\partial B_R(y) \times (0,\infty)$, it follows by the maximum principle that

$$w(x,t) \geq \phi_\infty(t+\epsilon) \qquad \forall (x,t) \in Q_\infty^{B_R(y)}.$$

Letting $\epsilon \to 0$ and using the fact that $W_R(x-y)$ remains uniformly bounded when $|x-y| \leq R'$, we derive

$$u(x,t) \geq \phi_\infty(t) - K_{R'} \qquad \forall (x,t) \in Q_\infty^{B'_R(y)}, \tag{4.19}$$

where $K_{R'} = \max\{w_R(x-y) : |x-y| \leq R'\}$. Combining this estimate with (2.24) yields (4.18). □

The following convergence lemma is obtained by using the arguments of Lemma 4.1

Proposition 4.8. *Assume f is nondecreasing and satisfies* (1.8) *and* (1.12). *Let $\{u_n\}$ be a sequence of positive solutions of* (1.1) *in Q_∞ with initial trace $(\mathcal{S}(u_n),\mu_n)$ such that $u_n \to u$ locally uniformly in Q_∞, and let A be an open subset of $\mathcal{R}(u_n) := \mathbb{R}^N \setminus \mathcal{S}(u_n)$. Then u is a positive solution of* (1.1) *in Q_∞, with initial trace denoted by $tr_{\mathbb{R}^N}(u) = (\mathcal{S},\mu)$. Furthermore, if $\mu_n(A)$ remains uniformly bounded, then $A \subset \mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ and $\chi_{_A}\mu_n \to \chi_{_A}\mu$ weakly. Conversely, if $A \subset \mathcal{R}(u)$, then $\mu_n(K)$ remains bounded independently of n, for every compact set $K \subset A$.*

Proof. The fact that u is a positive solution of (1.1) in Q_∞ is standard by the weak formulation of the equation. Assume now that $A \cap \mathcal{S} \neq \emptyset$. Let $z \in A \cap \mathcal{S}$ and $R > 0$ such that $\overline{B}_R(z) \subset A$. By convexity, u_n is bounded from above in $Q_\infty^{B_R(z)}$ by $v_n + w_R$, where $v_{n,z}$ satisfies

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0,\infty) \\ v(\cdot,0) = \chi_{_{B_R(z)}}\mu_n & \text{in } B_R(z), \end{cases} \tag{4.20}$$

and w_R is the maximal solution of (4.8). We can assume that, up to a subsequence, $\chi_{_{B_R(z)}}\mu_{n_k} \to \mu_z \in \mathfrak{M}_+^b(B_R(z))$ weakly, thus $v_{n_k,z} \to v_z$, where v_z is the solution of

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0,\infty) \\ v(\cdot,0) = \mu_z & \text{in } B_R(z), \end{cases} \tag{4.21}$$

Therefore,

$$u \leq v_z + w_R \qquad \text{in } Q_\infty^{B_R(z)}. \tag{4.22}$$

By Lemma 4.4, this implies that $u \in L^1(Q_T^{B_{R'}(z)})$ for any $0 < R' < R$. Furthermore, if (1.8) is satisfied, then for any positive constant $k, s \mapsto s^{-2-N/2}f(s^{-N/2}+k) \in L^1(0,1)$; thus, if v is such that $f(v) \in L^1(Q_T^{B_{R'}(z)})$, there holds $f(v+k) \in L^1(Q_T^{B_{R'}(z)})$. In particular, since $f(v_z) \in L^1(Q_T^{B_{R'}(z)})$, and if we take $k = \max\{w_R(x) : x \in B_{R'}(z)\}$, we derive that $f(u) \in L^1(Q_T^{B_{R'}(z)})$, and therefore $z \in \mathcal{R}$, which is a contradiction; thus $A \subset \mathcal{R}$. Next, there exist a subsequence $\{n_k\}$ and a bounded positive measure $\tilde{\mu}$, with support in A such that $\chi_{_A}\mu_{n_k} \to \tilde{\mu}$ weakly, and suppose $\overline{B}_R(z) \subset A$. Since $u_{n_k} \leq v_{n_k,z} + k$ and $f(u_{n_k}) \leq f(v_{n_k,z} + k)$ in $Q_T^{B_{R'}(z)}$ and $v_{n_k,z} + k$ and $f(v_{n_k,z} + k)$ are uniformly integrable in $Q_T^{B_{R'}(z)}$, it follows that u_{n_k} and

