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MAGNETOSTATIC SOLUTIONS FOR A SEMILINEAR PERTURBATION OF THE MAXWELL EQUATIONS

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Abstract. In this paper we consider a model introduced in [3] which describes the interaction between the matter and the electromagnetic field from a *unitarian standpoint*. This model is based on a semilinear perturbation of the Maxwell equations and, in the magnetostatic case, reduces to the following nonlinear elliptic degenerate equation:

$$\nabla \times (\nabla \times \mathbf{A}) = W'(|\mathbf{A}|^2)\mathbf{A},$$

where "$\nabla\times$" is the *curl* operator, $W : \mathbb{R} \to \mathbb{R}$ is a suitable nonlinear term, and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is the gauge potential associated with the magnetic field $\mathbf{H}$. We prove the existence of a nontrivial finite energy solution with a kind of cylindrical symmetry. The proof is carried out by using a suitable variational framework based on the Hodge decomposition, which is crucial in order to handle the strong degeneracy of the equation. Moreover, the use of a natural constraint and a concentration-compactness argument are also required.

1. Introduction

In [3] Benci and Fortunato introduced the following semilinear perturbation of the Maxwell equations:

$$\frac{\partial}{\partial t}\Big(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big) + \nabla \times (\nabla \times \mathbf{A}) = W'(|\mathbf{A}|^2 - \varphi^2)\mathbf{A} \tag{1.1}$$

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$$-\nabla \cdot \Big(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big) = W'(|\mathbf{A}|^2 - \varphi^2)\varphi, \tag{1.2}$$

where "$\nabla\times$" is the *curl* operator, $W : \mathbb{R} \to \mathbb{R}$ is a suitable nonlinear term, and $\mathbf{A} : \mathbb{R}^4 \to \mathbb{R}^3$, $\varphi : \mathbb{R}^4 \to \mathbb{R}$ are the gauge potentials associated with the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}$, respectively; i.e.,

$$\mathbf{H} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi.$$

Such a model originates from an idea of Born and Infeld ([5]) and describes the interaction between the matter and the electromagnetic field from a *unitarian standpoint*, in the sense that it allows the existence of a self-induced electromagnetic field; indeed, if we interpret the quantities

$$\rho = \rho\,(\mathbf{A}, \varphi) = W'\big(\,|\mathbf{A}|^2 - \varphi^2\big)\varphi,$$
$$\mathbf{J} = \mathbf{J}\,(\mathbf{A}, \varphi) = W'\big(\,|\mathbf{A}|^2 - \varphi^2\big)\mathbf{A}$$

as the charge density and the current density respectively, then the equations (1.1)–(1.2) formally become the Maxwell equations in the presence of matter and no external source is required (see [8] and [9]).

Looking for magnetostatic solutions, i.e., solutions of the type $(\mathbf{A}, 0)$ with $\mathbf{A}$ depending only on the space variable x, then (1.1)–(1.2) reduce to the following nonlinear elliptic equation:

$$\nabla \times (\nabla \times \mathbf{A}) = W'(|\mathbf{A}|^2)\mathbf{A}, \tag{1.3}$$

which gives rise to interesting approaches from a mathematical point of view. Indeed, the main difficulty in dealing with the equation (1.3) lies in its strongly degenerate nature due to the presence of the *curl* operator "$\nabla\times$"; consequently, its associated energy functional

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2\,dx - \int_{\mathbb{R}^3} W(|\mathbf{A}|^2)dx \tag{1.4}$$

is, in general, strongly indefinite in the sense that it is not bounded from below or from above and any possible critical point has infinite Morse index. Indeed, if $W(t^2)$ is strictly convex, the second variation of (1.4) is negatively defined on the infinite-dimensional subspace

$$\big\{\mathbf{A} = \nabla\phi : \phi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})\big\}.$$

The existence of a nontrivial solution $\mathbf{A}$ of (1.3) has been stated in [3]; however, the proof contains a gap. To overcome the difficulty due to the indefiniteness, in [1] a suitable subspace has been introduced in which to look for solutions of (1.3). Indeed, a nontrivial solution is found among the divergence-free vector fields, and precisely the vector fields $\mathbf{A}$ of the form

$$\mathbf{A} = A(r, x_3)\Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big), \tag{1.5}$$

where

$$r = r_x = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2} \tag{1.6}$$

and $A : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$. It is clear that for $\mathbf{A}$ of the form (1.5) we have that $\operatorname{div} \mathbf{A} = 0$ and, consequently, $\nabla \times (\nabla \times \mathbf{A}) = -\Delta \mathbf{A}$. Therefore the strong indefiniteness of (1.4) is avoided in such a subspace and (1.3) can be treated with standard methods of nonlinear analysis. In this paper we are interested in finding solutions with the following cylindrical symmetry:

$$\mathbf{A}(x) = A(r, x_3)\Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big) + A_3(r, x_3)(0, 0, 1), \tag{1.7}$$

where $A, A_3 : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$. Physical considerations impel some assumptions on the nonlinearity W; before stating our main result let us enumerate the hypotheses on the function W that will be steadily assumed:

- (W1) $W \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$; $W(0) = 0$;
- (W2) the map $t \in \mathbb{R} \mapsto W(t^2) \in \mathbb{R}$ is strictly convex;
- (W3) there are positive constants c_1, c_2, p, and q with $2 < p < 6 < q$ such that

$$\begin{aligned} c_1|t|^{p/2} &\le W(t) && \text{for } |t| \ge 1, \\ c_1|t|^{q/2} &\le W(t) && \text{for } |t| \le 1, \\ |W'(t)| &\le c_2|t|^{p/2-1} && \text{for } |t| \ge 1, \\ |W'(t)| &\le c_2|t|^{q/2-1} && \text{for } |t| \le 1. \end{aligned}$$

Now we proceed to state our main theorem.

Theorem 1.1. *Assume that the hypotheses* (W1)–(W3) *hold. Then the equation* (1.3) *has a nontrivial weak solution* $\mathbf{A}$ *having the form* (1.7)*. Furthermore,* $\mathbf{A}$ *is a finite-energy solution; i.e.,*

$$\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx, \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx < +\infty.$$

Since the vector fields of the type (1.7) are not divergence-free, in general, then the equation (1.3) restricted to the related subspace preserves its strongly degenerate nature; hence, a new approach is required. Following an idea introduced in [3], we use a new functional framework based on the

Hodge decomposition; more precisely, we split any vector field $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ as the sum

$$\mathbf{A} = \mathbf{u} + \nabla w, \tag{1.8}$$

where $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ is a divergence-free vector field ($\operatorname{div} \mathbf{u} = 0$) and ∇w is a potential vector field, with $w : \mathbb{R}^3 \to \mathbb{R}$. Using this decomposition the left-hand side of (1.3) becomes

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times (\nabla \times \mathbf{u}) = -\Delta \mathbf{u}.$$

Then we follow the following strategy. Fixing $\mathbf{u}$, the equation $W'(|\mathbf{u} + \nabla w|^2)(\mathbf{u} + \nabla w) = 0$ is uniquely solvable thanks to the strict convexity of $W(|t|^2)$; let us set $\Phi[\mathbf{u}]$ to be such a solution. By replacing (1.8) in (1.3) with $w = \Phi[\mathbf{u}]$, we get a new equation which depends only on $\mathbf{u}$ and can be studied by a constrained minimization technique. Another difficulty arises from the growth condition on the nonlinearity W which leads to studying the problem in a functional setting related to the Orlicz space $\mathbf{L}^p + \mathbf{L}^q$.

We point out that strongly indefinite functionals have been largely studied in relation to other problems arising from mathematical physics and have been treated with min-max methods; we recall, among many others, [2], [4], and [7]. However, min-max arguments do not seem to be applicable to the equation (1.3); therefore, we need to carry out a different approach.

The organization of the paper can be summarized as follows. In Section 2 we describe the functional settings in which we work and we provide some preliminary results. Section 3 is devoted to establishing the natural constraints for $\mathbf{u}$ and w, i.e., the suitable subspaces where we will look for solutions of the form $\mathbf{A} = \mathbf{u} + \nabla w$ of (1.3). In Section 4 we provide a concentration-compactness-type result; such a result, combined with a constrained minimization argument, leads to the proof of Theorem 1.1, which is carried out in Section 5.

Notation. Throughout the paper we will often use the notation C to denote generic positive constants. The value of C is allowed to vary from place to place.

2. THE VARIATIONAL SETTING

In this section we collect some preliminary results concerning the variational structure of the equation (1.3).

By using the decomposition (1.8), let us choose suitable functional spaces for $\mathbf{u}$ and w. Let $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm

$$\|\mathbf{u}\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx.$$

$\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ is a Hilbert space with the scalar product

$$\int_{\mathbb{R}^3} (\nabla \mathbf{u} | \nabla \mathbf{v}) dx, \quad \mathbf{u}, \mathbf{v} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3),$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $(\nabla \mathbf{u} | \nabla \mathbf{v}) = \sum_{i=1}^3 \nabla u_i \cdot \nabla v_i$, "$\cdot$" being the scalar product in $\mathbb{R}^3$. By the Sobolev inequalities $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ is continuously embedded into $\mathbf{L}^6 := L^6(\mathbb{R}^3, \mathbb{R}^3)$,

$$\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow \mathbf{L}^6. \tag{2.1}$$

Consequently, for every open and bounded domain $\Omega \subset \mathbb{R}^3$ we have

$$\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow H^1(\Omega, \mathbb{R}^3) \tag{2.2}$$

with continuous embedding.

Next denote by $\mathbf{L}^p + \mathbf{L}^q$ the Banach space made up of the vector fields $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with[1] $\mathbf{v}_1 \in \mathbf{L}^p$ and $\mathbf{v}_2 \in \mathbf{L}^q$, endowed with the norm

$$\|\mathbf{v}\|_{\mathbf{L}^p + \mathbf{L}^q} = \inf\{\|\mathbf{v}_1\|_{L^p} + \|\mathbf{v}_2\|_{L^q} \mid \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\}.$$

Finally, let $\mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R})$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ with respect to the norm

$$\|w\|_{\mathcal{D}^{p,q}} = \|\nabla w\|_{\mathbf{L}^p + \mathbf{L}^q}. \tag{2.3}$$

We begin by proving the following lemma, which summarizes some of the basic properties of the space $\mathbf{L}^p + \mathbf{L}^q$.

Lemma 2.1. *If $\mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q$, then[2] $\mathbf{v}\chi_{\{|\mathbf{v}| \le 1\}} \in \mathbf{L}^q$ and $\mathbf{v}\chi_{\{|\mathbf{v}| > 1\}} \in \mathbf{L}^p$. Furthermore, for every $M > 0$ there exists a constant $C = C(M)$ such that*

$$\|\mathbf{v}\|_{L^p_>}^p + \|\mathbf{v}\|_{L^q_\le}^q \le C \|\mathbf{v}\|_{\mathbf{L}^p + \mathbf{L}^q} \quad \forall \mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q \textit{ with } \|\mathbf{v}\|_{\mathbf{L}^p + \mathbf{L}^q} \le M,$$

where

$$\|\mathbf{v}\|_{L^p_>}^p := \int_{\{|\mathbf{v}| > 1\}} |\mathbf{v}|^p dx, \quad \|\mathbf{v}\|_{L^q_\le}^q := \int_{\{|\mathbf{v}| \le 1\}} |\mathbf{v}|^q dx.$$

Furthermore,

$$\mathbf{L}^6 \hookrightarrow \mathbf{L}^p + \mathbf{L}^q \hookrightarrow L^2(\Omega, \mathbb{R}^3) \tag{2.4}$$

[1] Hereafter $\mathbf{L}^s := L^s(\mathbb{R}^3, \mathbb{R}^3)$.

[2] In the following χ_A denotes the characteristic function of the set $A \subset \mathbb{R}^3$.

with continuous embeddings, and

$$H^1(\Omega,\mathbb{R}^3) \hookrightarrow\hookrightarrow \mathbf{L}^p+\mathbf{L}^q, \tag{2.5}$$

with compact embedding, where Ω *is any smooth and bounded domain in* $\mathbb{R}^3$.

