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WELL-POSEDNESS FOR THE FOURTH-ORDER SCHRÖDINGER EQUATIONS WITH QUADRATIC NONLINEARITY

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Abstract. This paper is concerned with one-dimensional quadratic semilinear fourth-order Schrödinger equations. Motivated by the quadratic Schrödinger equations in the pioneering work of Kenig-Ponce-Vega [12], three bilinearities uv, $\overline{uv}$, and $\overline{u}v$ for functions $u, v : \mathbb{R} \times [0,T] \mapsto \mathbb{C}$ are sharply estimated in function spaces $X_{s,b}$ associated to the fourth-order Schrödinger operator $i\partial_t + \Delta^2 - \varepsilon\Delta$. These bilinear estimates imply local wellposedness results for fourth-order Schrödinger equations with quadratic nonlinearity. To establish these bilinear estimates, we derive a fundamental estimate on dyadic blocks for the fourth-order Schrödinger from the $[k, Z]$-multiplier norm argument of Tao [20].

1. Introduction

This paper is mainly devoted to the local well-posedness of the initial value problems (IVP) for the fourth-order Schrödinger equation ($i = 1, 2, 3$)

$$\begin{cases} iu_t + \Delta^2 u - \varepsilon\Delta u \pm Q_i(u, \bar{u}) = 0, \ (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = u_0 \in H^s(\mathbb{R}), \end{cases} \tag{1.1}$$

where $Q_1(u, \bar{u}) = \bar{u}^2$, $Q_2(u, \bar{u}) = u^2$, $Q_3(u, \bar{u}) = u\bar{u}$, and $\varepsilon \in \{-1, 0, 1\}$.

The fourth-order Schrödinger equations have been introduced by Karpman [8] and Karpman and Shagalov [9] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations have been studied from the mathematical viewpoint in Fibich, Ilan, and

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Papanicolaou [7], who describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. A related reference is [2] by Ben-Artzi, Koch, and Saut, which gives sharp dispersive estimates for the biharmonic Schrödinger operator, which lead to the Strichartz estimates for the fourth-order Schrödinger equation; see also [16, 18, 19]. Concerning the Strichartz estimate for the high-order dispersive equation, we should point out that Kenig-Ponce-Vega [10] first established the sharp estimates involving gain of derivatives. We also can refer to Pausader [19] for the aim of finding a more completed result on the cubic fourth-order Schrödinger equation for initial data $u_0 \in H^2$ without radial assumption. In [21], we utilized the interaction Morawetz estimate to prove the scattering theory for the energy critical fourth-order Schrödinger equation with a subcritical perturbation. However, the scaling argument implies the nonlinear fourth-order Schrödinger equations mentioned in the above papers are H^s-critical with $s \geq 0$. We here are interested in the nonlinearities for which the Sobolev index s can take negative values, i.e., $s < 0$.

The main motivation of this paper is the pioneering work of Kenig-Ponce-Vega [12], in which they considered the Schrödinger equations with quadratic nonlinearities,

$$iu_t + \Delta u \pm N_i(u, \bar{u}) = 0, \qquad u(0) = u_0,$$

where $N_1(u, \bar{u}) = \bar{u}^2$, $N_2(u, \bar{u}) = u\bar{u}$, and $N_3(u, \bar{u}) = u^2$.

In one dimension, Kenig-Ponce-Vega [12] first proved for the nonlinearities $\bar{u}^2$ and u^2 well-posedness for $s > -\frac{3}{4}$, while for the nonlinearity $u\bar{u}$ well-posedness for $s > -\frac{1}{4}$. For the nonlinearity u^2, I. Bejenaru and T. Tao [3] showed the Cauchy problem was locally well-posed in $H^s(\mathbb{R})$ when $s \geq -1$ and ill-posed when $s < -1$. While for the nonlinearity $\bar{u}^2$, N. Kishimoto [14] was in same spirt as [3] in establishing that the IVP is locally well-posed in $H^s(\mathbb{R})$ when $s \geq -1$ and ill-posed when $s < -1$. For the dimension two case, we can refer to [6, 3, 4] for details.

The IVP for other equations, such as the generalized Korteweg-de Vries and the wave equations, can be seen in [11, 13, 15]. We remark that T. Tao systematically provided a multilinear estimate method in [20] to deal with these problems. It is interesting to compare these results with those known for other evolution models such as fourth-order Schrödinger equations. In this paper, we restrict ourselves to the one-dimensional case fourth-order Schrödinger equations with quadratic nonlinearities and study the well-posedness of these fourth-order Schrödinger equations. To this end, we first derive a fundamental estimate on dyadic blocks (see below) for the fourth-order Schrödinger equations by following the idea in the $[k, Z]$-multiplier norm method introduced by Tao [20]. We then apply this fundamental estimate to establish bilinear estimates in Bourgain spaces $X_{s,b}$ (see e.g. [1, 5, 15]), which will also be defined in the next part of this section. As applications of these estimates, we establish well-posedness of the IVP for the nonlinearities Q_1, Q_2, and Q_3 respectively.

With the definition of the Bourgain spaces $X_{s,b}$ in (2.1) (see below), we state our results in the following theorems:

Theorem 1.1. *Let $b = \frac{1}{2} + \eta$; we have*

$$\|Q_1(u, \bar{u})\|_{X_{s,b-1}} \lesssim \|u\|^2_{X_{s,b}}, \tag{1.2}$$

whenever $\eta > 0$ and $0 \geq s > -\frac{7}{4} + \frac{7}{2}\eta$, with the implicit constant depending on s and η.

Theorem 1.2. *Let $b = \frac{1}{2} + \eta$; we have*

$$\|Q_2(u, \bar{u})\|_{X_{s,b-1}} \lesssim \|u\|^2_{X_{s,b}}, \tag{1.3}$$

whenever $\eta > 0$ and $0 \geq s > -\frac{7}{4} + \frac{7}{2}\eta$, with the implicit constant depending on s and η.

Theorem 1.3. *Let $b = \frac{1}{2} + \eta$; we have*

$$\|Q_3(u, \bar{u})\|_{X_{s,b-1}} \lesssim \|u\|^2_{X_{s,b}}, \tag{1.4}$$

whenever $\eta > 0$ and $0 \geq s > -\frac{3}{4} + \eta$, with the implicit constant depending on s and η.

In the spirt of Kenig-Ponce-Vega [12, 13] and K. Nakanishi, H. Takaoka, and Y. Tsutsumi [17], we get the sharpness of Theorem 1.1–1.3 in the following theorem.

Theorem 1.4.
a) For any $s \leq -\frac{7}{4}$ and any $b \in \mathbb{R}$ the estimate (1.2)*and* (1.3) *fails.*
b) For any $s < -\frac{3}{4}$ and any $b \in \mathbb{R}$, or $s = -\frac{3}{4}$ and any $b \geq \frac{1}{2}$, the estimate (1.4) *fails.*

As a consequence of the first three theorems, we can make use of the technique used to prove Theorem 1.5 in [12] and also used in [13] to get the following results concerning the local wellposedness of the initial-value problems (1.1).

