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QUENCHING RATE OF SOLUTIONS FOR A SEMILINEAR PARABOLIC EQUATION

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Abstract. We study the behavior of solutions of the Cauchy problem for a semilinear parabolic equation with a singular absorption term. We discuss the convergence of solutions to a singular stationary solution from above as time goes to infinity, and show that in a supercritical case a sharp estimate of the quenching rate can be determined explicitly when a specific growth rate of initial data is given. We also obtain a universal lower bound of the quenching rate which implies the optimality of the results. Proofs are given by a comparison method that is based on matched asymptotic expansion. We first determine a quenching rate of solutions by a formal analysis. Based on the formal analysis, we give a rigorous proof by constructing appropriate super- and subsolutions with the desired quenching rate.

1. Introduction

In this paper, we investigate the behavior of solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta u - u^{-q}, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, Δ is the Laplace operator with respect to x, $q > 0$, $N \geq 3$, and $u_0 > 0$ is a given positive continuous function on $\mathbb{R}^N$ that grows to infinity as $|x| \to \infty$ at most of polynomial order. Problems that include singular nonlinear term like (1.1) arise from some applications in mechanics and physics, and have been studied in many papers since Kawarada studied the quenching problem in [23]. For further details and earlier studies, we can consult [1, 2, 3, 4, 6, 7, 12, 15, 16, 17, 26], Levine's survey [24, 25] and the references therein. There exist various interesting problems for (1.1),

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such as time global existence, stability, instability and so on. Among them, infinite-time quenching phenomenon in view of a stability of a stationary solution is one of these interesting problems, and we study the problem (1.1) along this line. For the sake of convenience, we first introduce the following definition.

Definition 1.1. *Let $u(x,t)$ be a classical solution of* (1.1)*. We say that $u(x,t)$ is an infinite-time quenching solution or infinite-time quenching occurs if $u(x,t)$ is a global-in-time solution of* (1.1) *and satisfies* $\inf_{x\in\mathbb{R}^N} u(x,t) \to 0$ *as $t\to\infty$.*

For a stability analysis, we begin to recall fundamental properties of the stationary solution for (1.1). Let $\varphi = \varphi_\alpha(r)$, $r = |x|$, and $\alpha > 0$, denote a radial stationary solution of (1.1), where $\varphi_\alpha(0) = \alpha$. Then $\varphi_\alpha(r)$ solves the initial-value problem

$$\begin{cases} \varphi_{\alpha,rr} + \frac{N-1}{r}\varphi_{\alpha,r} - \varphi_\alpha^{-q} = 0, \quad r > 0, \\ \varphi_\alpha(0) = \alpha, \quad \varphi_{\alpha,r}(0) = 0. \end{cases} \tag{1.2}$$

It is known that the solution φ_α is strictly increasing in $|x|$ and satisfies $\varphi_\alpha \to \infty$ as $|x| \to \infty$ for each $\alpha > 0$. Furthermore, a growth rate of φ_α is obtained in [16], and also in [12] for a more general problem.

The exponent

$$q_c := \begin{cases} \frac{N-2\sqrt{N-1}}{2\sqrt{N-1}-(N-4)} = \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & 3 \le N < 10, \\ \infty & N \ge 10, \end{cases}$$

which appeared first in [22] in a different context, is important for our problem. Guo and Wei [17] showed that for $q > q_c$, any pair of positive stationary solutions intersects each other, and that for $0 < q \le q_c$, the family of stationary solutions forms a simply ordered set; namely, φ_α is strictly increasing in α for each x by the same argument as that in [29]. We call it the ordering property of $\{\varphi_\alpha\}$. Moreover, the set connects a singular stationary solution and ∞ in the sense that φ_α satisfies

$$\lim_{\alpha\to\infty} \varphi_\alpha(|x|) = \infty, \quad \lim_{\alpha\to 0} \varphi_\alpha(|x|) = \varphi_0(|x|),$$

for each x, where $\varphi_0(|x|)$ is a singular stationary solution given by

$$\varphi_0(|x|) = L|x|^m, \quad x \in \mathbb{R}^N \setminus \{0\},$$

with $m = \frac{2}{q+1}$, $L = \{m(N-2+m)\}^{-1/(q+1)}$. It was also shown in [17] that each positive stationary solution has the expansion

$$\varphi_\alpha(|x|) = \begin{cases} L|x|^m + a_\alpha |x|^{-(\lambda_1 - m)} + \text{h.o.t.} & 0 < q < q_c, \\ L|x|^m + a_\alpha |x|^{-(\lambda_1 - m)} \log|x| + \text{h.o.t.} & q = q_c, \end{cases} \tag{1.3}$$

as $|x| \to \infty$, where $\lambda_1 > m$ is a positive constant given by

$$\lambda_1 = \lambda_1(N, q) := \frac{N - 2 + 2m - \sqrt{(N-2+2m)^2 - 8(N-2+m)}}{2},$$

and $a_\alpha = a(\alpha)$ is a positive number that is monotone increasing in α. Note that λ_1 is a smaller root of the quadratic equation

$$h(\lambda) := \lambda^2 - (N - 2 + 2m)\lambda + 2(N - 2 + m) = 0.$$

We define by

$$\lambda_2 = \lambda_2(N, q) := \frac{N - 2 + 2m + \sqrt{(N-2+2m)^2 - 8(N-2+m)}}{2}$$

a larger root of the quadratic equation.

The most important and related results for our stability analysis on (1.1) were recently obtained by Guo and Wei [17, 18, 19]. They showed that any regular positive radial stationary solution is unstable in any reasonable sense if $q > q_c$ and weakly asymptotically stable in a weighted L^∞ norm if $0 < q \le q_c$ in [17, 18]. Later, in [19], for $0 < q < q_c$, they improved the above results and showed global attractive properties of the stationary solutions. Namely, they showed that the solution approaches a given stationary solution for a wide class of the initial data. Moreover, they showed the existence of an infinite-time quenching solution of (1.1).

These results are obtained by developing the ideas of Wang [29], Gui-Ni-Wang [13, 14] and Poláčik-Yanagida [28] for the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

for which many papers have been published since the pioneer work of Fujita [10]. As relevant results to our problem, it is shown in [8, 9] with supercritical exponent in the Joseph and Lundgren sense that if the initial data is close enough to a singular stationary solution near the spatial infinity, then the solution grows up (i.e., converges to a singular stationary solution of (1.4)), and its rate, which only depends on the spatial decay rate of the initial data, was discussed. They also found a universal upper bound of the grow-up rate which applies to an initial data without assumption on a spatial decay rate in [9], and Mizoguchi improved their result in [27], which implies optimality of their results in [8].

Again for the problem (1.1), we recently obtained a sharp convergence rate of approach to a given regular stationary solution in [20] by using a technique similar to that for (1.4) in [21].

Now in view of their results for (1.1) and (1.4), it is expected that the singular stationary solution of (1.1) is asymptotically stable from above. Then, our purpose is to obtain a sufficient condition of the initial data that induce infinite-time quenching and a sharp estimate of the quenching rate together with the stability. In fact, we can show the following results.

Theorem 1.2. *Let $0 < q < q_c$. Suppose that u_0 satisfies*

$$\varphi_0(|x|) \le u_0(x) \le \varphi_0(|x|) + c_1(1+|x|)^{-l}, \quad x \in \mathbb{R}^N \setminus \{0\}$$

with some $c_1 > 0$.

(i) *If $\lambda_1 - m < l < \lambda_2 - m + 2$, then the solution of* (1.1) *is an infinite-time quenching solution and there exists a constant $C_1 > 0$ such that*

$$u(0,t) \le C_1(1+t)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}} \quad \text{for all } t > 0.$$

(ii) *If $l \ge \lambda_2 - m + 2$, then the solution of* (1.1) *is an infinite-time quenching solution and for any small $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$u(0,t) \le C_\varepsilon(1+t)^{-\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}+\varepsilon} \quad \text{for all } t > 0.$$

Our next theorem shows that if u_0 approaches φ_0 faster in space, then we have a slightly better estimate than that in Theorem 1.2 (ii).

Theorem 1.3. *Let $0 < q < q_c$. Suppose that u_0 satisfies*

$$\varphi_0(|x|) \le u_0(x) \le \varphi_0(|x|) + c_1 \exp(-\nu|x|^2), \quad x \in \mathbb{R}^N \setminus \{0\}$$

with some constants $c_1 > 0$ and $\nu > 0$. Then the solution of (1.1) *is an infinite-time quenching solution and there exists a constant $C_1 > 0$ such that*

$$u(0,t) \le C_1(1+t)^{-\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}} \quad \text{for all } t > 0.$$

The next result shows that the quenching rate is sharp.

Theorem 1.4. *Let $0 < q < q_c$ and $\lambda_1 - m < l < \lambda_2 - m + 2$. Suppose that u_0 satisfies $u_0(x) > \varphi_0(|x|)$ and*

$$u_0(x) \ge \varphi_0(|x|) + c_2|x|^{-l}, \quad x \in \mathbb{R}^N \setminus B_R(0)$$

with some $c_2 > 0$ *and* $R > 0$*, where* $B_R(0)$ *is a ball in* $\mathbb{R}^N$ *centered at the origin with radius* R*. Then there exists a constant* $C_2 > 0$ *such that*

$$u(0,t) \geq C_2(1+t)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}} \quad \text{for all } t > 0.$$

As mentioned above, Theorem 1.2 (i) is no longer valid for large l, as in the grow-up results for (1.4) and some convergence results for (1.1). In fact, we can find a universal lower bound for the quenching rate which applies to an initial data without any assumption on the spatial decay rate to the singular stationary solution.

Theorem 1.5. *Let* $0 < q < q_c$*. Suppose that* u_0 *satisfies*

$$u_0(x) > \varphi_0(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Then for any $\varepsilon > 0$*, there exists a constant* $\tilde{C}_\varepsilon > 0$ *such that*

$$u(0,t) \geq \tilde{C}_\varepsilon(1+t)^{-\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}-\varepsilon} \quad \text{for all } t > 0.$$

This lower bound implies that the quenching rate of Theorem 1.2 (i) is valid only up to the range $l < \lambda_2 - m + 2$.

Proofs of the above theorems are obtained by a comparison technique that is based on a matched asymptotic expansion. This expansion consists of two parts which are called the inner expansion and the outer expansion. The inner expansion is used to approximate the behavior of solutions near the origin, and the outer expansion is used to approximate the behavior of solutions near the spatial infinity. Then we intend to construct suitable super- and subsolutions by combining these inner and outer solutions. In fact, we can find a suitable supersolution similar in spirit to that in [9, 20], Moreover, we will construct a suitable subsolution directly and note that the method is different from the previous one in [5, 9]. That is to say, we do not use an intersection argument (see Section 4).

This paper is organized as follows. In Section 2, we calculate a formal analysis. The formal analysis in this section will give the idea of constructing super- and subsolutions, and a matching condition of these two expansions leads to an expected quenching rate. Sections 3 and 4 are devoted to deriving an upper bound and a lower bound of the quenching rate respectively. In Section 5, we derive a universal lower bound of the quenching rate.

2. Formal analysis and preliminary results

In this section, we carry out a formal asymptotic expansion of infinite-time quenching solutions that quench at the origin, and show some results which appear on formal analysis. Throughout this and the following sections, we always assume $0 < q < q_c$.

2.1. **Formal matched asymptotics.** We consider radial solutions $U = U(r,t)$, $r = |x|$ of (1.1). Namely, let $U(r,t)$ be a solution of

$$\begin{cases} U_t = U_{rr} + \frac{N-1}{r}U_r - U^{-q}, \quad r > 0, \quad t > 0, \\ U_r(0,t) = 0, \quad t > 0, \\ U(r,0) = U_0(r), \quad r \geq 0, \end{cases} \tag{2.1}$$

where U_0 is a positive continuous function that satisfies $U_0(r) \geq \varphi_0(r)$ and grows at most of polynomial order. From the comparison principle, we see that $U(r,t) \geq \varphi_0(r)$ for all $t > 0$. In the following, we calculate an asymptotic analysis for (2.1), which is only formal but will be useful in the rigorous analysis in subsequent sections.

First, we carry out a formal analysis near the origin. Using a method similar in spirit to that in [8, 9], we set $U(r,t) = \sigma(t)g(\xi,t)$, where $\sigma(t) := U(0,t)$ and $\xi = \xi(r,t)$. Substituting

$$U_t = \sigma_t g + \sigma(g_t + g_\xi \xi_t), \quad U_r = \sigma g_\xi \xi_r, \quad U_{rr} = \sigma g_{\xi\xi}\xi_r^2 + \sigma g_\xi \xi_{rr}$$

in (2.1), we obtain

$$g_{\xi\xi} + \frac{N-1}{r\xi_r}g_\xi - \frac{\sigma^{-(q+1)}}{\xi_r^2}g^{-q} = \frac{\sigma_t}{\sigma\xi_r^2}g + \frac{1}{\xi_r^2}g_t + \frac{\xi_t}{\xi_r^2}g_\xi - \frac{\xi_{rr}}{\xi_r^2}g_\xi. \tag{2.2}$$

Here we take ξ such that $r\xi_r = \xi$ and $\sigma^{-(q+1)}/\xi_r^2 = 1$; that is,

$$\xi = \sigma^{-1/m} r. \tag{2.3}$$

Note that for each $r > 0$ if $\sigma(t) \to 0$ as $t \to \infty$, then $\xi \to \infty$ as $t \to \infty$. From (2.3), it follows that (2.2) is rewritten as

$$g_{\xi\xi} + \frac{N-1}{\xi}g_\xi - g^{-q} = \sigma^{q+1}g_t + \sigma^q\sigma_t\Big(g - \frac{q+1}{2}\xi g_\xi\Big). \tag{2.4}$$

Assuming that

$$|\sigma^{q+1}g_t| \ll |\sigma^q\sigma_t| \ll 1,$$

we may put

$$g = \psi(\xi) + \sigma_t\sigma^q\Phi(\xi,t), \tag{2.5}$$

where ψ satisfies

$$\begin{cases} \psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi - \psi^{-q} = 0, \quad \xi > 0, \\ \psi(0) = 1, \quad \psi_\xi(0) = 0. \end{cases} \tag{2.6}$$

We note that the expansion like this was first used by Galaktionov and King [11]. By (1.3), $\psi(\xi) = \varphi_1(\xi)$ satisfies

$$\psi(\xi) = L\xi^m + a\xi^{-(\lambda_1 - m)} + \text{h.o.t.}, \quad \text{as } \xi \to \infty \tag{2.7}$$

with some constant $a = a_1 > 0$.

