

## ORBITAL STABILITY OF SOLITARY WAVES FOR A NONLINEAR SCHRÖDINGER SYSTEM

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**Abstract.** In this paper, we study the coupled nonlinear Schrödinger system

$$\begin{cases} iu_t + u_{xx} + (a|u|^2 + b|v|^2)u = 0 \\ iv_t + v_{xx} + (b|u|^2 + c|v|^2)v = 0 \end{cases}$$

where  $u, v$  are complex-valued functions of  $(x, t) \in \mathbb{R}^2$ , and  $a, b, c \in \mathbb{R}$ . Our work shows that, for this system of equations, the interplay between components of solutions in terms of the parameters  $a, b, c$  plays an important role in both the existence and stability of solitary waves. In particular, we prove that solitary wave solutions to this system are orbitally stable when either  $0 < b < \min\{a, c\}$ , or  $b > 0$  with  $b > \max\{a, c\}$  and  $b^2 > ac$ .

### 1. INTRODUCTION

It is well understood that the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} \pm |u|^2 u = 0, \tag{1.1}$$

where  $u$  is a complex-valued function of  $(x, t) \in \mathbb{R}^2$ , arises in a generic situation. The equation describes the evolution of small amplitude, slowly varying wave packets in a nonlinear media [7]. Indeed, it has been derived in such diverse fields as deep water waves [27], plasma physics [28], nonlinear optical fibers [13, 14], magneto-static spin waves [29], to name a few. The coupled nonlinear Schrödinger (CNLS) system

$$\begin{cases} iu_t + u_{xx} + (a|u|^2 + b|v|^2)u = 0 \\ iv_t + v_{xx} + (b|u|^2 + c|v|^2)v = 0 \end{cases} \tag{1.2}$$

where  $u, v$  are complex-valued functions of  $(x, t) \in \mathbb{R}^2$ , and  $a, b, c \in \mathbb{R}$ , arises physically under conditions similar to those described by (1.1) when there are two wavetrains moving with nearly the same group velocities [20,

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26]. The CNLS system also models physical systems in which the field has more than one component; for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. The CNLS system also arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states.

The system (1.2) has the conserved quantities

$$E(u, v) = \int_{-\infty}^{\infty} \left[ |u_x(x, t)|^2 + |v_x(x, t)|^2 - \frac{a}{2}|u(x, t)|^4 - \frac{c}{2}|v(x, t)|^4 - b|u(x, t)|^2|v(x, t)|^2 \right] dx, \quad (1.3)$$

$$Q(u) = \int_{-\infty}^{\infty} |u(x, t)|^2 dx \quad (1.4)$$

and

$$Q(v) = \int_{-\infty}^{\infty} |v(x, t)|^2 dx. \quad (1.5)$$

In other words, when applied to sufficiently regular solutions  $(u(x, t), v(x, t))$  of (1.2),  $E$  and  $Q$  are independent of  $t$ .

Bound-state solutions of (1.2) are, by definition, solutions of the form

$$\begin{aligned} u(x, t) &= e^{i(\omega_1 - \sigma^2)t + i\sigma x} \tilde{\phi}(x - 2\sigma t), \\ v(x, t) &= e^{i(\omega_2 - \sigma^2)t + i\sigma x} \tilde{\psi}(x - 2\sigma t), \end{aligned} \quad (1.6)$$

where  $\omega_1, \omega_2, \sigma \in \mathbb{R}$ , and  $\tilde{\phi}, \tilde{\psi} : \mathbb{R} \rightarrow \mathbb{R}$  are functions of one variable whose values are small when  $|\xi| = |x - 2\sigma t|$  is large. It is easy to see that  $(u(x, t), v(x, t))$  as defined in (1.6) are solutions of (1.2) if and only if  $(\tilde{\phi}, \tilde{\psi})$  is a critical point for the functional  $E(u, v)$ , when  $u(x)$  and  $v(x)$  are varied subject to the constraints that  $Q(u)$  and  $Q(v)$  be held constant. If  $(\tilde{\phi}, \tilde{\psi})$  is not only a critical point, but in fact a global minimizer of the constrained variational problem for  $E(u, v)$  then (1.6) is called a ground-state solution of (1.2). A brief discussion of what is currently known about bound-state solutions of (1.2) will be presented in Section 2 below.

To make the distinction between the terms “bound-state” and “ground-state” as clear as possible, we paraphrase the following from [1]. Bound-state and ground-state are traditional in the literature concerning the nonlinear Schrödinger equation (1.1) where the plus sign is taken for the nonlinear term. Bound-state solutions of (1.1) are solutions of the form

$$u(x, t) = e^{i(\omega - \sigma^2)t + i\sigma x} \phi(x - 2\sigma t) \quad (1.7)$$

where  $\omega, \sigma \in \mathbb{R}$  and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a function of one variable whose value is small when  $|\xi| = |x - 2\sigma t|$  is large. Equivalently, they are also minimizers of the Hamiltonian functional

$$\int_{-\infty}^{\infty} \left( |u_x|^2 - \frac{|u|^4}{2} \right) dx$$

subject to the constraint that  $\int_{-\infty}^{\infty} |u|^2 dx$  be held constant. It is a straightforward exercise to see that any bound-state solution of (1.1) must have a profile function of the form

$$\phi(x) = \sqrt{2\omega} e^{i\theta} \operatorname{sech}(\sqrt{\omega}x + x_0)$$

where  $x_0$  and  $\theta \in \mathbb{R}$ . In fact, these bound states are actually ground states (see [8]). Since  $|\phi(x)|$  decays monotonically to zero as  $x$  tends away from  $x_0$  to  $\infty$  or  $-\infty$ , bound-state solutions of (1.1) are often called solitary waves.

Solitary waves are localized nonlinear waves with remarkable stability properties, preserving their identity even after undergoing complex interactions. They go by the appellation of “soliton” because they exhibit strong stability properties, like those of particles. They are of special importance because of the distinguished role they sometimes play in the solution of the initial-value problem for the evolution equation in question. The stability theory is concerned with the persistence of these solitary waves under small perturbations of initial data. Thus one imagines a given initial wave form  $(u(x, 0), v(x, 0))$  for (1.2) specified for all values of  $x \in \mathbb{R}$ , and attention is given to the solution  $(u, v)$  of (1.2) emanating from  $(u(x, 0), v(x, 0))$ . The theory states that if the initial data  $(u(x, 0), v(x, 0))$  is near to the solitary wave profile  $(\tilde{\phi}, \tilde{\psi})$ , then the solution  $(u, v)$  emanating from the initial data will always be close to  $(\tilde{\phi}, \tilde{\psi})$  for all time, up to possible translations.

**Notation:** For  $1 \leq p \leq \infty$ , we denote by  $L^p = L^p(\mathbb{R})$  the space of all measurable functions  $f$  on  $\mathbb{R}$  for which the norm  $|f|_p = (\int_{-\infty}^{\infty} |f|^p dx)^{1/p}$  is finite for  $1 \leq p < \infty$  and  $|f|_{\infty}$  is the essential supremum of  $|f|$  on  $\mathbb{R}$ . Whether we intend the functions in  $L^p$  to be real-valued or complex-valued will be clear from the context.  $H_{\mathbb{C}}^1(\mathbb{R})$  is the usual Sobolev space consisting of measurable functions such that both  $f$  and  $f_x$  are in  $L^2$ . We define the space  $X$  to be the Cartesian product  $H_{\mathbb{C}}^1(\mathbb{R}) \times H_{\mathbb{C}}^1(\mathbb{R})$ . If  $T > 0$  and  $Y$  is any Banach space, we denote by  $\mathcal{C}([0, T], Y)$  the Banach space of continuous maps  $f : [0, T] \rightarrow Y$ , with norms given by  $\|f\|_{\mathcal{C}([0, T], Y)} = \sup_{t \in [0, T]} \|f(t)\|_Y$ .