Theorem 1.5. *Let $s \in (-\frac{7}{4}, 0]$. Then for any $u_0 \in H^s(\mathbb{R})$, there exists $T = T(\|u_0\|_{H^s})$ and a unique solution $u(t)$ of the IVP (1.1), with the nonlinear term $Q = Q_1$, satisfying $u \in C([-T,T]; H^s(\mathbb{R}))$ and $u \in X_{s,\frac{1}{2}+}$. In addition, the dependence of u on u_0 is Lipschitz.*

Theorem 1.6. *For the IVP (1.1) with nonlinear term $Q = Q_2$, the results in Theorem 1.5 hold for $s \in (-\frac{7}{4}, 0]$.*

Theorem 1.7. *For the IVP (1.1) with nonlinear term $Q = Q_3$, the results in Theorem 1.5 hold for $s \in (-\frac{3}{4}, 0]$.*

The proofs of the above theorems will be given in the subsequent subsection, provided the bilinear estimates in Theorem 1.1, Theorem 1.2, and Theorem 1.3 hold.

The paper is organized as follows. In Section 2, we introduce the linear estimates and Tao's $[k; Z]$-multiplier norm method and prove a fundamental estimate on dyadic blocks for the fourth-order Schrödinger equation. In Section 3, we prove the Theorems 1.1–1.4. In the appendix, Section 4, we present a detailed argument of the reductions, which are used in proving the fundamental estimate on dyadic blocks.

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2. Fundamental estimates

Let us start this section by providing the linear estimates for the fourth-order Schrödinger equation. Denote by $W(t)$ the unitary group generating the solution of the IVP for the linear equation

$$\begin{cases} iv_t + \Delta^2 v - \varepsilon \Delta v = 0, \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ v(x,0) = v_0(x); \end{cases}$$

then $v(x,t) = W(t)v_0(x) = S_t * v_0(x)$, where $\widehat{S_t} = e^{itq(\xi)}$ with $q(\xi) = \xi^4 + \varepsilon \xi^2$, or

$$S_t(x) = \int e^{i(x\xi + tq(\xi))} d\xi.$$

For $s, b \in \mathbb{R}$, let $X_{s,b}$ denote the completion of the functions in C_0^∞ with respect to the norm

$$\|f\|_{X_{s,b}} = \|\langle \xi \rangle^s \langle \tau - q(\xi) \rangle^b \widehat{f}(\xi, \tau)\|_{L^2_{\xi,\tau}}, \tag{2.1}$$

where $\langle \xi \rangle = 1 + |\xi|$.

Let $\phi \in C_0^\infty$ be a standard bump function and consider the following integral equation:

$$u(t) = \phi(\frac{t}{\delta})W(t)u_0 - \phi(\frac{t}{\delta})\int_0^t W(t-s)Q_i(u,\overline{u})(s)ds.$$

Denote the right-hand side by $\mathcal{T}(u)$. The goal is to show that $\mathcal{T}(u)$ is a contraction on the following complete metric space B, where

$$B = \{u \in X_{s,b} : \|u\|_{X_{s,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|u\|_{H^s}\}$$

with metric

$$d(u,v) = \|u-v\|_{X_{s,b}},\ u,v \in B.$$

Here c is the constant appearing in the following Proposition 2.1. For this purpose, we need two linear estimates.

Proposition 2.1. *For $s \in \mathbb{R}$ and $b \in (\frac{1}{2}, 1]$,*

$$\|\phi(\frac{t}{\delta})W(t)u_0\|_{X_{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|f\|_{H^s},$$

$$\|\phi(\frac{t}{\delta})\int_0^t W(t-s)f(s)ds\|_{X_{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|f\|_{X_{s,b-1}}.$$

The proof of these estimates follows directly from Kenig, Ponce, and Vega [11].

Once the estimates in Theorems 1.1–1.3 are available, a standard argument then yields that $\mathcal{T}(u)$ is a self-contained and contracted operator on B.

Let us introduce now Tao's $[k,Z]$-multiplier norm method and establish the fundamental estimate on dyadic blocks for the fourth-order Schrödinger equation.

Let Z be any abelian additive group with an invariant measure $d\xi$. In this paper, Z is the Euclidean space $\mathbb{R}\times\mathbb{R}$ with Lebesgue measure. For any integer $k \geq 2$, we let $\Gamma_k(Z)$ denote the "hyperplane"

$$\Gamma_k(Z) := \{(\xi_1,\ldots,\xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0\}$$

endowed with the measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1,\ldots,\xi_{k-1},-\xi_1-\cdots-\xi_{k-1})\ d\xi_1\cdots d\xi_{k-1}.$$

A $[k,Z]$-multiplier is defined to be any function $m : \Gamma_k(Z) \to \mathbb{C}$, which was introduced by Tao in [20]. And the multiplier norm $\|m\|_{[k,Z]}$ is defined to be the best constant such that the inequality

$$\Big|\int_{\Gamma_k(Z)} m(\xi)\prod_{j=1}^{k} f_j(\xi_j)\Big| \le c\prod_{j=1}^{k}\|f_j\|_{L^2(Z)} \tag{2.2}$$

holds for all test functions f_j on Z. To establish the fundamental estimate on dyadic blocks for the fourth-order Schrödinger equations, we introduce some notation.

We use $A \lesssim B$ to denote the statement that $A \le CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly use $A \ll B$ to denote the statement $A \le C^{-1}B$. We use $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. We will sometimes write $a+$ to denote $a+\eta$ for arbitrarily small $\eta > 0$.

Any summations over capitalized variables such as N_j, L_j, and H are presumed to be dyadic; i.e., these variables range over numbers of the form 2^k for $k \in \mathbb{Z}$. Let $N_1, N_2, N_3 > 0$. It will be convenient to define the quantities $N_{max} \ge N_{med} \ge N_{min}$ to be the maximum, median, and minimum of N_1, N_2, and N_3 respectively. Similarly define $L_{max} \ge L_{med} \ge L_{min}$ whenever $L_1, L_2, L_3 > 0$. We also adopt the following summation conventions. Any summation of the form $L_{max} \sim \cdots$ is a sum over the three dyadic variables $L_1, L_2, L_3 \ge 1$; thus, for instance,

$$\sum_{L_{max}\sim H} := \sum_{L_1,L_2,L_3\ge 1: L_{max}\sim H}.$$

Similarly, any summation of the form $N_{max} \sim \cdots$ is a sum over the three dyadic variables $N_1, N_2, N_3 > 0$; thus, for instance,

$$\sum_{N_{max}\sim N_{med}\sim N} := \sum_{N_1,N_2,N_3>0: N_{max}\sim N_{med}\sim N}.$$

If τ, ξ, and $q(\xi)$ are given, we define $\lambda := \tau - h(\xi)$, and similarly,

$$\lambda_j := \tau_j - h_j(\xi_j),\ j = 1,2,3,$$

where $h_j(\xi_j) = \pm q(\xi_j)$ $(j = 1,2,3)$. In addition, the quantity λ_j measures how close in frequency the j^{th} factor is to a free solution.

In this paper, we do not go further on the general framework of Tao's weighted convolution estimates. We focus our attention on the $[3;\mathbb{R}\times\mathbb{R}]$-multiplier norm estimate for the fourth-order Schrödinger equations. In the course of the estimate we need the resonance function

$$h(\xi) = h_1(\xi_1) + h_2(\xi_2) + h_3(\xi_3) = -\lambda_1 - \lambda_2 - \lambda_3, \tag{2.3}$$

where $h_j(\xi_j) = \pm q(\xi_j)$ $(j = 1, 2, 3)$. In addition, $h(\xi)$ measures to what extent the spatial frequencies ξ_1, ξ_2, and ξ_3 can resonate with each other.