Substituting (2.5) in (2.4), we have

$$\begin{aligned}\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \sigma_t\sigma^q\Big(\Phi_{\xi\xi} + \frac{N-1}{\xi}\Phi_\xi\Big) - \Big(\psi + \sigma_t\sigma^q\Phi\Big)^{-q} \\ = \sigma_t\sigma^q\Big(\psi - \frac{q+1}{2}\xi\psi_\xi\Big) + o(|\sigma_t\sigma^q|), \quad \text{as } \xi \to \infty.\end{aligned}$$

Here, by (2.6)

$$\begin{aligned}\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi - (\psi + \sigma_t\sigma^q\Phi)^{-q} = \psi^{-q} - (\psi + \sigma_t\sigma^q\Phi)^{-q} \\ = q\psi^{-(q+1)}\sigma_t\sigma^q\Phi + o(\sigma_t\sigma^q\psi^{-(q+1)}\Phi), \quad \text{as } \xi \to \infty.\end{aligned}$$

Hence Φ satisfies

$$\Phi_{\xi\xi} + \frac{N-1}{\xi}\Phi_\xi + q\psi^{-(q+1)}\Phi + o(\psi^{-(q+1)}\Phi) = \psi - \frac{q+1}{2}\xi\psi_\xi.$$

Here, by (2.7)

$$q\psi^{-(q+1)}(\xi) = \frac{qL^{-(q+1)}}{\xi^2} + \text{h.o.t.}, \quad \text{as } \xi \to \infty$$

and

$$\begin{aligned}\psi - \frac{q+1}{2}\xi\psi_\xi &= (L\xi^m + a\xi^{-(\lambda_1-m)} + \text{h.o.t.}) \\ &\quad - \frac{1}{m}\xi(mL\xi^{m-1} - a(\lambda_1 - m)\xi^{-(\lambda_1-m)-1} + \text{h.o.t.}) \\ &= a\xi^{-(\lambda_1-m)} + \frac{a}{m}(\lambda_1 - m)\xi^{-(\lambda_1-m)} + \text{h.o.t.} \\ &= \frac{a\lambda_1}{m}\xi^{-(\lambda_1-m)} + \text{h.o.t.}, \quad \text{as } \xi \to \infty.\end{aligned}$$

This implies that Φ is expanded as

$$\Phi = \mathrm{K}\xi^{2-(\lambda_1-m)} + \text{h.o.t.}, \quad \text{as } \xi \to \infty,$$

where K is a constant determined from

$$\mathrm{K}[(2+m-\lambda_1)(1+m-\lambda_1) + (N-1)(2+m-\lambda_1) + (2-m)(N-2+m)] = \frac{a\lambda_1}{m}.$$

After some computation, we obtain

$$\mathrm{K} = \frac{a\lambda_1}{2m(N+2m-2\lambda_1)} = \frac{a\lambda_1}{2m(\lambda_2-\lambda_1+2)} > 0.$$

Thus, for each $r>0$, the formal expansion near the origin is written as

$$\begin{aligned} U &= \sigma(\psi+\sigma_t\sigma^q\Phi) = \sigma(L\xi^m + a\xi^{-(\lambda_1-m)} + \mathrm{K}\sigma_t\sigma^q\xi^{2-(\lambda_1-m)} + \text{h.o.t.}) \\ &= Lr^m + a\sigma^{\frac{\lambda_1}{m}}r^{-(\lambda_1-m)} + \mathrm{K}\sigma_t\sigma^{\frac{\lambda_1-m}{m}}r^{2-(\lambda_1-m)} + \text{h.o.t.}, \end{aligned} \tag{2.8}$$

as $\xi = \sigma^{-1/m}r \to \infty$. Next, we consider a formal expansion of infinite-time quenching solutions near $r=\infty$. Setting

$$V(r,t) = U(r,t) - \varphi_0(r) = U - Lr^m,$$

we have

$$V_t = V_{rr} + \frac{N-1}{r}V_r + (Lr^m)^{-q} - (Lr^m+V)^{-q}.$$

By the binomial theorem, $V(r,t)$ satisfies approximately

$$V_t = V_{rr} + \frac{N-1}{r}V_r + \frac{qL^{-(q+1)}}{r^2}V, \quad r \simeq \infty. \tag{2.9}$$

Following [8, 9, 20, 21], we assume that V is of a self-similar form

$$V(r,t) = t^{-l/2}F(\eta), \quad \eta = t^{-1/2}r. \tag{2.10}$$

Then we see that F must satisfy

$$F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{\eta}{2}F_\eta + \frac{l}{2}F + \frac{qL^{-(q+1)}}{\eta^2}F = 0, \quad \eta > 0 \tag{2.11}$$

by substituting (2.10) into (2.9). The outer expansion matches with the inner solution (2.8) if $F(\eta)$ satisfies

$$\lim_{\eta\to 0}\eta^{\lambda_1-m}F(\eta) = \gamma_0 \tag{2.12}$$

in view of these spatial order, where γ_0 is an arbitrary positive constant depending on the initial data. On the other hand, F is required to satisfy

$$\lim_{\eta\to\infty}\eta^{l}F(\eta) = \gamma_\infty \tag{2.13}$$

with some positive constant γ_∞ in view of the linearity of (2.11) and spatial decay rate to the singular stationary solution. In the next subsection, we will see that (2.11) has a positive solution satisfying (2.12) and (2.13) for $\lambda_1 - m < l < \lambda_2 - m + 2$. In this case, we obtain the two-term expansion in the outer region

$$U \simeq \varphi_0 + t^{-l/2}F(\eta) = Lr^m + t^{-l/2}F(\eta) \quad \eta = t^{-l/2}r \simeq \infty. \tag{2.14}$$

In the inner region, where ψ dominates $\sigma_t \sigma^q \Psi$, matching the inner expansion (2.8) with the outer expansion (2.14) for $1 \ll r \ll \sqrt{t}$, we obtain

$$a\sigma^{\frac{\lambda_1}{m}} r^{-(\lambda_1 - m)} \simeq t^{-l/2} F(\eta).$$

Then, we have

$$\begin{aligned}\sigma^{\frac{\lambda_1}{m}} &\simeq r^{\lambda_1 - m} t^{-(l/2)} F(\eta) = r^{\lambda_1 - m} t^{-(\lambda_1 - m)/2} t^{(\lambda_1 - m)/2} t^{-(l/2)} F(\eta) \\ &= t^{-(l+m-\lambda_1)/2} \eta^{\lambda_1 - m} F(\eta) \simeq t^{-(l+m-\lambda_1)/2},\end{aligned}$$

which leads to the exact quenching rate given in Theorem 1.2 (i). Thus we obtain the quenching rate by the formal analysis.

2.2. **Properties of self-similar solutions.** In this subsection, we recall the behavior of solutions of (2.11) satisfying

$$\lim_{\eta \to 0} \eta^{\lambda_1 - m} F(\eta) = \gamma_0,$$

where $\gamma_0 > 0$ is an arbitrary constant. To this end, we set $f(\eta) = \eta^{\lambda_1 - m} F(\eta)$. Then from (2.11) we see that f must satisfy

$$\begin{cases} f_{\eta\eta} + \frac{N-1-2(\lambda_1 - m)}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{l-(\lambda_1 - m)}{2} f = 0, \quad \eta > 0, \\ f(0) = \gamma_0 > 0, \quad f_\eta(0) = 0. \end{cases} \tag{2.15}$$

The following lemma classifies the behavior of f as $\eta \to \infty$, and implies that $l = \lambda_2 - m + 2$ is critical.

Lemma 2.1. *Let f be the solution of* (2.15)*.*

(i) *If $\lambda_1 - m < l < \lambda_2 - m + 2$, then $f > 0$ and $f_\eta < 0$ for all $\eta > 0$. Moreover, for each $\eta' > 0$, there exist $d_-(\eta') > 0$ such that*

$$f(\eta) \geq d_-(\eta') \eta^{-(l+m-\lambda_1)} \quad \text{for} \quad \eta \geq \eta',$$

and $d_+ > 0$ such that

$$f(\eta) \leq d_+ \eta^{-(l+m-\lambda_1)} \quad \text{for all} \quad \eta > 0.$$

(ii) *If $l = \lambda_2 - m + 2$, then $f(\eta)$ is given explicitly by $f(\eta) = \gamma_0 e^{-\eta^2/4}$.*

(iii) *If $l > \lambda_2 - m + 2$, then $f(\eta)$ vanishes at some finite η.*

Proof. This proof proceeds word for word as Lemma 3.1 in [8], so we omit the proof here. We note that $l \in (\lambda_1 - m, \lambda_2 - m + 2)$ is equivalent to $\beta \in (0, n)$, which is exactly the same as that in [8], where $n := N - 2(\lambda_1 - m)$ and $\beta := l + m - \lambda_1$. This notation is used in the following sections. □

Remark 2.2. We have $f_\eta < 0$ as long as $f > 0$ in the case of Lemma 2.1 (iii). Indeed, we see from (2.15) that f does not attain a positive local minimum.

3. Upper bound

The aim of this section is to derive an upper bound of the quenching rate of solutions for (2.1) in the case of $\lambda_1 - m < l < \lambda_2 - m + 2$ by using an argument similar to that in [8, 9, 20]. That is to say, our strategy is to combine two supersolutions which have desired rates in some inner and outer regions.

3.1. Outer supersolution.

First, we give an outer supersolution $U^+_{\text{out}}(r,t)$.

Lemma 3.1. *If $l > \lambda_1 - m$, then*

$$U^+_{\text{out}}(r,t) := Lr^m + b(t+\tau)^{-\frac{l}{2}}F(\eta), \quad \eta = (t+\tau)^{-1/2}r,$$

is a supersolution of (2.1), where b and τ are arbitrary positive constants.

Proof. Using (2.11) and the fact that $Lr^m = \varphi_0$ is a stationary solution of (2.1), we have

$$\begin{aligned}
&U^+_{\text{out},t} - U^+_{\text{out},rr} - \frac{N-1}{r}U^+_{\text{out},r} + (U^+_{\text{out}})^{-q} \\
&= -\frac{bl}{2}(t+\tau)^{-\frac{l}{2}-1}F - \frac{b}{2}(t+\tau)^{-\frac{l}{2}-1}\eta F_\eta - b(t+\tau)^{-\frac{l}{2}-1}F_{\eta\eta} \\
&\quad - b(t+\tau)^{-\frac{l}{2}-1}\frac{N-1}{\eta}F_\eta - (Lr^m)_{rr} - \frac{N-1}{r}(Lr^m)_r \\
&\quad + \Big(Lr^m + b(t+\tau)^{-\frac{l}{2}}F(\eta)\Big)^{-q} \\
&\geq -b(t+\tau)^{-\frac{l}{2}-1}\Big(\frac{l}{2}F + \frac{\eta}{2}F_\eta + F_{\eta\eta} + \frac{N-1}{\eta}F_\eta\Big) - (Lr^m)_{rr} - \frac{N-1}{r}(Lr^m)_r \\
&\quad + (Lr^m)^{-q} - q(Lr^m)^{-(q+1)}b(t+\tau)^{-\frac{l}{2}}F \\
&= -b(t+\tau)^{-\frac{l}{2}-1}\Big(\frac{l}{2}F + \frac{\eta}{2}F_\eta + F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{qL^{-(q+1)}}{\eta^2}F\Big) \\
&\quad - \Big((Lr^m)_{rr} + \frac{N-1}{r}(Lr^m)_r - (Lr^m)^{-q}\Big) = 0
\end{aligned}$$

for all $r, t > 0$. This completes the proof. $\square$

Lemma 3.2. *Let $\lambda_1 - m < l < \lambda_2 - m + 2$. Suppose that $U_0 = U_0(r)$ is positive and satisfies*

$$U_0(r) \le Lr^m + c_1(1+r)^{-l} \quad \text{for } r \ge 0$$

with some constant $c_1 > 0$. Then it follows that the solution of (2.1) satisfies

$$U(r,t) \le U^+_{\mathrm{out}}(r,t) \quad \text{for all } r,t \ge 0$$

with some $b, \tau > 0$. In particular, there exists $c_0 > 0$ such that the solution of (2.1) satisfies

$$U(r,t) \le U^+_{\mathrm{out}}(r,t) \le Lr^m + c_0 r^{-l} \quad \text{for all } r,t \ge 0.$$