$f(u_{n_k})$ inherit this property. Therefore, if $\zeta \in C_c^2(B_R(z))$ we can assume that it vanishes outside $B_{R'}(z)$. Because

$$\begin{aligned}\int_{B_R(z)} \zeta(x) d\mu_{n_k}(x) &= \int_{B_R(z)} u_{n_k}(x,t)\zeta(x)dx \\ &+ \int_0^t \int_{B_R(z)} (-u_{n_k}\Delta\zeta + f(u_{n_k})\zeta)\, dx\, ds,\end{aligned} \tag{4.23}$$

we derive from Vitali's convergence theorem

$$\int_{B_R(z)} \zeta(x) d\tilde{\mu}(x) = \int_{B_R(z)} u(x,t)\zeta(x)dx + \int_0^t \int_{B_R(z)} (-u\Delta\zeta + f(u)\zeta)\, dx\, ds. \tag{4.24}$$

This implies that $\chi_{B_R(z)}\tilde{\mu} = \chi_{B_R(z)}\mu$ and, by a partition of unity, that $\tilde{\mu} = \chi_A \mu$.

Assume now that $K \subset \mathcal{R}$ is compact. If $\mu_n(K)$ is unbounded and up to a subsequence still denoted by $\{n\}$, there exists a point $y \in K$ such that for any neighborhood $\mathcal{O}$ of y, $\mathcal{O} \subset A$, $\mu_n(\mathcal{O}) \to \infty$ as $n \to \infty$. We can take $\mathcal{O} = B_r(y)$ and put $M_{n,r} = \mu_n(B_r(y))$. If $m \in \mathbb{N}^*$, there exists an integer $n = n(m,r)$ such that $m \leq M_{n,r}$, and $\lim_{r\to 0} n(m,r) = \infty$. Let $r_0 > r$ such that $B_{r_0}(y) \subset A$, and w_r be the solution of

$$\begin{cases} \partial_t w - \Delta w + f(w) = 0 & \text{in } Q_\infty^{B_{r_0}(y)} \\ w = 0 & \text{in } \partial B_\infty^{B_{r_0}(y)} \\ w(\cdot,0) = \chi_{B_r(y)}\mu_n & \text{in } B_{r_0}(y). \end{cases} \tag{4.25}$$

By the comparison principle, $w_r \leq u_n$ in $Q_\infty^{B_{r_0}(y)}$. Since $\chi_{B_r(y)}\mu_n \to m\delta_y$ as $r \to 0$ and $n \to \infty$, we derive $u_{y,m,B_{r_0}(y)} \leq u$ from Lemma 4.4 and the remark thereafter. Since m is arbitrary, $u_{y,\infty,B_{r_0}(y)} \leq u$. This implies that $y \in \mathcal{S}$, a contradiction. □

If A is an open subset of Ω and $\nu \in \mathfrak{M}_+(A)$, we define an extension $\underline{\nu}$ of ν to Ω by

$$\underline{\nu}(E) = \inf_{E \subseteq O} \nu(O \cap A) \tag{4.26}$$

for every Borel set $E \subset \Omega$, where the infimum is taken over the open subsets O; $\underline{\nu}$ is an outer regular Borel measure on Ω and $\nu = \underline{\nu}_{|A}$.

The following result which shows the existence of a minimal solution of (1.1) with a given initial trace in $\mathfrak{M}_+(A)$ for any open subset A in $\mathbb{R}^N$ is a straightforward adaptation of [5, Lemma 3.3].

Proposition 4.9. *Assume f is nondecreasing and satisfies* (1.8), (1.12), *and* (C2).