Proof. Define

$$f(t)=\begin{cases} t^q & \text{if } 0\le t\le 1,\\ t^p & \text{if } t>1.\end{cases}$$

It is immediate that $f(t)\le t^q$, $f(t)\le t^p\ \forall\, t\ge 0$, and

$$f(t_1+t_2)\le 2^q f(t_1)+2^q f(t_2)\quad \forall\, t_1,t_2\ge 0.$$

Fix $\mathbf{v}\in\mathbf{L}^p+\mathbf{L}^q$ and let $\mathbf{v}_1\in\mathbf{L}^p$ and $\mathbf{v}_2\in\mathbf{L}^q$ be such that $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$; then

$$\begin{aligned}\|\mathbf{v}\|^p_{L^p_>}+\|\mathbf{v}\|^q_{L^q_\le} &= \int_{\mathbb{R}^3} f(|\mathbf{v}|)dx=\int_{\mathbb{R}^3} f(|\mathbf{v}_1+\mathbf{v}_2|)dx\\ &\le 2^q\int_{\mathbb{R}^3} f(|\mathbf{v}_1|)dx+2^q\int_{\mathbb{R}^3} f(|\mathbf{v}_2|)dx\\ &\le 2^q\int_{\mathbb{R}^3}|\mathbf{v}_1|^p dx+2^q\int_{\mathbb{R}^3}|\mathbf{v}_2|^q dx<+\infty.\end{aligned}$$

Next fix $M>0$ and let $\mathbf{v}\in\mathbf{L}^p+\mathbf{L}^q$ be such that $\|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q}\le M$. Then there exist $\mathbf{v}_1\in\mathbf{L}^p$ and $\mathbf{v}_2\in\mathbf{L}^q$ such that $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ and $\|\mathbf{v}_1\|_{L^p}+\|\mathbf{v}_2\|_{L^q}\le 2M$. From the above computation we get

$$\begin{aligned}\|\mathbf{v}\|^p_{L^p_>}+\|\mathbf{v}\|^q_{L^q_\le} &\le 2^q\left(\|\mathbf{v}_1\|^p_{L^p}+\|\mathbf{v}_2\|^q_{L^q}\right)\le 4^q M^q\left(\frac{\|\mathbf{v}_1\|^p_{L^p}}{(2M)^p}+\frac{\|\mathbf{v}_2\|^q_{L^q}}{(2M)^q}\right)\\ &\le 4^q M^q\left(\frac{\|\mathbf{v}_1\|_{L^p}}{2M}+\frac{\|\mathbf{v}_2\|_{L^q}}{2M}\right)\end{aligned}$$

and, by taking the infimum over all $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\|\mathbf{v}\|^p_{L^p_>}+\|\mathbf{v}\|^q_{L^q_\le}\le 2^{2q-1}M^{q-1}\|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q}.$$

To prove the continuous embedding $\mathbf{L}^6\hookrightarrow\mathbf{L}^p+\mathbf{L}^q$, observe that, if $\mathbf{v}\in\mathbf{L}^6$, then

$$\|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q}\le\|\mathbf{v}\|_{L^p_>}+\|\mathbf{v}\|_{L^q_\le}\le\left(\int_{\{|\mathbf{v}|>1\}}|\mathbf{v}|^6dx\right)^{1/p}+\left(\int_{\{|\mathbf{v}|\le 1\}}|\mathbf{v}|^6dx\right)^{1/q}.$$

The continuous embedding $\mathbf{L}^p+\mathbf{L}^q\hookrightarrow L^2(\Omega,\mathbb{R}^3)$ follows from the Hölder's inequality. By the Sobolev inequalities it follows that, if $\Omega\subset\mathbb{R}^3$ is a smooth and bounded domain, then $H^1(\Omega,\mathbb{R}^3)$ is compactly embedded into $\mathbf{L}^p$; on the other hand, we clearly have $\mathbf{L}^p\hookrightarrow\mathbf{L}^p+\mathbf{L}^q$; therefore we deduce (2.5). □

The choice of the functional spaces for $\mathbf{u}$ and w is related to the growth properties of W. Indeed, according to the assumption (W3), we have

$$|W(t^2)| \le C|t|^p \text{ for } |t| \ge 1, \quad |W(t^2)| \le C|t|^q \text{ for } |t| \le 1. \tag{2.6}$$

Before going on let us notice that the function $t \ge 0 \mapsto W(t^2)$ is strictly convex and increasing by (W2)–(W3); therefore, the function

$$\mathbf{z} \in \mathbb{R}^3 \mapsto W(|\mathbf{z}|^2) \tag{2.7}$$

is strictly convex too.

Next lemma shows how $\mathbf{L}^p + \mathbf{L}^q$ is the natural space in which to define the functional $\mathbf{v} \mapsto \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)dx$. Hereafter p' and q' are the conjugate exponents of p and q.

Lemma 2.2. *The functional*

$$\mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q \mapsto \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)\, dx \in \mathbb{R} \tag{2.8}$$

is of class $\mathcal{C}^1$. Furthermore, for every $\mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q$ we have $W'(|\mathbf{v}|^2)\mathbf{v} \in \mathbf{L}^{p'} \cap \mathbf{L}^{q'}$, and the derivative of (2.8) *at $\mathbf{v}$ is given by*

$$\mathbf{w} \in \mathbf{L}^p + \mathbf{L}^q \longmapsto 2\int_{\mathbb{R}^3} W'(|\mathbf{v}|^2)\mathbf{v}\mathbf{w}\, dx. \tag{2.9}$$

Moreover, (2.8) *is coercive, Lipschitz on bounded sets, strictly convex, and consequently weakly lower semicontinuous.*

Proof. By (2.6) we get

$$\int_{\mathbb{R}^3} W(|\mathbf{v}|^2)\, dx \le C\|\mathbf{v}\|^p_{L^p_>} + C\|\mathbf{v}\|^q_{L^q_\le};$$

therefore, Lemma 2.1 implies that (2.8) is well defined and is bounded on the bounded sets of $\mathbf{L}^p + \mathbf{L}^q$. By using the assumption (W3) we get

$$\int_{\mathbb{R}^3} W(|\mathbf{v}|^2)\, dx \ge c_1\|\mathbf{v}\|^p_{L^p_>} + c_1\|\mathbf{v}\|^q_{L^q_\le},$$

and since by definition $\|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q} \le \|\mathbf{v}\|_{L^p_>} + \|\mathbf{v}\|_{L^q_\le}$, we obtain

$$\int_{\mathbb{R}^3} W(|\mathbf{v}|^2)\, dx \to +\infty \quad \text{as} \quad \|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q} \to +\infty.$$

The strict convexity of (2.7) implies the strict convexity of the map (2.8). Hence the weakly lower semicontinuity will follow immediately once we have proved the continuity.

We divide the remaining part of the proof into three steps.

Step 1. The map

$$\mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q \mapsto W'(|\mathbf{v}|^2)\mathbf{v} \in \mathbf{L}^{p'} \cap \mathbf{L}^{q'} \tag{2.10}$$

is continuous and bounded on bounded sets.

Indeed, fixing $M > 0$, by Lemma 2.1 there exists $C(M) > 0$ such that

$$\|\mathbf{v}\|^p_{L^p_>} + \|\mathbf{v}\|^q_{L^q_\le} \le C(M)\|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q} \qquad \forall \mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q \text{ with } \|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q} \le M.$$

Then, for any $\mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q$ with $\|\mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q} \le M$, by assumption (W3) we get

$$\begin{aligned} \int_{\mathbb{R}^3} |W'(|\mathbf{v}|^2)\mathbf{v}|^{p'} dx &\le c_2 \int_{\{|\mathbf{v}|>1\}} |\mathbf{v}|^{p'(p-1)} dx + c_2 \int_{\{|\mathbf{v}|\le 1\}} |\mathbf{v}|^{p'(q-1)} dx \\ &\le c_2 \int_{\{|\mathbf{v}|>1\}} |\mathbf{v}|^p dx + c_2 \int_{\{|\mathbf{v}|\le 1\}} |\mathbf{v}|^{q'(q-1)} dx \\ &= c_2 \|\mathbf{v}\|^p_{L^p_>} + c_2\|\mathbf{v}\|^q_{L^q_\le} \le c_2 C(M) M, \end{aligned} \tag{2.11}$$

where, in the second inequality, we have used that $q' < p'$. Analogously,

$$\int_{\mathbb{R}^3} |W'(|\mathbf{v}|^2)\mathbf{v}|^{q'} dx \le c_2 C(M) M. \tag{2.12}$$

This shows that the map (2.10) is well defined and bounded on bounded sets.

To prove the continuity, let $\mathbf{v}_n, \mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q$ be such that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}^p + \mathbf{L}^q$. Then, fixing $R > 0$, setting $B_R := \{|x| \le R\}$, the continuous embedding $\mathbf{L}^p + \mathbf{L}^q \hookrightarrow L^p(B_R, \mathbb{R}^3)$, which follows from the Hölder's inequality, implies $\mathbf{v}_n \to \mathbf{v}$ in $L^p(B_R, \mathbb{R}^3)$. According to assumption (W3) we have $|W'(t^2)| \le c|t|^{p-2}$ for every $t \in \mathbb{R}$; the continuity of the Nemitski operator leads to $W'(|\mathbf{v}_n|^2)\mathbf{v}_n \to W'(|\mathbf{v}|^2)\mathbf{v}$ in $L^{p'}(B_R, \mathbb{R}^3)$; consequently, since $q' < p'$, $W'(|\mathbf{v}_n|^2)\mathbf{v}_n \to W'(|\mathbf{v}|^2)\mathbf{v}$ in $L^{q'}(B_R, \mathbb{R}^3)$. Therefore,

$$\int_{B_R} |W'(|\mathbf{v}_n|^2)\mathbf{v}_n - W'(|\mathbf{v}|^2)\mathbf{v}|^{p'} dx \to 0,$$
$$\int_{B_R} |W'(|\mathbf{v}_n|^2)\mathbf{v}_n - W'(|\mathbf{v}|^2)\mathbf{v}|^{q'} dx \to 0.$$

Then take a sequence $R_n \to +\infty$ such that

$$\int_{B_{R_n}} |W'(|\mathbf{v}_n|^2)\mathbf{v}_n - W'(|\mathbf{v}|^2)\mathbf{v}|^{p'} dx \to 0,$$
$$\int_{B_{R_n}} |W'(|\mathbf{v}_n|^2)\mathbf{v}_n - W'(|\mathbf{v}|^2)\mathbf{v}|^{q'} dx \to 0.$$

It is immediate that $\mathbf{v}_n\chi_{\mathbb{R}^3\setminus B_{R_n}} \to 0$ in $\mathbf{L}^p + \mathbf{L}^q$ and then, by Lemma 2.1, $\|\mathbf{v}_n\chi_{\mathbb{R}^3\setminus B_{R_n}}\|_{L^p_>} \to 0$ and $\|\mathbf{v}_n\chi_{\mathbb{R}^3\setminus B_{R_n}}\|_{L^q_\le} \to 0$. Therefore, by applying (2.11)–(2.12) to $\mathbf{v}_n\chi_{\mathbb{R}^3\setminus B_{R_n}}$ we get

$$\int_{\mathbb{R}^3\setminus B_{R_n}} |W'(|\mathbf{v}_n|^2)\mathbf{v}_n|^{p'}dx \to 0, \qquad \int_{\mathbb{R}^3\setminus B_{R_n}} |W'(|\mathbf{v}_n|^2)\mathbf{v}_n|^{q'}dx \to 0.$$

Analogously by $\mathbf{v}\chi_{\mathbb{R}^3\setminus B_{R_n}} \to 0$ in $\mathbf{L}^p + \mathbf{L}^q$ we deduce

$$\int_{\mathbb{R}^3\setminus B_{R_n}} |W'(|\mathbf{v}|^2)\mathbf{v}|^{p'}dx \to 0, \qquad \int_{\mathbb{R}^3\setminus B_{R_n}} |W'(|\mathbf{v}|^2)\mathbf{v}|^{q'}dx \to 0,$$

and the conclusion follows.

Step 2. For any $\mathbf{v}, \mathbf{w} \in \mathbf{L}^p + \mathbf{L}^q$ the following holds:

$$\lim_{t\to 0}\int_{\mathbb{R}^3} \frac{W(|\mathbf{v}+t\mathbf{w}|^2) - W(|\mathbf{v}|^2)}{t}dx = 2\int_{\mathbb{R}^3} W'(|\mathbf{v}|^2)\mathbf{v}\mathbf{w}\,dx. \tag{2.13}$$

Since W is $\mathcal{C}^1$, if we show that $\frac{W(|\mathbf{v}+t\mathbf{w}|^2)-W(|\mathbf{v}|^2)}{t}$ is dominated by an $\mathbf{L}^1$-function independently on t, then by the Lebesgue theorem, we deduce the identity in (2.13). In what follows we assume $|t| < \frac{1}{2}$.

By the Lagrange theorem,

$$\left|\frac{W(|\mathbf{v}+t\mathbf{w}|^2) - W(|\mathbf{v}|^2)}{t}\right| = |2W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}|, \tag{2.14}$$

where $s = s(x,t)$ satisfies $0 < |s| < |t|$. Let us define the sets

$$A = \{x\in\mathbb{R}^3 : |\mathbf{v}| > 1, |\mathbf{w}| > 1\}, \quad B = \{x\in\mathbb{R}^3 : |\mathbf{v}| \le 1, |\mathbf{w}| > 1\},$$
$$C = \{x\in\mathbb{R}^3 : |\mathbf{v}| > 1, |\mathbf{w}| \le 1\}, \quad D = \{x\in\mathbb{R}^3 : |\mathbf{v}| \le 1, |\mathbf{w}| \le 1\}.$$

Then we are reduced to finding an L^1-function on each set which dominates the right-hand side in (2.14).

Let us consider the set A. By Lemma 2.1 $|\mathbf{w}|, |\mathbf{v}| \in L^p(A)$ and, consequently, by Hölder's inequality, $|\mathbf{v}|^{p-1}|\mathbf{w}| \in L^1(A)$. According to the hypothesis (W3) we distinguish two cases:

a) If $|\mathbf{v}+s\mathbf{w}| \le 1$, then

$$|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| \le c_2|\mathbf{v}+s\mathbf{w}|^{q-1}|\mathbf{w}| \le c_2|\mathbf{w}|,$$

and, since A has finite measure, we deduce $|\mathbf{w}| \in L^1(A)$.

b) If $|\mathbf{v}+s\mathbf{w}| \ge 1$, then

$$\begin{aligned}|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| &\le c_2|\mathbf{v}+s\mathbf{w}|^{p-1}|\mathbf{w}| \\ &\le c_2 2^{p-2}\left(|\mathbf{v}|^{p-1}|\mathbf{w}| + |\mathbf{w}|^p\right) \in L^1(A).\end{aligned}$$

We now consider the set B. Since B has finite measure, then $|\mathbf{w}| \in L^p(B) \cap L^1(B)$. Again we distinguish two cases:

a) If $|\mathbf{v}+s\mathbf{w}| \le 1$, then
$$|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| \le c_2|\mathbf{v}+s\mathbf{w}|^{q-1}|\mathbf{w}| \le c_2|\mathbf{w}| \in L^1(B).$$

b) If $|\mathbf{v}+s\mathbf{w}| \ge 1$, then
$$\begin{aligned}|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| &\le c_2|\mathbf{v}+s\mathbf{w}|^{p-1}|\mathbf{w}| \\ &\le c_2 2^{p-2}\left(1+|\mathbf{w}|^{p-1}\right)|\mathbf{w}| \in L^1(B).\end{aligned}$$

On C we have $|\mathbf{v}| \in L^p(C)$. Moreover, since C has finite measure, $|\mathbf{v}| \in L^{p-1}(C)$. So we have the following:

a) If $|\mathbf{v}+s\mathbf{w}| \le 1$, then
$$|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| \le c_2|\mathbf{v}+s\mathbf{w}|^{q-1}|\mathbf{w}| \le 1 \in L^1(C).$$

b) If $|\mathbf{v}+s\mathbf{w}| \ge 1$, then
$$|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| \le c_2|\mathbf{v}+s\mathbf{w}|^{p-1} \le c_2 2^{p-2}\left(|\mathbf{v}|^{p-1}+1\right) \in L^1(C).$$

Finally on D we have $|\mathbf{v}|, |\mathbf{w}| \in L^q(D)$ and again we have the following:

a) If $|\mathbf{v}+s\mathbf{w}| \le 1$, then
$$\begin{aligned}|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| &\le c_2|\mathbf{v}+s\mathbf{w}|^{q-1}|\mathbf{w}| \\ &\le c_2 2^{q-2}\left(|\mathbf{v}|^{q-1}|\mathbf{w}|+|\mathbf{w}|^q\right) \in L^1(D).\end{aligned}$$

b) If $|\mathbf{v}+s\mathbf{w}| \ge 1$, then, since $|s| \le |t| \le \frac{1}{2}$, we deduce $|\mathbf{v}| \ge \frac{1}{2}$ and $|\mathbf{v}| \ge \frac{|\mathbf{w}|}{2}$, by which
$$|W'(|\mathbf{v}+s\mathbf{w}|^2)(\mathbf{v}+s\mathbf{w})\mathbf{w}| \le c_2|\mathbf{v}+s\mathbf{w}|^{p-1}|\mathbf{w}| \le c_2 2^{p-1}|\mathbf{v}|^{p-1}|\mathbf{w}|$$
and, by Hölder's inequality,
$$|\mathbf{v}|^{p-1}|\mathbf{w}| = \frac{|2\mathbf{v}|^{p-1}}{2^{p-1}}|\mathbf{w}| \le \frac{|2\mathbf{v}|^{q-1}}{2^{p-1}}|\mathbf{w}| \in L^1(D).$$

Then the conclusion of step 2 follows.