Proof. For any $B_0 > 0$, we consider the supersolution $U^+_{\mathrm{out}}(r,t)$ given in Lemma 3.1 with $\tau = 1$ and $b = \max(\frac{c_1}{d_-(B_0)}, \frac{c_1 B_0^{\lambda_1 - m}}{f(B_0)})$, where f and $d_-(B_0) > 0$ are as in (2.15) and Lemma 2.1 respectively. Let us first show that U^+_{out} lies above U initially by dividing the interval of $r \ge 0$ into two parts as follows. For $r \ge B_0$, we can use Lemma 2.1 (i) to estimate

$$\begin{aligned} U^+_{\mathrm{out}}(r,0) &= Lr^m + bF(r) = Lr^m + br^{-(\lambda_1 - m)} f(r) \\ &\ge Lr^m + bd_-(B_0) r^{-l} \ge Lr^m + c_1 r^{-l} \\ &\ge Lr^m + c_1(1+r)^{-l} \ge U_0(r). \end{aligned}$$

On the other hand, for $0 \le r < B_0$, since f is monotone decreasing we see

$$\begin{aligned} U^+_{\mathrm{out}}(r,0) &= Lr^m + bF(r) = Lr^m + br^{-(\lambda_1 - m)} f(r) \\ &\ge Lr^m + bB_0^{-(\lambda_1 - m)} f(B_0) \ge Lr^m + c_1 \\ &\ge Lr^m + c_1(1+r)^{-l} \ge U_0(r). \end{aligned}$$

Thus, we have shown that $U^+_{\mathrm{out}}(r,0) \ge U_0(r)$ for all $r \ge 0$. Then it follows from Lemma 3.1 and the comparison principle that

$$U(r,t) \le U^+_{\mathrm{out}}(r,t) \quad \text{for all } r,t \ge 0.$$

In particular, we obtain

$$\begin{aligned} U^+_{\mathrm{out}}(r,t) &= Lr^m + b(t+1)^{-\frac{l}{2}} F(\eta) \\ &= Lr^m + b(t+1)^{-\frac{l}{2}} \eta^{-(\lambda_1 - m)} f(\eta) \\ &\le Lr^m + b(t+1)^{-\frac{l}{2}} \eta^{-(\lambda_1 - m)} d_+ \eta^{-(l+m-\lambda_1)} = Lr^m + bd_+ r^{-l} \end{aligned}$$

by using Lemma 2.1 (i) again. Now, we take $c_0 = bd_+$; then the proof is complete. ∎

Lemma 3.3. *Let $l = \lambda_2 - m + 2$. Suppose that $U_0 = U_0(r)$ is positive and satisfies*

$$U_0(r) \leq Lr^m + c_1 \exp(-\nu r^2) \quad \textit{for } r \geq 0$$

with some constants $c_1 > 0$ and $\nu > 0$. Then it follows that the solution of (2.1) *satisfies*

$$U(r,t) \leq U^+_{\mathrm{out}}(r,t) \quad \textit{for all } r, t \geq 0$$

with some $b, \tau > 0$. In particular, there exists $c_0 > 0$ such that the solution of (2.1) *satisfies*

$$U(r,t) \leq U^+_{\mathrm{out}}(r,t) \leq Lr^m + c_0 r^{-(\lambda_2 - m + 2)} \quad \textit{for all } r, t \geq 0.$$

Proof. Set $l = \lambda_2 - m + 2$. We choose $\tau > 0$ so large that $\tau > 1/(4\nu)$, and let $U^+_{\mathrm{out}}(r,t)$ be the supersolution given in Lemma 3.1. At $t = 0$, we see that

$$\begin{aligned} U^+_{\mathrm{out}}(r,0) &= Lr^m + b\tau^{-l/2} F(\tau^{-1/2} r) \\ &= Lr^m + b\tau^{-(l+m-\lambda_1)/2} r^{-(\lambda_1 - m)} \gamma_0 \exp(-r^2/4\tau) \\ &= Lr^m + b\tau^{-(l+m-\lambda_1)/2} r^{-(\lambda_1 - m)} \gamma_0 \exp((\nu - 1/4\tau) r^2) \exp(-\nu r^2) \\ &\geq Lr^m + b\tau^{-(l+m-\lambda_1)/2} m_0 \gamma_0 \exp(-\nu r^2), \end{aligned}$$

where m_0 is given by

$$m_0 := \min_{r>0} \left\{ r^{-(\lambda_1 - m)} \exp\{(\nu - 1/4\tau) r^2\} \right\} = \left(\frac{2(\nu - 1/4\tau)e}{\lambda_1 - m} \right)^{(\lambda_1 - m)/2}.$$

Hence, if we take $b > 0$ such that $b\tau^{-(l+m-\lambda_1)/2} m_0 \gamma_0 \geq c_1$, then

$$\begin{aligned} U^+_{\mathrm{out}}(r,0) &\geq Lr^m + b\tau^{-(l+m-\lambda_1)/2} m_0 \gamma_0 \exp(-\nu r^2) \\ &\geq Lr^m + c_1 \exp(-\nu r^2) \geq U_0(r) \end{aligned}$$

for all $r > 0$. Therefore we see by the comparison principle that

$$U^+_{\mathrm{out}}(r,t) \geq U(r,t) \quad \text{for all } r, t \geq 0.$$

In particular, we obtain

$$\begin{aligned} U^+_{\mathrm{out}}(r,t) &= Lr^m + b(t+\tau)^{-\frac{l}{2}} F(\eta) \\ &= Lr^m + b(t+\tau)^{-\frac{l}{2}} \eta^{-(\lambda_1 - m)} f(\eta) \\ &\leq Lr^m + b(t+\tau)^{-\frac{l}{2}} \eta^{-(\lambda_1 - m)} \gamma_0 \exp(-\eta^2/4) \\ &= Lr^m + b\ \gamma_0 \eta^{(l+m-\lambda_1)} \exp(-\eta^2/4) r^{-l} \\ &\leq Lr^m + b\ \gamma_0 M_0 r^{-l} \quad \text{for all } r, t \geq 0, \end{aligned}$$

where

$$M_0 := \max_{\eta>0}\left\{\eta^{l+m-\lambda_1}\exp(-\eta^2/4)\right\} = \left(\frac{2(l+m-\lambda_1)}{e}\right)^{(l+m-\lambda_1)/2}.$$

Taking $c_0 = b\gamma_0 M_0$, we finish the proof. □

3.2. **Inner supersolution and matching.** Next, we construct an inner supersolution $U_{\text{in}}^+(r,t)$ in the same way as [9]. To construct the inner supersolution, we need an auxiliary result for the initial-value problem

$$\begin{cases} \Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + q\psi^{-(q+1)}\Psi = \psi - \frac{q+1}{2}\xi\psi_\xi + \frac{1}{1+\xi^{\lambda_1-m}}, \quad \xi > 0, \\ \Psi(0) = -1, \quad \Psi_\xi(0) = 0. \end{cases} \tag{3.1}$$

Lemma 3.4. *The solution* Ψ *of* (3.1) *has the following properties*:

(i) *There exists a constant* $K > 0$ *such that*

$$\Psi(\xi) = K\xi^{2+m-\lambda_1} + o(\xi^{2+m-\lambda_1}),$$
$$\Psi_\xi(\xi) = K(2+m-\lambda_1)\xi^{1+m-\lambda_1} + o(\xi^{1+m-\lambda_1}) \quad \textit{as } \xi \to \infty.$$

(ii) *There exists a constant* $\xi_0 > 0$ *such that* $\Psi(\xi) \le 0$ *for* $0 \le \xi \le \xi_0$.

Proof. The procedures of proofs are similar to those in Lemma 4.1 of [9]. We begin with an observation on $\psi(\xi)$. From the expansion

$$\psi(\xi) = L\xi^m + (a + \phi_1(\xi))\xi^{-(\lambda_1-m)} \tag{3.2}$$

with $\phi_1(\xi) \to 0$ as $\xi \to \infty$, we have

$$\begin{aligned} \frac{1}{\xi^{N-1}}(\xi^{N-1}\psi_\xi)_\xi &= \left(L\xi^m + (a+\phi_1)\xi^{-(\lambda_1-m)}\right)^{-q} \\ &= L^{-q}\xi^{m-2} - qL^{-(q+1)}(a+\phi_2(\xi))\xi^{m-\lambda_1-2} \end{aligned}$$

with $\phi_2(\xi) \to 0$ as $\xi \to \infty$. Hence, for arbitrary $\xi_1 > 0$, we have

$$\begin{aligned} \xi^{N-1}\psi_\xi(\xi) = \xi_1^{N-1}\psi_\xi(\xi_1) &+ \frac{L^{-q}}{N-2+m}(\xi^{N-2+m} - \xi_1^{N-2+m}) \\ &- \frac{qL^{-(q+1)}a}{N-2+m-\lambda_1}(\xi^{m-\lambda_1+N-2} - \xi_1^{m-\lambda_1+N-2}) \\ &- qL^{-(q+1)}\int_{\xi_1}^{\xi}\phi_2(\tau)\tau^{m-\lambda_1+N-3}d\tau. \end{aligned}$$

Therefore, from

$$L^{-q} = LL^{-(q+1)} = Lm(N-2+m), \quad \frac{qL^{-(q+1)}}{N-2+m-\lambda_1} = \lambda_1 - m,$$

we obtain

$$\begin{aligned}
&\psi_\xi(\xi) - mL\xi^{m-1} + a(\lambda_1 - m)\xi^{m-\lambda_1-1} \\
&\quad = (\xi_1^{N-1}\psi_\xi(\xi_1) - mL\xi_1^{m+N-2} - a(\lambda_1 - m)\xi_1^{m-\lambda_1+N-2})\xi^{1-N} \\
&\qquad - \Big(qL^{-(q+1)}\int_{\xi_1}^{\xi} \phi_2(\tau)\tau^{m-\lambda_1+N-3}d\tau\Big)\xi^{1-N} =: I_1(\xi) + I_2(\xi).
\end{aligned}$$

Now, $\lambda_1 < N-2+m$ implies $1-N < m-\lambda_1-1$ and then $I_1(\xi) = o(\xi^{m-\lambda_1-1})$. From $\phi_2(\xi) \to 0$ as $\xi \to \infty$, we also see that $I_2(\xi) = o(\xi^{m-\lambda_1-1})$. Thus, we have shown that

$$\psi_\xi(\xi) = mL\xi^{m-1} - a(\lambda_1 - m)\xi^{m-\lambda_1-1} + o(\xi^{m-\lambda_1-1})$$

as $\xi \to \infty$. This yields the expansion

$$\psi - \frac{1}{m}\xi\psi_\xi + \frac{1}{1+\xi^{\lambda_1-m}} = \Big(\frac{a\lambda_1}{m} + 1\Big)\xi^{-(\lambda_1-m)} + g_2(\xi),$$

where $\xi^{\lambda_1-m}g_2(\xi) \to 0$ as $\xi \to \infty$. Since by (3.2)

$$q\psi^{-(q+1)} = \frac{qL^{-(q+1)}}{\xi^2} + g_1(\xi)$$

with $\xi^2 g_1(\xi) \to 0$ as $\xi \to \infty$, we may apply Lemma 4.2 in [9] to complete the proof of (i). Assertion (ii) immediately follows from the continuity of $\Psi(\xi)$. □

Lemma 3.5. *Let $l > \lambda_1 - m$. Put $\sigma(t) := A(t + A^\upsilon)^{-k}$, $k := \frac{m(l+m-\lambda_1)}{2\lambda_1}$ with $A > 0$, and*

$$\frac{2\lambda_1}{m(l+m)} < \upsilon < \frac{2\lambda_1}{m(l+m-\lambda_1)}. \tag{3.3}$$

If $A > 0$ is sufficiently large, then

$$U_{\rm in}^+(r,t) := \sigma(t)(\psi(\xi) + \sigma_t\sigma^q\Psi(\xi)), \quad \xi = \sigma^{-1/m}r$$

is a supersolution of (2.1) for all $r, t > 0$.