Up to symmetry, there are only two possibilities for the h_j: the $(+++)$ case,

$$h_1(\xi) = h_2(\xi) = h_3(\xi) = q(\xi) = \xi^4 + \varepsilon\xi^2, \tag{2.4}$$

and the $(++-)$ case,

$$h_1(\xi) = h_2(\xi) = q(\xi), h_3(\xi) = -q(\xi). \tag{2.5}$$

The $(+++)$ case corresponds to estimates of the form $Q_1(u, \bar{u}) = \bar{u}^2$,

$$\|\overline{u_1 u_2}\|_{X_{s,b}} \lesssim \|u_1\|_{X_{s_1,b_1}} \|u_2\|_{X_{s_2,b_2}},$$

while the $(++-)$ case and its permutations are similar but treat $Q_2(u, \bar{u}) = u^2$ and $Q_3(u, \bar{u}) = u\bar{u}$ instead of $\bar{u}^2$.

By dyadic decomposition of the variables ξ_j and λ_j, as well as the resonance function $h(\xi)$, we may assume that $|\xi_j| \sim N_j$, $|\lambda_j| \sim L_j$, and $|h(\xi)| \sim H$. By the translation invariance of the $[k; Z]$-multiplier norm (see Lemma 3.4 (8) in Tao [20]), we can always restrict our estimate to

$$\lambda_j \geq 2 \tag{2.6}$$

and

$$\max(N_1, N_2, N_3) \geq 2. \tag{2.7}$$

We will provide the detailed proof of this reduction in the Appendix. One is led to consider

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]}, \tag{2.8}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3} := \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j|\sim N_j} \chi_{|\lambda_j|\sim L_j}. \tag{2.9}$$

From the identities $\xi_1 + \xi_2 + \xi_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + h = 0$, on the support of the multiplier, we see that $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{max} \sim N_{med}, \tag{2.10}$$

and

$$L_{max} \sim \max(H, L_{med}). \tag{2.11}$$

Thus we may implicitly assume (2.10) and (2.11) in the summations.

We now consider the $(+++)$ case (2.4). Because the resonance function

$$h(\xi) = \xi_1^4 + \varepsilon\xi_1^2 + \xi_2^4 + \varepsilon\xi_2^2 + \xi_3^4 + \varepsilon\xi_3^2,$$

and $\max(N_1, N_2, N_3) \geq 2$, we see that we may assume that

$$H \sim N_{max}^4, \tag{2.12}$$

since the multiplier in (2.8) vanishes otherwise.

Proposition 2.2. *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ obey (2.6), (2.7), (2.10), (2.11), and (2.12), and let the dispersion relations be given by (2.4). Then we have*

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} \min\{N_{max}N_{min}, \frac{L_{med}}{N_{max}^2}\}^{\frac{1}{2}}, \tag{2.13}$$

except in the exceptional case $N_{max} \sim N_{min}$, $L_{max} \sim H$, in which case

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} L_{med}^{\frac{1}{4}}. \tag{2.14}$$

Proof. We will prove this proposition by using the tools developed in [20].

In the high modulation case $L_{max} \sim L_{med} \gg H$, we have by an elementary estimate employed by Tao (see (37), p. 861 in [20])

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} (\frac{L_{med}}{N_{max}^2})^{\frac{1}{2}}.$$

For the low modulation case, $L_{max} \sim H$, by symmetry we may assume that $L_1 \geq L_2 \geq L_3$; hence, $L_1 \sim N_{max}^4$. By Corollary 4.2 in Tao's paper [20], we have

$$\begin{aligned}(2.8) \lesssim L_3^{\frac{1}{2}} |\{\xi_2 : |\xi_2 - \xi_2^0| \ll N_{min};\ |\xi - \xi_2 - \xi_3^0| \ll N_{min};\\ \xi_2^4 + \varepsilon\xi_2^2 + (\xi - \xi_2)^4 + \varepsilon(\xi - \xi_2)^2 = \tau + O(L_2)\}|^{\frac{1}{2}},\end{aligned} \tag{2.15}$$

for some $\tau \in \mathbb{R}$ and ξ, ξ_1^0, ξ_2^0, and ξ_3^0 satisfying

$$|\xi_j^0| \sim N_j, (j = 1, 2, 3);\ |\xi - \xi_1^0| \ll N_{min};\ |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min}.$$

The identity

$$\xi_2^4 + \varepsilon\xi_2^2 + (\xi - \xi_2)^4 + \varepsilon(\xi - \xi_2)^2 = \frac{1}{2}(\xi_2 - \frac{\xi}{2})^2\big[(\xi - 2\xi_2)^2 + 6\xi^2 + 4\varepsilon\big] + \frac{\xi^4}{8} + \frac{\varepsilon\xi^2}{2} \tag{2.16}$$

together with (2.15) implies it suffices to show that

$$\begin{aligned}&\big|\{\xi_2 : |\xi_2 - \xi_2^0| \ll N_{min};\ (\xi_2 - \frac{\xi}{2})^2\big[(\xi - 2\xi_2)^2 + 6\xi^2 + 4\varepsilon\big]\\ &= 2\tau - \frac{\xi^4}{4} - \varepsilon\xi^2 + O(L_2)\}\big| \lesssim N_{max}^{-1} \min\{N_{min}N_{max};\ \frac{L_2}{N_{max}^2}\}\end{aligned} \tag{2.17}$$

with the right-hand side replaced by $N_{max}^{-1} L_2^{\frac{1}{2}}$ in the exceptional case $N_{max} \sim N_{min}$.

We need to consider three cases: $N_1 \sim N_2 \sim N_3$, $N_1 \sim N_2 \gg N_3$, and $N_1 \sim N_3 \gg N_2$. (The case $N_2 \sim N_3 \gg N_1$ then follows by symmetry.)

If $N_1 \sim N_2 \sim N_3$, we see from (2.16) that the variable ξ_2 is contained in the union of two intervals of length $O(N_1^{-1} L_2^{\frac{1}{2}})$ at worst, and (2.14) follows.

If $N_1 \sim N_2 \gg N_3$, we must have $|\xi_2 - \frac{\xi}{2}| \sim N_1$, so (2.16) shows that ξ_2 is contained in the union of two intervals of length $O(N_1^{-3} L_2)$. But ξ_2 is also contained in an interval of length $\ll N_3$. So (2.13) follows.

If $N_1 \sim N_3 \gg N_2$, then we must have $|\xi_2 - \frac{\xi}{2}| \sim N_1$, so (2.16) shows that ξ_2 is contained in the union of two intervals of length $O(N_1^{-3} L_2)$. But ξ_2 is also contained in an interval of length $\ll N_2$. The claim (2.13) follows. $\square$

We now consider the $(+ + -)$ case (2.5). In this case, the resonance function

$$\begin{aligned} h(\xi) :&= (\xi_1^4 + \varepsilon \xi_1^2) + (\xi_2^4 + \varepsilon \xi_2^2) - (\xi_3^4 + \varepsilon \xi_3^2) \\ &= -4\xi_1 \xi_2 (\xi_1^2 + \frac{3}{2}\xi_1 \xi_2 + \xi_2^2 + \frac{1}{2}\varepsilon), \end{aligned}$$

so we may assume that

$$H \sim N_1 N_2 \max\{N_1, N_2\}^2, \tag{2.18}$$

since the multiplier in (2.8) vanishes otherwise.