Logically, prior to a discussion of stability as formulated above in terms of perturbations of the initial data there should be a theory for the initial-valued

problem itself. This is a subject that has attracted considerable attention and it is not our purpose here to provide a survey of the theory. It has been proved (see, for example, Chapter 4 in [8]) that, for any  $(u(x, 0), v(x, 0)) \in X$ , there exists a unique solution  $(u(x, t), v(x, t))$  of (1.2) in  $\mathcal{C}(\mathbb{R}; X)$  emanating from  $(u(x, 0), v(x, 0))$ , and  $(u(x, t), v(x, t))$  satisfies

$$Q(u(x, t)) = Q(u(x, 0)), \quad Q(v(x, t)) = Q(v(x, 0)),$$

$$E(u(x, t), v(x, t)) = E(u(x, 0), v(x, 0)).$$

This manuscript is organized as follows. In Section 2, known results for the coupled nonlinear Schrödinger system are first summarized, then the main contributions of this manuscript are presented. Section 3 contains the proof of the relative compactness of minimizing sequences for the variational problem which defines the solitary wave solutions of (1.2). An immediate consequence of this fact is that *the set of minimizers* of this variational problem is stable. Finally, Section 4 discusses the existence and properties of solitary wave solutions of the system (1.2), including their stability properties.

## 2. STATEMENT OF RESULTS

In the case when  $a = c > -1$ ,  $b = 1$  (also known as the symmetric case), the nonlinear Schrödinger system (1.2) is known to have explicit solitary wave solutions of the form (see, for example, [19])

$$(u_\omega(x, t), v_\omega(x, t)) = \left( e^{i(\omega - \sigma^2)t + i\sigma x + im} \phi_\omega(x - 2\sigma t), e^{i(\omega - \sigma^2)t + i\sigma x + in} \phi_\omega(x - 2\sigma t) \right) \quad (2.1)$$

where  $\omega > 0$ ,  $\sigma, m, n$  are real constants, and

$$\phi_\omega(\xi) = \sqrt{\frac{2\omega}{a+1}} \operatorname{sech}(\sqrt{\omega}\xi). \quad (2.2)$$

For this particular form of solitary wave, stability has been proved by Ohta [19]. In particular, his result is as follows.

*Let  $a = c > -1$  and  $b = 1$ . Then, for any  $\omega > 0$ , the solitary wave solution  $(u_\omega(t), v_\omega(t))$  given by (2.1)-(2.2) is orbitally stable in the following sense: for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $(u_0, v_0) \in X$  satisfies*

$$\|u_0 - \phi_\omega\|_{H_{\mathbb{C}}^1(\mathbb{R})} + \|v_0 - \phi_\omega\|_{H_{\mathbb{C}}^1(\mathbb{R})} < \delta,$$

*then the solution  $(u(x, t), v(x, t))$  with  $(u(0), v(0)) = (u_0, v_0)$  satisfies*

$$\sup_{t \in \mathbb{R}} \inf_{\theta, \eta, y \in \mathbb{R}} \left\{ \|u(\cdot, t) - e^{i\theta} \tau_y \phi_\omega\|_{H_{\mathbb{C}}^1(\mathbb{R})} + \|v(\cdot, t) - e^{i\eta} \tau_y \phi_\omega\|_{H_{\mathbb{C}}^1(\mathbb{R})} \right\} < \epsilon,$$

where  $\tau_y f(x) = f(x - y)$ .

Notice that if  $(u(x, t), v(x, t))$  as defined in (1.6) is a bound-state solution of (1.2), then  $(\tilde{\phi}, \tilde{\psi})$  solves the equations

$$\begin{cases} -\tilde{\phi}_{xx} + \omega_1 \tilde{\phi} = a|\tilde{\phi}|^2 \tilde{\phi} + b|\tilde{\psi}|^2 \tilde{\phi} \\ -\tilde{\psi}_{xx} + \omega_2 \tilde{\psi} = c|\tilde{\psi}|^2 \tilde{\psi} + b|\tilde{\phi}|^2 \tilde{\psi}. \end{cases} \tag{2.3}$$

It was observed (e.g., [5, 4]) that when  $\omega_1 = \omega_2 = \omega$ , and  $a, c > 0$  with  $b > \max\{a, c\}$  or  $-\sqrt{ac} < b < \min\{a, c\}$ , the pair

$$(\tilde{\phi}, \tilde{\psi}) = \left( \sqrt{\frac{b-c}{b^2-ac}} \phi, \sqrt{\frac{b-a}{b^2-ac}} \phi \right)$$

where

$$\phi(\xi) = \sqrt{2\omega} \operatorname{sech}(\sqrt{\omega}\xi) \tag{2.4}$$

is a positive solution of (2.3). Furthermore, it is proved in [25] that for  $b > \max\{a, c\}$  and  $\omega_1 = \omega_2 = \omega$  this solution is also the unique positive solution of (2.3).

In this paper, Ohta's result will be extended to include more general settings, namely, the nonsymmetric case  $a \neq c$ . We assume that either

- (A1)  $0 < b < \min\{a, c\}$ ; or
- (A2)  $b > 0$  with  $b > \max\{a, c\}$  and  $b^2 > ac$ .

For these values of  $a, b$  and  $c$ , let

$$\begin{aligned} (\Phi(x, t), \Psi(x, t)) = & \left( e^{i(\omega-\sigma^2)t+i\sigma x+im} \sqrt{\frac{b-c}{b^2-ac}} \phi(x-2\sigma t), \right. \\ & \left. e^{i(\omega-\sigma^2)t+i\sigma x+in} \sqrt{\frac{b-a}{b^2-ac}} \phi(x-2\sigma t) \right) \end{aligned} \tag{2.5}$$

where  $\omega > 0$ ,  $\sigma, m, n$  are real constants and  $\phi(\xi)$  as defined in (2.4).

We will show that the solitary-wave solution (2.5) is orbitally stable. However, notice that since (1.2) is invariant under the Galilean transformations

$$(u(x, t), v(x, t)) \rightarrow (e^{-i\sigma^2 t + idx} u(x-2\sigma t, t), e^{-i\sigma^2 t + idx} v(x-2\sigma t, t)), \quad d, \sigma \in \mathbb{R},$$

and the phase transformations

$$(u(x, t), v(x, t)) \rightarrow (e^{i\alpha} u(x, t), e^{i\beta} v(x, t)), \quad \alpha, \beta \in \mathbb{R},$$

we may consider the case when  $\sigma = m = n = 0$  in (2.5). Therefore, from here on we put

$$(\Phi(x, t), \Psi(x, t)) = \left( e^{i\omega t} \sqrt{\frac{b-c}{b^2-ac}} \phi(x), e^{i\omega t} \sqrt{\frac{b-a}{b^2-ac}} \phi(x) \right). \tag{2.6}$$

The precise statement of our result is as follows.

**Theorem 2.1.** *Let  $a, b$  and  $c$  be real numbers such that either (A1) or (A2) is satisfied. Then, for any  $\omega > 0$ , the solitary wave solution  $(\Phi(x, t), \Psi(x, t))$  given by (2.6)-(2.4) is orbitally stable in the following sense: for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $(u_0, v_0) \in X$  satisfies*

$$\inf_{\theta, \eta, y \in \mathbb{R}} \left\{ \left\| u_0 - \sqrt{\frac{b-c}{b^2-ac}} e^{i\theta} \tau_y \phi \right\|_{H^1_{\mathbb{C}}(\mathbb{R})} + \left\| v_0 - \sqrt{\frac{b-a}{b^2-ac}} e^{i\eta} \tau_y \phi \right\|_{H^1_{\mathbb{C}}(\mathbb{R})} \right\} < \delta,$$

then the solution  $(u(x, t), v(x, t))$  with  $(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0)$  satisfies

$$\begin{aligned} \sup_{t \in \mathbb{R}} \inf_{\theta, \eta, y \in \mathbb{R}} \left\{ \left\| u(\cdot, t) - \sqrt{\frac{b-c}{b^2-ac}} e^{i\theta} \tau_y \phi \right\|_{H^1_{\mathbb{C}}(\mathbb{R})} \right. \\ \left. + \left\| v(\cdot, t) - \sqrt{\frac{b-a}{b^2-ac}} e^{i\eta} \tau_y \phi \right\|_{H^1_{\mathbb{C}}(\mathbb{R})} \right\} < \epsilon, \end{aligned}$$

where  $\tau_y f(x) = f(x - y)$ .