Proof. First, we note that $U_{\rm in}^+(r,t) > 0$ for all $r, t \geq 0$ if we take sufficiently large $A > 0$ (see Remark 3.6 for a proof). Since

$$\begin{aligned}
(U_{\rm in}^+(r,t))_t &= \sigma_t\psi + \sigma\psi_\xi\xi_t + (\sigma_t\sigma^q\Psi(\xi))_t, \\
(U_{\rm in}^+(r,t))_r &= \sigma^{1-1/m}\psi_\xi + \sigma_t\sigma^{q+1-1/m}\Psi_\xi, \\
(U_{\rm in}^+(r,t))_{rr} &= \sigma^{1-2/m}\psi_{\xi\xi} + \sigma_t\sigma^{q+1-2/m}\Psi_{\xi\xi}
\end{aligned}$$

and $\sigma\xi_t = -\frac{1}{m}\xi\sigma_t$, $q+1-\frac{2}{m}=0$, we obtain

$$\begin{aligned}
&(U_{\rm in}^+)_t - (U_{\rm in}^+)_{rr} - \frac{N-1}{r}(U_{\rm in}^+)_r + (U_{\rm in}^+)^{-q}\\
&= \sigma_t\psi + \sigma\psi_\xi\xi_t + (\sigma_t\sigma^q\Psi(\xi))_t - \sigma^{1-2/m}\psi_{\xi\xi} + \sigma_t\sigma^{q+1-2/m}\Psi_{\xi\xi}\\
&\qquad - \frac{N-1}{r}\Big(\sigma^{1-1/m}\psi_\xi + \sigma_t\sigma^{q+1-1/m}\Psi_\xi\Big) + \sigma^{-q}(\psi+\sigma_t\sigma^q\Psi)^{-q}\\
&= \sigma_t\Big(\psi - \frac{1}{m}\xi\psi_\xi\Big) + \big(\sigma_t\sigma^{q+1}\Psi\big)_t - \sigma_t\Big(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi\Big)\\
&\qquad + \sigma^{-q}((\psi+\sigma_t\sigma^q\Psi)^{-q} - \psi^{-q})\\
&\geq \sigma_t\Big(\psi - \frac{1}{m}\xi\psi_\xi\Big) + \big(\sigma_t\sigma^{q+1}\Psi\big)_t - \sigma_t\Big(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + q\psi^{-(q+1)}\Psi\Big)\\
&= -\sigma_t\frac{1}{1+\xi^{\lambda_1-m}} + \big(\sigma_t\sigma^{q+1}\Psi\big)_t =: J(r,t).
\end{aligned}$$

Using

$$(\sigma_t\sigma^{q+1}\Psi)_t = (\sigma_t\sigma^{q+1})_t\Psi + \sigma_t\sigma^{q+1}\Psi_\xi\xi_t = (\sigma_t\sigma^{q+1})_t\Psi - \frac{1}{m}\sigma_t^2\sigma^q\xi\Psi_\xi,$$

$$\begin{aligned}
(\sigma_t\sigma^{q+1})_t &= (-kA(t+A^\upsilon)^{-k-1}A^{q+1}(t+A^\upsilon)^{-k(q+1)})_t\\
&= -kA^{q+2}\left((t+A^\upsilon)^{-(q+2)k-1}\right)_t\\
&= k((q+2)k+1)A^{q+2}(t+A^\upsilon)^{-(q+2)k-2},
\end{aligned}$$

$$\sigma_t^2\sigma^q = \big(-kA(t+A^\upsilon)^{-k-1}\big)^2A^q(t+A^\upsilon)^{-kq} = k^2A^{q+2}(t+A^\upsilon)^{-(q+2)k-2},$$

and with Lemma 3.4, we claim that there exists $\xi_1>0$ such that $J\geq 0$ for $\xi\geq\xi_1$. Indeed, we see at $\xi\simeq\infty$ that

$$\begin{aligned}
J &= -\frac{\sigma_t}{1+\xi^{\lambda_1-m}} + \big(\sigma_t\sigma^{q+1}\Psi\big)_t\\
&= \frac{kA(t+A^\upsilon)^{-(k+1)}}{1+\xi^{\lambda_1-m}} + (\sigma_t\sigma^{q+1})_t\Psi - \frac{1}{m}\sigma_t^2\sigma^q\xi\Psi_\xi\\
&= \frac{kA(t+A^\upsilon)^{-(k+1)}}{1+\xi^{\lambda_1-m}} + \Big(k\{(q+2)k+1\}\Psi - \frac{1}{m}k^2\xi\Psi_\xi\Big)\\
&\qquad \times A^{q+2}(t+A^\upsilon)^{-(q+2)k-2}\\
&= \frac{kA(t+A^\upsilon)^{-(k+1)}}{1+\xi^{\lambda_1-m}} + [k\{(q+2)k+1\}(K\xi^{2+m-\lambda_1} + o(\xi^{2+m-\lambda_1}))
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{k^2}{m}\xi(K(2+m-\lambda_1)\xi^{1+m-\lambda_1} + o(\xi^{1+m-\lambda_1}))]A^{q+2}(t+A^{\upsilon})^{-(q+2)k-2} \\
&= \frac{kA(t+A^{\upsilon})^{-(k+1)}}{1+\xi^{\lambda_1-m}} + K\Big(\Big[(q+1)k^2 + k + \frac{k^2}{m}(\lambda_1-2)\Big]\xi^{2+m-\lambda_1} \\
&\qquad + o(\xi^{2+m-\lambda_1})\Big)A^{q+2}(t+A^{\upsilon})^{-(q+2)k-2}.
\end{aligned}
$$

Since $\lambda_1 > 2$ from the monotonicity of $\lambda_1(N,m)$ with respect to m, the above claim holds. On the other hand, due to the smoothness of Ψ, there exist $D_0 > 0$ and $D_1 > 0$ such that

$$
-D_0 \le \Psi(\xi) \quad \text{and} \quad \xi\Psi_{\xi}(\xi) \le D_1
$$

for $0 \le \xi \le \xi_1$. Note furthermore that for $0 \le \xi \le \xi_1$

$$
-\frac{\sigma_t}{1+\xi^{\lambda_1-m}} = \frac{kA(t+A^{\upsilon})^{-(k+1)}}{1+\xi^{\lambda_1-m}} \ge \frac{kA(t+A^{\upsilon})^{-(k+1)}}{1+\xi_1^{\lambda_1-m}}.
$$

Thus we find

$$
\begin{aligned}
J &= -\frac{\sigma_t}{1+\xi^{\lambda_1-m}} + \big(\sigma_t\sigma^{q+1}\Psi\big)_t \\
&= \frac{kA(t+A^{\upsilon})^{-(k+1)}}{1+\xi^{\lambda_1-m}} + (\sigma_t\sigma^{q+1})_t\Psi - \frac{1}{m}\sigma_t^2\sigma^q\xi\Psi_{\xi} \\
&\ge kA(t+A^{\upsilon})^{-(k+1)}\Big[\frac{1}{1+\xi_1^{\lambda_1-m}} - (\{(q+2)k+1\}D_0 \\
&\qquad + \frac{k}{m}D_1)A^{q+1}(t+A^{\upsilon})^{-(q+1)k-1}\Big] \ge 0
\end{aligned}
$$

for $0 \le \xi \le \xi_1$, provided that

$$
\frac{1}{1+\xi_1^{\lambda_1-m}} - \Big(\{(q+2)k+1\}D_0 + \frac{k}{m}D_1\Big)A^{-([(q+1)k+1]\upsilon-(q+1))} \ge 0,
$$

which is true if $A^{-([(q+1)k+1]\upsilon-(q+1))}$ is small. Since

$$
\upsilon > \frac{2\lambda_1}{m(l+m)} = \frac{q+1}{(q+1)k+1}
$$

due to (3.3), the proof is complete. □

Remark 3.6. We claim that $U_{\mathrm{in}}^{+}(r,t)$ is positive for all $r, t \ge 0$ if $A > 0$ is sufficiently large. Indeed, we see that there exists ξ_0 such that $\Psi \le 0$ for $0 \le \xi \le \xi_0$ and the claim is trivial in that range. On the other hand, for $\xi > \xi_0$ from Lemma 3.4 we have

$$
U_{\mathrm{in}}^{+}(r,t) = \sigma(t)(\psi(\xi) + \sigma_t\sigma^q\Psi(\xi)) \ge \sigma(t)(\varphi_0(\xi) + \sigma_t\sigma^q\Psi(\xi))
$$

$$\geq \sigma(t)(L\xi^m + K^+\sigma_t\sigma^q\xi^{m-\lambda_1+2}) = \sigma(t)\xi^m(L + K^+\sigma_t\sigma^q\xi^{-(\lambda_1-2)})$$
$$\geq \sigma(t)\xi^m(L + K^+\sigma_t\sigma^q\xi_0^{-(\lambda_1-2)})$$

since $\lambda_1 > 2$, where K^+ is a positive constant determined in (3.4). We can take $A > 0$ so large that

$$\sigma_t\sigma^q = -kA^{q+1}(t + A^\upsilon)^{-k(q+1)-1} \geq -kA^{-([(q+1)k+1]\upsilon-(q+1))} \geq -\frac{L\xi_0^{\lambda_1-2}}{2K^+}$$

holds for all $t \geq 0$ by (3.3). Hence we see that

$$U_{\rm in}^+(r,t) \geq \frac{L}{2}\sigma(t)\xi^m > 0$$

for $\xi > \xi_0$ and the claim holds.

In view of the expansion of Ψ in Lemma 3.4, there exists $K^+ > 0$ such that

$$\Psi(\xi) \leq K^+\xi^{2+m-\lambda_1} \quad \text{for } \xi > 0. \tag{3.4}$$

We choose $\xi_0 > 0$ such that $\Psi(\xi) \leq 0$ for $0 \leq \xi \leq \xi_0$ and

$$\xi_0^{-m} \geq 2L \tag{3.5}$$

and then take $a^-(\xi_0) > 0$ such that

$$\psi(\xi) \geq L\xi^m + a^-(\xi_0)\xi^{-(\lambda_1-m)} \quad \text{for } \xi \geq \xi_0. \tag{3.6}$$

We now fix a constant $B_0 > 0$ such that

$$0 < B_0 < \xi_0, \tag{3.7}$$

$$K^+kB_0^2 < \frac{a^-(\xi_0)}{2}, \tag{3.8}$$

and put for any $c_0 > 0$,

$$r_0 = \left(\frac{c_0}{L}\right)^{1/(l+m)}. \tag{3.9}$$

Next, we show some properties of $U_{\rm in}^+(r,t)$ with the constants defined above.

Lemma 3.7. *Let $l > \lambda_1 - m$ and suppose B_0 satisfies* (3.7)*. For any $c_1 > 0$, there exists $A_1 > 0$ such that if $A > A_1$, then $U_{\rm in}^+$ satisfies*

$$U_{\rm in}^+(r,0) \geq Lr^m + c_1(1+r)^{-l} \quad \textit{for } r \in [0, B_0A^{(1-\upsilon k)/m}].$$

Proof. We note that $r \le B_0 A^{(1-vk)/m}$ is equivalent to $\xi = \sigma^{-1/m} r = A^{-(1-vk)/m} r \le B_0$ at $t = 0$. By (3.7), this implies that $\Psi \le 0$ at $t = 0$ for $0 \le r \le B_0 A^{(1-vk)/m}$. We show the desired inequality by splitting the above interval into two parts for any A satisfying

$$B_0 A^{(1-vk)/m} \ge \left(\frac{c_1}{L}\right)^{\frac{1}{l+m}} =: r_1.$$

First, if $r_1 \le r \le B_0 A^{(1-vk)/m}$, we obtain

$$\begin{aligned}
&Lr^m + c_1(1+r)^{-l} \le Lr^m + c_1 r^{-l} = Lr^m + c_1 r^{-(l+m)} r^m \\
&\le Lr^m + c_1 r_1^{-(l+m)} r^m = 2Lr^m \le 2L\big(B_0 A^{(1-vk)/m}\big)^m \\
&\le 2L\big(\xi_0 A^{(1-vk)/m}\big)^m = 2L\xi_0^m A^{1-vk} \le A^{1-vk} = \sigma(0) \\
&\le \sigma(0)\psi(\xi) \le \sigma(0)(\psi(\xi) + \sigma_t \sigma^q \Psi(\xi)) = U_{\text{in}}^+(r,0)
\end{aligned}$$

from (3.5), $\psi(\xi) = \varphi_1(\xi) \ge 1$ and $\sigma_t \sigma^q \Psi(\xi) \ge 0$. Next, if $0 \le r \le r_1$, then we can take $A > 0$ large enough that

$$U_{\text{in}}^+(r,0) \ge A^{1-vk} \ge \max_{\rho \in [0, r_1]} \{L\rho^m + c_1(1+\rho)^{-l}\}$$

from the fact that $Lr^m + c_1(1+r)^{-l}$ remains bounded on the interval. The proof is now complete. □

Lemma 3.8. *Let $l > \lambda_1 - m$. Suppose that B_0 satisfies* (3.7) *and* (3.8)*. For any $c_0 > 0$, there exists $A_2 > 0$ such that if $A > A_2$, then*

$$U_{\text{in}}^+(r,t) \ge Lr^m + c_0 r^{-l} \quad \text{at } r = B_0(t + A^{2(1-vk)/m})^{1/2}$$

for all $t \ge 0$.

Proof. We choose large $A > 1$ such that $B_0 A^{(1-vk)/m} \ge r_0$, and hence

$$r = B_0(t + A^{2(1-vk)/m})^{1/2} \ge r_0 \quad \text{for all } t \ge 0.$$

If $0 \le \xi \le \xi_0$, then $\sigma = \xi^{-m} r^m \ge \xi_0^{-m} r^m$, and we have

$$\begin{aligned}
U_{\text{in}}^+(r,t) &= \sigma(t)(\psi(\xi) + \sigma_t \sigma^q \Psi(\xi)) \ge \sigma(t)\psi(\xi) \ge \sigma(t) \ge \xi_0^{-m} r^m \\
&\ge 2Lr^m = Lr^m + Lr^{l+m} r^{-l} \ge Lr^m + Lr_0^{l+m} r^{-l} = Lr^m + c_0 r^{-l}
\end{aligned}$$

from the negativity of σ_t and Ψ, the monotonicity of ψ, (3.5), and (3.9).