Step 3. End of the proof.

Since, by Step 1, $W'(|\mathbf{v}|^2)\mathbf{v} \in \mathbf{L}^{p'} \cap \mathbf{L}^{q'}$, it is immediate to prove that (2.9) is linear and continuous. Therefore, by Step 2 we deduce that the map (2.8) is Gâteaux differentiable at $\mathbf{v}$ and its Gâteaux derivative is given by (2.9). The continuity of (2.10) implies that the Gâteaux derivative is a continuous map $\mathbf{L}^p+\mathbf{L}^q \mapsto \mathcal{L}(\mathbf{L}^p+\mathbf{L}^q, \mathbb{R})$, and hence the $\mathcal{C}^1$ regularity of (2.8) follows. Finally, using that (2.10) is bounded on bounded sets, by applying the Lagrange theorem one easily obtains that the map (2.8) is Lipschitz on bounded sets. □

We conclude this section with the following result, which is a consequence of the strict convexity of $W(t^2)$.

Lemma 2.3. *Let $\mathbf{v}_n, \mathbf{v} \in \mathbf{L}^p + \mathbf{L}^q$ be such that $\mathbf{v}_n \rightharpoonup \mathbf{v}$ weakly in $\mathbf{L}^p + \mathbf{L}^q$ and*

$$\int_{\mathbb{R}^3} W(|\mathbf{v}_n|^2)dx \to \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)dx. \tag{2.15}$$

Then $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}^p + \mathbf{L}^q$.

Proof. The first object is to prove that

$$\mathbf{v}_n \to \mathbf{v} \text{ a.e. in } \mathbb{R}^3. \tag{2.16}$$

Since $\frac{\mathbf{v}_n+\mathbf{v}}{2} \rightharpoonup \mathbf{v}$, from the weak lower semicontinuity and the convexity of the map (2.8) we get

$$\begin{aligned} \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)dx &\leq \liminf_{n\to+\infty} \int_{\mathbb{R}^3} W\Big(\Big|\frac{\mathbf{v}_n+\mathbf{v}}{2}\Big|^2\Big)dx \\ &\leq \frac{1}{2}\limsup_{n\to+\infty}\Big(\int_{\mathbb{R}^3} W(|\mathbf{v}_n|^2)dx + \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)dx\Big) = \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)dx, \end{aligned}$$

by which

$$\int_{\mathbb{R}^3} W\Big(\Big|\frac{\mathbf{v}_n+\mathbf{v}}{2}\Big|^2\Big)dx \to \int_{\mathbb{R}^3} W(|\mathbf{v}|^2)dx. \tag{2.17}$$

Observe that the sequence $\int_{\mathbb{R}^3} W(|\mathbf{v}_n|^2)dx$ is bounded, while $W(R^2) \to +\infty$ as $R \to +\infty$; hence, fixing $\varepsilon, r > 0$, there exists $R > r$ such that

$$\text{meas}\{x \in \mathbb{R}^3 : |\mathbf{v}| \geq R\},\ \text{meas}\{x \in \mathbb{R}^3 : |\mathbf{v}_n| \geq R\} \leq \varepsilon \quad \forall n. \tag{2.18}$$

The strict convexity of (2.7) implies

$$a := \min_{\mathbf{z}_1,\mathbf{z}_2\in\mathbb{R}^3,\, r\leq|\mathbf{z}_1-\mathbf{z}_2|,\, |\mathbf{z}_1|,|\mathbf{z}_2|\leq R}\Big(\frac{1}{2}W(|\mathbf{z}_1|^2)+\frac{1}{2}W(|\mathbf{z}_2|^2)-W\Big(\Big|\frac{\mathbf{z}_1+\mathbf{z}_2}{2}\Big|^2\Big)\Big)>0.$$

Now compute

$$\begin{aligned} &a\,\text{meas}\{x \in \mathbb{R}^3 : |\mathbf{v}_n - \mathbf{v}| \geq r,\ |\mathbf{v}_n|, |\mathbf{v}| \leq R\} \\ &\qquad \leq \int_{\mathbb{R}^3}\Big(\frac{1}{2}W(|\mathbf{v}_n|^2) + \frac{1}{2}W(|\mathbf{v}|^2) - W\Big(\Big|\frac{\mathbf{v}_n+\mathbf{v}}{2}\Big|^2\Big)\Big)dx \to 0 \end{aligned}$$

by (2.15) and (2.17). Then by (2.18) we obtain

$$\limsup_{n\to+\infty} \text{meas}\{x \in \mathbb{R}^3 : |\mathbf{v}_n - \mathbf{v}| \geq r\} \leq 2\varepsilon$$

and, by the arbitrariness of ε and r, (2.16) follows.

We are going to prove that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}^p + \mathbf{L}^q$. Fix $\varepsilon > 0$ and consider $R, \delta > 0$ such that

$$\int_{\{|x|\geq R\}} W(|\mathbf{v}|^2)dx \leq \varepsilon, \qquad \int_A W(|\mathbf{v}|^2)dx \leq \varepsilon \tag{2.19}$$

for any measurable set $A \subset \mathbb{R}^3$ such that $\mathrm{meas}(A) \leq \delta$. By Egoroff's theorem, setting $B_R := \{|x| \leq R\}$, there exists a measurable $E \subset B_R$ such that

$$\mathrm{meas}\, E \leq \delta, \quad \mathbf{v}_n \to \mathbf{v} \text{ uniformly in } B_R \setminus E,$$

which implies

$$\int_{B_R\setminus E} W(|\mathbf{v}_n|^2)dx \to \int_{B_R\setminus E} W(|\mathbf{v}|^2)dx, \quad \int_{B_R\setminus E} W(|\mathbf{v}_n - \mathbf{v}|^2)dx \to 0. \tag{2.20}$$

Combining (2.15) with (2.19)–(2.20) we compute

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus B_R} W(|\mathbf{v}_n|^2)dx + \int_E W(|\mathbf{v}_n|^2)dx \\ &\leq \int_{\mathbb{R}^3}\Big(W(|\mathbf{v}_n|^2) - W(|\mathbf{v}|^2)\Big)dx - \int_{B_R\setminus E}\Big(W(|\mathbf{v}_n|^2) - W(|\mathbf{v}|^2)\Big)dx + 2\varepsilon \to 2\varepsilon.\end{aligned}$$

Notice that by the hypothesis (W3) and (2.6) we get $W((2t)^2) \leq CW(t^2)$ for some $C > 0$, by which, using the convexity of (2.7), $\forall\, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^3$,

$$W(|\mathbf{z}_1+\mathbf{z}_2|^2) \leq \frac{1}{2}W(|2\mathbf{z}_1|^2)+\frac{1}{2}W(|2\mathbf{z}_2|^2) \leq CW(|\mathbf{z}_1|^2)+CW(|\mathbf{z}_2|^2). \tag{2.21}$$

Finally, by (2.21) we deduce

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{\mathbb{R}^3} W(|\mathbf{v}_n - \mathbf{v}|^2)dx \\ &\leq \limsup_{n\to+\infty}\int_{B_R\setminus E} W(|\mathbf{v}_n - \mathbf{v}|^2)dx + C\limsup_{n\to+\infty}\int_E \Big(W(|\mathbf{v}_n|^2) + W(|\mathbf{v}|^2)\Big)dx \\ &\quad + C\limsup_{n\to+\infty}\int_{\mathbb{R}^3\setminus B_R}\Big(W(|\mathbf{v}_n|^2) + W(|\mathbf{v}|^2)\Big)dx \leq 2C\varepsilon.\end{aligned}$$

The arbitrariness of ε gives $\int_{\mathbb{R}^3} W(|\mathbf{v}_n - \mathbf{v}|^2)dx \to 0$. By using again the assumption (W3) we have $\|\mathbf{v}_n - \mathbf{v}\|_{L^p_>} \to 0$, $\|\mathbf{v}_n - \mathbf{v}\|_{L^q_\leq} \to 0$ and we conclude

$$\|\mathbf{v}_n - \mathbf{v}\|_{\mathbf{L}^p+\mathbf{L}^q} \leq \|\mathbf{v}_n - \mathbf{v}\|_{L^p_>} + \|\mathbf{v}_n - \mathbf{v}\|_{L^q_\leq} \to 0. \qquad \square$$

3. A natural constraint

Since we are interested in finding solutions to (1.3) with some kind of cylindrical symmetry, we consider the following subspaces of $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R})$:

$$\mathcal{U} = \left\{ \mathbf{u} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) : \mathbf{u}(x) = u(r, x_3)\Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big) + u_3(r, x_3)(0, 0, 1) \right\},$$

$$\mathcal{V} = \left\{ w \in \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R}) : w(x) = w(r, x_3) \right\},$$

where $r = r_x$ has been defined in (1.6). It is obvious that $\mathcal{U}$ and $\mathcal{V}$ are closed subspaces of $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R})$, respectively.

The first step is to solve a minimization problem.

Lemma 3.1. *For every $\mathbf{u} \in \mathbf{L}^p + \mathbf{L}^q$ there exists a unique $\Phi[\mathbf{u}] \in \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R})$ such that*

$$\int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla\Phi[\mathbf{u}]|^2)dx = \inf_{w \in \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R})} \int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla w|^2)dx.$$

Furthermore, the map $\Phi : \mathbf{u} \in \mathbf{L}^p + \mathbf{L}^q \mapsto \Phi[\mathbf{u}] \in \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R})$ is continuous and $\mathbf{u} \in \mathcal{U} \implies \Phi[\mathbf{u}] \in \mathcal{V}$.

Proof. Fixing $\mathbf{u} \in \mathbf{L}^p + \mathbf{L}^q$, we want to apply Weierstrass' theorem to the functional

$$w \in \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R}) \longmapsto \int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla w|^2)dx. \tag{3.1}$$

First of all, it is well known that $\mathbf{L}^p + \mathbf{L}^q$ and, consequently, $\mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R}^3)$, are reflexive spaces (see [6]). Thanks to Lemma 2.2, (3.1) is coercive, strictly convex, and weakly lower semicontinuous. Then, according to Weierstrass' theorem, (3.1) possesses a minimizer. The strict convexity implies that the minimizer $\Phi[\mathbf{u}]$ is unique.

Now take $w \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ and $\mathbf{u} \in \mathcal{U}$ of the form

$$\mathbf{u}(x) = u(r, x_3)\Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big) + u_3(r, x_3)(0, 0, 1) \text{ with } u, u_3 \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}) \tag{3.2}$$

and define

$$\tilde{w}(r, x_3) = \frac{1}{2\pi r} \int_{S_{r,x_3}} w(y)\, d\sigma(y) = \frac{1}{2\pi} \int_{S_{1,x_3}} w(ry_1, ry_2, y_3)\, d\sigma(y),$$

where S_{r,x_3} denotes the circle $S_{r,x_3} := \{(y_1, y_2, x_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 = r^2\}$. It is immediate that $\tilde{w} \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}) \cap \mathcal{V}$ and

$$\frac{\partial \tilde{w}}{\partial x_i}(x) = \frac{x_i}{2\pi r^2} \int_{S_{r,x_3}} \left(\frac{\partial w}{\partial x_1}(y)\frac{y_1}{r} + \frac{\partial w}{\partial x_2}(y)\frac{y_2}{r} \right) d\sigma(y)$$

$$= \frac{x_i}{2\pi r^2}\int_{S_{r,x_3}} \frac{\partial w}{\partial r}(y)d\sigma(y) \quad \text{for } i=1,2,$$
$$\frac{\partial \tilde w}{\partial x_3}(x) = \frac{1}{2\pi r}\int_{S_{r,x_3}} \frac{\partial w}{\partial x_3}(y)\,d\sigma(y).$$

Since

$$u(x) = \frac{1}{2\pi r}\int_{S_{r,x_3}} u(y)\,d\sigma(y), \quad u_3(x) = \frac{1}{2\pi r}\int_{S_{r,x_3}} u_3(y)\,d\sigma(y),$$

we derive

$$\mathbf{u}(x)+\nabla\tilde w(x) = \frac{(x_1,x_2,0)}{2\pi r^2}\int_{S_{r,x_3}}\Big(u(y)+\frac{\partial w}{\partial r}(y)\Big)d\sigma(y) + \frac{(0,0,1)}{2\pi r}\int_{S_{r,x_3}}\Big(u_3(y)+\frac{\partial w}{\partial x_3}(y)\Big)d\sigma(y).$$