We break the proof of Theorem 2.1 into two parts. First, we show that the set of minimizers of a certain variational problem is stable (see Section 3.) Next, we show that this set consists of just singletons given by (2.6)-(2.4) (see Section 4). Most of the crucial ideas have already appeared in the paper of Ohta [19] which in turn follows closely in spirit ideas from the classical work of Cazenave-Lions [9] for single equations (see [8, 24] for more references). Our work shows that, for this system of equations, the interplay between components of solutions in terms of the parameters  $a, b, c$  plays an important role in both the existence and stability of solitary waves.

### 3. VARIATIONAL PROBLEM

Recall that the system (1.2) has the conserved quantities (1.3)-(1.5). Let  $a, b$  and  $c$  be real numbers such that either (A1) or (A2) is satisfied. Define:

$$E_1(u) = \int_{-\infty}^{\infty} \left[ |u_x(x, t)|^2 - \frac{b^2 - ac}{2(b - c)} |u(x, t)|^4 \right] dx \tag{3.1}$$

and

$$E_2(u) = \int_{-\infty}^{\infty} \left[ |u_x(x, t)|^2 - \frac{b^2 - ac}{2(b - a)} |u(x, t)|^4 \right] dx. \tag{3.2}$$

For fixed  $\omega > 0$ , let

$$\lambda = 4\sqrt{\omega} \frac{b - c}{b^2 - ac} \tag{3.3}$$

and

$$\mu = 4\sqrt{\omega} \frac{b - a}{b^2 - ac}. \tag{3.4}$$

For fixed  $\omega > 0$  (hence  $\lambda = \lambda(\omega) > 0$  and  $\mu = \mu(\omega) > 0$  are also fixed), define the real numbers  $I, I_1$  and  $I_2$  as follows:

$$I(\lambda, \mu) = \inf\{E(u, v) : u, v \in H_{\mathbb{C}}^1(\mathbb{R}), |u|_2^2 = \lambda, |v|_2^2 = \mu\}, \tag{3.5}$$

$$I_1(\lambda) = \inf\{E_1(u) : u \in H_{\mathbb{C}}^1(\mathbb{R}), |u|_2^2 = \lambda\} \tag{3.6}$$

and

$$I_2(\mu) = \inf\{E_2(u) : u \in H_{\mathbb{C}}^1(\mathbb{R}), |u|_2^2 = \mu\}. \tag{3.7}$$

The sets of minimizers for  $I(\lambda, \mu), I_1(\lambda)$  and  $I_2(\mu)$  are, respectively,

$$G(\lambda, \mu) = \{(u, v) \in X : I(\lambda, \mu) = E(u, v), |u|_2^2 = \lambda, |v|_2^2 = \mu\},$$

$$G_1(\lambda) = \{u \in H_{\mathbb{C}}^1(\mathbb{R}) : I_1(\lambda) = E_1(u), |u|_2^2 = \lambda\},$$

and

$$G_2(\mu) = \{u \in H_{\mathbb{C}}^1(\mathbb{R}) : I_2(\mu) = E_2(u), |u|_2^2 = \mu\}.$$

From now on, we always assume that  $\omega > 0$  is fixed (hence so are the positive numbers  $\lambda$  and  $\mu$ ) and that either (A1) or (A2) is satisfied.

**Remark.** 1) In the last several years there has been intensive work in studying the existence of standing waves for nonlinear Schrödinger systems of the form studied in this paper, for example, see [2, 3, 4, 5, 6, 11, 12, 15, 18, 21] and references therein. Most of these papers are concerned with the corresponding nonlinear elliptic systems and various methods have been employed to construct solutions for various parameter regimes. In order to study the stability questions here, we have to tackle a different variational formulation.

2) There is another paper related to what is investigated here. In [10] a different variational setting than the one in this paper was used to prove the stability of solitary waves for (1.2), namely using the sum of the  $L^2$ -norms of the two components. The two variational problems can have different solitary-wave solutions. In fact, the last two pages of [10] show that, in the case when  $a = c = \alpha$  and  $b < \alpha$ , the solitary waves which solve the variational problem in our paper are not the same as the solitary waves which solve the variational problem in [10].

3) One referee brought to our attention the papers [22, 23]. In [22, 23] Song proves stability of standing waves to a system of Schrödinger equations with combined power-type nonlinearities, which includes our equation (1.2) when the dimension  $n = 1$ . However, due to the nature of the problem being posed in higher dimension, uniqueness of the ground-state solutions was not

studied in [22, 23]; moreover, the range of stability for the coefficients  $a, b, c$  is smaller compared to the one obtained by us. (Our result allows for  $a, c$  to be negative as well.)

4) It was shown in [4, 5] that, when  $b < 0$ , there are ground-state solutions still. However, our method breaks down when  $b < 0$ .

**Lemma 3.1.** *For all  $\lambda, \mu > 0$ , one has  $-\infty < I(\lambda, \mu) < 0$ .*

**Proof.** To see that  $I(\lambda, \mu) < 0$ , first choose any function  $f \in H^1_{\mathbb{C}}(\mathbb{R})$  such that  $|f|_2^2 = \lambda$  and let  $g = \sqrt{\frac{\mu}{\lambda}}f$  so  $|g|_2^2 = \mu$ . For each  $r > 0$ , let  $f_r(x) = \sqrt{r}f(rx)$  and  $g_r(x) = \sqrt{r}g(rx)$ . Then for all  $r$  we have  $|f_r|_2^2 = |f|_2^2 = \lambda$ ,  $|g_r|_2^2 = |g|_2^2 = \mu$  and

$$E(f_r, g_r) = \int_{-\infty}^{\infty} \left[ r^2(|f'|^2 + |g'|^2) - r\left(\frac{a}{2}|f|^4 + \frac{c}{2}|g|^4 + b|f|^2|g|^2\right) \right] dx.$$

Substituting  $g = \sqrt{\frac{\mu}{\lambda}}f$  into the above expression, we have

$$E(f_r, g_r) = \int_{-\infty}^{\infty} \left[ r^2\left(1 + \frac{\mu}{\lambda}\right)|f'|^2 - r\frac{(b^2 - ac)(2b - a - c)}{2(b - c)^2}|f|^4 \right] dx.$$

Under either (A1) or (A2) we have  $(b^2 - ac)(2b - a - c) > 0$ . Hence by taking  $r = r_0$  small enough, we have  $E(f_{r_0}, g_{r_0}) < 0$  and consequently  $I(\lambda, \mu) < 0$ .

To prove that  $I(\lambda, \mu) > -\infty$ , it suffices to bound  $E(f, g)$  from below by a number which is independent of  $f$  and  $g$ . Notice first that, from the Gagliardo-Nirenberg inequality, for any  $h \in H^1_{\mathbb{C}}(\mathbb{R})$  one has

$$\int_{-\infty}^{\infty} |h|^4 dx \leq C \left( \int_{-\infty}^{\infty} |h_x|^2 dx \right)^{1/2} \left( \int_{-\infty}^{\infty} |h|^2 dx \right)^{3/2} \leq \epsilon |h_x|_2^2 + C_{\epsilon} |h|_2^6,$$

where  $\epsilon > 0$  is arbitrary and  $C_{\epsilon}$  depends on  $\epsilon$  but not on  $h$ . Hence

$$\begin{aligned} E(f, g) &= \int_{-\infty}^{\infty} \left( |f'|^2 + |g'|^2 - \frac{a}{2}|f|^4 - \frac{c}{2}|g|^4 - b|f|^2|g|^2 \right) dx \\ &\geq \int_{-\infty}^{\infty} \left( (|f'|^2 + |g'|^2) - C(|f|^4 + |g|^4) \right) dx \\ &\geq \|f\|_1^2 + \|g\|_1^2 - \epsilon(\|f\|_1^2 + \|g\|_1^2) - C_{\epsilon, \lambda, \mu} > -\infty. \quad \square \end{aligned}$$

The following lemma guarantees that minimizing sequences must be bounded *uniformly* in  $H^1_{\mathbb{C}}$ , and the  $L^4$ -norms bounded away from zero for all large  $n$ .