On the other hand, if $\xi > \xi_0$, then (3.4), (3.6), and (3.8) yield

$$\begin{aligned}
U_{\text{in}}^+(r,t) &= \sigma(\psi(\xi) + \sigma_t \sigma^q \Psi(\xi)) \\
&\ge \sigma(L\xi^m + a^-(\xi_0)\xi^{-(\lambda_1 - m)} + K^+ \sigma_t \sigma^q \xi^{2+m-\lambda_1}) \\
&= Lr^m + \sigma^{\lambda_1/m} r^{-(\lambda_1 - m)}(a^-(\xi_0) + K^+ \frac{\sigma_t}{\sigma} r^2)
\end{aligned}$$

$$
\begin{aligned}
&= Lr^m + \sigma^{\lambda_1/m} r^{-(\lambda_1-m)}\Big(a^-(\xi_0) - K^+ k B_0^2 \frac{t + A^{2(1-\upsilon k)/m}}{t + A^\upsilon}\Big) \\
&\geq Lr^m + \sigma^{\lambda_1/m} r^{-(\lambda_1-m)} \left(a^-(\xi_0) - K^+ k B_0^2\right) \\
&\geq Lr^m + \sigma^{\lambda_1/m} r^{-(\lambda_1-m)} \frac{a^-(\xi_0)}{2},
\end{aligned} \tag{3.10}
$$

where we used $\upsilon > 2(1-\upsilon k)/m$ and $A > 1$.

Since $(l + m - \lambda_1)/2 = \lambda_1 k/m$ and $\upsilon k < 1$, we find

$$
\begin{aligned}
&\frac{\frac{a^-(\xi_0)}{2}\sigma^{\lambda_1/m} r^{-(\lambda_1-m)}}{c_0 r^{-l}} = \frac{a^-(\xi_0)}{2c_0} A^{\lambda_1/m}(t + A^\upsilon)^{-\frac{\lambda_1 k}{m}} r^{l+m-\lambda_1} \\
&= \frac{a^-(\xi_0) B_0^{l+m-\lambda_1}}{2c_0} A^{\lambda_1/m}\Big(\frac{t + A^{2(1-\upsilon k)/m}}{t + A^\upsilon}\Big)^{\frac{\lambda_1 k}{m}} \\
&\geq \frac{a^-(\xi_0) B_0^{l+m-\lambda_1}}{2c_0} A^{\frac{\lambda_1}{m} - \frac{\lambda_1 k}{m}(\upsilon - \frac{2(1-\upsilon k)}{m})} \\
&\geq \frac{a^-(\xi_0) B_0^{l+m-\lambda_1}}{2c_0} A^{\frac{\lambda_1}{m}(1+\frac{2k}{m})(1-\upsilon k)} \geq 1
\end{aligned}
$$

for all sufficiently large $A > 1$. Combined with (3.10), this shows that

$$U_{\text{in}}^+(r,t) \geq Lr^m + c_0 r^{-l} \quad \text{at } \; r = B_0(t + A^{2(1-\upsilon k)/m})^{1/2}.$$

This completes the proof. □

Proposition 3.9. *Suppose that $\lambda_1 - m < l < \lambda_2 - m + 2$ and*

$$U_0(r) \leq Lr^m + c_1(1+r)^{-l} \quad \text{for } \; r \geq 0$$

with some $c_1 > 0$. Then there exists a constant $C_1 > 0$ such that the solution of (2.1) *satisfies*

$$U(0,t) \leq C_1(1+t)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}} \quad \text{for all } \; t \geq 0.$$

Proof. We recall that we choose $\xi_0 > 0$ satisfying $\Psi(\xi) \leq 0$ for $0 \leq \xi \leq \xi_0$ and $a^- = a^-(\xi_0) > 0$ satisfying $\psi(\xi) \geq L\xi^m + a^-\xi^{-(\lambda_1-m)}$ for $\xi \geq \xi_0$. Next, we fix a constant $B_0 > 0$ satisfying (3.7) and (3.8). That is to say, $0 < B_0 < \xi_0$ and $K^+ k B_0^2 < \frac{a^-(\xi_0)}{2}$. By the assumption on the initial data, we can apply Lemma 3.2 to show that the solution of (2.1) satisfies

$$U(r,t) \leq U_{\text{out}}^+(r,t) \leq Lr^m + c_0 r^{-l} \quad \text{for all } \; r,t \geq 0$$

with some $b, \tau, c_0 > 0$. Then for large $A > 1$, there exists a supersolution $U_{\text{in}}^+(r,t)$ from Lemmas 3.5, 3.7, and 3.8 such that

$$U_{\text{in}}^+(r,0) \geq Lr^m + c_1(1+r)^{-l} \quad \text{for } r \in [0, B_0 A^{(1-\upsilon k)/m}] \tag{3.11}$$

and

$$U_{\text{in}}^{+}(r,t) \geq Lr^{m} + c_0 r^{-l} \quad \text{at } \ r = B_0(t + A^{2(1-\upsilon k)/m})^{1/2}$$

for all $t \geq 0$, which shows

$$U_{\text{out}}^{+}(r,t) \leq U_{\text{in}}^{+}(r,t) \quad \text{at } \ r = B_0(t + A^{2(1-\upsilon k)/m})^{1/2} \tag{3.12}$$

for all $t \geq 0$. We set $r^*(t) := \sup\{r > 0 : U_{\text{in}}^{+}(\rho,t) < U_{\text{out}}^{+}(\rho,t) \text{ for } \rho \in [0,r)\}$ and see that $r^*(t) > 0$ from the fact that $U_{\text{in}}^{+}(r,t) < U_{\text{out}}^{+}(r,t)$ for small $r > 0$.

Then it follows that $r^*(t)$ is well defined from (3.12) and the fact mentioned above. It is clear that $U_r^{+}(0,t) = U_{\text{in},r}^{+}(0,t) = 0$, $t > 0$. Then,

$$U^{+}(r,t) := \begin{cases} U_{\text{in}}^{+}(r,t) & \text{for } r < r^*(t), \\ U_{\text{out}}^{+}(r,t) & \text{for } r \geq r^*(t), \end{cases}$$

defines a supersolution of (2.1) that initially lies above U_0. Indeed, it is clear that the initial data is below the outer solution $U_{\text{out}}^{+}(r,0)$ from the construction of the super outer solution in Lemma 3.2. To see that the initial data is below the inner solution $U_{\text{in}}^{+}(r,0)$ in the interval $r < r^*(0)$, we use (3.11) and $r^*(0) \leq B_0 A^{(1-\upsilon k)/m}$ from (3.12). Hence by the comparison principle, we have $U(r,t) \leq U^{+}(r,t)$ for all $r,t \geq 0$. Moreover, since $\upsilon > 2\lambda_1/(m(l+m)) = (q+1)/((q+1)k+1)$ we may take A so large that

$$\begin{aligned} 1 - \sigma_t \sigma^q &= 1 + kA^{q+1}(t + A^{\upsilon})^{-k(q+1)-1} \\ &\leq 1 + kA^{-([(q+1)k+1]\upsilon - (q+1))} \leq 2 \ \text{ for all } \ t \geq 0. \end{aligned} \tag{3.13}$$

In particular, by (3.13) we have

$$\begin{aligned} U(0,t) &\leq U^{+}(0,t) = U_{\text{in}}^{+}(0,t) = \sigma(t)(\psi(0) + \sigma_t \sigma^q \Psi(0)) \\ &= \sigma(t)(1 - \sigma_t\sigma^q) \leq 2\sigma(t) = 2A(t + A^{\upsilon})^{-k} \leq C_1(t+1)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}} \end{aligned}$$

for all $t \geq 0$ with some $C_1 > 0$. This finishes the proof. □

Proposition 3.10. *Suppose that U_0 satisfies $U_0(r) \leq Lr^m + c_1 \exp(-\nu r^2)$ for $r \geq 0$ with some $c_1 > 0$ and $\nu > 0$. Then there exists a constant $C_1 > 0$ such that the solution of* (2.1) *satisfies*

$$U(0,t) \leq C_1(1+t)^{-\frac{m(\lambda_2 - \lambda_1 + 2)}{2\lambda_1}} \quad \textit{for all } \ t \geq 0.$$

Proof. We choose $\xi_0 > 0$ satisfying $\Psi(\xi) \leq 0$ for $0 \leq \xi \leq \xi_0$ and $a^- = a^-(\xi_0) > 0$ satisfying

$$\psi(\xi) \geq L\xi^m + a^- \xi^{-(\lambda_1 - m)} \quad \text{for } \ \xi \geq \xi_0$$

and fix B_0 as in (3.7) and (3.8) again, namely in the same way as in Proposition 3.9. Then we see from Lemma 3.3 that the solution of (2.1) satisfies

$$U(r,t) \le U^+_{\text{out}}(r,t) \le Lr^m + c_0 r^{-(\lambda_2 - m+2)} \quad \text{for all } t, r \ge 0$$

with some $c_0 > 0$. We note that we can take $c_2 > 0$ such that

$$U_0(r) \le Lr^m + c_2(1+r)^{-(\lambda_2-m+2)} \quad \text{for all } r \ge 0.$$

If we take $A > 1$ large enough, then there exists a supersolution $U^+_{\text{in}}(r,t)$ from Lemmas 3.5, 3.7, and 3.8 such that

$$U^+_{\text{in}}(r,0) \ge Lr^m + c_2(1+r)^{-(\lambda_2-m+2)} \quad \text{for } r \in [0, B_0 A^{(1-\upsilon k)/m}]$$

and

$$U^+_{\text{in}}(r,t) \ge Lr^m + c_0 r^{-(\lambda_2-m+2)} \quad \text{at } r = B_0(t + A^{2(1-\upsilon k)/m})^{1/2}$$

for all $t \ge 0$, which shows

$$U^+_{\text{out}}(r,t) \le U^+_{\text{in}}(r,t) \quad \text{at } r = B_0(t + A^{2(1-\upsilon k)/m})^{1/2}$$

for all $t \ge 0$. The remaining part of the proof is obtained in a similar manner to Proposition 3.9. □

Now, let us complete the proofs of Theorems 1.2–1.3.

Proof of Theorem 1.2 (i). Taking $U_0(r) = Lr^m + c_1(1+r)^{-l}$, we have by assumption $u_0(x) \le U_0(|x|)$, $x \in \mathbb{R}^N$. By the comparison principle and Proposition 3.9, this implies

$$u(0,t) \le U(0,t) \le C_1(1+t)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}}$$

for all $t > 0$ with some constant $C_1 > 0$. □

Proof of Theorem 1.2 (ii). Given any small $\varepsilon > 0$, we set $\hat{l} := \lambda_2 - m + 2 - \frac{2\lambda_1}{m}\varepsilon$ and define $\hat{U}_0(r) := Lr^m + c_1(1+r)^{-\hat{l}}$. By $\hat{U}$ we denote the solution of (2.1) with the initial data $\hat{U}_0$. Then $u_0 < \hat{U}_0$, and it follows from the comparison principle that $u(r,t) < \hat{U}(r,t)$ for all $r, t > 0$. On the other hand, by Theorem 1.2 (i), $\hat{U}(r,t)$ satisfies

$$\hat{U}(0,t) \le C_\varepsilon(1+t)^{-\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}+\varepsilon}$$

for all $t > 0$ with some constant $C_\varepsilon > 0$. □

Proof of Theorem 1.3. Taking $U_0(r) = Lr^m + c_1\exp(-\nu r^2)$, we have by the assumption $u_0(x) \le U_0(|x|)$, $x \in \mathbb{R}^N$. By the comparison principle and Proposition 3.10, this implies

$$u(0,t) \le U(0,t) \le C_1(1+t)^{-\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}}$$

for all $t > 0$ with some constant $C_1 > 0$. □

4. Lower bound

In this section, we derive a lower bound of the quenching rate for (2.1). Toward this end, we will construct an appropriate subsolution U^- of (2.1). Similar to the procedure in the preceding section, our approach is to combine two subsolutions in some inner and outer regions. Our key ideas are to construct a subsolution explicitly in the inner region and to modify the outer solution defined in Lemma 3.1 by using two self-similar solutions with different decay rates in a manner similar to our methods in [20]. Then we will see that the inner solution has a desired quenching rate and the outer solution has desired properties; namely, the outer solution is identically equal to φ_0 near the origin and has a desired rate near spatial infinity.

4.1. Outer subsolution.

In this subsection, we construct a suitable outer subsolution of (2.1) with a desired property mentioned above.

First, we recall the initial-value problem (2.15):

$$\begin{cases} f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f = 0, \quad \eta > 0, \\ f(0) = \gamma_0 > 0, \quad f_\eta(0) = 0, \end{cases}$$

where $n = N - 2(\lambda_1 - m)$, $\beta = l - (\lambda_1 - m)$.

Next, we modify this initial-value problem as follows:

$$\begin{cases} \tilde{f}_{\eta\eta} + \frac{n-1}{\eta} \tilde{f}_\eta + \frac{\eta}{2} \tilde{f}_\eta + \frac{\tilde{\beta}}{2} \tilde{f} = 0, \quad \eta > 0, \\ \tilde{f}(0) = \gamma_0, \quad \tilde{f}_\eta(0) = 0, \end{cases} \tag{4.1}$$

where $\tilde{\beta} = l - (\lambda_1 - m) + \delta$ with some constant $\delta > 0$ that satisfies

$$0 < \delta < \min(\lambda_2 - m + 2 - l, 2). \tag{4.2}$$

Lemma 4.1. *Let $\lambda_1 - m < l < \lambda_2 - m + 2$ and $\delta > 0$ satisfying (4.2) be fixed. Let f and $\tilde{f}$ be the solutions of (2.15) and (4.1) respectively. We define*

$$U^-_{\mathrm{out}}(r,t) := \begin{cases} Lr^m + b_1(t+\tau)^{-\frac{l}{2}}(F(\eta) - b_2\tilde{F}(\eta)), & \text{for } \eta > \eta_*, \\ Lr^m, & \text{for } 0 < \eta \le \eta_* \end{cases}$$

with $\eta = (t+\tau)^{-1/2} r$, where $\eta_* := \inf\{\eta > 0 : f(\eta) > b_2 \tilde{f}(\eta)\}$, $F(\eta) := \eta^{-(\lambda_1 - m)} f(\eta)$, and $\tilde{F}(\eta) := \eta^{-(\lambda_1 - m)} \tilde{f}(\eta)$. For any $b_1 > 0$, if $b_2 > 0$ is sufficiently large, then U^-_{out} is a subsolution of (2.1).