Since the function (2.7) is convex, Jensen's inequality applies and gives

$$\begin{aligned} W(|\mathbf{u}(x)+\nabla\tilde w(x)|^2) &\le \frac{1}{2\pi r}\int_{S_{r,x_3}} W\Big(\Big|\frac{(x_1,x_2,0)}{r}\Big(u(y)+\frac{\partial w}{\partial r}(y)\Big) \\ &\quad + (0,0,1)\Big(u_3(y)+\frac{\partial w}{\partial x_3}(y)\Big)\Big|^2\Big)d\sigma(y) \\ &= \frac{1}{2\pi r}\int_{S_{r,x_3}} W\Big(\Big|u(y)+\frac{\partial w}{\partial r}(y)\Big|^2+\Big|u_3(y)+\frac{\partial w}{\partial x_3}(y)\Big|^2\Big)d\sigma(y). \end{aligned} \tag{3.3}$$

By using the cylindrical coordinates (r,θ,x_3), where $\theta = \arctan\frac{x_2}{x_1}$, we have

$$\nabla w(y) = \Big(\frac{y_1}{r}\frac{\partial w}{\partial r}(y) - \frac{y_2}{r^2}\frac{\partial w}{\partial \theta}(y), \frac{y_2}{r}\frac{\partial w}{\partial r}(y)+\frac{y_1}{r^2}\frac{\partial w}{\partial\theta}(y), \frac{\partial w}{\partial x_3}(y)\Big) \quad \forall y\in S_{r,x_3},$$

by which for any $y\in S_{r,x_3}$

$$\begin{aligned} |\mathbf{u}(y)+\nabla w(y)|^2 &= \Big|\frac{(y_1,y_2,0)}{r}\Big(u(y)+\frac{\partial w}{\partial r}(y)\Big) \\ &\quad + (0,0,1)\Big(u_3(y)+\frac{\partial w}{\partial x_3}(y)\Big) + \frac{(-y_2,y_1,0)}{r^2}\frac{\partial w}{\partial\theta}(y)\Big|^2 \\ &= \Big|u(y)+\frac{\partial w}{\partial r}(y)\Big|^2 + \Big|u_3(y)+\frac{\partial w}{\partial x_3}(y)\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial w}{\partial\theta}(y)\Big|^2 \\ &\ge \Big|u(y)+\frac{\partial w}{\partial r}(y)\Big|^2 + \Big|u_3(y)+\frac{\partial w}{\partial x_3}(y)\Big|^2. \end{aligned} \tag{3.4}$$

Let us recall that the function $t\ge 0\mapsto W(t^2)$ is increasing by (W2); consequently, the function $t\ge 0\mapsto W(t)$ is increasing too. Therefore by (3.3) and (3.4) we deduce

$$W(|\mathbf{u}(x)+\nabla\tilde{w}(x)|^2)\le\frac{1}{2\pi r}\int_{S_{r,x_3}}W\big(|\mathbf{u}(y)+\nabla w(y)|^2\big)d\sigma(y)$$

and after integration

$$\int_{\mathbb{R}^3}W(|\mathbf{u}+\nabla\tilde{w}|^2)dx\le\int_{\mathbb{R}^3}W(|\mathbf{u}+\nabla w|^2)dx. \tag{3.5}$$

We have thus proved that (3.5) holds for any $w\in\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R})$ and any $\mathbf{u}\in\mathcal{U}$ of the type (3.2). Now take $\mathbf{u}(x)=u(r,x_3)(\frac{x_1}{r},\frac{x_2}{r},0)+u_3(r,x_3)(0,0,1)\in\mathcal{U}$ and let $u_n,u_{3,n},w_n\in\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R})$ such that $u_n\to u$, $u_{3,n}\to u_3$ in $L^6(\mathbb{R}^3)$, and $w_n\to\Phi[\mathbf{u}]$ in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$. Without loss of generality we may assume that[3] $u_n(x)=u_n(r,x_3)$ and $u_{3,n}(x)=u_{3,n}(r,x_3)$. Then (3.5) holds for $\mathbf{u}_n=u_n(r,x_3)(\frac{x_1}{r},\frac{x_2}{r},0)+u_{3,n}(r,x_3)(0,0,1)$ and w_n. Observe that $\mathbf{u}_n\to\mathbf{u}$ in $\mathbf{L}^6$ and, consequently, in $\mathbf{L}^p+\mathbf{L}^q$ thanks to Lemma 2.1; therefore,

$$\int_{\mathbb{R}^3}W(|\mathbf{u}_n+\nabla\tilde{w}_n|^2)dx\le\int_{\mathbb{R}^3}W(|\mathbf{u}_n+\nabla w_n|^2)dx\to\int_{\mathbb{R}^3}W(|\mathbf{u}+\nabla\Phi[\mathbf{u}]|^2)dx$$

by Lemma 2.2. The coerciveness of the map (2.8) implies that $\tilde{w}_n$ is bounded in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$; hence, up to a subsequence, $\tilde{w}_n\rightharpoonup\phi$ weakly in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ for some $\phi\in\mathcal{V}$. Then by using again Lemma 2.2

$$\begin{aligned}\int_{\mathbb{R}^3}W(|\mathbf{u}+\nabla\phi|^2)dx&\le\liminf_{n\to+\infty}\int_{\mathbb{R}^3}W(|\mathbf{u}_n+\nabla\tilde{w}_n|^2)dx\\&\le\int_{\mathbb{R}^3}W(|\mathbf{u}+\nabla\Phi[\mathbf{u}]|^2)dx.\end{aligned}$$

By uniqueness $\phi=\Phi[\mathbf{u}]\in\mathcal{V}$.

Finally, we are going to show that the map $\Phi:\mathbf{u}\in\mathbf{L}^p+\mathbf{L}^q\to\Phi[\mathbf{u}]\in\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ is continuous. Let $\mathbf{u}_n\to\mathbf{u}$ in $\mathbf{L}^p+\mathbf{L}^q$. Since

$$0\le\int_{\mathbb{R}^3}W\big(|\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n]|^2\big)dx\le\int_{\mathbb{R}^3}W\big(|\mathbf{u}_n|^2\big)dx\to\int_{\mathbb{R}^3}W\big(|\mathbf{u}|^2\big)dx,$$

we deduce that $\mathbf{u}_n+\nabla\Phi(\mathbf{u}_n)$ is bounded in $\mathbf{L}^p+\mathbf{L}^q$, and consequently $\Phi(\mathbf{u}_n)$ is bounded in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$. Then there exists $w\in\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ such that (up to a subsequence) $\Phi(\mathbf{u}_n)\rightharpoonup w$ weakly in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$, by which

$$\mathbf{u}_n+\nabla\Phi(\mathbf{u}_n)\rightharpoonup\mathbf{u}+\nabla w\ \text{ weakly in }\mathbf{L}^p+\mathbf{L}^q;$$

---

[3]Indeed, we first approximate u in $L^6(\mathbb{R}^3)$ by a compactly supported function $v=v(r,z)$ (for example, by multiplying u by a smooth radially symmetric cut-off function), and next we approximate v in $L^6(\mathbb{R}^3)$ by the convolution by a smooth and radially symmetric mollifier ρ_ε, i.e., $(v*\rho_\varepsilon)(x):=\int_{\mathbb{R}^3}v(y)\rho_\varepsilon(x-y)dy\in\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R})$.

so, using the weak lower semicontinuity of the map (2.8), we have

$$\begin{aligned}\int_{\mathbb{R}^3} W\big(|\mathbf{u}+\Phi[\mathbf{u}]|^2\big)dx &= \lim_{n\to+\infty}\int_{\mathbb{R}^3} W\big(|\mathbf{u}_n+\Phi[\mathbf{u}]|^2\big)dx \\ &\geq \liminf_{n\to+\infty}\int_{\mathbb{R}^3} W\big(|\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n]|^2\big)dx \\ &\geq \int_{\mathbb{R}^3} W\big(|\mathbf{u}+\nabla w|^2\big)dx \geq \int_{\mathbb{R}^3} W\big(|\mathbf{u}+\nabla\Phi[\mathbf{u}]|^2\big)dx,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^3} W\big(|\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n]|^2\big)\,dx \longrightarrow \int_{\mathbb{R}^3} W\big(|\mathbf{u}+\nabla\Phi[\mathbf{u}]|^2\big)\,dx.$$

Lemma 2.3 applies and gives

$$\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n] \longrightarrow \mathbf{u}+\nabla\Phi[\mathbf{u}] \text{ in } \mathbf{L}^p+\mathbf{L}^q.$$

So we conclude $\Phi[\mathbf{u}_n]\to\Phi[\mathbf{u}]$ in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ and then Φ is continuous. □

The introduction of the functional set $\mathcal{U}\times\mathcal{V}$ has a crucial role in dealing with the strong-indefiniteness of the equation (1.3). Before going on we need some properties of the space $\mathcal{U}$.

Let $\mathcal{O}(2)$ be the orthogonal group of the rotation matrices on the plane x_1x_2; i.e.,

$$\mathcal{O}(2) := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta\in[0,2\pi] \right\}.$$

Then for any $g\in\mathcal{O}(2)$ consider the following action of the group $\mathcal{O}(2)$ on $\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$:

$$T_g(\mathbf{u})(x) := g^{-1}\mathbf{u}(gx).$$

Now we set $\mathrm{Fix}\,\mathcal{O}(2) = \{\mathbf{u}\in\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3) : T_g\mathbf{u}=\mathbf{u}\ \forall g\in\mathcal{O}(2)\}$. Observe that $\mathcal{U}\subset\mathrm{Fix}\,\mathcal{O}(2)$. We will use a decomposition of the functions in $\mathrm{Fix}\,\mathcal{O}(2)$ provided by the following lemma.

Lemma 3.2. *For every $\mathbf{u}\in\mathrm{Fix}\,\mathcal{O}(2)$ there exists a unique set of three functions $\mathbf{u}_\rho,\mathbf{u}_\tau,\mathbf{u}_\zeta\in\mathrm{Fix}\,\mathcal{O}(2)$ such that $\mathbf{u}=\mathbf{u}_\rho+\mathbf{u}_\tau+\mathbf{u}_\zeta$ and*

$$\begin{aligned}&\mathbf{u}_\rho = u_\rho(r,x_3)\Big(\frac{x_1}{r},\frac{x_2}{r},0\Big), \quad \mathbf{u}_\tau = u_\tau(r,x_3)\Big(-\frac{x_2}{r},\frac{x_1}{r},0\Big),\\ &\mathbf{u}_\zeta = u_\zeta(r,x_3)(0,0,1).\end{aligned}$$

Furthermore, for almost every $x\in\mathbb{R}^3$,

$$\big(\nabla\mathbf{u}_\rho(x)|\nabla\mathbf{u}_\tau(x)\big) = \big(\nabla\mathbf{u}_\rho(x)|\nabla\mathbf{u}_\zeta(x)\big) = \big(\nabla\mathbf{u}_\tau(x)|\nabla\mathbf{u}_\zeta(x)\big) = 0. \tag{3.6}$$

Proof. Set $\mathbf{u} = (u_1, u_2, u_3) \in \operatorname{Fix}\mathcal{O}(2)$ and set

$$\mathbb{R}_{x_3} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}.$$

Denote by $\mathbf{u}_\rho$, $\mathbf{u}_\tau$, and $\mathbf{u}_\zeta$ the projections of the vector field $\mathbf{u}$ along the directions $\boldsymbol{\rho}(x) = (\frac{x_1}{r}, \frac{x_2}{r}, 0)$, $\boldsymbol{\tau}(x) = (-\frac{x_2}{r}, \frac{x_1}{r}, 0)$, and $\boldsymbol{\zeta}(x) = (0, 0, 1)$. By some computations we get $\mathbf{u}_\rho = u_\rho \boldsymbol{\rho}$, $\mathbf{u}_\tau = u_\tau \boldsymbol{\tau}$, and $\mathbf{u}_\zeta = u_\zeta \boldsymbol{\zeta}$, where

$$u_\rho(x) = \frac{u_1 x_1 + u_2 x_2}{r}, \quad u_\tau(x) = \frac{-u_1 x_2 + u_2 x_1}{r}, \quad u_\zeta = u_3. \tag{3.7}$$

By construction we have $\mathbf{u} = \mathbf{u}_\rho + \mathbf{u}_\tau + \mathbf{u}_\zeta$. Furthermore, the uniqueness of the decomposition $(\mathbf{u}_\rho, \mathbf{u}_\tau, \mathbf{u}_\zeta)$ follows from the orthogonality of the vectors $\boldsymbol{\rho}$, $\boldsymbol{\tau}$, and $\boldsymbol{\zeta}$. We are going to prove that u_ρ, u_τ, and u_ζ have cylindrical symmetry; i.e., $u_\rho = u_\rho(r, x_3)$, $u_\tau = u_\tau(r, x_3)$, and $u_\zeta = u_\zeta(r, x_3)$. For every $x \in \mathbb{R}^3 \setminus \mathbb{R}_{x_3}$ consider

$$\theta_x = \begin{pmatrix} \frac{x_1}{r} & \frac{x_2}{r} & 0 \\ -\frac{x_2}{r} & \frac{x_1}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}(2).$$

Observe that, since $\theta_x \mathbf{u}(x) = (u_\rho(x), u_\tau(x), u_\zeta(x))$, it is enough to show that the vector field $x \in \mathbb{R}^3 \to \theta_x \mathbf{u}(x)$ is cylindrically symmetric; that is, $\theta_{gx}\mathbf{u}(gx) = \theta_x \mathbf{u}(x)$ for every $g \in \mathcal{O}(2)$. Indeed, since $\theta_{gx} = \theta_x g^{-1}$ for every $g \in \mathcal{O}(2)$, then we get

$$\theta_{gx}\mathbf{u}(gx) = \theta_x g^{-1}\mathbf{u}(gx) = \theta_x T_g \mathbf{u}(x) = \theta_x \mathbf{u}(x).$$