**Lemma 3.2.** *If  $\{(u_n, v_n)\}$  is a minimizing sequence for  $I(\lambda, \mu)$ , then there exist constants  $B > 0$  and  $\delta > 0$  such that*

- i)  $\|u_n\|_1 + \|v_n\|_1 \leq B$  for all  $n$ , and*
- ii)  $|u_n|_4^4 + |v_n|_4^4 \geq \delta$  for all sufficiently large  $n$ .*

**Proof.** To see i), we first write

$$\begin{aligned} \|u_n\|_1^2 + \|v_n\|_1^2 &= E(u_n, v_n) + \lambda + \mu + \int_{-\infty}^{\infty} \left( \frac{a}{2}|u_n|^4 + \frac{c}{2}|v_n|^4 + b|u_n|^2|v_n|^2 \right) dx \\ &\leq \sup_n E(u_n, v_n) + \lambda + \mu + C \int_{-\infty}^{\infty} (|u_n|^4 + |v_n|^4) dx \\ &\leq A + C \left( \int_{-\infty}^{\infty} |u_n'|^2 dx \right)^{1/2} + C \left( \int_{-\infty}^{\infty} |v_n'|^2 dx \right)^{1/2} \end{aligned}$$

where an application of the Gagliardo-Nirenberg inequality, followed by the facts that  $|u_n|_2^2 = \lambda$  and  $|v_n|_2^2 = \mu$  for all  $n$ , have been used. Since  $\|u_n\|_1^2 + \|v_n\|_1^2$  is now shown to be bounded by a smaller power, the existence of the advertised bound  $B$  follows.

To see ii), suppose no such constant  $\delta$  exists, then

$$\liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|u_n|^4 + |v_n|^4) dx = 0.$$

Consequently, we have

$$I(\lambda, \mu) = \lim_{n \rightarrow \infty} E(u_n, v_n) \geq -C \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|u_n|^4 + |v_n|^4) dx = 0,$$

a contradiction to Lemma 3.1. □

Let  $\{(u_n, v_n)\} \in X$  be a minimizing sequence for  $E$  and consider a sequence of nondecreasing functions  $M_n : [0, \infty) \rightarrow [0, \lambda + \mu]$  as follows:

$$M_n(s) = \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} (|u_n(x)|^2 + |v_n(x)|^2) dx.$$

As  $M_n(s)$  is a uniformly bounded sequence of nondecreasing functions in  $s$ , one can show that it has a subsequence, which is still denoted as  $M_n$ , that converges point-wise to a nondecreasing limit function  $M(s) : [0, \infty) \rightarrow [0, \lambda + \mu]$ . Let

$$\rho = \lim_{s \rightarrow \infty} M(s) := \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} M_n(s) = \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} (|u_n(x)|^2 + |v_n(x)|^2) dx.$$

Then  $0 \leq \rho \leq \lambda + \mu$ .

Lions' concentration compactness lemma [16, 17] shows that there are three possibilities for the value of  $\rho$ :

- Case 1: (*Vanishing*)  $\rho = 0$ . Since  $M(s)$  is non-negative and non-decreasing, this is equivalent to saying

$$M(s) = \lim_{n \rightarrow \infty} M_n(s) = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} (|u_n(x)|^2 + |v_n(x)|^2) dx = 0$$

for all  $s < \infty$ , or

- Case 2: (*Dichotomy*)  $\rho \in (0, \lambda + \mu)$ , or
- Case 3: (*Compactness*)  $\rho = \lambda + \mu$ , which implies that there exists  $\{y_n\}_{n=1} \in \mathbb{R}$  such that  $|u_n(\cdot + y_n)|^2 + |v_n(\cdot + y_n)|^2$  is tight; namely, for all  $\epsilon > 0$ , there exists  $s < \infty$  such that

$$\int_{y_n-s}^{y_n+s} (|u_n(x)|^2 + |v_n(x)|^2) dx \geq (\lambda + \mu) - \epsilon.$$

The next lemma will find use in ruling out the vanishing of minimizing sequences in Proposition 3.7. This is indeed just a special case of a general result established in [16, 17].

**Lemma 3.3.** *Suppose  $B > 0$  and  $\delta > 0$  are given. Then there exists  $C = C(B, \delta) > 0$  such that, if  $f, g \in H^1_{\mathbb{C}}$  with  $\|f\|_1 + \|g\|_1 \leq B$  and  $|u|_4^4 + |v|_4^4 \geq \delta$ , then*

$$\sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} (|f|^4 + |g|^4) dx \geq C.$$

**Proof.** The following proof is modified from the one presented in [1]. We have

$$\begin{aligned} \|f\|_1^2 + \|g\|_1^2 &= \sum_{j \in \mathbb{Z}} \int_{j-1/2}^{j+1/2} (|f_x|^2 + |f|^2 + |g_x|^2 + |g|^2) dx \leq B^2 \\ &= \frac{B^2}{|f|_4^4 + |g|_4^4} (|f|_4^4 + |g|_4^4) = \sum_{j \in \mathbb{Z}} \frac{B^2}{|f|_4^4 + |g|_4^4} \int_{j-1/2}^{j+1/2} (|f|^4 + |g|^4) dx. \end{aligned}$$

Hence there exists some  $j_0 \in \mathbb{Z}$  for which

$$\int_{j_0-1/2}^{j_0+1/2} (|f_x|^2 + |f|^2 + |g_x|^2 + |g|^2) dx \leq \frac{B^2}{|f|_4^4 + |g|_4^4} \int_{j_0-1/2}^{j_0+1/2} (|f|^4 + |g|^4) dx.$$

On the other hand, from the Sobolev embedding theorem, there exists a constant  $A$  independent of  $f$  and  $g$  such that

$$\int_{j_0-1/2}^{j_0+1/2} (|f|^4 + |g|^4) dx \leq A \left( \int_{j_0-1/2}^{j_0+1/2} (|f_x|^2 + |f|^2 + |g_x|^2 + |g|^2) dx \right)^2.$$

Thus, we must have

$$\int_{j_0-1/2}^{j_0+1/2} (|f|^4 + |g|^4) dx \geq \frac{\delta^2}{AB^4} = C > 0,$$

and the lemma is proved. □

**Lemma 3.4.** *For any fixed  $\omega > 0$ , we have*

- 1)  $I_1(\lambda(\omega)) = -\frac{1}{3} \left[ \frac{b^2-ac}{4(b-c)} \right]^2 \lambda^3.$
- 2)  $I_2(\mu(\omega)) = -\frac{1}{3} \left[ \frac{b^2-ac}{4(b-a)} \right]^2 \mu^3.$
- 3)  $I(\lambda, \mu) = I_1(\lambda) + I_2(\mu).$

**Proof.** To prove statements 1) and 2), notice that, by theorem 8.1.7 in [8], we have for any fixed  $\omega > 0$

$$G_1(\lambda(\omega)) = \{e^{i\alpha} \sqrt{\frac{b-c}{b^2-ac}} \phi(\cdot + y) : \alpha, y \in \mathbb{R}\}$$

and

$$G_2(\mu(\omega)) = \{e^{i\beta} \sqrt{\frac{b-a}{b^2-ac}} \phi(\cdot + z) : \beta, z \in \mathbb{R}\}$$

where  $\phi = \sqrt{2\omega} \operatorname{sech}(\sqrt{\omega}x)$ , which implies that  $I_1(\lambda(\omega)) = E_1(\sqrt{\frac{b-c}{b^2-ac}} \phi)$  and  $I_2(\mu(\omega)) = E_2(\sqrt{\frac{b-a}{b^2-ac}} \phi)$ . Using the facts that

$$\left| \frac{d}{dx} \phi(x) \right|^2 = 2\omega^2 (\operatorname{sech}^2(\sqrt{\omega}x) - \operatorname{sech}^4(\sqrt{\omega}x)),$$

$$\int_{-\infty}^{\infty} \operatorname{sech}^2 x \, dx = 2, \quad \text{and} \quad \int_{-\infty}^{\infty} \operatorname{sech}^4 x \, dx = \frac{4}{3},$$

straightforward calculations give 1) and 2).