Proof. The proof is carried out along a line similar to our previous proof in [20] except for a treatment of the nonlinearity.

It is trivial that $Lr^m = \varphi_0(r)$ is a subsolution of (2.1). Thus, we only need to verify the range of $\eta > \eta_*$. First, by Lemma 2.1 we find that there exist positive constants $\tilde{d}_-(1)$ and d_+ such that

$$\tilde{f}(\eta) \geq \tilde{d}_-(1)\eta^{-(l+m-\lambda_1)-\delta} \quad \text{for } \eta \geq 1 \tag{4.3}$$

and

$$f(\eta) \leq d_+\eta^{-(l+m-\lambda_1)} \quad \text{for all } \eta > 0. \tag{4.4}$$

Then we choose $b_2 > 0$ so large that

$$f(1) - b_2\tilde{f}(1) < 0 \tag{4.5}$$

and

$$\frac{b_2\delta}{2}\tilde{d}_-(1) - qL^{-(q+1)}d_+ > 0. \tag{4.6}$$

We note that $\eta_* < \infty$. Indeed, we see from Lemma 2.1 again that there exist positive constants $\tilde{d}_+$ and $d_-(1)$ such that

$$f(\eta) \geq d_-(1)\eta^{-(l+m-\lambda_1)} \quad \text{for } \eta \geq 1 \tag{4.7}$$

and

$$\tilde{f}(\eta) \leq \tilde{d}_+\eta^{-(l+m-\lambda_1)-\delta} \quad \text{for all } \eta > 0. \tag{4.8}$$

By (4.7) and (4.8), we obtain for $\eta \geq 1$ that

$$\begin{aligned} f(\eta) - b_2\tilde{f}(\eta) &\geq (d_-(1)\eta^{-(l+m-\lambda_1)} - b_2\tilde{d}_+\eta^{-(l+m-\lambda_1)-\delta}) \\ &= (d_-(1) - b_2\tilde{d}_+\eta^{-\delta})\eta^{-(l+m-\lambda_1)}. \end{aligned}$$

This implies that $\eta_* < \infty$ since $f(\eta) > b_2\tilde{f}(\eta)$ for sufficiently large η from the above inequality.

We recall that F and $\tilde{F}$ satisfy

$$F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{\eta}{2}F_\eta + \frac{l}{2}F + \frac{qL^{-(q+1)}}{\eta^2}F = 0$$

and

$$\tilde{F}_{\eta\eta} + \frac{N-1}{\eta}\tilde{F}_\eta + \frac{\eta}{2}\tilde{F}_\eta + \frac{l+\delta}{2}\tilde{F} + \frac{qL^{-(q+1)}}{\eta^2}\tilde{F} = 0$$

respectively. We define U_1^- and U_2^- by

$$U_1^- := (t+\tau)^{-\frac{l}{2}}F(\eta), \quad U_2^- = (t+\tau)^{-\frac{l}{2}}\tilde{F}(\eta)$$

and a linear operator $\mathcal{D}U$ by

$$\mathcal{D}U := U_{rr} + \frac{N-1}{r}U_r.$$

Then we see that

$$\begin{aligned} U^-_{1,t} - \mathcal{D}U^-_1 &= -(t+\tau)^{-\frac{l}{2}-1}\Big(\frac{l}{2}F + \frac{\eta}{2}F_\eta + F_{\eta\eta} + \frac{N-1}{\eta}F_\eta\Big) \\ &= q(t+\tau)^{-\frac{l}{2}-1}L^{-(q+1)}\eta^{-2}F, \\ U^-_{2,t} - \mathcal{D}U^-_2 &= -(t+\tau)^{-\frac{l}{2}-1}\Big(\frac{l+\delta}{2}\tilde{F} + \frac{\eta}{2}\tilde{F}_\eta + \tilde{F}_{\eta\eta} + \frac{N-1}{\eta}\tilde{F}_\eta - \frac{\delta}{2}\Big) \\ &= (t+\tau)^{-\frac{l}{2}-1}\Big(qL^{-(q+1)}\eta^{-2} + \frac{\delta}{2}\Big)\tilde{F}. \end{aligned}$$

Since $b_1(t+\tau)^{-\frac{l}{2}}(F(\eta) - b_2\tilde{F}(\eta)) = b_1(U_1^- - b_2U_2^-)$ is nonnegative by the definition, $Lr^m = \varphi_0$, and $\mathcal{D}\varphi_0 - \varphi_0^{-q} = 0$, we obtain

$$\begin{aligned} &U^-_{\mathrm{out},t} - U^-_{\mathrm{out},rr} - \frac{N-1}{r}U^-_{\mathrm{out},r} + (U^-_{\mathrm{out}})^{-q} = U^-_{\mathrm{out},t} - \mathcal{D}U^-_{\mathrm{out}} + (U^-_{\mathrm{out}})^{-q} \\ &= b_1\big((U^-_{1,t} - \mathcal{D}U^-_1) - b_2(U^-_{2,t} - \mathcal{D}U^-_2)\big) - \mathcal{D}Lr^m + (Lr^m + b_1(U_1^- - b_2U_2^-))^{-q} \\ &\le b_1\big((U^-_{1,t} - \mathcal{D}U^-_1) - b_2(U^-_{2,t} - \mathcal{D}U^-_2)\big) - \mathcal{D}Lr^m + (Lr^m)^{-q} \\ &= b_1(t+\tau)^{-\frac{l}{2}-1}\Big[qL^{-(q+1)}\eta^{-2}F - b_2\big(qL^{-(q+1)}\eta^{-2} + \frac{\delta}{2}\big)\tilde{F}\Big] - \big(\mathcal{D}\varphi_0 - \varphi_0^{-q}\big) \\ &\le b_1(t+\tau)^{-\frac{l}{2}-1}\Big(qL^{-(q+1)}\eta^{-2}F - b_2\frac{\delta}{2}\tilde{F}\Big) \\ &= -b_1(t+\tau)^{-\frac{l}{2}-1}\eta^{-2}\Big(\frac{b_2\delta}{2}\eta^2\tilde{F} - qL^{-(q+1)}F\Big) \\ &= -b_1(t+\tau)^{-\frac{l}{2}-1}\eta^{-2}\Big(\frac{b_2\delta}{2}\eta^2\tilde{f} - qL^{-(q+1)}f\Big)\eta^{-(\lambda_1-m)}. \end{aligned}$$

Therefore, U^-_{out} becomes a subsolution provided that

$$\frac{b_2\delta}{2}\eta^2\tilde{f} - qL^{-(q+1)}f \ge 0$$

is satisfied. Let us prove this inequality. Using (4.2), (4.3), (4.4), and noting $\eta_* > 1$ from (4.5), we obtain

$$\begin{aligned} \frac{b_2\delta}{2}\eta^2\tilde{f} - qL^{-(q+1)}f &\ge \Big(\frac{b_2\delta}{2}\tilde{d}_-(1)\eta^{2-\delta} - qL^{-(q+1)}d_+\Big)\eta^{-(l+m-\lambda_1)} \\ &\ge \Big(\frac{b_2\delta}{2}\tilde{d}_-(1)1 - qL^{-(q+1)}d_+\Big)\eta^{-(l+m-\lambda_1)} > 0 \end{aligned}$$

for $\eta > \eta_*$ from (4.6). Thus we have the conclusion. □

4.2. **Inner subsolution and matching.** We use an inner solution similar to that in [9], but the method is different. That is to say, we do not use an argument of the nonincreasing property of the intersection number and will construct a subsolution of (2.1) directly.

Lemma 4.2. *For any $k > 0$,*

$$U^-_{\text{in}}(r,t) := \sigma\psi(\xi) = \varphi_\sigma(r), \quad \xi = \sigma^{-1/m} r$$

is a subsolution of (2.1) *for all $r, t > 0$. Here $\sigma = \sigma_\epsilon := \epsilon(t+\tau)^{-k}$, and ϵ and τ are arbitrary positive constants.*

Proof. Since $\sigma\psi(\xi) = \varphi_\sigma(|x|)$ is a solution of

$$\begin{cases} \Delta\varphi - \varphi^{-q} = 0, & x \in \mathbb{R}^N, \\ \varphi(0) = \sigma, \end{cases}$$

we have

$$\begin{aligned} U^-_{\text{in},t} - \big(\Delta U^-_{\text{in}} - (U^-_{\text{in}})^{-q}\big) &= (\varphi_\sigma)_t - \big(\Delta\varphi_\sigma - (\varphi_\sigma)^{-q}\big) = \sigma_t\psi(\xi) + \sigma\psi_\xi(\xi)\xi_t \\ &= \sigma_t\psi(\xi) + \sigma\psi_\xi(\xi)\Big(-\frac{1}{m}\sigma^{-\frac{1}{m}-1}\sigma_t r\Big) = \sigma_t\Big(\psi(\xi) - \frac{\xi}{m}\psi_\xi(\xi)\Big). \end{aligned}$$

Then, we claim that

$$\psi(\xi) - \frac{\xi}{m}\psi_\xi(\xi) \geq 0, \quad \text{for all } \xi \geq 0.$$

To see the inequality holds, we rewrite the equation $\Delta\psi - \psi^{-q} = 0$ as

$$\xi^{1-N}(\xi^{N-1}\psi_\xi)_\xi - \psi^{-q} = 0, \quad \xi > 0.$$

Integrating over $[0, \xi]$, we have

$$\begin{aligned} \xi^{N-1}\psi_\xi &= \int_0^\xi \xi^{N-1}(\psi(\xi))^{-q} d\xi \leq \int_0^\xi \xi^{N-1}(\varphi_0(\xi))^{-q} d\xi \\ &= \int_0^\xi \xi^{N-1}(L\xi^m))^{-q} d\xi = \int_0^\xi \xi^{N-1} L^{-q}\xi^{m-2} d\xi \\ &= \frac{L^{-q}}{N-2+m}\xi^{N-2+m} = mL\xi^{N-2+m}. \end{aligned}$$

Here, we use $L^{-q} = LL^{-(q+1)} = Lm(N-2+m)$. Then we obtain $\psi_\xi \leq mL\xi^{m-1}$ and we see that

$$\psi(\xi) - \frac{\xi}{m}\psi_\xi(\xi) \geq \varphi_0(\xi) - \frac{\xi}{m}(mL\xi^{m-1}) = 0.$$

Since $\sigma_t < 0$ is trivial, this finishes the proof. □

Lemma 4.3. *We take $\lambda_1 - m < l < \lambda_2 - m + 2$ and $k = m(l + m - \lambda_1)/2\lambda_1$ in Lemma 4.2. For any $b_1, b_2 > 0$ and $\tau > 0$, there exist $B_1 > 0$ and $\epsilon > 0$ such that*

$$U_{\text{in}}^-(r,t) < U_{\text{out}}^-(r,t) \quad \text{at } r = B_1(t+\tau)^{1/2}$$

for all $t \geq 0$.

Proof. We take $B_1 > \eta_*$. Then, we see that $f(B_1) - b_2 f(B_1) > 0$ holds from the definition of η_* and there exist a constant a^+ such that

$$\psi(\xi) \leq L\xi^m + a^+\xi^{-(\lambda_1 - m)} \quad \text{for } \xi > 0$$

from (1.3). If we choose small $\epsilon > 0$ such that

$$b_1(f(B_1) - b_2 f(B_1)) - a^+\epsilon^{\frac{\lambda_1}{m}} > 0,$$

we obtain at $r = B_1(t+\tau)^{1/2}$, namely at $\eta = B_1$,

$$\begin{aligned} U_{\text{out}}^-(r,t) - U_{\text{in}}^-(r,t) &= Lr^m + b_1(t+\tau)^{-\frac{l}{2}}(F - b_2\tilde{F}) - \sigma\psi(\xi) \\ &\geq Lr^m + b_1(t+\tau)^{-\frac{l}{2}}(F - b_2\tilde{F}) - \left(Lr^m + a^+\sigma^{\frac{\lambda_1}{m}} r^{-(\lambda_1 - m)}\right) \\ &= b_1(t+\tau)^{-\frac{l}{2}}(F - b_2\tilde{F}) - a^+\epsilon^{\frac{\lambda_1}{m}}(t+\tau)^{-\frac{l+m-\lambda_1}{2}} r^{-(\lambda_1 - m)} \\ &= (t+\tau)^{-\frac{l}{2}}\left(b_1(F(B_1) - b_2\tilde{F}(B_1)) - a^+\epsilon^{\frac{\lambda_1}{m}} B_1^{-(\lambda_1 - m)}\right) \\ &= (t+\tau)^{-\frac{l}{2}}\left(b_1(f(B_1) - b_2\tilde{f}(B_1)) - a^+\epsilon^{\frac{\lambda_1}{m}}\right) B_1^{-(\lambda_1 - m)} > 0. \end{aligned}$$