(i) Let A be an open subset of $\mathbb{R}^N$ and let $\nu \in \mathfrak{M}_+(A)$ with associated extension $\underline{\nu}$. Then there exists a positive solution of (1.1) *in Q_∞ denoted by $\underline{u}_\nu$ satisfying $Tr_{\mathbb{R}^N}(\underline{u}_\nu) = \underline{\nu}$ and such that $\underline{u}_\nu \leq v$ for every positive solution v of* (1.1) *in Q_∞ such that $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$ and $\chi_A \mu \geq \nu$.*

(ii) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary and u_n be the solution of the problem

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) & = 0 \qquad \text{in } Q_T^\Omega \\ u_n & = n \qquad \text{on } \partial\Omega \times (0,\infty) \\ u_n(\cdot,0) & = n \qquad \text{in } \Omega. \end{cases} \tag{4.27}$$

Denote $U_{\infty,\Omega} := \lim_{n\to\infty} u_n$. Then $U_{\infty,\Omega}$ is the maximal solution of (1.1) *in Q_∞^Ω in the sense that the following relation holds in Q_T^Ω for every positive solution v of* (1.1):

$$U_{\infty,\Omega} \geq v. \tag{4.28}$$

Taking $A = \mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$, we obtain the existence of a minimal positive solution of (1.1) with a given positive Radon measure $\mu \in \mathfrak{M}_+(\mathcal{R})$ as the regular part of the initial trace.

Corollary 4.10. *Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^N$, $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, and $\mu \in \mathfrak{M}_+(\mathcal{R})$. Then there exists a positive solution $\underline{u}_\mu$ of* (1.1) *such that $Tr_{\mathbb{R}^N}(\underline{u}_\mu) = \underline{\mu}$ and $\underline{u}_\mu \leq v$ for every positive solution v of* (1.1) *in Q_∞ such that $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$.*

As a counterpart of Theorem 1.11 we have the following existence theorem.

Proof of Theorem 1.12.

Step 1: Construction of the minimal solution. Let $\underline{u}_{\mathcal{S}}$ and $\underline{u}_\mu$ be the minimal solution constructed in Corollary 4.6 and Corollary 4.10. Then $\underline{\check{u}}_{\mathcal{S},\mu} := \sup\{\underline{u}_{\mathcal{S}}, \underline{u}_\mu\}$ is a subsolution of (1.1) in Q_∞ while $\underline{\hat{u}}_{\mathcal{S},\mu} := \underline{u}_{\mathcal{S}} + \underline{u}_\mu$ is a supersolution. Furthermore, $\underline{\check{u}}_{\mathcal{S},\mu} \leq \underline{\hat{u}}_{\mathcal{S},\mu}$. Therefore the set of solutions u in Q_∞ such that $\underline{\tilde{u}}_{\mathcal{S},\mu} \leq u \leq \underline{\hat{u}}_{\mathcal{S},\mu}$ is not empty, and we denote by $\underline{u}_{\mathcal{S},\mu}$ the smallest solution larger than $\underline{\check{u}}_{\mathcal{S},\mu}$; it is a solution with initial trace $(\mathcal{S},\mu)$. If u is any other positive solution with the same initial trace, it is larger than $\underline{u}_{\mathcal{S}}$ and $\underline{u}_\mu$ by Corollary 4.6 and Corollary 4.10. Therefore it is larger than $\underline{\check{u}}_{\mathcal{S},\mu}$ and consequently larger than $\underline{u}_{\mathcal{S},\mu}$.

Step 2: Construction of the maximal solution. The proof is somewhat similar to the one on [5, Theorem 3-4], but we give it for the sake of completeness. We denote, for $\delta > 0$,

$$\mathcal{S}^\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \mathcal{S}) \leq \delta\} \text{ and } \mathcal{R}^\delta := \mathbb{R}^N \setminus \mathcal{S}^\delta$$

and let μ_δ be the measure given by

$$\mu_\delta(E) = \mu(\mathcal{R}_\delta \cap E) \quad \forall E \subset \mathbb{R}^N,\ E \text{ Borel}.$$

We denote by $u_{\mathcal{S}^\delta}$ a solution of (1.1) in Q_∞ with initial trace $(\mathcal{S}^\delta, 0)$: a solution is easily constructed as the limit when $R, k \to \infty$ of the solution $v = v_{k,R}$ of

$$\left\{\begin{array}{ll} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(.,0) = k\chi_{(\overline{B}_R \cap \mathcal{S}^\delta)\cup(\overline{B}_R \cap \overline{B}^c_{R-\delta})}. & \end{array}\right. \tag{4.29}$$