To prove 3), we claim first that  $I(\lambda, \mu) \geq I_1(\lambda) + I_2(\mu)$ . This is because, from the Cauchy-Schwartz inequality, one can see that

$$2 \int_{\Omega} |u|^2 |v|^2 dx \leq 2 \left( \frac{b-a}{b-c} \right)^{1/2} |u^2|_2 \left( \frac{b-c}{b-a} \right)^{1/2} |v^2|_2 \leq \frac{b-a}{b-c} |u|_4^4 + \frac{b-c}{b-a} |v|_4^4.$$

Then from the definition of  $E(u, v)$  as defined in (1.3), one has

$$\begin{aligned} E(u, v) &\geq |u_x|_2^2 + |v_x|_2^2 - \frac{a}{2}|u|_4^4 - \frac{c}{2}|v|_4^4 - \frac{b}{2}\left(\frac{b-a}{b-c}|u|_4^4 + \frac{b-c}{b-a}|v|_4^4\right) \\ &= |u_x|_2^2 + |v_x|_2^2 - \frac{b^2-ac}{2(b-c)}|u|_4^4 - \frac{b^2-ac}{2(b-a)}|v|_4^4 = E_1(u) + E_2(v). \end{aligned}$$

The claim follows from taking the infima on both sides of the above inequality. Now, because  $|\sqrt{\frac{b-c}{b^2-ac}}\phi|_2^2 = \lambda$  and  $|\sqrt{\frac{b-a}{b^2-ac}}\phi|_2^2 = \mu$  for any  $\omega > 0$  fixed, we therefore have

$$\begin{aligned} I(\lambda, \mu) &\leq E\left(\sqrt{\frac{b-c}{b^2-ac}}\phi, \sqrt{\frac{b-a}{b^2-ac}}\phi\right) \\ &= E_1\left(\sqrt{\frac{b-c}{b^2-ac}}\phi\right) + E_2\left(\sqrt{\frac{b-a}{b^2-ac}}\phi\right) = I_1(\lambda) + I_2(\mu) \leq I(\lambda, \mu) \end{aligned}$$

by the claim. Hence  $I(\lambda, \mu) = I_1(\lambda) + I_2(\mu)$  and the lemma is proved.  $\square$

**Remark.** Notice that the argument used in the proof of part 3) shows that the inequality  $I(p, q) \geq I_1(p) + I_2(q)$  actually holds for all  $p, q > 0$ ; that is, it holds without the assumption that  $\frac{p}{q} = \frac{b-c}{b-a}$ .

**Corollary 3.5.** *For any  $\omega > 0$  fixed,*

$$\left\{ \left( e^{i\alpha} \sqrt{\frac{b-c}{b^2-ac}}\phi(\cdot+y), e^{i\beta} \sqrt{\frac{b-a}{b^2-ac}}\phi(\cdot+y) \right) : \alpha, \beta, y \in \mathbb{R} \right\} \subset G(\lambda(\omega), \mu(\omega)).$$

The following lemma provides *strict sub-additivity* of the function  $I$  needed to rule out the dichotomy of minimizing sequences. Notice that when we write  $I(0, \mu)$ , we intend for the first component of the minimizing sequence to vanish *in the sense of concentration compactness introduced above*.

**Lemma 3.6.** *For any  $\alpha \in [0, \lambda]$  and  $\beta \in [0, \mu]$  satisfying  $0 < \alpha + \beta < \lambda + \mu$ , we have*

$$I(\lambda, \mu) < I(\alpha, \beta) + I(\lambda - \alpha, \mu - \beta).$$

**Proof.** We consider separately the following three cases.

**Case 1:**  $\alpha \in (0, \lambda)$  and  $\beta \in (0, \mu)$ . From 3) in Lemma 3.4, we have

$$I(\lambda, \mu) = I_1(\lambda) + I_2(\mu);$$

and from Theorem 8.1.7 in [8], we have

$$I_1(\lambda) < I_1(\alpha) + I_1(\lambda - \alpha) \quad \text{and} \quad I_2(\mu) < I_2(\beta) + I_2(\mu - \beta).$$

Consequently, we obtain that

$$I(\lambda, \mu) < I_1(\alpha) + I_1(\lambda - \alpha) + I_2(\beta) + I_2(\mu - \beta). \tag{3.8}$$

Using the Remark following Lemma 3.4, we get

$$I_1(\alpha) + I_2(\beta) \leq I(\alpha, \beta) \quad \text{and} \quad I_1(\lambda - \alpha) + I_2(\mu - \beta) \leq I(\lambda - \alpha, \mu - \beta). \tag{3.9}$$

Combining (3.8) and (3.9), we arrive at

$$I(\lambda, \mu) < I(\alpha, \beta) + I(\lambda - \alpha, \mu - \beta),$$

as desired.

**Case 2:**  $\alpha = 0$  and  $\beta \in (0, \mu]$ . The variational problem in this case becomes

$$I(0, \beta) = \inf\{|v_x|_2^2 - \frac{c}{2}|v|_4^4 : |v|_2^2 = \beta\}.$$

In case  $c \leq 0$ , we have  $I(0, \beta) \geq 0$ . In case  $c > 0$ , the problem gives (see, for example [8]) the unique solution, up to translation,

$$v(x) = \sqrt{\frac{2\gamma_1}{c}} \operatorname{sech}(\sqrt{\gamma_1}x)$$

with  $\sqrt{\gamma_1} = \beta c/4$ . A straightforward calculation as in 1) and 2) of Lemma 3.4 then shows that

$$I(0, \beta) \begin{cases} = -\frac{1}{3}\left(\frac{c}{4}\right)^2 \beta^3, & \text{if } c > 0; \\ \geq 0, & \text{if } c \leq 0. \end{cases} \tag{3.10}$$

On the other hand, part 2) in Lemma 3.4 says that

$$I_2(\beta) = -\frac{1}{3}\left(\frac{b^2 - ac}{4(b - a)}\right)^2 \beta^3. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain that  $I(0, \beta) > I_2(\beta)$ . Thus,

$$I(\lambda, \mu) = I_1(\lambda) + I_2(\mu) \leq I_1(\lambda) + I_2(\beta) + I_2(\mu - \beta) < I(0, \beta) + I(\lambda, \mu - \beta)$$

proving case 2.