Proposition 2.3. *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ obey* (2.6), (2.7), (2.10), (2.11), *and* (2.12), *and let the dispersion relations be given by* (2.5). *Then we have*

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} \min\{N_{max} N_{min}, \frac{L_{med}}{N_{max}^2}\}^{\frac{1}{2}}, \tag{2.19}$$

except in the exceptional cases $N_1 \sim N_{min}$, $L_1 \sim L_{max} \sim H$ or $N_2 \sim N_{min}$, $L_2 \sim L_{max} \sim H$, in which cases

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} \min\{N_{max} N_{min}, \frac{L_{med}}{N_{max} N_{min}}\}^{\frac{1}{2}}. \tag{2.20}$$

Proof. In the high modulation case $L_{max} \sim L_{med} \gg H$, we have by an elementary estimate employed by Tao (see (37), p. 861 in [20])

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} (\frac{L_{med}}{N_{max}^2})^{\frac{1}{2}}.$$

For the low modulation case, $L_{max} \sim H$, by symmetry we only consider three cases: $N_1 \sim N_2 \gg N_3$; $N_2 \sim N_3 \gtrsim N_1$, $L_1 \ll L_{max}$; and $N_2 \sim N_3 \gtrsim N_1$, $L_1 \sim L_{max} \sim H$.

Case 1:((++) high-high interactions) $N_1 \sim N_2 \gg N_3$.

By symmetry, it suffices to consider the two cases $L_1 \geq L_2, L_3$ and $L_3 \geq L_1, L_2$.

Case 1(a): $L_1 \geq L_2, L_3$. By Corollary 4.2 in Tao's paper [20], we have

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} |\{\xi_2 : |\xi_2 - \xi_2^0| \ll N_3;\ |\xi - \xi_2 - \xi_3^0| \ll N_3;\ \xi_2^4 + \varepsilon\xi_2^2 - (\xi - \xi_2)^4 - \varepsilon(\xi - \xi_2)^2 = \tau + O(L_{med})\}|^{\frac{1}{2}} \quad (2.21)$$

for some $\tau \in \mathbb{R}$ and ξ, ξ_1^0, ξ_2^0, and ξ_3^0 satisfying

$$|\xi_j^0| \sim N_j, (j = 1, 2, 3);\ |\xi - \xi_1^0| \ll N_3;\ |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_3.$$

Combining (2.21) with the identity

$$\xi_2^4 + \varepsilon\xi_2^2 - (\xi - \xi_2)^4 - \varepsilon(\xi - \xi_2)^2 = \xi(\xi_2 - \frac{\xi}{2})(\xi^2 + (\xi - 2\xi_2)^2 + 2\varepsilon), \quad (2.22)$$

it suffices to show that

$$\left|\{\xi_2 : |\xi_2 - \xi_2^0| \ll N_3; \xi(\xi_2 - \frac{\xi}{2})(\xi^2 + (\xi - 2\xi_2)^2 + 2\varepsilon) = \tau + O(L_{med})\}\right| \lesssim N_{max}^{-1} \min\{N_3 N_{max}; \frac{L_{med}}{N_{max}^2}\}. \quad (2.23)$$

We see from the left-hand side of (2.23) that the variable ξ_2 is contained in the union of two intervals of length $O(N_{max}^{-3} L_{med})$ at worst. But ξ_2 is also contained in an interval of length $\ll N_3$. The claim (2.23) follows.

Case 1(b): $L_3 \geq L_1, L_2$. By Corollary 4.2 in Tao's paper [20], we have

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} |\{\xi_1 : |\xi_1 - \xi_1^0| \ll N_1;\ \xi_1^4 + \varepsilon\xi_1^2 + (\xi - \xi_1)^4 + \varepsilon(\xi - \xi_1)^2 = \tau + O(L_{med})\}|^{\frac{1}{2}} \quad (2.24)$$

for some $\tau \in \mathbb{R}$ and ξ, ξ_1^0, ξ_2^0, and ξ_3^0 satisfying

$$|\xi_j^0| \sim N_j, (j = 1, 2, 3);\ |\xi - \xi_3^0| \ll N_3;\ |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_3.$$

With the observation that the set of (2.24) is the same as the set of (2.15), then one can follow the argument as before to get (2.13), which is exactly (2.19).

Case 2: (High-low interactions) $N_2 \sim N_3 \gtrsim N_1$, $L_1 \ll L_{max}$. In this case, we either have $L_3 \geq L_1, L_2$ or $L_2 \geq L_1, L_3$.

Case 2(a):((++) case) $L_3 \geq L_1, L_2$. By Corollary 4.2 in Tao's paper [20], we have

$$(2.8) \lesssim L_{min}^{\frac{1}{2}} |\{\xi_1 : |\xi_1 - \xi_1^0| \ll N_1;\ \xi_1^4 + \varepsilon\xi_1^2 + (\xi - \xi_1)^4 + \varepsilon(\xi - \xi_1)^2 = \tau + O(L_{med})\}|^{\frac{1}{2}}$$

for some $\tau \in \mathbb{R}$ and ξ, ξ_1^0, ξ_2^0, and ξ_3^0,

$$|\xi_j^0| \sim N_j, (j=1,2,3);\ |\xi-\xi_3^0| \ll N_1;\ |\xi_1^0+\xi_2^0+\xi_3^0| \ll N_1.$$

Then (2.19) follows from the argument as the case of (2.24).

Case 2(b):((+−) case) $L_2 \geq L_1, L_3$. By Corollary 4.2 in Tao's paper [20], we have

$$\begin{aligned}(2.8) &\lesssim L_{min}^{\frac{1}{2}}|\{\xi_1 : |\xi_1-\xi_1^0| \ll N_1;\ \xi_1^4+\varepsilon\xi_1^2-(\xi-\xi_1)^4-\varepsilon(\xi-\xi_1)^2 \\ &= \tau + O(L_{med})\}|^{\frac{1}{2}}\end{aligned}$$

for some $\tau \in \mathbb{R}$ and ξ, ξ_1^0, ξ_2^0, and ξ_3^0,

$$|\xi_j^0| \sim N_j\ (j=1,2,3);\ |\xi-\xi_2^0| \ll N_1;\ |\xi_1^0+\xi_2^0+\xi_3^0| \ll N_1.$$

Hence we can get (2.19) by following the argument of (2.21).

Case 3:((+−) high-high interactions) $N_2 \sim N_3 \gtrsim N_1,\ L_1 \sim L_{max} \sim H$**.** By Corollary 4.2 in Tao's paper [20], we have

$$\begin{aligned}(2.8) &\lesssim L_{min}^{\frac{1}{2}}|\{\xi_2 : |\xi_2-\xi_2^0| \ll N_{min};\ \xi_2^4+\varepsilon\xi_2^2-(\xi-\xi_2)^4-\varepsilon(\xi-\xi_2)^2 \\ &= \tau + O(L_{med})\}|^{\frac{1}{2}}\end{aligned}$$

for some $\tau \in \mathbb{R}$ and ξ, ξ_1^0, ξ_2^0, and ξ_3^0,

$$|\xi_j^0| \sim N_j\ (j=1,2,3);\ |\xi-\xi_1^0| \ll N_1;\ |\xi_1^0+\xi_2^0+\xi_3^0| \ll N_1.$$

As before, we will handle this case if we can show that

$$\begin{aligned}&\big|\{\xi_2 : |\xi_2-\xi_2^0| \ll N_{min}; \xi(\xi_2-\frac{\xi}{2})(\xi^2+(\xi-2\xi_2)^2+2\varepsilon) = \tau + O(L_{med})\}\big| \\ &\qquad \lesssim N_{max}^{-1}\min\{N_1N_{max}; \frac{L_{med}}{N_{max}N_{min}}\}. \qquad (2.25)\end{aligned}$$

We see from the left-hand side of (2.25) that the variable ξ_2 is contained in the union of two intervals of length $O(N_{max}^{-2}N_{min}^{-1}L_{med})$ at worst. But ξ_2 is also contained in an interval of length $\ll N_1$. The claim (2.25) follows. □

3. Proofs of Theorems 1.1–1.4

In this section, we prove Theorems 1.1–1.3 by using the fundamental estimate on dyadic blocks in Proposition 2.2 and Proposition 2.3.