By Proposition 4.7, there holds, for any $0 < \delta' < \delta$ and $\epsilon > 0$,

$$\lim_{t\to 0} \frac{u_{\mathcal{S}^\delta}(x,t)}{\phi_\infty(t)} = 1 \quad \text{uniformly on } \mathcal{S}_{\delta'}. \tag{4.30}$$

Let u_{μ_δ} be the solution of (1.1) in Q_∞ with initial trace $(\emptyset, \mu_\delta)$. This solution is constructed by approximation, as the limit, when $R \to \infty$, of the solution $u = u_{\chi_{B_R}\mu_\delta}$ of

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty \\ u(\cdot,0) = \chi_{B_R}\mu_\delta & \text{in } \mathbb{R}^N. \end{array}\right. \tag{4.31}$$

For $\tau > 0$, let $u_{\delta,\tau}$ be the solution of (1.1) in Q_∞ with initial data $m_{\delta,\tau}$ defined by

$$m_{\delta,\tau}(x) = \left\{\begin{array}{ll} \phi_\infty(\tau) & \text{if } x \in \mathcal{S}_\delta \\ u_{\mu_\delta}(x,\tau) & \text{if } x \in \mathcal{R}_\delta. \end{array}\right.$$

Then $u(\cdot,\tau) \leq m_{\delta,\tau}$ in $\mathcal{S}_\delta$ and $u(\cdot,\tau) \geq m_{\delta,\tau}$ in $\mathcal{R}_\delta$ by Proposition 4.9. Therefore,

$$\lim_{\tau\to 0}(u(\cdot,\tau) - m_{\delta,\tau}(\cdot))_+ = 0$$

in the weak sense of measures. Furthermore, this solution does not depend on u, but only on $\mathcal{S}_\delta$ and μ_δ. The set of functions $\{u_{\delta,\tau}\}_{\tau>0}$ is locally uniformly bounded in Q_∞. By the regularity theory for parabolic equations, there exists a subsequence $\{\tau_k\}$ and a positive solution u^*_δ of (1.1) in Q_∞ such that $u_{\delta,\tau_k} \to u^*_\delta$ locally uniformly in Q_∞. By Proposition 4.8 and Proposition 4.9, $tr_{\mathbb{R}^N}(u^*_\delta) = (\mathcal{S}^\delta, \mu_\delta)$. Let $\omega_{\delta,\tau}$ be the solution of (1.1) in Q_∞ with initial data $(u(\cdot,\tau) - m_{\delta,\tau}(\cdot))_+$ (it is constructed in the same way as $\underline{u}_\mu$ in Proposition 4.9-(i)). By Theorem 1.8-(ii), $\lim_{\tau\to 0}\omega_{\delta,\tau} = 0$, locally uniformly. Since $u \leq u_{\delta,\tau} + \omega_{\delta,\tau}$ in $(\tau,\infty) \times \mathbb{R}^N$, we obtain $u \leq u^*_\delta$. If

$0 < \delta' < \delta$, we can compare similarly $u_{\delta,\tau}$ with the solution $u_{\delta',\tau}$ of (1.1) with initial data

$$m_{\delta',\tau}(x) = \begin{cases} \phi_\infty(\tau) & \text{if } x \in \mathcal{S}'_\delta \\ u_{\mu_{\delta'}}(x,\tau) & \text{if } x \in \mathcal{R}'_\delta. \end{cases}$$

If $u^*_{\delta'}$ is the limit of any sequence $\{u_{\delta',\tau_{k'}}\}$, it satisfies $0 < u^*_{\delta'} \leq u^*_\delta$ and has initial trace $(\mathcal{S}^{\delta'}, \mu_{\delta'})$. If we take in particular $\delta = \delta_n = 2^{-n}$, we construct a decreasing sequence of positive solutions $\{u^*_{2^{-n}}\}$ of (1.1) in Q_∞, with $tr_{\mathbb{R}^N}(u^*_{2^{-n}}) = (\mathcal{S}^{2^{-n}}, \mu_{2^{-n}})$, satisfying

$$u \leq u^*_{2^{-n}} \qquad \text{in } Q_\infty.$$

Clearly the limit $\overline{u}_{\mathcal{S},\mu}$ of the sequence $\{u^*_{2^{-n}}\}$ is a positive solution of (1.1) in Q_∞ with initial trace $(\mathcal{S}, \mu)$ and is independent of u. It is the maximal solution of the equation with this initial trace. □