**Case 3:**  $\alpha \in (0, \lambda]$  and  $\beta = 0$ . The variational problem in this case becomes

$$I(\alpha, 0) = \inf\{|u_x|_2^2 - \frac{a}{2}|u|_4^4 : |u|_2^2 = \alpha\}.$$

Again in case  $a \leq 0$ , we have  $I(\alpha, 0) \geq 0$  and in case  $a > 0$  the problem gives (see again, [8]) the unique solution, up to translation,

$$u(x) = \sqrt{\frac{2\gamma_2}{a}} \operatorname{sech}(\sqrt{\gamma_2}x)$$

with  $\sqrt{\gamma_2} = \alpha a/4$ . A simple calculation shows that

$$I(\alpha, 0) \begin{cases} = -\frac{1}{3} \left(\frac{a}{4}\right)^2 \alpha^3, & \text{if } a > 0; \\ \geq 0, & \text{if } c \leq 0. \end{cases} \tag{3.12}$$

On the other hand, part 1) in Lemma 3.4 says that

$$I_1(\alpha) = -\frac{1}{3} \left(\frac{b^2 - ac}{4(b - c)}\right)^2 \alpha^3. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain that  $I(\alpha, 0) > I_1(\alpha)$ . Thus,

$$I(\lambda, \mu) = I_1(\lambda) + I_2(\mu) \leq I_1(\alpha) + I_1(\lambda - \alpha) + I_2(\mu) < I(\alpha, 0) + I(\lambda - \alpha, \mu)$$

proving case 3. □

With all the calculations in place, we now proceed to show that if  $\{(u_n, v_n)\} \in X$  is any minimizing sequence for  $E$  then the only possibility is case 3, and hence minimizing sequences must be compact. We first rule out the vanishing case with the following proposition.

**Proposition 3.7.** *For every minimizing sequence,  $\rho > 0$ .*

**Proof.** Using Lemmas 3.2 and 3.3, we conclude that there must exist a  $C > 0$  and a sequence of real numbers  $\{y_n\}$  such that

$$\int_{y_n-1/2}^{y_n+1/2} (|u_n|^4 + |v_n|^4) dx \geq C$$

for all  $n$ . Therefore,

$$\begin{aligned} C &\leq |u_n|_\infty^2 \int_{y_n-1/2}^{y_n+1/2} |u_n|^2 dx + |v_n|_\infty^2 \int_{y_n-1/2}^{y_n+1/2} |v_n|^2 dx \\ &\leq A \int_{y_n-1/2}^{y_n+1/2} (|u_n|^2 + |v_n|^2) dx, \end{aligned}$$

where  $A$  denotes the various Sobolev constants in the embedding of  $H^1$  into  $L^\infty$  whose precise values are not of importance. It follows that

$$\rho = \lim_{s \rightarrow \infty} M(s) \geq M\left(\frac{1}{2}\right) = \lim_{n \rightarrow \infty} M_n\left(\frac{1}{2}\right) \geq \frac{C}{A} > 0. \tag{□}$$

**Remark.** Notice that while Proposition 3.7 guarantees that  $\rho > 0$  for every minimizing sequence  $\{(u_n, v_n)\}$ , it does not rule out the possibility

that either  $\{u_n\}$  or  $\{v_n\}$  could vanish. However, if this happens, say  $\{u_n\}$  vanishes, then by case 2 of Lemma 3.6, we obtain that  $I(0, \beta) > I_2(\beta)$  and  $I(\lambda, \mu) = I_1(\lambda) + I_2(\mu) \leq I_1(\lambda) + I_2(\beta) + I_2(\mu - \beta) < I(0, \beta) + I(\lambda, \mu - \beta)$  which will lead to a contradiction presented in Proposition 3.9 later. The case for vanishing  $\{v_n\}$  can be handled similarly.

We now choose a function  $\Gamma \in C_0^\infty[-2, 2]$  such that  $\Gamma \equiv 1$  on  $[-1, 1]$ , and let  $\Pi \in C^\infty(\mathbb{R})$  be such that  $\Gamma^2 + \Pi^2 \equiv 1$  on  $\mathbb{R}$ . For each  $r \in \mathbb{R}$ , define  $\Gamma_r(x) = \Gamma(x/r)$  and  $\Pi_r(x) = \Pi(x/r)$ . Given an  $\epsilon > 0$ , for all sufficiently large  $r$  we have

$$\rho - \epsilon < M(r) \leq M(2r) \leq \rho.$$

Assume for the moment that such a value of  $r$  has been chosen. Then one can choose  $N$  so large that

$$\rho - \epsilon < M_n(r) \leq M_n(2r) < \rho + \epsilon$$

for all  $n \geq N$ . Consequently, for each  $n \geq N$ , one can find  $y_n$  such that

$$\int_{y_n-r}^{y_n+r} (|u_n|^2 + |v_n|^2) dx > \rho - \epsilon \tag{3.14}$$

and

$$\int_{y_n-2r}^{y_n+2r} (|u_n|^2 + |v_n|^2) dx < \rho + \epsilon. \tag{3.15}$$

Define

$$\begin{aligned} u_{n,1}(x) &= \Gamma_r(x - y_n)u_n(x), & u_{n,2}(x) &= \Pi_r(x - y_n)u_n(x), \\ v_{n,1}(x) &= \Gamma_r(x - y_n)v_n(x), & v_{n,2}(x) &= \Pi_r(x - y_n)v_n(x). \end{aligned} \tag{3.16}$$

The next lemma describes the behavior of minimizing sequences in the case  $0 < \rho < \lambda + \mu$ .

**Lemma 3.8.** *For every  $\epsilon > 0$  given, there exists an  $N > 0$  such that, for every  $n \geq N$ ,*

- (1)  $|Q(u_{n,1}) + Q(v_{n,1}) - \rho| < \epsilon.$
- (2)  $|Q(u_{n,2}) + Q(v_{n,2}) - (\lambda + \mu - \rho)| < \epsilon.$
- (3)  $E(u_n, v_n) \geq E(u_{n,1}, v_{n,1}) + E(u_{n,2}, v_{n,2}) - C\epsilon$ , for some constant  $C > 0$  independent of  $n$ .

**Proof.** Statements 1 and 2 follow immediately from the definitions of the functions  $u_{n,1}, u_{n,2}, v_{n,1}$  and  $v_{n,2}$  given in (3.16). To see 3), notice that

$$E(u_{n,1}, v_{n,1}) + E(u_{n,2}, v_{n,2}) = \int_{-\infty}^{\infty} \left[ \Gamma_r^2(u'_n)^2 + 2\Gamma_r\Gamma'_r u_n u'_n + (\Gamma'_r)^2 u_n^2 \right] dx$$

$$\begin{aligned}
 & + \int_{-\infty}^{\infty} \left[ \Pi_r^2 (u_n')^2 + 2\Pi_r \Pi_r' u_n u_n' + (\Pi_r')^2 u_n^2 \right] dx \\
 & + \int_{-\infty}^{\infty} \left[ \Gamma_r^2 (v_n')^2 + 2\Gamma_r \Gamma_r' v_n v_n' + (\Gamma_r')^2 v_n^2 \right] dx \\
 & + \int_{-\infty}^{\infty} \left[ \Pi_r^2 (v_n')^2 + 2\Pi_r \Pi_r' v_n v_n' + (\Pi_r')^2 v_n^2 \right] dx - \frac{a}{2} \int_{-\infty}^{\infty} \Gamma_r^2 |u_n|^4 dx \\
 & - \frac{a}{2} \int_{-\infty}^{\infty} \Pi_r^2 |u_n|^4 dx + \frac{a}{2} \int_{-\infty}^{\infty} (\Gamma_r^2 - \Gamma_r^4) |u_n|^4 dx + \frac{a}{2} \int_{-\infty}^{\infty} (\Pi_r^2 - \Pi_r^4) |u_n|^4 dx \\
 & - \frac{c}{2} \int_{-\infty}^{\infty} \Gamma_r^2 |v_n|^4 dx - \frac{c}{2} \int_{-\infty}^{\infty} \Pi_r^2 |v_n|^4 dx + \frac{c}{2} \int_{-\infty}^{\infty} (\Gamma_r^2 - \Gamma_r^4) |v_n|^4 dx \\
 & + \frac{c}{2} \int_{-\infty}^{\infty} (\Pi_r^2 - \Pi_r^4) |v_n|^4 dx - b \int_{-\infty}^{\infty} \Gamma_r^2 |u_n|^2 |v_n|^2 dx - b \Pi_r^2 |u_n|^2 |v_n|^2 dx \\
 & + b \int_{-\infty}^{\infty} (\Gamma_r^2 - \Gamma_r^4) |u_n|^2 |v_n|^2 dx + b \int_{-\infty}^{\infty} (\Pi_r^2 - \Pi_r^4) |u_n|^2 |v_n|^2 dx,
 \end{aligned}$$