3.1. **Proof of Theorem 1.1.**

Proof. Suppose $b = \frac{1}{2} + \eta$.

By Plancherel's theorem, (1.2) is reduced to showing that

$$\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \langle \xi_3 \rangle^{s}}{\langle \tau_1 - q(\xi_1) \rangle^{b} \langle \tau_2 - q(\xi_2) \rangle^{b} \langle \tau_3 - q(\xi_3) \rangle^{1-b}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.1}$$

By dyadic decomposition of the variables ξ_j and λ_j $(j = 1,2,3)$ and $h(\xi)$, we may assume that $|\xi_j| \sim N_j$, $|\lambda_j| \sim L_j$ $(j = 1,2,3)$, and $|h(\xi)| \sim H$. By the translation invariance of the $[k;Z]$-multiplier norm, we can always restrict our estimate to $L_j \gtrsim 1$ $(j = 1,2,3)$ and $\max(N_1, N_2, N_3) \gtrsim 1$. For details of this reduction, see the Appendix. The comparison principle and orthogonality (see Schur's test in [20], p. 851) imply the multiplier norm estimate (3.1) can be reduced to proving

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_1,L_2,L_3\gtrsim 1} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle N_3 \rangle^{-s} L_1^b L_2^b L_3^{1-b}} \|X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{3.2}$$

and

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{H\ll L_{med}} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle N_3 \rangle^{-s} L_1^b L_2^b L_3^{1-b}} \|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{3.3}$$

for all $N \gtrsim 1$. These will be accomplished by Proposition 2.2 and some tedious summation.

Fix $N \gtrsim 1$. We first prove (3.3). We may assume (2.12). By (2.13) we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}\gtrsim N^4} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle N_3 \rangle^{-s} L_1^b L_2^b L_3^{1-b}} L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \lesssim 1. \tag{3.4}$$

Estimating

$$\frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle N_3 \rangle^{-s}} \lesssim \frac{N^{-2s}}{\langle N_{min} \rangle^{-s}}, \; L_1^b L_2^b L_3^{1-b} \gtrsim L_{min}^b L_{med}^b L_{max}^{1-b},$$

and then performing the L summations, we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N} \frac{N^{-2s-4+\eta} N_{min}^{\frac{1}{2}}}{\langle N_{min} \rangle^{-s}} \lesssim 1,$$

which is true for $s \in (-\frac{7}{4}, 0]$.

Now we show the low modulation case (3.2). We may again assume $L_{max} \sim N^4$. We first deal with the contribution where (2.14) holds. In this case we have $N_1, N_2, N_3 \sim N \gtrsim 1$, so we reduce to

$$\sum_{L_1,L_2,L_3 \gtrsim 1} \frac{N^{-s}}{L_{min}^b L_{med}^b L_{max}^{1-b}} L_{min}^{\frac{1}{2}} N^{-\frac{1}{2}} L_{med}^{\frac{1}{4}} \lesssim 1,$$

but this is easily verified.

Now we deal with the cases where (2.13) applies.We reduce by (2.13) to

$$\sum_{N_{min} \ll N} \sum_{L_{max} \sim N^4} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}}{\langle N_3\rangle^{-s} L_{min}^b L_{med}^b L_{max}^{1-b}} L_{min}^{\frac{1}{2}} N^{-\frac{1}{2}} \min\{N N_{min}, \frac{L_{med}}{N^2}\}^{\frac{1}{2}} \lesssim 1. \tag{3.5}$$

Since

$$\min\{N N_{min}, \frac{L_{med}}{N^2}\}^{\frac{1}{2}} \leq (N N_{min})^{\eta} (\frac{L_{med}}{N^2})^{\frac{1}{2}-\eta},$$

and

$$\frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}}{\langle N_3\rangle^{-s}} \lesssim \frac{N^{-2s}}{\langle N_{min}\rangle^{-s}},$$

we can reduce (3.5) to

$$\sum_{N_{min} \ll N} \sum_{L_{max} \sim N^4} \frac{N^{-2s}}{\langle N_{min}\rangle^{-s} L_{min}^b L_{med}^b L_{max}^{1-b}} L_{min}^{\frac{1}{2}} N^{-\frac{1}{2}} (N N_{min})^{\eta} (\frac{L_{med}}{N^2})^{\frac{1}{2}-\eta} \lesssim 1.$$

performing the L summations, we reduce to

$$\sum_{N_{min} \ll N} \frac{N^{-2s-\frac{7}{2}+7\eta} N_{min}^{\eta}}{\langle N_{min}\rangle^{-s}} \lesssim 1,$$

which is true for $s > -\frac{7}{4} + \frac{7}{2}\eta$. This completes the proof of the estimate (1.2). □

3.2. **Proof of Theorem 1.2.**

Proof. By Plancherel, it suffices to show that

$$\left\| \frac{\langle \xi_1\rangle^{-s}\langle \xi_2\rangle^{-s}\langle \xi_3\rangle^{s}}{\langle \tau_1 - q(\xi_1)\rangle^b \langle \tau_2 - q(\xi_2)\rangle^b \langle \tau_3 + q(\xi_3)\rangle^{1-b}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.6}$$

Replacing Proposition 2.2 by Proposition 2.3, an argument similar to that in the proof of Theorem 1.1 gives the estimate (1.3). Although a different proposition was used, the restriction of regularity s is still in the same range $(-\frac{7}{4}, 0]$. □

3.3. **Proof of Theorem 1.3.**

Proof. By Plancherel's theorem, the estimate (1.4) reduces to showing that

$$\Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{s}\langle\xi_3\rangle^{-s}}{\langle\tau_1-q(\xi_1)\rangle^{b}\langle\tau_2-q(\xi_2)\rangle^{1-b}\langle\tau_3+q(\xi_3)\rangle^{b}}\Big\|_{[3;\mathbb{R}\times\mathbb{R}]}\lesssim 1. \tag{3.7}$$

We follow the steps as for (3.1), and the worst case is $N_2 \sim N_{min}$, $L_2 \sim L_{max} \sim H$, and H satisfying (2.18), in which case (3.7) is reduced to

$$\sum_{N_2\lesssim N}\sum_{1\lesssim L_1,L_3\lesssim N^3N_2}\frac{N^{-2s}}{\langle N_2\rangle^{-s}L_1^b(N^3N_2)^{1-b}L_3^b} L_{min}^{\frac{1}{2}}N^{-\frac{1}{2}}\min\{NN_2,\frac{L_{med}}{NN_2}\}^{\frac{1}{2}}\lesssim 1,$$

where we make use of (2.20) in Proposition 2.3. To sum the left-hand side, we break down this estimate into two subcases. In the case of $\frac{L_{med}}{NN_2} < NN_2$, we need to prove

$$\sum_{N^{-1}\lesssim N_2\lesssim N}\sum_{1\lesssim L_1,L_3\lesssim N^3N_2}\frac{N^{-2s}}{\langle N_2\rangle^{-s}L_1^b(N^3N_2)^{1-b}L_3^b}L_{min}^{\frac{1}{2}}N^{-\frac{1}{2}}(\frac{L_{med}}{NN_2})^{\frac{1}{2}}\lesssim 1;$$

by performing N_2 summation, we reduce to

$$\sum_{1\lesssim L_1,L_3\lesssim N^4}\frac{N^{-2s-\frac{3}{2}+2\eta}L_{min}^{\frac{1}{2}}L_{med}^{\frac{1}{2}}}{L_1^bL_3^b}\lesssim 1,$$

which is true for $s > -\frac{3}{4}+\eta$. On the other hand, in the case of $\frac{L_{med}}{NN_2} \geq NN_2$, it is easy to get

$$\sum_{N_2\lesssim N}\sum_{1\lesssim L_1,L_3\lesssim N^3N_2}\frac{N^{-2s}}{\langle N_2\rangle^{-s}L_1^b(N^3N_2)^{1-b}L_3^b}L_{min}^{\frac{1}{2}}N^{-\frac{1}{2}}(NN_2)^{\frac{1}{2}}\lesssim 1.$$