It remains to prove that

$$\mathbf{u}_\rho, \mathbf{u}_\tau, \mathbf{u}_\zeta \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3), \tag{3.8}$$

which is not immediate because of the presence of the singular term $\frac{1}{r}$ in the definitions (3.7). Notice that, once we have proved (3.8), we conclude immediately $\mathbf{u}_\rho, \mathbf{u}_\tau, \mathbf{u}_\zeta \in \operatorname{Fix}\mathcal{O}(2)$ since $\mathbf{u}_\rho$, $\mathbf{u}_\tau$, and $\mathbf{u}_\zeta$ are fixed points for the action $\mathcal{O}(2)$. By the definition of $\mathbf{u}_\rho$, $\mathbf{u}_\tau$, and $\mathbf{u}_\zeta$ we easily deduce $\mathbf{u}_\rho, \mathbf{u}_\tau, \mathbf{u}_\zeta \in L^6(\mathbb{R}^3, \mathbb{R}^3) \cap H^1_{loc}(\mathbb{R}^3 \setminus \mathbb{R}_{x_3}, \mathbb{R}^3)$. Denote by $\nabla \mathbf{u}_\rho$, $\nabla \mathbf{u}_\tau$, and $\nabla \mathbf{u}_\zeta$ the functions defined almost everywhere in $\mathbb{R}^3$ representing the gradient in the sense of distributions of $\mathbf{u}_\rho$, $\mathbf{u}_\tau$, and $\mathbf{u}_\zeta$ in $\mathbb{R}^3 \setminus \mathbb{R}_{x_3}$. A direct computation shows that for almost every $x \in \mathbb{R}^3$,

$$(\nabla \mathbf{u}_\rho(x) | \nabla \mathbf{u}_\tau(x)) = (\nabla \mathbf{u}_\rho(x) | \nabla \mathbf{u}_\zeta(x)) = (\nabla \mathbf{u}_\tau(x) | \nabla \mathbf{u}_\zeta(x)) = 0.$$

Indeed, the equalities $(\nabla \mathbf{u}_\rho(x) | \nabla \mathbf{u}_\zeta(x)) = (\nabla \mathbf{u}_\tau(x) | \nabla \mathbf{u}_\zeta(x)) = 0$ are immediate and

$$(\nabla \mathbf{u}_\rho(x) | \nabla \mathbf{u}_\tau(x)) = -\nabla\Big(u_\rho \frac{x_1}{r}\Big) \cdot \nabla\Big(u_\tau \frac{x_2}{r}\Big) + \nabla\Big(u_\rho \frac{x_2}{r}\Big) \cdot \nabla\Big(u_\tau \frac{x_1}{r}\Big)$$

$$= -\Big(x_1\nabla\big(\frac{u_\rho}{r}\big) + \Big(\frac{u_\rho}{r}, 0, 0\Big)\Big) \cdot \Big(x_2\nabla\big(\frac{u_\tau}{r}\big) + \Big(0, \frac{u_\tau}{r}, 0\Big)\Big)$$
$$+ \Big(x_2\nabla\big(\frac{u_\rho}{r}\big) + \Big(0, \frac{u_\rho}{r}, 0\Big)\Big) \cdot \Big(x_1\nabla\big(\frac{u_\tau}{r}\big) + \Big(\frac{u_\tau}{r}, 0, 0\Big)\Big)$$
$$= -\frac{x_1}{r}u_\tau\frac{\partial(r^{-1}u_\rho)}{\partial x_2} - \frac{x_2}{r}u_\rho\frac{\partial(r^{-1}u_\tau)}{\partial x_1} + \frac{x_2}{r}u_\tau\frac{\partial(r^{-1}u_\rho)}{\partial x_1} + \frac{x_1}{r}u_\rho\frac{\partial(r^{-1}u_\tau)}{\partial x_2} = 0$$

since $r^{-1}u_\rho$ and $r^{-1}u_\tau$ have a cylindrical symmetry. This implies

$$|\nabla\mathbf{u}|^2 = |\nabla\mathbf{u}_\rho|^2 + |\nabla\mathbf{u}_\tau|^2 + |\nabla\mathbf{u}_\zeta|^2 \quad \text{a.e. in } \mathbb{R}^3$$

and then $\nabla\mathbf{u}_\rho, \nabla\mathbf{u}_\tau, \nabla\mathbf{u}_\zeta \in \mathbf{L}^2$. So (3.8) will follow if we show that $\nabla\mathbf{u}_\rho$, $\nabla\mathbf{u}_\tau$, and $\nabla\mathbf{u}_\zeta$ actually coincide with the distributional gradient of $\mathbf{u}_\rho$, $\mathbf{u}_\tau$, and $\mathbf{u}_\zeta$ in the whole of $\mathbb{R}^3$; in other words, considering the component $\mathbf{u}_\rho$ (the computations for the other components are similar), we have to show that, for any $\mathbf{v} \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, the following holds:

$$\int_{\mathbb{R}^3} \frac{\partial\mathbf{u}_\rho}{\partial x_i} \cdot \mathbf{v}\, dx = -\int_{\mathbb{R}^3} \mathbf{u}_\rho \cdot \frac{\partial\mathbf{v}}{\partial x_i}\, dx. \tag{3.9}$$

Now for all $\varepsilon > 0$ consider a function $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$\eta_\varepsilon = 0 \text{ for } r \le \frac{\varepsilon}{2}, \quad \eta_\varepsilon = 1 \text{ for } r \ge \varepsilon, \quad 0 \le \eta_\varepsilon \le 1, \quad |\nabla\eta_\varepsilon| \le \frac{4}{\varepsilon}.$$

Set $\mathbf{v}_\varepsilon(x) = \mathbf{v}(x)\eta_\varepsilon(x) \in \mathcal{C}_0^\infty(\mathbb{R}^3 \setminus \mathbb{R}_{x_3})$. We have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\partial\mathbf{u}_\rho}{\partial x_i} \cdot \mathbf{v}_\varepsilon\, dx &= -\int_{\mathbb{R}^3} \mathbf{u}_\rho \cdot \frac{\partial\mathbf{v}_\varepsilon}{\partial x_i}\, dx \\ &= -\int_{\mathbb{R}^3} \eta_\varepsilon\mathbf{u}_\rho \cdot \frac{\partial\mathbf{v}}{\partial x_i}\, dx - \int_{\mathbb{R}^3} \frac{\partial\eta_\varepsilon}{\partial x_i}\mathbf{u}_\rho \cdot \mathbf{v}\, dx.\end{aligned} \tag{3.10}$$

Now, by Lebesgue's theorem

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^3} \frac{\partial\mathbf{u}_\rho}{\partial x_i} \cdot \mathbf{v}_\varepsilon\, dx = \int_{\mathbb{R}^3} \frac{\partial\mathbf{u}_\rho}{\partial x_i} \cdot \mathbf{v}\, dx,$$
$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^3} \eta_\varepsilon\mathbf{u}_\rho \cdot \frac{\partial\mathbf{v}}{\partial x_i}\, dx = \int_{\mathbb{R}^3} \mathbf{u}_\rho \cdot \frac{\partial\mathbf{v}}{\partial x_i}\, dx.$$

Let $R > 0$ be such that $\mathbf{v} = 0$ for $|x| \ge R$ and set Ω_ε to be the cylinder $\Omega_\varepsilon := \{|x_3| \le R\} \cap \{r \le \varepsilon\}$. Observe that $\text{meas}(\Omega_\varepsilon) = 2\pi R\varepsilon^2$, so

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \frac{\partial\eta_\varepsilon}{\partial x_i}\mathbf{u}_\rho \cdot \mathbf{v}\, dx\Big| &\le \|\mathbf{v}\|_{L^\infty}\frac{4}{\varepsilon}\int_{\Omega_\varepsilon} |\mathbf{u}_\rho|\, dx \\ &\le \|\mathbf{v}\|_{L^\infty}\frac{4}{\varepsilon}\big(\text{meas}(\Omega_\varepsilon)\big)^{\frac{5}{6}}\Big(\int_{\mathbb{R}^3} |\mathbf{u}_\rho|^6\, dx\Big)^{1/6} \to 0\end{aligned}$$

as ε goes to 0. Letting ε go to 0 in (3.10) the equality (3.9) follows. $\square$

We are going to show that $\mathcal{U} \times \mathcal{V}$ is a *natural* subspace in which to find solutions of (1.3). More precisely, let us define the functional $J : \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ by

$$J(\mathbf{u}, w) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla w|^2) \, dx, \tag{3.11}$$

which is of class $\mathcal{C}^1$ according to Lemma 2.2. We will prove that $\mathcal{U} \times \mathcal{V}$ is a *natural* constraint for J, i.e., the critical points of $J\big|_{\mathcal{U}\times\mathcal{V}}$ are also critical points for J, and next we will show that the critical points of J give rise to solutions to (1.3). The following proposition is the key result of this section.

Proposition 3.3. *Let* $(\mathbf{u}_0, w_0) \in \mathcal{U} \times \mathcal{V}$ *be a critical point of* $J_{|\mathcal{U}\times\mathcal{V}}$*. Then* $\mathbf{A} = \mathbf{u}_0 + \nabla w_0$ *is a weak solution to* (1.3)*.*

Proof. First of all we are going to show that

$$w_0 \text{ is a critical point of } J(\mathbf{u}_0, \cdot) : \mathcal{D}^{p,q}(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R} \tag{3.12}$$

and

$$\mathbf{u}_0 \text{ is a critical point of } J(\cdot, w_0) : \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}. \tag{3.13}$$

Indeed, by Lemma 2.2 it follows that $J(\mathbf{u}_0, \cdot)\big|_{\mathcal{V}}$ is strictly convex; therefore, w_0 is a global minimum for $J(\mathbf{u}_0, \cdot)\big|_{\mathcal{V}}$. Lemma 3.1 implies $w_0 = \Phi[\mathbf{u}_0]$; consequently, w_0 is a global minimum for $J(\mathbf{u}_0, \cdot)$ and hence a critical point of $J(\mathbf{u}_0, \cdot)$. To prove (3.13), let us recall the following Principle of Symmetric Criticality of Palais ([12]):

Principle of symmetric criticality. Assume that there exists a topological group of transformations $\mathcal{G}$ which acts isometrically on a Hilbert space X and define

$$\operatorname{Fix} \mathcal{G} := \{\mathbf{A} \in X : G\mathbf{A} = \mathbf{A} \ \ \forall G \in \mathcal{G}\}. \tag{3.14}$$

If $\mathcal{J} \in \mathcal{C}^1(X, \mathbb{R})$ is invariant under $\mathcal{G}$, i.e.,

$$\mathcal{J}(G\mathbf{A}) = \mathcal{J}(\mathbf{A}) \ \ \forall G \in \mathcal{G}, \forall \mathbf{A} \in X, \tag{3.15}$$

and if $\mathbf{A}$ is a critical point of $\mathcal{J}_{|\operatorname{Fix}\mathcal{G}}$, then $\mathbf{A}$ is a critical point of $\mathcal{J}$.

We are going to apply the above principle to the functional $J(\cdot, w_0)$ with respect to the group $\mathcal{O}(2)$. It is immediate that the action of $\mathcal{O}(2)$ on $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ is isometric. Furthermore, for any $g \in \mathcal{O}(2)$ the cylindrical symmetry of w_0 implies $w_0(gx) = w_0(x)$, by which $g^{-1}\nabla w_0(gx) = \nabla w_0(x)$;

therefore, for every $\mathbf{u} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} W(|T_g\mathbf{u} + \nabla w_0|^2)dx = \int_{\mathbb{R}^3} W(|g^{-1}\mathbf{u}(gx) + g^{-1}\nabla w_0(gx))|^2)dx$$
$$= \int_{\mathbb{R}^3} W(|\mathbf{u}(gx) + \nabla w_0(gx)|^2)dx = \int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla w_0|^2)dx.$$

Then we easily deduce that $J(\cdot, w_0)$ is invariant with respect to $\mathcal{O}(2)$. According to the Principle of Symmetric Criticality every critical point $\mathbf{u}$ of $J(\cdot, w_0)_{|\,\mathrm{Fix}\,\mathcal{O}(2)}$ is a critical point of $J(\cdot, w_0)$. Next we will introduce a new group $\mathcal{G}$ acting on $\mathrm{Fix}\,\mathcal{O}(2)$ and we will apply again the Principle of Symmetric Criticality to the functional $J_{|\,\mathrm{Fix}\,\mathcal{O}(2)}$ with respect to the group $\mathcal{G}$. According to Lemma 3.2, let $\mathcal{S}$ be the action on $\mathrm{Fix}\,\mathcal{O}(2)$ defined by

$$\mathcal{S}(\mathbf{u}) = \mathcal{S}(\mathbf{u}_\rho + \mathbf{u}_\tau + \mathbf{u}_\zeta) = (\mathbf{u}_\rho - \mathbf{u}_\tau + \mathbf{u}_\zeta).$$

We set $\mathcal{G}$ to be the group generated by $\mathcal{S}$; since $\mathcal{S}^2 = id$, we have $\mathcal{G} \approx \mathbb{Z}_2$. By (3.6) we get

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx = \int_{\mathbb{R}^3} |\nabla \mathbf{u}_\rho|^2 dx + \int_{\mathbb{R}^3} |\nabla \mathbf{u}_\tau|^2 dx + \int_{\mathbb{R}^3} |\nabla \mathbf{u}_\zeta|^2 dx = \int_{\mathbb{R}^3} |\nabla \mathcal{S}\mathbf{u}|^2 dx;$$

then the action of $\mathcal{G}$ on $\mathrm{Fix}\,\mathcal{O}(2)$ is isometric. Moreover, we immediately compute $|\mathbf{u}|^2 = |\mathbf{u}_\rho|^2 + |\mathbf{u}_\tau|^2 + |\mathbf{u}_\zeta|^2 = |\mathcal{S}\mathbf{u}|^2$ and $\mathbf{u}_\tau \nabla w_0 = 0$ for all $\mathbf{u} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, which leads to $\int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla w_0|^2)dx = \int_{\mathbb{R}^3} W(|\mathcal{S}\mathbf{u} + \nabla w_0|^2)dx$. Then $J_{|\,\mathrm{Fix}\,\mathcal{O}(2)}$ is invariant under the action of $\mathcal{G}$.