Remark. When $f(r) = |r|^{q-1}r$ with $1 < q < 1 + 2/N$, precise expansion of $u_{\infty\delta}(x,t)$ when $t \to 0$ allows one to prove uniqueness. Even when $f(r) = r\ln^\alpha(r+1)$ with $\alpha > 2$, uniqueness is not known. The first step would be to prove that uniqueness holds if $tr_\Omega(u) = (\{a\}, 0)$ for some $a \in \Omega$. However, if $\mathcal{S} = \emptyset$, uniqueness holds from Theorem 1.8-(ii).

4.3. **The Keller-Osserman condition does not hold.** In this section we assume that (1.12) does not hold but (1.8) is satisfied.

Lemma 4.11. *Assume* (1.10) *and* (1.16) *are satisfied and* $\lim_{k\to\infty} u_k = \phi_\infty$. *If u is a positive solution of* (1.1) *in* Q_∞ *which satisfies*

$$\limsup_{t\to 0} \int_G u(x,t)dx = \infty, \tag{4.32}$$

for some bounded open subset $G \subset \mathbb{R}^N$, *then* $u(x,t) \geq \phi_\infty(t)$. *This holds in particular if* (C1) *and* (C3) *are satisfied.*

Proof. By assumption, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n\to\infty} \int_G u(x,t_n)dx = \infty. \tag{4.33}$$

If (4.32) holds, we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\operatorname{diam}(G_k) = \epsilon_k \to 0$ when $k \to \infty$, and

$$\lim_{n\to\infty} \int_{G_k} u(x,t_n)dx = \infty \qquad \forall k \in \mathbb{N}. \tag{4.34}$$

Furthermore, there exists a unique $a \in \cap_k G_k$. We set

$$\int_{G_k} u(x,t_n)dx = M_{n,k}.$$

Since $\lim_{n\to\infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any k, there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x,t_{n(k)})dx \geq m. \tag{4.35}$$

By induction, we define $n(1)$ as the smallest integer n such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer n larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any k, there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x,t_{n(k)});\ell\}dx = m, \tag{4.36}$$

and we set

$$V_k(x) = \inf\{u(x,t_{n(k)});\ell\}\chi_{G_k}(x).$$

Let $v_k = v$ be the unique bounded solution of

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(\cdot,0) = V_k & \text{in } \mathbb{R}^N. \end{cases} \tag{4.37}$$

Since $v(x,0) \leq u(x,t_{n(k)})$, we derive

$$u(x,t+t_{n(k)}) \geq v_k(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{4.38}$$

When $k \to \infty$, $V_k \to m\delta_a$; thus, $v_k \to u_{m\delta_a}$ by Lemma 4.4. Therefore $u \geq u_{m\delta_a}$. Since m is arbitrary and $u_{m\delta_a} \to \phi_\infty$ when $m \to \infty$ by Theorem 1.3, it follows that $u \geq \phi_\infty$. □

Lemma 4.12. *Assume* (1.15) *and let* $\lim_{k\to\infty} u_k = \infty$ *hold. There exists no positive solution* u *of* (1.1) *in* Q_∞ *which satisfies* (4.32) *for some bounded open subset* $G \subset \mathbb{R}^N$*. This holds in particular if* (C1) *and* (C3) *are satisfied.*

Proof. If we assume that such a u exists, we proceed as in the proof of the previous lemma. Since Lemma 4.4 holds, we derive that $u \geq u_{m\delta_a}$ for any m. Since $\lim_{m\to\infty} u_{m\delta_a}(x,t) = \infty$ for all $(x,t) \in Q_\infty$, we are led to a contradiction. □

Thanks to these results, we can characterize the initial trace of positive solutions of (1.1) when the Keller-Osserman condition does not hold.