Therefore, this ends the proof of Theorem 1.3. □

3.4. **Examples demonstrating sharpness in Theorem 1.4.** The idea is similar to the Kenig, Ponce, and Vega presented in the proof of Theorem 1.4 in [12, 13] and K. Nakanishi, H. Takaoka, and Y. Tsutsumi [17]. Recall the notation χ_E denotes the characteristic function of the set E.

Proof of Theorem 1.4(a). We begin by considering the estimate (1.2). Define the set

$$R_N = \{(\xi,\tau) : |\xi - N| \le 1, |\tau + \xi^4 + \varepsilon\xi^2| \le 1\}.$$

Choose $\hat{\hat{u}} = \chi_{R_N}$ and $\hat{\hat{v}} = \chi_{R_{-N}}$. We can show that for large N, $\widehat{(\overline{uv})} \sim \frac{1}{N^3}\chi_{Q_N}$, where $Q_N = \{(\xi,\tau) : |\xi| \lesssim 1,\ |\tau - \frac{1}{2}N^4| \lesssim N^3\}$. Indeed, $Q_N \sim R_N + R_{-N}$ and a translate of R_N overlaps R_{-N} in a set of size at most $|\frac{1}{N^3} \times 1|$. Note that $|Q_N| \sim |1 \times N^3| = N^3$. So

$$\|\overline{uv}\|_{X_{s,b-1}} \sim \frac{1}{N^3}(N^4)^{b-1}N^{\frac{3}{2}} \sim N^{4(b-1)-\frac{3}{2}}$$

and $\|u\|_{X_{s,b}} = \|v\|_{X_{s,b}} \sim N^s$. Therefore, the estimate (1.2) implies that

$$N^{-4(1-b)-\frac{3}{2}} \lesssim N^{2s}. \tag{3.8}$$

Now, taking $\hat{\hat{w}}(\xi,\tau) = \chi_{T_N}(\xi,\tau)$, where $T_N = \{(\xi,\tau) : |\xi + N| \le 1,\ |\tau - \xi^4 - \varepsilon\xi^2| \le 1\}$. We have that for N large $(\hat{\hat{u}} * \hat{\hat{w}})(\xi,\tau) \sim c\chi_S(\xi,\tau)$, where S is the rectangle of dimensions $cN^3 \times N^{-3}$ centered at the origin with the longest side pointing in the $(1, 4N^3)$ direction.

Thus, (1.2) implies that

$$N^{3(b-1)} \lesssim N^{2s+4b}. \tag{3.9}$$

Together with (3.8) and (3.9), and letting N tend to infinity, this gives

$$s \ge \max\{-2(1-b) - \tfrac{3}{4}, -\tfrac{b+3}{2}\}, \tag{3.10}$$

which completes that when $s < -\frac{7}{4}$ and any $b \in \mathbb{R}$, the estimate (1.2) fails.

Now we consider the endpoint case $s = -\frac{7}{4}$. By duality and the Plancherel theorem, the estimate (1.2) is equivalent to the following estimate:

$$\langle \hat{u}_3, \hat{\hat{u}}_1 * \hat{\hat{u}}_2 \rangle \le C\|u_3\|_{X_{-s,1-b}}\|u_1\|_{X_{s,b}}\|u_2\|_{X_{s,b}}. \tag{3.11}$$

By (3.10) we know that if the estimate (3.11) holds with $s = -\frac{7}{4}$, we must have $b = \frac{1}{2}$. Therefore we need only to consider the case $s = -\frac{7}{4}$ and $b = \frac{1}{2}$.

Let η be a sufficiently small positive number independent of N. We define three functions $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ as follows:

$$\hat{\hat{u}}_1 = \chi_{R_N}(\tau,\xi),\ \hat{\hat{u}}_2 = \chi_{T_N}(\tau,\xi),\ \hat{u}_3 = (1+\tau)^{-1}\chi_{Y_N}(\tau,\xi),$$

where R_N and T_N are defined above and $Y_N = \{(\tau,\xi) : |\tau + 4N^3\xi| \le \eta,\ 1 \le \tau \le \eta N^3\}$. We have that for large N,

$$\{(\tau,\xi) : |\tau + 4N^3\xi| \le \eta,\ 1 \le \tau \le \eta N^3\} \subset \mathrm{supp}(\hat{u}_1 * \hat{u}_2).$$

Hence

$$\langle \hat{u}_3, \hat{\hat{u}}_1 * \hat{\hat{u}}_2 \rangle \leq C \int_1^{\eta N^3} (1+\tau)^{-1} \int_{-\frac{\tau}{4N^3}-\frac{\eta}{4N^3}}^{-\frac{\tau}{4N^3}+\frac{\eta}{4N^3}} d\xi \, d\tau$$

$$\sim N^{-3} \int_1^{\eta N^3} (1+\tau)^{-1} d\tau \sim N^{-3} \log N. \qquad (3.12)$$

On the other hand, since $|\xi| \leq C$ for $(\xi, \tau) \in \operatorname{supp} \hat{u}_3$, by simple calculations we get

$$\|u_3\|_{X_{\frac{7}{4},\frac{1}{2}}} \sim \Big(\int_1^{\eta N^3} \int_{-\frac{\tau}{4N^3}-\frac{\eta}{4N^3}}^{-\frac{\tau}{4N^3}+\frac{\eta}{4N^3}} (1+\tau)^{-2} \langle \xi \rangle^{\frac{7}{2}} \langle \tau - \xi^4 - \varepsilon \xi^2 \rangle \, d\xi \, d\tau \Big)^{\frac{1}{2}}$$

$$\sim N^{-\frac{3}{2}} \Big(\int_1^{\eta N^3} (1+\tau)^{-1} d\tau \Big)^{\frac{1}{2}} \sim N^{-\frac{3}{2}} (\log N)^{\frac{1}{2}} \qquad (3.13)$$

$$\|u_1\|_{X_{-\frac{7}{4},\frac{1}{2}}} \sim N^{-\frac{7}{4}}, \ \|u_2\|_{X_{-\frac{7}{4},\frac{1}{2}}} \sim N^{-\frac{7}{4}} N^2 \sim N^{\frac{1}{4}}. \qquad (3.14)$$

If the estimate (3.11) holds with $s = -\frac{7}{4}$ with $b = \frac{1}{2}$, we must have by (3.12)–(3.14)

$$N^{-3} \log N \leq C N^{-\frac{3}{2}} (\log N)^{\frac{1}{2}} \times N^{-\frac{7}{4}} N^{\frac{1}{4}} = C N^{-3} (\log N)^{\frac{1}{2}}, \qquad (3.15)$$

where C is a positive constant independent of N. Therefore, we let $N \to \infty$ in (3.15) to obtain a contradiction, which completes that when $s = -\frac{7}{4}$ and $b = \frac{1}{2}$, the estimate (3.11) fails.