where, for ease of notation, we have written the functions  $\Gamma_r(x - y_n)$  and  $\Pi_r(x - y_n)$  simply as  $\Gamma_r$  and  $\Pi_r$ . Since  $\Gamma_r^2 + \Pi_r^2 \equiv 1$ ,  $|\Gamma_r'|_{\infty} = |\Gamma'|_{\infty}/r$  and  $|\Pi_r'|_{\infty} = |\Pi'|_{\infty}/r$ , an application of Hölder's inequality gives

$$\begin{aligned}
 E(u_{n,1}, v_{n,1}) + E(u_{n,2}, v_{n,2}) & = E(u_n, v_n) + O\left(\frac{1}{r}\right) \\
 & + \frac{a}{2} \int_{-\infty}^{\infty} [(\Gamma_r^2 - \Gamma_r^4) + (\Pi_r^2 - \Pi_r^4)] |u_n|^4 dx \\
 & + \frac{c}{2} \int_{-\infty}^{\infty} [(\Gamma_r^2 - \Gamma_r^4) + (\Pi_r^2 - \Pi_r^4)] |v_n|^4 dx \\
 & + b \int_{-\infty}^{\infty} [(\Gamma_r^2 - \Gamma_r^4) + (\Pi_r^2 - \Pi_r^4)] |u_n|^2 |v_n|^2 dx,
 \end{aligned}$$

where  $O(1/r)$  denotes a term bounded in absolute value by  $C/r$  with  $C$  independent of  $r$  and  $n$ . Using (3.14) and (3.15), one can see that

$$\begin{aligned}
 \left| \int_{-\infty}^{\infty} [(\Gamma_r^2 - \Gamma_r^4) + (\Pi_r^2 - \Pi_r^4)] |u_n|^4 dx \right| & \leq |u_n|_{\infty}^2 \int_{r \leq |x-y_n| \leq 2r} 2|u_n|^2 dx \leq C\epsilon, \\
 \left| \int_{-\infty}^{\infty} [(\Gamma_r^2 - \Gamma_r^4) + (\Pi_r^2 - \Pi_r^4)] |v_n|^4 dx \right| & \leq |v_n|_{\infty}^2 \int_{r \leq |x-y_n| \leq 2r} 2|v_n|^2 dx \leq C\epsilon,
 \end{aligned}$$

and

$$\left| \int_{-\infty}^{\infty} [(\Gamma_r^2 - \Gamma_r^4) + (\Pi_r^2 - \Pi_r^4)] |u_n|^2 |v_n|^2 dx \right| \leq |u_n|_{\infty}^2 \int_{r \leq |x-y_n| \leq 2r} 2|v_n|^2 dx \leq C\epsilon,$$



where again  $C$  denotes various constants independent of  $r$  and  $n$ .

We now choose  $r$  large enough so that  $|O(1/r)| \leq \epsilon$ . Hence

$$E(u_n, v_n) \geq E(u_{n,1}, v_{n,1}) + E(u_{n,2}, v_{n,2}) - C\epsilon$$

for all  $n \geq N(r)$ . Hence the lemma is proved. □

The dichotomy of minimizing sequences can now be prevented by the next proposition.

**Proposition 3.9.** *For every minimizing sequence,  $\rho \notin (0, \lambda + \mu)$ .*

**Proof.** Notice first that if  $u, v \in H^1_{\mathbb{C}}$  such that  $|Q(u) - \alpha| < \epsilon/2$  and  $|Q(v) - \beta| < \epsilon/2$ , then  $Q(\nu_1 u) = \alpha$ ,  $Q(\nu_2 v) = \beta$  where  $\nu_1 = \sqrt{\frac{\alpha}{Q(u)}}$ ,  $\nu_2 = \sqrt{\frac{\beta}{Q(v)}}$  satisfying  $|\nu_1 - 1| < C_1\epsilon$ ,  $|\nu_2 - 1| < C_2\epsilon$  with  $C_1$  and  $C_2$  independent of  $u, v$  and  $\epsilon$ . Thus, for any  $\alpha \in [0, \lambda]$  and  $\beta \in [0, \mu]$  satisfying  $0 < \alpha + \beta = \rho < \lambda + \mu$ , we have

$$|Q(u) + Q(v) - \rho| < \epsilon, \quad Q(\nu_1 u) + Q(\nu_2 v) = \rho,$$

and

$$I(\alpha, \beta) \leq E(\nu_1 u, \nu_2 v) \leq E(u, v) + C_3\epsilon,$$

where  $C_3$  depends only on  $C_1, C_2, \|u\|_1$  and  $\|v\|_1$ . A similar result holds for functions  $f, g \in H^1_{\mathbb{C}}$  such that  $|Q(f) - (\lambda - \alpha)| < \epsilon/2$  and  $|Q(g) - (\mu - \beta)| < \epsilon/2$ .

Using these facts and Lemma 3.8, it follows that there exist a subsequence  $\{u_{n_k}, v_{n_k}\}$  of  $\{u_n, v_n\}$  and corresponding functions  $u_{n_k,1}, u_{n_k,2}, v_{n_k,1}$  and  $v_{n_k,2}$  such that, for all  $k$ ,

$$\begin{aligned} E(u_{n_k,1}, v_{n_k,1}) &\geq I(\alpha, \beta) - \frac{1}{k}, \\ E(u_{n_k,2}, v_{n_k,2}) &\geq I(\lambda - \alpha, \mu - \beta) - \frac{1}{k}, \\ E(u_{n_k}, v_{n_k}) &\geq E(u_{n_k,1}, v_{n_k,1}) + E(u_{n_k,2}, v_{n_k,2}) - \frac{1}{k}. \end{aligned}$$

Consequently, by taking the limit of both sides as  $k \rightarrow \infty$ , we obtain that

$$I(\lambda, \mu) \geq I(\alpha, \beta) + I(\lambda - \alpha, \mu - \beta),$$

a contradiction to Lemma 3.6. □

**Remark.** Combining Proposition 3.9 with the Remark following Proposition 3.7, we can rule out the vanishing of the minimizing sequence  $\{(u_n, v_n)\}$

in the true sense; that is,

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} |u_n(x)|^2 dx > 0$$

and

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} |v_n(x)|^2 dx > 0.$$

As we have ruled out both the vanishing and dichotomy cases, it follows from Lions' concentration compactness lemma [16, 17] that every minimizing sequence must be compact; i.e.,  $\rho = \lambda + \mu$ . Thus we have the following.

**Proposition 3.10.** *Let  $\{u_n, v_n\}$  be a minimizing sequence for  $E(u, v)$ . Then there exists a sequence of real numbers  $\{y_n\}$  such that*

- (1) *For every  $\epsilon > 0$ , there exists an  $s = s(\epsilon) < \infty$  such that*

$$\int_{y_n-s}^{y_n+s} (|u_n(x)|^2 + |v_n(x)|^2) dx \geq (\lambda + \mu) - \epsilon$$

*for all sufficiently large  $n$ .*

- (2) *The sequence  $\{(u_n(x + y_n), v_n(x + y_n))\}$  for  $x \in \mathbb{R}$  has a subsequence which converges in  $X$ -norm to a function  $(\Phi, \Psi) \in G(\lambda, \mu)$ . In particular,  $G(\lambda, \mu)$  is non-empty.*