Finally, we have $\mathcal{U} = \mathrm{Fix}\,\mathcal{G} := \{\mathbf{u} \in \mathrm{Fix}\,\mathcal{O}(2) : \mathcal{S}(\mathbf{u}) = \mathbf{u}\}$; indeed, the inclusion $\mathcal{U} \subset \mathrm{Fix}\,\mathcal{G}$ is obvious. On the other hand, if $\mathbf{u} \in \mathrm{Fix}\,\mathcal{G}$, then the invariance under $\mathcal{S}$ implies $\mathbf{u}_\tau = 0$, and consequently $\mathbf{u} = \mathbf{u}_\rho + \mathbf{u}_\zeta \in \mathcal{U}$. Therefore (3.13) follows by applying twice the Principle of Symmetric Criticality.

By (3.12)–(3.13) it follows that $(\mathbf{u}_0, w_0)$ is a critical point of J, and consequently the following equation is satisfied in the weak sense:

$$-\Delta \mathbf{u}_0 = W'(|\mathbf{u}_0 + \nabla w_0|^2)(\mathbf{u}_0 + \nabla w_0).$$

Therefore, for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ we have

$$\int_{\mathbb{R}^3} \nabla \mathbf{u}_0 \nabla(\nabla\varphi)dx = \int_{\mathbb{R}^3} W'(|\mathbf{u}_0 + \nabla w_0|^2)(\mathbf{u}_0 + \nabla w_0)\nabla\varphi = \frac{\partial J(\mathbf{u}_0, w_0)}{\partial w}[\varphi] = 0,$$

by which, using the result of the next Lemma 3.4, $\mathrm{div}\, \mathbf{u}_0 = 0$. Then we conclude

$$-\Delta \mathbf{u}_0 = \nabla \times (\nabla \times \mathbf{u}_0) = \nabla \times (\nabla \times (\mathbf{u}_0 + \nabla w_0))$$

and the conclusion follows. □

Lemma 3.4. *Let* $\mathbf{u} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ *be such that*

$$\int_{\mathbb{R}^3} \nabla\mathbf{u}\nabla(\nabla\varphi)\,dx = 0 \quad \forall\,\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}).$$

Then $\operatorname{div}\mathbf{u} = 0$.

Proof. After integration by parts we have

$$0 = \int_{\mathbb{R}^3} \nabla\mathbf{u}\nabla(\nabla\varphi)\,dx = \int_{\mathbb{R}^3} \operatorname{div}\mathbf{u}\Delta\varphi\,dx \quad \forall\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}).$$

Now let $f \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ and consider $g \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ the solution of

$$\Delta g = f.$$

According to the representation formula we have

$$|g(x)| = \Big|\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)dy\Big| \le \frac{C}{|x|} \quad \forall x \in \mathbb{R}^3$$

for some $C > 0$ and, analogously,

$$|\nabla g(x)| \le \frac{C}{|x|} \quad \forall x \in \mathbb{R}^3.$$

Consider a sequence $\chi_n \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ such that $|\chi_n| \le 1$, $\chi_n = 1$ if $|x| \le n$, $\chi_n = 0$ if $|x| \ge n+1$, $|\nabla\chi_n| \le C$, $|\Delta\chi_n| \le C$ and compute

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\operatorname{div}\mathbf{u}\nabla\chi_n\nabla g|\,dx \\
&\le C\Big(\int_{\{n\le|x|\le n+1\}} |\operatorname{div}\mathbf{u}|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\{n\le|x|\le n+1\}} \frac{1}{|x|^2} dx\Big)^{\frac{1}{2}} \to 0
\end{aligned}$$

and, analogously,

$$\int_{\mathbb{R}^3} \operatorname{div}\mathbf{u}\Delta\chi_n g\,dx \to 0.$$

Then we conclude

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} \operatorname{div}\mathbf{u}\Delta(\chi_n g)\,dx \\
&= \int_{\mathbb{R}^3} \operatorname{div}\mathbf{u}\chi_n f\,dx + 2\int_{\mathbb{R}^3} \operatorname{div}\mathbf{u}\nabla\chi_n\nabla g\,dx + \int_{\mathbb{R}^3} \operatorname{div}\mathbf{u} g\Delta\chi_n\,dx \to \int_{\mathbb{R}^3} \operatorname{div}\mathbf{u} f\,dx
\end{aligned}$$

by which $\int_{\mathbb{R}^3} \operatorname{div}\mathbf{u} f\,dx = 0$ and, by the arbitrariness of f, $\operatorname{div}\mathbf{u} = 0$. □

According to Proposition 3.3 we can solve the equation (1.3) by looking directly for critical points of $J\big|_{\mathcal{U}\times\mathcal{V}}$; in this way we have avoided the strong indefiniteness of the equation (1.3) and we will deal with the functional J, which can be treated with standard methods of nonlinear analysis.

4. A concentration compactness result

The action of the space translation along the x_3-axis causes a lack of compactness in the Sobolev embeddings for the spaces $\mathcal{U}$ and $\mathcal{V}$. To overcome this difficulty we will use the following two lemmas, which provide some results in the spirit of the Concentration Compactness Principle developed by P.L. Lions ([10]–[11]).

Lemma 4.1. *Suppose that $(\mathbf{u}_n)_n$ is bounded in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ and there is $R > 0$ such that*

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,R)} |\mathbf{u}_n|^2 \, dx = 0. \tag{4.1}$$

Then $\mathbf{u}_n \to 0$ in $\mathbf{L}^p + \mathbf{L}^q$.

Proof. Fix $\varepsilon \in (0, 1)$ and for every n consider the new sequence of functions

$$f_n := \begin{cases} |\mathbf{u}_n| & \text{if } |\mathbf{u}_n| \geq \varepsilon \\ |\mathbf{u}_n|^3 \varepsilon^{-2} & \text{if } |\mathbf{u}_n| \leq \varepsilon \end{cases}.$$

It is immediate that

$$|f_n|^2 \leq \varepsilon^{-4} |\mathbf{u}_n|^6, \quad |f_n|^2 \leq |\mathbf{u}_n|^2, \tag{4.2}$$
$$|\nabla f_n|^2 \leq 9 |\nabla |\mathbf{u}_n||^2 \leq 9 |\nabla \mathbf{u}_n|^2.$$

In particular, $f_n \in H^1(\mathbb{R}^3)$ and, using (2.1),

$$\|f_n\|^2_{H^1(\mathbb{R}^3)} \leq \varepsilon^{-4} \int_{\mathbb{R}^3} |\mathbf{u}_n|^6 dx + 9 \int_{\mathbb{R}^3} |\nabla \mathbf{u}_n|^2 dx \leq C\varepsilon^{-4}. \tag{4.3}$$

Furthermore, combining (4.1) with (4.2), we get

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,R)} |f_n|^2 \, dx = 0.$$

A well-known result by P.L. Lions gives

$$f_n \to 0 \text{ in } L^s(\mathbb{R}^3) \ \ \forall\ 2 < s < 6. \tag{4.4}$$

Then, setting $\Omega_n := \{x \in \mathbb{R}^3 : |\mathbf{u}_n| \geq \varepsilon\}$, we conclude

$$\|\mathbf{u}_n\|_{\mathbf{L}^p+\mathbf{L}^q} \leq \|\mathbf{u}_n \chi_{\Omega_n}\|_{L^p} + \|\mathbf{u}_n \chi_{\mathbb{R}^3\setminus\Omega_n}\|_{L^q} \leq \|f_n\|_{L^p} + \varepsilon^{(q-6)/q} \|\mathbf{u}_n\|^{6/q}_{L^6}$$

by which $\limsup_n \|\mathbf{u}_n\|_{\mathbf{L}^p+\mathbf{L}^q} \le C\varepsilon^{\frac{q-6}{q}}$. The arbitrariness of ε gives the result. □

Lemma 4.2. *Let $\mathbf{u}_n, \mathbf{u}_0 \in \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ be such that $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ weakly in $\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$. Then, up to a subsequence, the following holds:*

$$\mathbf{u}_n \to \mathbf{u}_0, \quad \nabla\Phi[\mathbf{u}_n] \to \nabla\Phi[\mathbf{u}_0] \quad \textit{in} \quad (\mathbf{L}^p+\mathbf{L}^q)_{loc}.$$

Proof. We are going to show the existence of sequences $(\bar{\mathbf{u}}_i)_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ and $(x_n^i)_{i,n\in\mathbb{N},\, n\ge i} \subset \mathbb{R}^3$ such that

$$\lim_{n\to+\infty}\Big(\mathbf{u}_n - \sum_{i=0}^{n} \bar{\mathbf{u}}_i(\cdot - x_n^i)\Big) = 0 \ \text{ in } \mathbf{L}^p+\mathbf{L}^q. \tag{4.5}$$

Set $\bar{\mathbf{u}}_0 := \mathbf{u}_0$ and $x_n^0 = 0$. By combining (2.2) and (2.5), up to a subsequence

$$\mathbf{u}_n\chi_{B(0,n)} \to \bar{\mathbf{u}}_0 \text{ in } \mathbf{L}^p+\mathbf{L}^q \text{ and a.e. in } \mathbb{R}^3. \tag{4.6}$$

By (2.1) and (2.4) we get $\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3) \hookrightarrow \mathbf{L}^p+\mathbf{L}^q$; then $(\mathbf{u}_n)_n$ is bounded in $\mathbf{L}^p+\mathbf{L}^q$. Moreover, according to Lemma 2.2, the map (2.8) is Lipschitz and, consequently, bounded on bounded sets; therefore,

$$\int_{\mathbb{R}^3} W(|\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n]|^2)dx \le \int_{\mathbb{R}^3} W(|\mathbf{u}_n|^2)dx \le C.$$

The coerciveness of (2.8) implies that $(\Phi[\mathbf{u}_n])_n$ is bounded in $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$. Since $\mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ is a reflexive space, we may also assume

$$\nabla\Phi[\mathbf{u}_n] \rightharpoonup \nabla\Phi_0 \text{ weakly in } \mathbf{L}^p+\mathbf{L}^q$$

for some $\Phi_0 \in \mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$.

If (4.1) holds for $\mathbf{u}_n - \bar{\mathbf{u}}_0$, then by Lemma 4.1 we deduce $\mathbf{u}_n \to \mathbf{u}_0$ in $\mathbf{L}^p+\mathbf{L}^q$. Otherwise, possibly passing to a subsequence, there exist $c_1 > 0$ and points $x_n^1 \in \mathbb{R}^3$ such that

$$\int_{B(x_n^1,1)} |\mathbf{u}_n - \bar{\mathbf{u}}_0|^2\,dx \ge c_1 \ge \frac{1}{2}\sup_{y\in\mathbb{R}^3}\int_{B(y,1)} |\mathbf{u}_n - \bar{\mathbf{u}}_0|^2\,dx > 0 \text{ for every } n. \tag{4.7}$$

By (4.6) we deduce that, up to a subsequence, $|x_n^1| \ge n-2$. Otherwise $B(x_n^1,1) \subset B(0,n)$ and then, by (4.6), $\|(\mathbf{u}_n - \bar{\mathbf{u}}_0)\chi_{B(x_n^1,1)}\|_{\mathbf{L}^p+\mathbf{L}^q} \to 0$; (2.4) would lead to $\|\mathbf{u}_n - \bar{\mathbf{u}}_0\|_{L^2(B(x_n^1,1))} \to 0$ in contradiction with (4.7).

Proceeding as above, we may assume the existence of $\bar{\mathbf{u}}_1 \in \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ and $\Phi_1 \in \mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ such that

$$\mathbf{u}_n(\cdot + x_n^1) \rightharpoonup \bar{\mathbf{u}}_1 \text{ weakly in } \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3),$$

$$\mathbf{u}_n(\cdot + x_n^1)\chi_{B(0,n)} \to \bar{\mathbf{u}}_1 \text{ in } \mathbf{L}^p + \mathbf{L}^q \text{ and a.e. in } \mathbb{R}^3$$

or, equivalently,

$$\mathbf{u}_n(\cdot + x_n^1)\chi_{B(0,n)} - \bar{\mathbf{u}}_0(\cdot + x_n^1) \to \bar{\mathbf{u}}_1 \text{ in } \mathbf{L}^p + \mathbf{L}^q \text{ and a.e. in } \mathbb{R}^3, \tag{4.8}$$

$$\nabla\Phi[\mathbf{u}_n](\cdot + x_n^1) \rightharpoonup \nabla\Phi_1 \text{ weakly in } \mathbf{L}^p + \mathbf{L}^q.$$

If (4.1) holds for $\mathbf{u}_n - \bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_1(\cdot - x_n^1)$, then Lemma 4.1 implies $\mathbf{u}_n - \bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_1(\cdot - x_n^1) \to 0$ in $\mathbf{L}^p + \mathbf{L}^q$. Let us consider the case that (4.1) does not hold; then, possibly passing to a subsequence, there exist $c_2 > 0$ and points $x_n^2 \in \mathbb{R}^3$ such that for every n

$$\int_{B(x_n^2,1)} |\mathbf{u}_n - \bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_1(\cdot - x_n^1)|^2\,dx \tag{4.9}$$
$$\geq c_2 \geq \frac{1}{2}\sup_{y\in\mathbb{R}^3}\int_{B(y,1)} |\mathbf{u}_n - \bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_1(\cdot - x_n^1)|^2\,dx > 0.$$

Proceeding as above, by using (4.8), we prove that, up to a subsequence, $|x_n^2| \geq n-2$. We also state that, up to a subsequence, $|x_n^1 - x_n^2| \geq n-2$. Otherwise, $B(x_n^2,1) \subset B(x_n^1,n)$ and then, by (4.8), $\|(\mathbf{u}_n - \bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_1(\cdot - x_n^1))\chi_{B(x_n^2,1)}\|_{\mathbf{L}^p+\mathbf{L}^q} \to 0$; (2.4) would lead to $\|\mathbf{u}_n - \bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_1(\cdot - x_n^1)\|_{L^2(B(x_n^2,1))} \to 0$ in contradiction with (4.9).