This completes the proof. □

Lemma 4.4. *Let $\lambda_1 - m < l < \lambda_2 - m + 2$. If $U_0(r)$ satisfies*

$$U_0(r) \geq Lr^m + c_2 r^{-l} \quad \text{for } r \geq R$$

with some $c_2 > 0$ and $R > 0$, then for any $b_2 > 0$, there exists $b_1 > 0$ such that

$$U_0(r) \geq U_{\text{out}}^-(r,0) \quad \text{for } r \geq R.$$

Proof. We only need to check the range $\eta > \eta_*$ by the definition of the outer solution. By Lemma 2.1, we see that there exists a constant $d_+ > 0$ such that

$$f(\eta) < d_+\eta^{-(l+m-\lambda_1)} \quad \text{for all } \eta > 0.$$

We take b_1 so small that $c_2 > b_1 d_+$; then we see

$$\begin{aligned} U_0(r) - U_{\text{out}}^-(r,0) &\geq Lr^m + c_2 r^{-l} - \left(Lr^m + b_1\tau^{-\frac{l}{2}}(F - b_2\tilde{F})\right) \\ &\geq c_2 r^{-l} - b_1\tau^{-\frac{l}{2}} F \geq c_2 r^{-l} - b_1\tau^{-\frac{l}{2}}\eta^{-(\lambda_1 - m)} d_+\eta^{-(l+m-\lambda_1)} \end{aligned}$$

$$= c_2 r^{-l} - b_1 d_+ \tau^{-\frac{l}{2}} \eta^{-l} = (c_2 - b_1 d_+) r^{-l} > 0$$

for $r > \max(R, \eta_* \tau^{1/2})$. This finishes the proof. □

Lemma 4.5. *For each $\tau > 0$ and arbitrary $R' > 0$, we have $U_{\mathrm{in}}^-(r, 0) \to \varphi_0(r)$ as $\epsilon \to 0$ uniformly in $r \in [0, R']$.*

Proof. First, we note that $U_{\mathrm{in}}^-(r, 0) = \varphi_{\epsilon\tau^{-k}} > \varphi_0$ for each $\epsilon, \tau > 0$ from the ordering property of the stationary solutions. We set $\gamma := \sigma(0) = \epsilon\, \tau^{-k}$ and see $\gamma \to 0$ as $\epsilon \to 0$. Note that the range of $r \in [0, R']$ corresponds to the range $0 \le \xi \le \gamma^{-1/m} R'$ and $\gamma^{-1/m} R' > 1$ for sufficiently small $\epsilon > 0$. For $1 \le \xi \le \gamma^{-1/m} R'$, we have

$$\begin{aligned} 0 &< U_{\mathrm{in}}^-(r, 0) - \varphi_0(r) = \gamma\psi(\xi) - L r^m \\ &\le \gamma(L\xi^m + a^+ \xi^{-(\lambda_1 - m)}) - L r^m \le (L r^m + a^+ \gamma) - L r^m = a^+ \gamma. \end{aligned}$$

On the other hand, for $0 \le \xi < 1$ we have

$$0 < U_{\mathrm{in}}^-(r, 0) - \varphi_0(r) = \gamma\psi(\xi) - L r^m \le \gamma\psi(1).$$

Letting $\epsilon \to 0$, we obtain the assertion of the lemma. □

Proposition 4.6. *Suppose that $\lambda_1 - m < l < \lambda_2 - m + 2$, $U_0(r) > L r^m$, and*

$$U_0(r) \ge L r^m + c_2 r^{-l} \quad \textit{for } r \ge R$$

with some $c_2 > 0$ and $R > 0$. Then there exists a constant $C_2 > 0$ such that the solution of (2.1) *satisfies*

$$U(0, t) \ge C_2 (1 + t)^{-\frac{m(l + m - \lambda_1)}{2\lambda_1}} \quad \textit{for all } t \ge 0.$$

Proof. We take $k := \frac{m(l+m-\lambda_1)}{2\lambda_1}$ in Lemma 4.2, $\delta > 0$ as in Lemma 4.1, and $b_1 > 0$ satisfying $c_2 > b_1 d_+$ as in Lemma 4.4. We take $b_2 > 0$ such that

$$f(1) - b_2 \tilde{f}(1) < 0$$

and

$$\frac{b_2 \delta}{2} \tilde{d}_-(1) - q L^{-(q+1)} d_+ > 0.$$

Then by Lemma 4.1 U_{out}^- becomes a subsolution of (2.1). Next, we choose $B_1 > \eta_* > 1$ and $\tau > 0$ so large that

$$\tau^{1/2} > R. \tag{4.9}$$

Finally, we take $\epsilon > 0$ so small that

$$b_1 (f(B_1) - b_2 f(B_1)) - a^+ \epsilon^{\frac{\lambda_1}{m}} > 0$$

as in Lemma 4.3 and

$$U_{\text{in}}^-(r,0) \le U_0(r) \quad \text{for } r \in [0, B_1\tau^{1/2}] \tag{4.10}$$

by applying Lemma 4.5.

Now we can construct a subsolution $U^-(r,t)$ of (2.1) that initially lies below U_0 as follows. Let $U_{\text{in}}^-(r,t)$ and $U_{\text{out}}^-(r,t)$ be as in Lemmas 4.2 and 4.1 respectively. From Lemmas 4.2 and 4.1,

$$U^-(r,t) := \begin{cases} U_{\text{in}}^-(r,t) & \text{for } r < r_*(t), \\ U_{\text{out}}^-(r,t) & \text{for } r \ge r_*(t) \end{cases}$$

defines a subsolution of (2.1), where

$$r_*(t) := \sup\{r > 0 : U_{\text{in}}(\rho,t) > U_{\text{out}}(\rho,t) \text{ for } \rho \in [0,r)\}.$$

It is clear that $U_r^-(0,t) = U_{\text{in},r}^-(0,t) = 0$, $t > 0$. We note that $r_*(t)$ is well defined from the fact that $U_{\text{in}}^- > \varphi_0$ and $U_{\text{out}}^- = \varphi_0$ for small $r \ge 0$ and Lemma 4.3.

In the following, we verify that $U^-(r,t)$ lies below U_0 initially. From Lemma 4.3, we see that $r_*(0) < B_1\tau^{1/2}$, and then find that (4.10) yields $U_{\text{in}}^-(r,0) \le U_0(r)$ for $0 \le r \le r_*(0)$. On the other hand, noting $\eta = \eta_*$ corresponds to $r = \tau^{1/2}\eta_*$ at $t = 0$, we find $r_*(0) > \tau^{1/2}\eta_* > R$ from (4.9) and $\eta_* > 1$ which is derived by (4.5). Therefore, we obtain $U_{\text{out}}^-(r,0) \le U_0(r)$ for $r > r_*(0)$ from Lemma 4.4.

Now, by the comparison principle, we obtain $U(r,t) \ge U^-(r,t)$ for all $r, t \ge 0$, and in particular

$$\begin{aligned} U(0,t) &\ge U^-(0,t) = U_{\text{in}}^-(0,t) = \varphi_{\sigma(t)}(0) \\ &= \sigma(t) = \epsilon\ (t+\tau)^{-k} \ge C_2(t+1)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}} \end{aligned}$$

for all $t \ge 0$ with some $C_2 > 0$. This completes the proof of Proposition 4.6. □

Proof of Theorem 1.4. Taking $U_0(r) := \min_{r=|x|} u_0(x)$ we have

$$u_0(x) \ge U_0(|x|), \quad x \in \mathbb{R}^N$$

and by assumption

$$U_0(r) \ge Lr^m + c_2 r^{-l}, \quad \text{for} \quad r \ge R.$$

Then by the comparison principle and Proposition 4.6, we have

$$u(0,t) \ge U(0,t) \ge C_2(1+t)^{-\frac{m(l+m-\lambda_1)}{2\lambda_1}}$$

for all $t > 0$ with some constant $C_2 > 0$. □

5. Universal lower bound

In this section, we prove that there exists a universal lower bound of the quenching rate which applies to any initial data that is close to a singular stationary solution from above. Our strategy is similar in spirit to that of our previous paper [20, 21].

5.1. **Outer subsolution.** In this subsection, following [20, 21], we construct a suitable subsolution of (2.1) with a desired property; U_{out} is identically equal to φ_0 near $\eta = 0$ and $\eta = \infty$.

First, we recall the initial-value problem (2.15):

$$\begin{cases} f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f = 0, & \eta > 0, \\ f(0) = \gamma_0 > 0, \quad f_\eta(0) = 0, \end{cases}$$

where $n = N - 2(\lambda_1 - m)$, $\beta = l + m - \lambda_1$, and throughout this section, l is fixed to $l = \lambda_2 - m + 2 + \tilde{\varepsilon}$, with a constant $\tilde{\varepsilon} > 0$. Then we see that f vanishes at some finite η_0 and $f > 0$ for $0 < \eta < \eta_0$ from Lemma 2.1.

Next, we modify this initial-value problem as follows:

$$\begin{cases} \tilde{f}_{\eta\eta} + \frac{n-1}{\eta} \tilde{f}_\eta + \frac{\eta}{2} \tilde{f}_\eta + \frac{\tilde{\beta}}{2} \tilde{f} = 0, & \eta > 0, \\ \tilde{f}(\eta_0/2) = f(\eta_0/2), \quad \tilde{f}_\eta(\eta_0/2) = f_\eta(\eta_0/2), \end{cases} \tag{5.1}$$

where $\tilde{\beta} = l + m - \lambda_1 + \delta$ with any constant $\delta > 0$. Then, we see in [20] that the solution of (5.1) has a vanishing property as follows:

Lemma 5.1. *There exist two vanishing points of $\tilde{f}$ (denoted by η_1 and η_2) such that $0 < \eta_1 < \eta_0/2 < \eta_2 < \eta_0$ and $0 < \tilde{f}(\eta) < f(\eta)$ for $\eta_1 < \eta < \eta_2$.*

Proof. This proof proceeds word for word like that of our previous proof of Lemma 4.1 in [21]. See Lemma 5.1 in [20] or Lemma 4.1 in [21] for a proof. □

Remark 5.2. *The function $\tilde{f}$ satisfies $\tilde{f}(\eta) < f(0)$ for $\eta_1 < \eta < \eta_2$ from Lemma* 5.1 *and Remark* 2.2.

Lemma 5.3. *Let $\tilde{\varepsilon}$ be a positive constant satisfying $\tilde{\varepsilon} > \delta > 0$, $b_1 > 0$ be an arbitrary constant, and define*

$$U_{\text{out}}(r,t) := \begin{cases} Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}} \tilde{F}(\eta) & \eta_1 \le \eta \le \eta_2, \\ Lr^m & \textit{otherwise}, \end{cases}$$

with $\eta = (t+\tau)^{-1/2} r$, where$\tilde{F}(\eta) = \eta^{-(\lambda_1 - m)} \tilde{f}(\eta)$ and $\tilde{f}$ is defined in Lemma 5.1. *If $\tau > 0$ is sufficiently large, then U_{out} is a subsolution of* (2.1).

Proof. It is trivial that $Lr^m = \varphi_0$ is a subsolution of (2.1). Thus, we only need to verify the range of $\eta_1 \le \eta \le \eta_2$. We note that $\tilde{F}$ satisfies

$$\tilde{F}_{\eta\eta} + \frac{N-1}{\eta}\tilde{F}_\eta + \frac{\eta}{2}\tilde{F}_\eta + \frac{l+\delta}{2}\tilde{F} + \frac{qL^{-(q+1)}}{\eta^2}\tilde{F} = 0.$$

We see that

$$\begin{aligned}
&U_{\mathrm{out},t} - U_{\mathrm{out},rr} - \frac{N-1}{r}U_{\mathrm{out},r} + (U_{\mathrm{out}})^{-q} \\
&= -\frac{b_1(l+\tilde{\varepsilon})}{2}(t+\tau)^{-\frac{\tilde{\varepsilon}}{2}-1}\tilde{F} - \frac{b_1}{2}(t+\tau)^{-\frac{\tilde{\varepsilon}}{2}-1}\eta\tilde{F}_\eta - b_1(t+\tau)^{-\frac{\tilde{\varepsilon}}{2}-1}\tilde{F}_{\eta\eta} \\
&\qquad - b_1(t+\tau)^{-\frac{\tilde{\varepsilon}}{2}-1}\frac{N-1}{\eta}\tilde{F}_\eta - \mathcal{D}Lr^m + \left(Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right)^{-q} \\
&= -b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}-1}\Big(\frac{l+\tilde{\varepsilon}}{2}\tilde{F} + \frac{\eta}{2}\tilde{F}_\eta + \tilde{F}_{\eta\eta} + \frac{N-1}{\eta}\tilde{F}_\eta\Big) \\
&\qquad - \mathcal{D}\varphi_0 + \left(Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right)^{-q} \\
&\le -b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}-1}\Big(\tilde{F}_{\eta\eta} + \frac{N-1}{\eta}\tilde{F}_\eta + \frac{\eta}{2}\tilde{F}_\eta + \frac{l+\delta}{2}\tilde{F} - \frac{\delta-\tilde{\varepsilon}}{2}\tilde{F}\Big) \\
&\qquad - \mathcal{D}\varphi_0 + \varphi_0^{-q} - q\left(Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right)^{-(q+1)} b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta) \\
&= -b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}-1}\Big(\frac{\tilde{\varepsilon}-\delta}{2} - \frac{qL^{-(q+1)}}{\eta^2} + q(t+\tau) \\
&\qquad \times \left(Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right)^{-(q+1)}\Big)\tilde{F}.
\end{aligned}$$

Therefore, U_{out} becomes a subsolution provided that

$$\frac{\tilde{\varepsilon}-\delta}{2} - \frac{qL^{-(q+1)}}{\eta^2} + q(t+\tau)\left(Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right)^{-(q+1)} \ge 0$$

is satisfied.