The same analysis shows the necessity of the conditions $s > -\frac{7}{4}$ for (1.3) to hold. Indeed, consider $\hat{u} = \chi_{R_N}$, $\hat{v} = \chi_{R_{-N}}$, and $\hat{w} = \chi_{T_N}$, along with $\hat{u}_1 = \chi_{R_N}$, $\hat{u}_2 = \chi_{T_N}$, and $\hat{u}_3 = \chi_{Y_N}$ for the endpoint case $s = -\frac{7}{4}$ and $b = \frac{1}{2}$.

Proof of Theorem 1.4(b): Let $\hat{\hat{u}} = \chi_{T_N}$ and $\hat{v} = \chi_{R_N}$. Thus for N large, we have $(\hat{\hat{u}} * \hat{v})(\xi, \tau) \sim c\chi_S(\xi, \tau)$, where S is the rectangle of dimensions $cN^3 \times N^{-3}$ centered at the origin with the longest side pointing in the $(1, 4N^3)$ direction.

Finally, from (1.4) we can get that for N large

$$N^{2s} \geq c \Big(\int_{|\xi|\leq 1} \int_{|\tau| \leq 1} \chi_S(\xi, \tau) d\xi \, d\tau \Big)^{\frac{1}{2}} \geq c N^{-\frac{3}{2}},$$

which completes showing that when $s < -\frac{3}{4}$ and any $b \in \mathbb{R}$, the estimate (1.4) fails. Now we consider the endpoint case $s = -\frac{3}{4}$. By duality and the Plancherel theorem, the estimate (1.4) is equivalent to the following estimate:

$$\langle \hat{u}_3, \hat{u}_1 * \hat{\bar{u}}_2 \rangle \leq C\|u_3\|_{X_{\frac{3}{4},1-b}} \|u_1\|_{X_{-\frac{3}{4},b}} \|u_2\|_{X_{-\frac{3}{4},b}}. \tag{3.16}$$

Let $\hat{u}_1 = \chi_{T_N}(\tau,\xi)$, $\hat{\bar{u}}_2 = \chi_{R_N}(\tau,\xi)$, and $\hat{u}_3 = \chi_{B_N}(\tau,\xi)$, where R_N and T_N are defined in the above and $B_N = \{(\tau,\xi) : |\tau + 4N^3\xi| \leq \frac{1}{2},\ |\xi| \leq 1\}$. So we have that for large N, $(\hat{u}_1 * \hat{\bar{u}}_2)(\tau,\xi) \sim 1$, $(\tau,\xi) \in A$, where $A = \{(\tau,\xi) : |\tau + 4N^3\xi| \leq \frac{1}{10},\ |\xi| \leq \frac{1}{10}\}$. Since $A \subset \operatorname{supp} \hat{u}_3$, we get

$$\langle \hat{u}_3, \hat{u}_1 * \hat{u}_2 \rangle \geq C|A| \sim 1. \tag{3.17}$$

On the other hand,

$$\|u_3\|_{X_{\frac{3}{4},1-b}} \sim \left[|\operatorname{supp} \hat{u}_3| \times N^{6(1-b)}\right]^{\frac{1}{2}} \sim N^{3(1-b)}, \tag{3.18}$$

$$\|u_1\|_{X_{-\frac{3}{4},b}}, \|u_2\|_{X_{-\frac{3}{4},b}} \sim N^{-\frac{3}{4}}. \tag{3.19}$$

If the estimate (3.16) holds, then we must have by (3.17)–(3.18)

$$1 \leq CN^{3(1-b)} \times (N^{-\frac{3}{4}})^{\frac{1}{2}} = CN^{3(\frac{1}{2}-b)}, \tag{3.20}$$

where C is a positive constant independent of N. Since N is an arbitrary large positive number, we conclude by (3.20) that $b \leq \frac{1}{2}$.

Hence, it remains only to exclude the case $b = \frac{1}{2}$. Let η be a sufficiently small positive number independent of N. We choose $\hat{u}_1 = \chi_{T_N}(\tau,\xi)$, $\hat{\bar{u}}_2 = \chi_{R_N}(\tau,\xi)$, and $\hat{u}_3 = (1+\tau)^{-1}\chi_{Y_N}(\tau,\xi)$, where R_N, T_N, and Y_N are defined above. Therefore,

$$\begin{aligned} \langle \hat{u}_3, \hat{u}_1 * \hat{\bar{u}}_2 \rangle &\sim \int_1^{\eta N^3} (1+\tau)^{-1} \int_{-\frac{\tau}{4N^3}-\frac{\eta}{4N^3}}^{-\frac{\tau}{4N^3}+\frac{\eta}{4N^3}} d\xi\, d\tau \\ &\sim N^{-3} \int_1^{\eta N^3} (1+\tau)^{-1} d\tau \sim N^{-3}\log N. \end{aligned} \tag{3.21}$$

On the other hand, since $|\xi| \leq C$ for $(\tau,\xi) \in \operatorname{supp} \hat{u}_3$, we have

$$\begin{aligned} \|u_3\|_{X_{\frac{3}{4},\frac{1}{2}}} &\sim \Big(\int_1^{\eta N^3} \int_{-\frac{\tau}{4N^3}-\frac{\eta}{4N^3}}^{-\frac{\tau}{4N^3}+\frac{\eta}{4N^3}} (1+\tau)^{-2} \langle\xi\rangle^{\frac{3}{2}} \langle \tau - \xi^4 - \varepsilon\xi^2 \rangle\, d\xi\, d\tau \Big)^{\frac{1}{2}} \\ &\sim N^{-\frac{3}{2}} \Big(\int_1^{\eta N^3} (1+\tau)^{-1} d\tau \Big)^{\frac{1}{2}} \sim N^{-\frac{3}{2}} (\log N)^{\frac{1}{2}} \end{aligned} \tag{3.22}$$

$$\|u_1\|_{X_{-\frac{3}{4},\frac{1}{2}}},\ \|u_2\|_{X_{-\frac{3}{4},\frac{1}{2}}} \sim N^{-\frac{3}{4}}. \tag{3.23}$$

Hence, if the estimate (3.16) holds with $b = \frac{1}{2}$, we must have by (3.21)–(3.23)

$$N^{-3}\log N \leq CN^{-\frac{3}{2}} (\log N)^{\frac{1}{2}} \times (N^{-\frac{3}{4}})^2 = CN^{-3} (\log N)^{\frac{1}{2}}, \tag{3.24}$$

where C is a positive constant independent of N. Therefore, we let $N \to \infty$ in (3.24) to obtain a contradiction, which completes showing that when $s = -\frac{3}{4}$ and $b = \frac{1}{2}$, the estimate (3.16) fails.

4. Appendix

This appendix is devoted to the proof of the rationality of the assumption (2.6) and (2.7). To this end, we firstly give a fundamental lemma.

Lemma 4.1. *For all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, we have*

$$\int_0^{12} \chi_{|\lambda_1-\tau|\geq 2}(\tau)\chi_{|\lambda_2-\tau|\geq 2}(\tau)\chi_{|\lambda_3+2\tau|\geq 2}(\tau)\ d\tau \geq 2,$$

and similarly for all $\xi_1, \xi_2 \in \mathbb{R}$, we have

$$\int_0^{10} \chi_{|\xi_1-\xi|\geq 2}(\xi)\chi_{|\xi_2+\xi|\geq 2}(\xi)\ d\xi \geq 2.$$

Now we give the proof for the assumption (2.6) for the estimate (3.1); to be precise, we have the following Proposition 4.1.