If this alternative process stops in a finite number of steps, then (4.5) follows by taking the functions $\bar{\mathbf{u}}_i = 0$ from a certain index on. Otherwise, by a diagonal process, we construct a subsequence of $(\mathbf{u}_n)_n$, which we continue to denote by $(\mathbf{u}_n)_n$, a sequence of points $(x_n^0,\dots,x_n^n)_n \subset \mathbb{R}^3$, a sequence of positive numbers $(c_i)_{i\in\mathbb{N}}$, and two sequences of functions $(\bar{\mathbf{u}}_i)_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ and $(\Phi_i)_{i\in\mathbb{N}} \subset \mathcal{D}^{p,q}(\mathbb{R}^3,\mathbb{R})$ such that for every $i,n\in\mathbb{N}$ with $n \geq i$,

$$\int_{B(x_n^{i+1},1)} \Big|\mathbf{u}_n - \sum_{j=0}^{i}\bar{\mathbf{u}}_j(\cdot - x_n^j)\Big|^2 dx \geq c_{i+1}$$
$$\geq \frac{1}{2}\sup_{y\in\mathbb{R}^3}\int_{B(y,1)} \Big|\mathbf{u}_n - \sum_{j=0}^{i}\bar{\mathbf{u}}_j(\cdot - x_n^j)\Big|^2 dx > 0,$$

$$|x_n^i - x_n^j| \geq n-2 \quad \forall i,\, j = 0,\dots,n \text{ with } i \neq j; \tag{4.10}$$

$$\mathbf{u}_n(\cdot + x_n^i) \rightharpoonup \bar{\mathbf{u}}_i \text{ weakly in } \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3), \tag{4.11}$$

$$\mathbf{u}_n(\cdot + x_n^i)\chi_{B(0,n)} \to \bar{\mathbf{u}}_i \text{ in } \mathbf{L}^p + \mathbf{L}^q \text{ and a.e. in } \mathbb{R}^3, \tag{4.12}$$

$$\nabla\Phi[\mathbf{u}_n](\cdot + x_n^i) \rightharpoonup \nabla\Phi_i \text{ weakly in } \mathbf{L}^p + \mathbf{L}^q. \tag{4.13}$$

Moreover, by (4.10), we may also assume

$$\int_{\mathbb{R}^3} \Big|\sum_{i=0}^{n} |\bar{\mathbf{u}}_i(x - x_n^i)|\Big|^6 dx - \sum_{i=0}^{n} \int_{\mathbb{R}^3} |\bar{\mathbf{u}}_i|^6 dx \le \frac{1}{2^n}, \tag{4.14}$$

$$\int_{\mathbb{R}^3} W\Big(\Big|\sum_{i=0}^{n} \big(\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]\big)(\cdot - x_n^i)\Big|^2\Big) dx - \sum_{i=0}^{n} \int_{\mathbb{R}^3} W\big(|\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]|^2\big) dx \le \frac{1}{2^n}. \tag{4.15}$$

By (2.1) and (4.11) it follows that $\mathbf{u}_n(\cdot + x_n^i) \rightharpoonup \bar{\mathbf{u}}_i$ in $\mathbf{L}^6$ for every i; on the other hand, the balls $B(x_n^i, \frac{n-2}{2})$ and $B(x_n^j, \frac{n-2}{2})$ are disjoint for $i \neq j$ by (4.10). Hence, fixing $k \in \mathbb{N}$, using the weak lower semicontinuity we get

$$\begin{aligned}\sum_{i=0}^{k} \int_{\mathbb{R}^3} |\bar{\mathbf{u}}_i|^6 dx &\le \sum_{i=0}^{k} \liminf_{n\to+\infty} \int_{B(0,\frac{n-2}{2})} |\mathbf{u}_n(\cdot + x_n^i)|^6 dx \\ &= \sum_{i=0}^{k} \liminf_{n\to+\infty} \int_{B(x_n^i,\frac{n-2}{2})} |\mathbf{u}_n|^6 dx \le \liminf_{n\to+\infty} \int_{\cup_{i=0}^k B(x_n^i,\frac{n-2}{2})} |\mathbf{u}_n|^6 dx \le C\end{aligned}$$

for some $C > 0$. In a similar way, by using (4.13) and the weak lower semicontinuity of the map (2.8),

$$\begin{aligned}&\sum_{i=0}^{k} \int_{\mathbb{R}^3} W\big(|\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]|^2\big) dx \\ &\le \liminf_{n\to+\infty} \int_{\cup_{i=0}^k B(x_n^i,\frac{n-2}{2})} W\big(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2\big) dx \le C\end{aligned}$$

for some $C > 0$. Letting $k \to +\infty$ in the above inequalities, (4.14), (4.15), and the coerciveness of the map (2.8) imply that the sequences

$$\sum_{i=0}^{n} |\bar{\mathbf{u}}_i(\cdot - x_n^i)| \ \text{ and } \ \sum_{i=0}^{n} \nabla\Phi[\bar{\mathbf{u}}_i](\cdot - x_n^i) \ \text{ are bounded in } \mathbf{L}^p + \mathbf{L}^q. \tag{4.16}$$

We are proving that $c_i \le 2c_j$ if $i > j$. Indeed, let α denote the volume of the unit ball $B(0,1)$ and compute

$$\begin{aligned}\sqrt{c_{i+1}} &\le \Big\|\mathbf{u}_n - \sum_{k=0}^{i} \bar{\mathbf{u}}_k(\cdot - x_n^k)\Big\|_{L^2(B(x_n^{i+1},1))} \\ &\le \Big\|\mathbf{u}_n - \sum_{k=0}^{j} \bar{\mathbf{u}}_k(\cdot - x_n^k)\Big\|_{L^2(B(x_n^{i+1},1))} + \sum_{k=j+1}^{i} \|\bar{\mathbf{u}}_k(\cdot - x_n^k)\|_{L^2(B(x_n^{i+1},1))}\end{aligned}$$

$$\leq \sqrt{2c_{j+1}} + \alpha^{1/3} \sum_{k=j+1}^{i} \|\bar{\mathbf{u}}_k\|_{L^6(B(x_n^{i+1}-x_n^k,1))}.$$

The above inequalities hold for every n; letting $n \to +\infty$ and using (4.10) we deduce $c_{i+1} \leq 2c_{j+1}$. We observe that for every $n \geq 4$

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^3} \Big\|\mathbf{u}_n - \sum_{i=0}^{n} \bar{\mathbf{u}}_i(\cdot - x_n^i)\Big\|^6_{L^2(B(x,1))} \leq 2^3 c_{n+1}^3 \leq \frac{2^6}{n}\sum_{i=1}^{n} c_i^3 \\
&\leq \frac{2^6}{n}\sum_{i=1}^{n} \Big\|\mathbf{u}_n - \sum_{j=0}^{i-1} \bar{\mathbf{u}}_j(\cdot - x_n^j)\Big\|^6_{L^2(B(x_n^i,1))} \\
&\leq \frac{2^6}{n}\alpha^2\sum_{i=1}^{n} \Big\|\mathbf{u}_n - \sum_{j=0}^{i-1} \bar{\mathbf{u}}_j(\cdot - x_n^j)\Big\|^6_{L^6(B(x_n^i,1))} \\
&\leq \frac{2^6}{n}\alpha^2\sum_{i=1}^{n} \Big\||\mathbf{u}_n| + \sum_{j=0}^{n} |\bar{\mathbf{u}}_j(\cdot - x_n^j)|\Big\|^6_{L^6(B(x_n^i,1))} \\
&= \frac{2^6}{n}\alpha^2 \Big\||\mathbf{u}_n| + \sum_{j=0}^{n} |\bar{\mathbf{u}}_j(\cdot - x_n^j)|\Big\|^6_{L^6(\cup_{i=1}^n B(x_n^i,1))},
\end{aligned}$$

where the last equality holds since by (4.10) the balls $(B(x_n^i,1))_{i=1,\dots,n}$ are disjoint for $n \geq 4$. Then by using (4.16) we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^3} \Big\|\mathbf{u}_n - \sum_{i=0}^{n} \bar{\mathbf{u}}_i(\cdot - x_n^i)\Big\|^6_{L^2(B(x,1))} \\
&\leq \frac{2^{12}}{n}\alpha^2\|\mathbf{u}_n\|^6_{L^6} + \frac{2^{12}}{n}\alpha^2\Big\|\sum_{j=0}^{n} |\bar{\mathbf{u}}_j(\cdot - x_n^j)|\Big\|^6_{L^6} \to 0
\end{aligned}$$

as $n \to +\infty$. Lemma 4.1 applies and gives (4.5).

It remains to show how by (4.5) the conclusion of the lemma easily follows. Indeed, fixing $\varepsilon > 0$, for n sufficiently large we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \leq \int_{\mathbb{R}^3} W\Big(\Big|\mathbf{u}_n + \sum_{i=0}^{n} \nabla\Phi[\bar{\mathbf{u}}_i](\cdot - x_n^i)\Big|^2\Big)dx \\
&\leq \int_{\mathbb{R}^3} W\Big(\Big|\sum_{i=0}^{n} \big(\bar{\mathbf{u}}_i(\cdot - x_n^i) + \nabla\Phi[\bar{\mathbf{u}}_i](\cdot - x_n^i)\big)\Big|^2\Big)dx + \varepsilon
\end{aligned}$$

$$\leq \sum_{i=0}^{n} \int_{\mathbb{R}^3} W(|\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]|^2)dx + \varepsilon + \frac{1}{2^n},$$

where we have used (4.15) and, taking into account (4.16), the uniform continuity of the map (2.8) on bounded sets (see Lemma 2.2). The arbitrariness of ε gives

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \leq \sum_{i=0}^{+\infty} \int_{\mathbb{R}^3} W(|\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]|^2)dx. \quad (4.17)$$

Since $\mathbf{u}_n(\cdot + x_n^i) + \nabla\Phi[\mathbf{u}_n](\cdot + x_n^i) \rightharpoonup \bar{\mathbf{u}}_i + \nabla\Phi_i$ in $\mathbf{L}^p + \mathbf{L}^q$ by (4.12) and (4.13), for every i the weak lower semicontinuity of (2.8) gives

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{B(x_n^i, \frac{n-2}{2})} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \qquad (4.18)\\ &\geq \liminf_{n\to+\infty} \int_{B(x_n^i, \frac{n-2}{2})} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \\ &\geq \int_{\mathbb{R}^3} W(|\bar{\mathbf{u}}_i + \nabla\Phi_i|^2)dx \geq \int_{\mathbb{R}^3} W(|\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]|^2)dx. \end{aligned}$$

We claim that the inequalities in (4.18) are actually equalities. Indeed, assume for the sake of contradiction that for some $\hat{\imath} \in \mathbb{N}$ and $a > 0$ one has

$$\limsup_{n\to+\infty} \int_{B(x_n^{\hat{\imath}}, \frac{n-2}{2})} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \geq \int_{\mathbb{R}^3} W(|\bar{\mathbf{u}}_{\hat{\imath}} + \nabla\Phi[\bar{\mathbf{u}}_{\hat{\imath}}|^2)dx + a.$$

Then, for any $k > \hat{\imath}$, by (4.10) we deduce

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \qquad (4.19)\\ &\geq \sum_{i=0}^{k} \liminf_{n\to+\infty} \int_{B(x_n^i, \frac{n-2}{2})} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \\ &\geq \sum_{i=0}^{k} \int_{\mathbb{R}^3} W(|\bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i]|^2)dx + a. \end{aligned}$$

By comparing (4.17) and (4.19) and taking into account the arbitrariness of k, we get a contradiction. Hence we conclude that the inequalities in (4.18) are actually equalities. In particular, $\Phi_i = \Phi[\bar{\mathbf{u}}_i]$ for all i and, by applying Lemma 2.3,

$$(\mathbf{u}_n(\cdot + x_n^i) + \nabla\Phi[\mathbf{u}_n](\cdot + x_n^i))\chi_{B(0,\frac{n-2}{2})} \to \bar{\mathbf{u}}_i + \nabla\Phi[\bar{\mathbf{u}}_i] \text{ in } \mathbf{L}^p + \mathbf{L}^q \ \ \forall i,$$

by which, using (4.12),

$$\nabla\Phi[\mathbf{u}_n](\cdot + x_n^i)\chi_{B(0,\frac{n-2}{2})} \to \nabla\Phi[\bar{\mathbf{u}}_i] \text{ in } \mathbf{L}^p + \mathbf{L}^q \ \forall i.$$

The conclusion follows by taking $i = 0$. □

5. Proof of the main theorem

According to Proposition 3.3 a natural method to solve (1.3) would be to look for critical points of $J|_{\mathcal{U}\times\mathcal{V}}$. Anyway, rather than working directly on $J|_{\mathcal{U}\times\mathcal{V}}$, first we will consider a constrained minimization method. Then set

$$\Sigma := \left\{\mathbf{u} \in \mathcal{U} : \int_{\mathbb{R}^3} W(|\mathbf{u} + \nabla\Phi[\mathbf{u}]|^2)\, dx = 1\right\}.$$

Observe that Σ is not empty. Indeed, an easy computation shows that for any $\mathbf{u} \in \mathcal{U}$ there exists a suitable rescaling parameter $\sigma > 0$ such that the function $\mathbf{u}(\sigma x)$ belongs to Σ. Now consider the following constrained minimization problem:

$$\min\left\{\int_{\mathbb{R}^3} |\nabla\mathbf{u}|^2\, dx : \mathbf{u} \in \Sigma\right\}. \tag{5.1}$$

We will show that the solutions of the problem (5.1) give rise, modulo rescaling, to solutions of the equation (1.3).

Let us begin with analyzing the behaviour of the minimizing sequences for (5.1).

Proposition 5.1. *There exists a minimizing sequence of* (5.1) *which weakly converges to a function* $\mathbf{u} \in \mathcal{U} \setminus \{0\}$.