Let us show that this inequality holds for sufficiently large $\tau > 0$. Noting that $\eta \ge \eta_1$ implies $r \ge \tau^{1/2}\eta_1$, it follows that

$$\begin{aligned}
&\frac{\tilde{\varepsilon}-\delta}{2} - \frac{qL^{-(q+1)}}{\eta^2} + q(t+\tau)\left(Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right)^{-(q+1)} \\
&\ge \frac{\tilde{\varepsilon}-\delta}{2} - \frac{qL^{-(q+1)}}{\eta^2} \\
&\qquad + q(t+\tau)\left[(Lr^m)^{-(q+1)} - (q+1)(Lr^m)^{-(q+2)} b_1(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\tilde{\varepsilon}-\delta}{2} - b_1 q(q+1)(t+\tau)\left(Lr^m\right)^{-(q+2)}(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta) \\
&= \frac{\tilde{\varepsilon}-\delta}{2} - b_1 q(q+1)(t+\tau)\left(Lr^m\right)^{-(q+1)}\left(Lr^m\right)^{-1}(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\tilde{F}(\eta) \\
&= \frac{\tilde{\varepsilon}-\delta}{2} - b_1 q(q+1)\frac{L^{-(q+1)}}{\eta^2}(Lr^m)^{-1}(t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}}\eta^{-(\lambda_1-m)}\tilde{f}(\eta) \\
&\geq \frac{\tilde{\varepsilon}-\delta}{2} - b_1 q(q+1)\frac{L^{-(q+1)}}{\eta_1^2}\frac{L^{-1}}{(\tau^{1/2}\eta_1)^m}\tau^{-\frac{l+\tilde{\varepsilon}}{2}}\eta_1^{-(\lambda_1-m)}f(0) \geq 0,
\end{aligned}
$$

if we take sufficiently large $\tau > 0$. Thus we have reached the conclusion. $\square$

5.2. **Inner subsolution and matching.** We use the same inner subsolution $U_{\text{in}}^{-}(r,t)$ as that in Lemma 4.2.

Lemma 5.4. *For any $b_1 > 0$ and $\tau > 0$, there exist $B_2 > 0$ and $\epsilon > 0$ such that*

$$
U_{\text{in}}^{-}(r,t) < U_{\text{out}}(r,t) \quad \text{at } r = B_2(t+\tau)^{1/2}
$$

for all $t \geq 0$.

Proof. We take B_2 satisfying $\eta_1 < B_2 < \eta_2$ and choose small $\epsilon > 0$ such that

$$
b_1\tilde{f}(B_2) - a^{+}\epsilon^{\frac{\lambda_1}{m}} > 0.
$$

Then we obtain at $r = B_2(t+\tau)^{1/2}$, namely at $\eta = B_2$,

$$
\begin{aligned}
U_{\text{out}}(r,t) - U_{\text{in}}^{-}(r,t) &= Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\epsilon}}{2}}\tilde{F} - \sigma\psi(\xi) \\
&\geq Lr^m + b_1(t+\tau)^{-\frac{l+\tilde{\epsilon}}{2}}\eta^{-(\lambda_1-m)}\tilde{f} - \left(Lr^m + a^{+}\sigma^{\frac{\lambda_1}{m}}r^{-(\lambda_1-m)}\right) \\
&= b_1(t+\tau)^{-\frac{l+\tilde{\epsilon}}{2}}B_2^{-(\lambda_1-m)}\tilde{f}(B_2) - a^{+}\epsilon^{\frac{\lambda_1}{m}}(t+\tau)^{-\frac{l+m-\lambda_1+\tilde{\epsilon}}{2}}r^{-(\lambda_1-m)} \\
&= (t+\tau)^{-\frac{l+\tilde{\epsilon}}{2}}B_2^{-(\lambda_1-m)}(b_1\tilde{f}(B_2) - a^{+}\epsilon^{\frac{\lambda_1}{m}}) > 0.
\end{aligned}
$$

Thus we have the conclusion. $\square$

Lemma 5.5. *For any $b_1 > 0$, if $U_0(r)$ satisfies*

$$
U_0(r) > Lr^m \quad \text{for all } r \geq 0,
$$

then there exists large $\tau > 0$ such that

$$
U_0(r) \geq U_{\text{out}}(r,0) \quad \text{for all } r \geq 0.
$$

Proof. We only need to consider the range $\eta_1 < \eta < \eta_2$ from the definition of U_{out}. Let

$$
m_1 = \min_{r\in[\tau^{1/2}\eta_1,\tau^{1/2}\eta_2]}(U_0(r) - Lr^m).
$$

We choose $\tau > 0$ so large that

$$b_1 \tau^{-\frac{l+\tilde{\varepsilon}}{2}} \eta_1^{-(\lambda_1 - m)} f(0) < m_1.$$

Then we see that

$$\begin{aligned} U_{\mathrm{out}}^-(r,0) &= Lr^m + b_1 \tau^{-\frac{l+\tilde{\varepsilon}}{2}} \eta^{-(\lambda_1-m)} \tilde{f}(\eta) \\ &\le Lr^m + b_1 \tau^{-\frac{l+\tilde{\varepsilon}}{2}} \eta_1^{-(\lambda_1-m)} f(0) < Lr^m + m_1 \le U_0(r). \end{aligned}$$

This finishes the proof. □

Proposition 5.6. *Suppose that* $U_0(r)$ *satisfies*

$$U_0(r) > Lr^m \quad \textit{for all } r \ge 0.$$

Then for any small $\varepsilon > 0$*, there exists a constant* $\tilde{C}_\varepsilon > 0$ *such that the solution of* (2.1) *satisfies*

$$U(0,t) \ge \tilde{C}_\varepsilon (1+t)^{-\frac{m(\lambda_2 - \lambda_1 + 2)}{2\lambda_1} - \varepsilon} \quad \textit{for all } t > 0.$$

Proof. We put $\tilde{\varepsilon} = \varepsilon\lambda_1/m$ and $\delta = \tilde{\varepsilon}/2$ in Lemma 5.3 and

$$k = \frac{m(l + m - \lambda_1 + \tilde{\varepsilon})}{2\lambda_1} = \frac{m(\lambda_2 - \lambda_1 + 2)}{2\lambda_1} - \varepsilon$$

in Lemma 4.2 and fix $b_1 > 0$, and $B_2 > 0$ satisfying $\eta_1 < B_2 < \eta_2$. We can take $\tau > 0$ so large that U_{out} becomes a subsolution of (2.1) by Lemma 5.3 and U_{out} lies below U_0 initially by Lemma 5.5. Finally, we take $\epsilon > 0$ small such that the claim in Lemma 5.4 holds and $U_{\mathrm{in}}^-(r,0)$ lies below U_0 for $r \in [0, \tau^{1/2}\eta_2]$ by Lemma 4.5. Now, we can construct a subsolution $U^-(r,t)$ of (2.1)

Let $U_{\mathrm{out}}^- := U_{\mathrm{out}}$ in Lemma 5.3 and use U_{in}^- in Lemma 4.2. We define $U^-(r,t)$ to be exactly the same function as in Proposition 4.6. We note again that $r_*(t)$ is well-defined by the same argument as in Proposition 4.6. Thus it is shown that $U^-(r,t)$ is a subsolution of (2.1) which satisfies

$$U^-(0,t) = U_{\mathrm{in}}^-(0,t) = \varphi_{\sigma(t)}(0) = \sigma(t) = \epsilon(t+\tau)^{-k} \ge \tilde{C}_\varepsilon (t+1)^{-k}$$

for all $t \ge 0$ with some $\tilde{C}_\varepsilon > 0$. On the other hand, from Lemmas 4.5 and 5.5, we see that

$$U_0(r) \ge U^-(r,0) \quad \text{for } r \ge 0.$$

By the comparison principle, we obtain

$$U(r,t) \ge U^-(r,t) \quad \text{for all } r,t \ge 0.$$

In particular, we obtain

$$U(0,t) \ge U^-(0,t) \ge \tilde{C}_\varepsilon (1+t)^{-k}.$$

We have arrived at the conclusion. □

Proof of Theorem 1.5. We take

$$U_0(r) := \min_{|x|=r} u_0(x).$$

Then by Proposition 5.6, we have

$$u(0,t) \geq U(0,t) \geq \tilde{C}_\varepsilon (1+t)^{-\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1} - \varepsilon}$$

for all $t > 0$ with some constant $\tilde{C}_\varepsilon > 0$. □

Remark 5.7. *We can relax the condition on the initial data, noting that Proposition* 5.6 *holds for the case $U_0 = \varphi_0$ for sufficiently large $r > 0$.*

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References

[1] A. Acker and W. Walter, *The quenching problem for nonlinear parabolic differential equations*, Springer Lecture Notes in Mathematics 564, 1–12, Springer, New York, 1976.

[2] A. Acker and W. Walter, *On the global existence of solutions of parabolic differential equations with a singular nonlinear term*, Nonlinear Anal., 2 (1978), 499–504.

[3] C.Y. Chan and M.K. Kwong, *Quenching phenomena for a singular nonlinear parabolic equations*, Nonlinear Anal., 12 (1988), 1377–1383.

[4] K. Deng and H.A. Levine, *On the blowup of u_t at quenching*, Proc. Amer. Math. Soc., 106 (1989), 1049–1056.

[5] X.-Y. Chen and P. Poláčik, *Asymptotic periodicity of positive solutions of reaction-diffusion equations on a ball*, J. Reine Angew. Math., 472 (2006), 17–51.

[6] Q. Dai and Y. Gu, *A short note on quenching phenomena for semilinear parabolic equations*, J. Differential Equations, 137 (1997), 240–250.

[7] M. Fila and B. Kawohl, *Is quenching in infinite time possible?*, Quart. Appl. Math., 48 (1990), 531–534.

[8] M. Fila, J.R. King, M. Winkler, and E. Yanagida, *Optimal lower bound of the grow-up rate for a supercritical parabolic equation*, J. Differential Equations, 228 (2006), 339–356.

[9] M. Fila, M. Winkler, and E. Yanagida, *Grow-up rate of solutions for a supercritical semilinear diffusion equation*, J. Differential Equations, 205 (2004), 365–389.

[10] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1966), 109–124.

[11] V.A. Galaktionov and J.R. King, *Composite structure of global unbounded solutions of nonlinear heat equation with critical Sobolev exponents*, J. Differential Equations, 189 (2003), 199–233.

[12] H. Guo, Z. Guo, and K. Li, *Positive solutions of a semilinear elliptic equation with singular nonlinearity*, J. Math. Anal. Appl., 323 (2006), 344–359.

[13] C. Gui, W.-M. Ni, and X. Wang, *On the stability and instability of positive steady state of a semilinear heat equation in* $\mathbb{R}^n$, Comm. Pure Appl. Math., 45 (1992), 1153–1181.

[14] C. Gui, W.-M. Ni, and X. Wang, *Further study on a nonlinear heat equation*, J. Differential Equations, 169 (2001), 588–613.

[15] Y. Guo, Z. Pan, and M.J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J. Appl. Math., 66 (2005), 309–338.

[16] Z. Guo and J. Wei, *Symmetry of non-negative solutions of a semilinear elliptic equation with singular nonlinearity*, Proc. Roy. Soc. Edinburgh, 137A (2007), 963–994.

[17] Z. Guo and J. Wei, *On the Cauchy problem for a reaction-diffusion equation with a singular nonlinearity*, J. Differential Equations, 240 (2007), 279–323.

[18] Z. Guo and J. Wei, *On Cauchy problem for a reaction-diffusion equation with singular nonlinearity II*, preprint.

[19] Z. Guo and J. Wei, *On global solutions of a Cauchy problem with singular nonlinearity*, preprint.

[20] M. Hoshino, *Convergence rate for a semilinear parabolic equation with a singular nonlinearity*, Nonlinear Anal., 71 (2009), 2236–2256.

[21] M. Hoshino and E. Yanagida, *Sharp estimates of the convergence rate for a semilinear parabolic equation with supercritical nonlinearity*, Nonlinear Anal., 69 (2008), 3136–3152.

[22] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational. Mech. Anal., 49 (1972–73), 241–269.

[23] H. Kawarada, *On solution of initial-boundary problem for* $u_t = u_{xx} + \frac{1}{1-u}$, Publ. RIMS, Kyoto Univ., 10 (1975), 729–736.

[24] H.A. Levine, *The phenomenon of quenching: A survey,* in "Proceedings, VIth International Conference on Trends in the Theory and Practice of Nonlinear Analysis," North Holland, New York, 1985.

[25] H.A. Levine, *Advances in quenching,* in "Progress in Nonlinear Differential Equations and Their Applications," Vol. 7, pp. 319–346, Birkhäuser, Boston-Cambridge, MA, 1992.

[26] H.A. Levine and J.T. Montgomery, *The quenching of solutions of some nonlinear parabolic equations*, SIAM J. Math. Anal., 11 (1980), 842–847.

[27] N. Mizoguchi, *Growup of solutions for a semilinear heat equation with supercritical nonlinearity*, J. Differential Equations, 227 (2006), 652–669.

[28] P. Poláčik and E. Yanagida, *On bounded and unbounded global solutions of supercritical semilinear heat equation*, Math. Ann., 327 (2003), 745–771.

[29] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc, 337 (1993), 549–590.