Proposition 4.1.

$$\Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}\langle\lambda_3\rangle^{1-b}}\Big\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim \Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\prod_{j=1}^{3}\chi_{|\lambda_j|\geq 2}}{\langle\xi_3\rangle^{-s}\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}\langle\lambda_3\rangle^{1-b}}\Big\|_{[3;\mathbb{R}\times\mathbb{R}]}.$$

Proof. Applying Lemma 4.1 to the right-hand side of the above, we have

$$\begin{aligned}
&\Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}\langle\lambda_3\rangle^{1-b}}\Big\|_{[3;\mathbb{R}\times\mathbb{R}]}\\
&\lesssim \Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}\langle\lambda_3\rangle^{1-b}}\int_0^{12}\chi_{|\lambda_1-\tau|\geq 2}(\tau)\chi_{|\lambda_2-\tau|\geq 2}(\tau)\chi_{|\lambda_3+2\tau|\geq 2}(\tau)\ d\tau\Big\|_{[3;\mathbb{R}\times\mathbb{R}]}\\
&= \Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\xi_3\rangle^{-s}}\int_0^{12}\frac{\langle\lambda_1-\tau\rangle^{b}\langle\lambda_2-\tau\rangle^{b}\langle\lambda_3+2\tau\rangle^{1-b}}{\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}\langle\lambda_3\rangle^{1-b}}\\
&\qquad\frac{\chi_{|\lambda_1-\tau|\geq 2}(\tau)\chi_{|\lambda_2-\tau|\geq 2}(\tau)\chi_{|\lambda_3+2\tau|\geq 2}(\tau)}{\langle\lambda_1-\tau\rangle^{b}\langle\lambda_2-\tau\rangle^{b}\langle\lambda_3+2\tau\rangle^{1-b}}\ d\tau\Big\|_{[3;\mathbb{R}\times\mathbb{R}]},
\end{aligned}\tag{4.1}$$

since if $|\tau| \leq M$, then $\langle\lambda-\tau\rangle \sim_M \langle\lambda\rangle$. Therefore,

$$\begin{aligned}
(4.1) &\lesssim \Big\|\int_0^{12}\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\xi_3\rangle^{-s}}\frac{\chi_{|\lambda_1-\tau|\geq 2}(\tau)\chi_{|\lambda_2-\tau|\geq 2}(\tau)\chi_{|\lambda_3+2\tau|\geq 2}(\tau)}{\langle\lambda_1-\tau\rangle^{b}\langle\lambda_2-\tau\rangle^{b}\langle\lambda_3+2\tau\rangle^{1-b}}\ d\tau\Big\|_{[3;\mathbb{R}\times\mathbb{R}]}\\
&\lesssim \Big\|\frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\xi_3\rangle^{-s}}\frac{\chi_{|\lambda_1|\geq 2}\chi_{|\lambda_2|\geq 2}\chi_{|\lambda_3|\geq 2}}{\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}\langle\lambda_3\rangle^{1-b}}\Big\|_{[3;\mathbb{R}\times\mathbb{R}]}.
\end{aligned}\tag{4.2}$$

We use time-translation invariance (see Lemma 3.4 (8) in Tao [20]) of the $[k, Z]$-multiplier norm in the last inequality. This completes the proof. □

Finally, we give the proof of rationality of the assumption (2.7) for the estimate (3.1); to be precise, we have the following Proposition 4.2.

Proposition 4.2.

$$\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \prod_{j=1}^3 \chi_{|\xi_j| \le 2}}{\langle \xi_3 \rangle^{-s} \langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b \langle \lambda_3 \rangle^{1-b}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim \left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \prod_{j=1}^2 \chi_{|\xi_j| \ge 2}}{\langle \xi_3 \rangle^{-s} \langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b \langle \lambda_3 \rangle^{1-b}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]}.$$

Proof. Applying Lemma 4.1 to the right-hand side of the above, we have

$$\begin{aligned}
&\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \prod_{j=1}^3 \chi_{|\xi_j| \le 2}}{\langle \xi_3 \rangle^{-s} \langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b \langle \lambda_3 \rangle^{1-b}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \qquad (4.3)\\
&\lesssim \left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \prod_{j=1}^3 \chi_{|\xi_j| \le 2}}{\langle \xi_3 \rangle^{-s} \langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b \langle \lambda_3 \rangle^{1-b}} \int_0^{10} \chi_{|\xi_1-\xi|\ge 2}(\xi) \chi_{|\xi_2+\xi|\ge 2}(\xi) \; d\xi \right\|_{[3;\mathbb{R}\times\mathbb{R}]}\\
&\le \left\| \int_0^{10} \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s}}{\langle \xi_1 - \xi \rangle^{-s} \langle \xi_2 + \xi \rangle^{-s}} \frac{\langle \tau_1 - q(\xi_1 - \xi) \rangle^b \langle \tau_2 - q(\xi_2 + \xi) \rangle^b \prod_{j=1}^2 \chi_{|\xi_j| \le 2}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} \right.\\
&\qquad \left. \frac{\langle \xi_1 - \xi \rangle^{-s} \langle \xi_2 + \xi \rangle^{-s}}{\langle \xi_3 \rangle^{-s}} \frac{\chi_{|\xi_1-\xi|\ge 2} \chi_{|\xi_2+\xi|\ge 2}}{\langle \tau_1 - q(\xi_1 - \xi) \rangle^b \langle \tau_2 - q(\xi_2 + \xi) \rangle^b \langle \lambda_3 \rangle^{1-b}} \; d\xi \right\|_{[3;\mathbb{R}\times\mathbb{R}]}.
\end{aligned}$$

Since $\langle \xi_j - \xi \rangle \sim \langle \xi_j \rangle, \; |\xi| \le 10$, and

$$\langle \tau_j - q(\xi_j - \xi) \rangle \sim \langle \tau_j - q(\xi_j) \rangle = \langle \lambda_j \rangle, \; |\xi| \le 10, \; |\xi_j| \le 2,$$

where $q(\xi) = \xi^4 + \varepsilon \xi^2$ and $j = 1, 2$. So

$$\begin{aligned}
(4.3) &\lesssim \left\| \int_0^{10} \frac{\langle \xi_1 - \xi \rangle^{-s} \langle \xi_2 + \xi \rangle^{-s}}{\langle \xi_3 \rangle^{-s}} \right.\\
&\qquad \left. \times \frac{\chi_{|\xi_1-\xi|\ge 2} \chi_{|\xi_2+\xi|\ge 2}}{\langle \tau_1 - q(\xi_1 - \xi) \rangle^b \langle \tau_2 - q(\xi_2 + \xi) \rangle^b \langle \lambda_3 \rangle^{1-b}} d\xi \right\|_{[3;\mathbb{R}\times\mathbb{R}]}\\
&\lesssim \left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \prod_{j=1}^2 \chi_{|\xi_j| \ge 2}}{\langle \xi_3 \rangle^{-s} \langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b \langle \lambda_3 \rangle^{1-b}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]}.
\end{aligned}$$

We make use of the space-translation invariance (see Lemma 3.4 (8) in Tao [20])of the $[k, Z]$-multiplier norm in the last inequality. This completes the proof. □

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