Proof. Let $(\mathbf{u}_n)_n \subset \Sigma$ be a minimizing sequence of (5.1). We claim that for every $R > 0$

$$\liminf_{n\to+\infty} \sup_{x\in\mathbb{R}^3} \int_{B(x,R)} |\mathbf{u}_n|^2\, dx > 0. \tag{5.2}$$

Otherwise, by Lemma 4.1, up to as subsequence $\mathbf{u}_n \to 0$ in $\mathbf{L}^p + \mathbf{L}^q$ and, consequently, by Lemma 2.2, $\int_{\mathbb{R}^3} W(|\mathbf{u}_n|^2)dx \to 0$, which implies

$$\int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)dx \le \int_{\mathbb{R}^3} W(|\mathbf{u}_n|^2)dx \to 0,$$

which contradicts $\mathbf{u}_n \in \Sigma$. From (5.2) we deduce the existence of $R > 0$, $\varepsilon > 0$, and a sequence $(x_n)_n = ((x_{n,1}, x_{n,2}, x_{n,3}))_n$ in $\mathbb{R}^3$ such that

$$\int_{B(x_n,R)} |\mathbf{u}_n|^2\, dx \ge \varepsilon, \quad \forall n.$$

From the Hölder's inequality we get

$$\int_{B(x_n,R)} |\mathbf{u}_n|^6\,dx \geq \delta, \quad \forall n$$

for a suitable $\delta > 0$. We claim that $(x_n)_n$ is bounded in the directions x_1 and x_2. Indeed, the cylindrical symmetry of $|\mathbf{u}_n|$ implies that $\int_{B(x,R)} |\mathbf{u}_n|^6\,dx \geq \delta$ for every $x = (x_1, x_2, x_{n,3})$ such that, using the notation (1.6), $r_x = \sqrt{x_1^2 + x_2^2} = \sqrt{x_{n,1}^2 + x_{n,2}^2} = r_{x_n}$. Geometric arguments assure that the number of disjoint balls of the kind $B(x,R)$ with $r_x = r_{x_n}$ grows as r_{x_n} grows. Then the boundedness of $(\mathbf{u}_n)_n$ in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ and, consequently, in $\mathbf{L}^6$ put an upper bound to the sequence $(r_{x_n})_n$. If we relabel $(\mathbf{u}_n)_n$ the sequence obtained making the translation in the x_3-direction, i.e., $\mathbf{u}_n(\cdot + x_{n,3}\ell_3)$ (with $\ell_3 = (0,0,1)$), we obtain a new minimizing sequence in $\mathcal{U}$ satisfying, possibly increasing the radius R,

$$\int_{B(0,R)} |\mathbf{u}_n|^2\,dx \geq \varepsilon \quad \forall n \in \mathbb{N}. \tag{5.3}$$

Since $(\mathbf{u}_n)_n$ is bounded in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, there exists $\mathbf{u} \in \mathcal{U}$ such that, up to a subsequence, $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$. On the other hand, by (2.4) we have $\mathbf{u}_n \to \mathbf{u}$ in $L^2(B(0,R), \mathbb{R}^3)$, by which $\int_{B(0,R)} |\mathbf{u}|^2 dx \geq \varepsilon$, and we conclude that $\mathbf{u} \neq 0$. $\square$

Proof of Theorem 1.1 According to Proposition 5.1, let $(\mathbf{u}_n)_n \subset \Sigma$ be a minimizing sequence for (5.1) and let $\mathbf{u}_0 \in \mathcal{U} \setminus \{0\}$ be such that $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ weakly in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$. First we introduce the following notation:

$$\forall \mathbf{u} \in \mathcal{U},\ \forall \mu > 0:\ \mathbf{u}_\mu(x) = \mathbf{u}(\mu x).$$

An easy computation shows that $\Phi[\mathbf{u}_\mu](x) = \mu^{-1}\Phi[\mathbf{u}](\mu x)$. Consider $\mathbf{v} \in \mathcal{U}$, $\mathbf{v}$ compactly supported. For every $n \in \mathbb{N}$ and $t > 0$ set

$$\mu_{n,t} := \Big(\int_{\mathbb{R}^3} W(|\mathbf{u}_n + t\mathbf{v} + \nabla\Phi[\mathbf{u}_n + t\mathbf{v}]|^2)\,dx \Big)^{\frac{1}{3}},$$

so that $(\mathbf{u}_n + t\mathbf{v})_{\mu_{n,t}} \in \Sigma$. Since $(\mathbf{u}_n)_n$ is a minimizing sequence, for every $t > 0$ we deduce

$$\begin{aligned} \liminf_{n\to+\infty} & \frac{\displaystyle\int_{\mathbb{R}^3} |\nabla(\mathbf{u}_n + t\mathbf{v})_{\mu_{n,t}}|^2\,dx - \int_{\mathbb{R}^3} |\nabla\mathbf{u}_n|^2\,dx}{t} \\ & = \frac{\displaystyle\liminf_{n\to+\infty} \int_{\mathbb{R}^3} |\nabla(\mathbf{u}_n + t\mathbf{v})_{\mu_{n,t}}|^2\,dx - \theta}{t} \geq 0, \end{aligned} \tag{5.4}$$

where $\theta = \inf_{\mathbf{u}\in\Sigma} \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx$. Consider two sequences $t_n, \varepsilon_n > 0$, $t_n \to 0^+, \varepsilon_n \searrow 0^+$; then for every $n \geq 1$ by (5.4) there exists $\mathbf{u}_{k_n}$ such that

$$\frac{\displaystyle\int_{\mathbb{R}^3} |\nabla(\mathbf{u}_{k_n} + t_n \mathbf{v})_{\mu_{k_n,t_n}}|^2\, dx - \int_{\mathbb{R}^3} |\nabla \mathbf{u}_{k_n}|^2\, dx}{t_n} \geq -\varepsilon_n.$$

As a consequence we can extract a subsequence $(\mathbf{u}_{k_n})_n$ from $(\mathbf{u}_n)_n$, which we relabel $(\mathbf{u}_n)_n$, such that, setting $\mu_n = \mu_{k_n,t_n}$,

$$\liminf_{n\to+\infty} \frac{\displaystyle\int_{\mathbb{R}^3} |\nabla(\mathbf{u}_n + t_n \mathbf{v})_{\mu_n}|^2\, dx - \int_{\mathbb{R}^3} |\nabla \mathbf{u}_n|^2\, dx}{t_n} \geq 0. \tag{5.5}$$

On the other hand, for every n we have

$$\begin{aligned}
&\frac{\displaystyle\int_{\mathbb{R}^3} |\nabla(\mathbf{u}_n + t_n \mathbf{v})_{\mu_n}|^2\, dx - \int_{\mathbb{R}^3} |\nabla \mathbf{u}_n|^2\, dx}{t_n} \\
&= \frac{\displaystyle\int_{\mathbb{R}^3} \frac{1}{\mu_n} |\nabla(\mathbf{u}_n + t_n \mathbf{v})|^2\, dx - \int_{\mathbb{R}^3} |\nabla \mathbf{u}_n|^2\, dx}{t_n} \\
&= \frac{1}{\mu_n}\Big[\frac{(1-\mu_n)}{t_n} \int_{\mathbb{R}^3} |\nabla \mathbf{u}_n|^2 dx + t_n \int_{\mathbb{R}^3} |\nabla \mathbf{v}|^2\, dx + 2\int_{\mathbb{R}^3} (\nabla \mathbf{u}_n | \nabla \mathbf{v})\, dx\Big].
\end{aligned} \tag{5.6}$$

By using the minimizing property of Φ and Lemma 2.2, we get

$$\begin{aligned}
&2t_n \int_{\mathbb{R}^3} W'(|\mathbf{u}_n + s_n\mathbf{v} + \nabla\Phi[\mathbf{u}_n + t_n\mathbf{v}]|^2)(\mathbf{u}_n + s_n\mathbf{v} + \nabla\Phi[\mathbf{u}_n + t_n\mathbf{v}])\mathbf{v}\, dx \\
&= \int_{\mathbb{R}^3} W(|\mathbf{u}_n + t_n\mathbf{v} + \nabla\Phi[\mathbf{u}_n + t_n\mathbf{v}]|^2)\, dx - \int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n + t_n\mathbf{v}]|^2)\, dx \\
&\leq \int_{\mathbb{R}^3} W(|\mathbf{u}_n + t_n\mathbf{v} + \nabla\Phi[\mathbf{u}_n + t_n\mathbf{v}]|^2)\, dx - \int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)\, dx \\
&\leq \int_{\mathbb{R}^3} W(|\mathbf{u}_n + t_n\mathbf{v} + \nabla\Phi[\mathbf{u}_n]|^2)\, dx - \int_{\mathbb{R}^3} W(|\mathbf{u}_n + \nabla\Phi[\mathbf{u}_n]|^2)\, dx \\
&= 2t_n \int_{\mathbb{R}^3} W'(|\mathbf{u}_n + s_n'\mathbf{v} + \nabla\Phi[\mathbf{u}_n]|^2)(\mathbf{u}_n + s_n'\mathbf{v} + \nabla\Phi[\mathbf{u}_n])\mathbf{v}\, dx
\end{aligned}$$

for suitable $s_n, s_n' \in (0, t_n)$. Since $\mathbf{u}_n + t_n\mathbf{v} \rightharpoonup \mathbf{u}_0$ in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, by applying Lemma 4.2 we get

$$\begin{aligned}
&(\mathbf{u}_n + s_n\mathbf{v} + \nabla\Phi[\mathbf{u}_n + t_n\mathbf{v}])\chi_{\operatorname{supp}\mathbf{v}},\ (\mathbf{u}_n + s_n'\mathbf{v} + \nabla\Phi[\mathbf{u}_n])\chi_{\operatorname{supp}\mathbf{v}} \\
&\to (\mathbf{u}_0 + \nabla\Phi[\mathbf{u}_0])\chi_{\operatorname{supp}\mathbf{v}} \quad \text{in} \quad \mathbf{L}^p + \mathbf{L}^q.
\end{aligned}$$

Therefore, by using again Lemma 2.2, we arrive at

$$\frac{1}{t_n}\int_{\mathbb{R}^3}\Big(W(|\mathbf{u}_n+t_n\mathbf{v}+\nabla\Phi[\mathbf{u}_n+t_n\mathbf{v}]|^2)\,dx-W(|\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n]|^2)\Big)\,dx$$
$$\to 2\int_{\mathbb{R}^3}W'(|\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0]|^2)(\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0])\mathbf{v}\,dx.$$

Since $\mathbf{u}_n\in\Sigma$, we compute

$$\begin{aligned}\frac{1-\mu_n}{t_n}&=\frac{1-\mu_n^3}{t_n(1+\mu_n+\mu_n^2)}\\&=\frac{\int_{\mathbb{R}^3}W(|\mathbf{u}_n+\nabla\Phi[\mathbf{u}_n]|^2)\,dx-\int_{\mathbb{R}^3}W(|\mathbf{u}_n+t_n\mathbf{v}+\nabla\Phi[\mathbf{u}_n+t_n\mathbf{v}]|^2)\,dx}{t_n(1+\mu_n+\mu_n^2)}\\&=-\frac{2\int_{\mathbb{R}^3}W'(|\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0]|^2)(\mathbf{u}_0+\nabla\Phi_0)\mathbf{v}\,dx+o(1)}{(1+\mu_n+\mu_n^2)}.\end{aligned}\tag{5.7}$$

By (5.7), since $t_n\to 0^+$, we have

$$\lim_{n\to+\infty}\mu_n=1\tag{5.8}$$

and then, again from (5.7),

$$\lim_{n\to+\infty}\frac{1-\mu_n}{t_n}=-\frac{2}{3}\int_{\mathbb{R}^3}W'(|\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0]|^2)(\mathbf{u}_0+\nabla\Phi_0)\cdot\mathbf{v}\,dx.\tag{5.9}$$

Inserting (5.8)–(5.9) in (5.6) and using (5.5), we deduce that

$$\int_{\mathbb{R}^3}(\nabla\mathbf{u}_0|\nabla\mathbf{v})\,dx-\frac{\theta}{3}\int_{\mathbb{R}^3}W'(|\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0]|^2)(\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0])\cdot\mathbf{v}\,dx\geq 0.$$

Replacing $\mathbf{v}$ by $-\mathbf{v}$ and repeating the same arguments as before, we have that the opposite inequality holds too, and then

$$\int_{\mathbb{R}^3}(\nabla\mathbf{u}_0|\nabla\mathbf{v})\,dx-\frac{\theta}{3}\int_{\mathbb{R}^3}W'(|\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0]|^2)(\mathbf{u}_0+\nabla\Phi[\mathbf{u}_0])\cdot\mathbf{v}\,dx=0.\tag{5.10}$$

We have thus proved that (5.10) holds for every $\mathbf{v}\in\mathcal{U}$ compactly supported. Hence by density we obtain that (5.10) holds for every $\mathbf{v}\in\mathcal{U}$. The constrained minimization method has caused a Lagrange multiplier θ to appear in (5.10). We remark that $\theta>0$; otherwise, by (5.10), taking $\mathbf{v}=\mathbf{u}_0$, we would obtain $\int_{\mathbb{R}^3}|\nabla\mathbf{u}_0|^2dx=0$; i.e., $\mathbf{u}_0=0$, which is a contradiction. The Lagrange multiplier θ can be removed by a rescaling argument: set

$$\bar{\mathbf{u}}(x)=\mathbf{u}_0\Big(\sqrt{\tfrac{3}{\theta}}x\Big)\in\mathcal{U}.$$

By (5.10) it is easy to verify that

$$\int_{\mathbb{R}^3} (\nabla \bar{\mathbf{u}} | \nabla \mathbf{v}) dx = \int_{\mathbb{R}^3} W'(|\bar{\mathbf{u}} + \nabla\Phi[\bar{\mathbf{u}}]|^2)(\bar{\mathbf{u}} + \nabla\Phi[\bar{\mathbf{u}}]) \cdot \mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathcal{U},$$

which means that $\bar{\mathbf{u}}$ satisfies

$$\frac{\partial J_{|\mathcal{U}}}{\partial \mathbf{u}}(\bar{\mathbf{u}}, \Phi[\bar{\mathbf{u}}]) = 0.$$

On the other hand the definition of Φ implies

$$\frac{\partial J_{|\mathcal{U}}}{\partial w}(\bar{\mathbf{u}}, \Phi[\bar{\mathbf{u}}]) = 0;$$

i.e., $(\bar{\mathbf{u}}, \Phi[\bar{\mathbf{u}}])$ is a critical point of $J_{|\mathcal{U}}$ and, consequently, by Proposition 3.3, $\bar{\mathbf{u}} + \Phi[\bar{\mathbf{u}}]$ is a weak solution of (1.3).

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