Proof of Theorem 1.13. If there exists some open subset G of $\mathbb{R}^N$ with the property (4.32), then $u \geq \phi_\infty$ and the initial trace of u is the Borel measure ν_∞. Next we assume that for any bounded open subset G of $\mathbb{R}^N$ there holds

$$\limsup_{t\to 0} \int_G u(x,t)dx < \infty. \tag{4.39}$$

If $\mathcal{S}(u) \neq \emptyset$, there exist $z \in \mathbb{R}^N$ and a bounded open neighborhood G of z such that

$$\int_0^T \int_G f(u)\, dx\, dxt = \infty.$$

By (4.39), $u \in L^\infty(0,T;L^1(G)) \subset L^1(Q_T^G)$. Then, by Lemma 4.2, (4.4) holds, which contradicts (4.39). Thus $\mathcal{S}(u) = \emptyset$ and $\mathcal{R}(u) = \mathbb{R}^N$. It follows from Proposition 1.9 that there exists a positive Radon measure μ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(\mathbb{R}^N). \tag{4.40}$$

□

Because of the lack of uniqueness from Theorem 1.6 it is difficult to give a complete characterization of admissible initial data for solutions of (1.1) under the assumptions of Theorem 1.13. However, we have the result as in Proposition 1.14.

Proof of Proposition 1.14. We first notice that $\max\{\phi_\infty(t); w_b(|x|)\}$ is a subsolution of (1.1) which is dominated by the supersolution $\phi_\infty(t)+w_b(|x|)$. The construction is standard: for $\tau > 0$ we set

$$\psi(x,\tau) = \frac{1}{2}\left(\max\{\phi_\infty(t); w_b(|x|)\} + \phi_\infty(t) + w_b(|x|)\right).$$

There exists a function $u = u_\tau \in C(\overline{Q_\infty})$ a solution of (1.1) in Q_∞ satisfying $u_\tau(\cdot,0) = \psi(\cdot,\tau)$. Furthermore,

$$\max\{\phi_\infty(t+\tau); w_b(|x|)\} \leq u_\tau(x,t) \leq \phi_\infty(t+\tau) + w_b(|x|) \quad \forall (x,t) \in Q_\infty. \tag{4.41}$$

By the parabolic equation regularity theory, the set $\{u_\tau\}_{\tau>0}$ is locally equicontinuous in Q_∞. Thus there exist a subsequence $\{\tau_n\}$ and $u \in C(Q_\infty)$ such that $u_{\tau_n} \to u$ on any compact subset of Q_∞. Clearly u is a weak, thus a strong solution of (1.1) and it satisfies (1.28). Since any solution u with initial trace ν_∞ dominates ϕ_∞ by Lemma 4.11, it follows that ϕ_∞ is the minimal one. □

Proof of Theorem 1.15. As in the proof of Theorem 1.13 and because of Lemma 4.12, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$, and the proof follows from Proposition 1.9. □

Remark. Under the assumptions of Theorem 1.13, it is clear, from the proof of Proposition 3.1, that for any $0 < a < b$ and any initial data $u_0 \in C(\mathbb{R}^N)$ satisfying

$$w_a(x) \leq u_0(x) \leq w_b(x) \qquad \forall x \in \mathbb{R}^N$$

there exists a solution $u \in C(\overline{Q_\infty})$ of (1.1) in Q_∞ satisfying $u(\cdot, 0) = u_0$ and

$$w_a(x) \leq u(x,t) \leq w_b(x) \qquad \forall (x,t) \in Q_\infty.$$

We conjecture that for any positive measure μ on $\mathbb{R}^N$ which satisfies, for some $b > 0$,

$$\int_{B_R} d\mu(x) \leq \int_{B_R} w_b(x)dx \qquad \forall R > 0, \tag{4.42}$$

there exists a positive solution u of (1.1) in Q_∞ with initial trace μ. Another interesting open problem is to see if there exist local solutions in Q_T with an initial trace μ satisfying

$$\lim_{R\to\infty} \frac{\int_{B_R} d\mu(x)}{\int_{B_R} w_b(x)dx} = \infty \qquad \forall b > 0. \tag{4.43}$$

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