**Proof.** Statement 1) is just a consequence of Lions' concentration compactness lemma [16, 17]. To see 2), notice first that 1) implies that, for every  $k \in \mathbb{N}$ , there exists  $s_k \in \mathbb{R}$  such that for all sufficiently large  $n$ ,

$$\int_{-s_k}^{s_k} (|\tilde{u}_n(x)|^2 + |\tilde{v}_n(x)|^2) dx \geq (\lambda + \mu) - \frac{1}{k},$$

where, for ease of notation, we denote  $u_n(x + y_n)$  and  $v_n(x + y_n)$  by  $\tilde{u}_n(x)$  and  $\tilde{v}_n(x)$  respectively. But  $\|\tilde{u}_n\|_1 + \|\tilde{v}_n\|_1 \leq B$  for all  $n$  by i) in Lemma 3.2, hence from the compact embedding of  $H^1(\Omega)$  into  $L^2(\Omega)$  on bounded intervals  $\Omega$ , it follows that some subsequence of  $\{(\tilde{u}_n, \tilde{v}_n)\}$  converges in  $L^2[-s_k, s_k]$ -norm to a pair of limit functions  $(\Phi, \Psi) \in L^2[-s_k, s_k]$  satisfying

$$\int_{-s_k}^{s_k} (|\Phi|^2 + |\Psi|^2) dx \geq (\lambda + \mu) - \frac{1}{k}.$$

Using a Cantor diagonalization process, together with the fact that

$$\int_{-\infty}^{\infty} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2) dx = \lambda + \mu \quad \text{for all } n,$$

we conclude that some subsequence of  $\{(\tilde{u}_n, \tilde{v}_n)\}$  converges in  $L^2(\mathbb{R})$ -norm to the functions  $(\Phi, \Psi) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$  satisfying

$$\int_{-\infty}^{\infty} (|\Phi|^2 + |\Psi|^2) dx = \lambda + \mu.$$

Furthermore, from the weak compactness of the unit sphere, we also have that  $\{(\tilde{u}_n, \tilde{v}_n)\}$  converges weakly to  $(\Phi, \Psi)$  in  $X$  and that

$$\|(\Phi, \Psi)\|_X \leq \liminf_{n \rightarrow \infty} \|(\tilde{u}_n, \tilde{v}_n)\|_X.$$

Now, from the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} |\tilde{u}_n - \Phi|_4^4 &\leq C \left( \int_{-\infty}^{\infty} |\tilde{u}'_n - \Phi'|^2 dx \right)^{1/2} \left( \int_{-\infty}^{\infty} |\tilde{u}_n - \Phi|^2 dx \right)^{3/2} \\ &\leq C \left( \int_{-\infty}^{\infty} |\tilde{u}_n - \Phi|^2 dx \right)^{3/2}, \end{aligned}$$

where  $C$  denotes various constants independent of  $n$ . Similarly,

$$\begin{aligned} |\tilde{v}_n - \Psi|_4^4 &\leq C \left( \int_{-\infty}^{\infty} |\tilde{v}'_n - \Psi'|^2 dx \right)^{1/2} \left( \int_{-\infty}^{\infty} |\tilde{v}_n - \Psi|^2 dx \right)^{3/2} \\ &\leq C \left( \int_{-\infty}^{\infty} |\tilde{v}_n - \Psi|^2 dx \right)^{3/2}. \end{aligned}$$

Hence  $\tilde{u}_n \rightarrow \Phi$  and  $\tilde{v}_n \rightarrow \Psi$  in  $L^4(\mathbb{R})$  also. Consequently, it follows that

$$E(\Phi, \Psi) \leq \lim_{n \rightarrow \infty} E(\tilde{u}_n, \tilde{v}_n) = I(\lambda, \mu),$$

whence  $E(\Phi, \Psi) = I(\lambda, \mu)$  and  $(\Phi, \Psi) \in G(\lambda, \mu)$ . Since

$$E(\Phi, \Psi) = \lim_{n \rightarrow \infty} E(\tilde{u}_n, \tilde{v}_n), \quad |\Phi|_4 = \lim_{n \rightarrow \infty} |\tilde{u}_n|_4, \quad |\Psi|_4 = \lim_{n \rightarrow \infty} |\tilde{v}_n|_4, \quad |\Phi|_2 = \lim_{n \rightarrow \infty} |\tilde{u}_n|_2 \text{ and } |\Psi|_2 = \lim_{n \rightarrow \infty} |\tilde{v}_n|_2,$$

we conclude that

$$\|(\Phi, \Psi)\|_X = \lim_{n \rightarrow \infty} \|(\tilde{u}_n, \tilde{v}_n)\|_X.$$

As  $X = H^1_{\mathbb{C}} \times H^1_{\mathbb{C}}$  is a Hilbert space, it is easy to see that  $(\tilde{u}_n, \tilde{v}_n)$  must converge strongly to  $(\Phi, \Psi)$  in  $X$ -norm. □

**Corollary 3.11.** *If  $\{u_n, v_n\}$  is any minimizing sequence for  $I(\lambda, \mu)$ , then there exists a sequence of real numbers  $\{y_n\}$  such that*

- (1)  $\lim_{n \rightarrow \infty} \inf_{(\Phi, \Psi) \in G, y \in \mathbb{R}} \|(u_n(\cdot + y), v_n(\cdot + y)) - (\Phi, \Psi)\|_X = 0.$
- (2)  $\lim_{n \rightarrow \infty} \inf_{(\Phi, \Psi) \in G} \|(u_n, v_n) - (\Phi, \Psi)\|_X = 0.$

**Proof.** To prove 1), suppose that it is false; then there exists a subsequence  $\{(u_{n_k}, v_{n_k})\}$  of  $\{(u_n, v_n)\}$  and a number  $\epsilon > 0$  such that

$$\inf_{(\Phi, \Psi) \in G, y \in \mathbb{R}} \|(u_{n_k}(\cdot + y), v_{n_k}(\cdot + y)) - (\Phi, \Psi)\|_X \geq \epsilon \text{ for all } k \in \mathbb{N}.$$

As  $\{(u_{n_k}, v_{n_k})\}$  is itself a minimizing sequence for  $I(\lambda, \mu)$ , there exist a sequence of real numbers  $\{y_k\}$  and  $(\Phi_0, \Psi_0) \in G(\lambda, \mu)$  such that

$$\liminf_{k \rightarrow \infty} \|(u_{n_k}(\cdot + y_k), v_{n_k}(\cdot + y_k)) - (\Phi_0, \Psi_0)\|_X = 0,$$

a contradiction.

Because the functionals  $E$  and  $Q$  are all invariant under translations,  $G(\lambda, \mu)$  therefore contains any translation of  $(\Phi, \Psi)$  if it contains  $(\Phi, \Psi)$ . Consequently, 2) follows from 1).  $\square$

An immediate consequence of the above result is that the set of minimizers  $G(\lambda, \mu)$  is stable. Precisely, we have the following.

**Theorem 3.12.** *For every  $\epsilon > 0$  given, there exists  $\delta > 0$  such that, if*

$$\inf_{(\Phi, \Psi) \in G} \|(u_0, v_0) - (\Phi, \Psi)\|_X < \delta,$$

*then the solution  $(u(x, t), v(x, t))$  of (1.2) with  $(u(x, 0), v(x, 0)) = (u_0, v_0)$  satisfies*

$$\inf_{(\Phi, \Psi) \in G} \|(u(\cdot, t), v(\cdot, t)) - (\Phi, \Psi)\|_X < \epsilon \text{ for all } t \in \mathbb{R}.$$

**Proof.** Suppose the theorem is false. Then there exist a number  $\epsilon > 0$ , a sequence of times  $\{t_n\}$  and a sequence  $\{(u_n(x, 0), v_n(x, 0))\} \in X$  such that

$$\inf_{(\Phi, \Psi) \in G} \|(u_n(x, 0), v_n(x, 0)) - (\Phi, \Psi)\|_X < \frac{1}{n}$$

and

$$\inf_{(\Phi, \Psi) \in G} \|(u_n(\cdot, t), v_n(\cdot, t)) - (\Phi, \Psi)\|_X \geq \epsilon$$

for all  $n$  where  $(u_n(x, t), v_n(x, t))$  solves (1.2) with initial data  $(u_n(x, 0), v_n(x, 0))$ . Since  $(u_n(x, 0), v_n(x, 0))$  converges to an element in  $G(\lambda, \mu)$  in  $X$ -norm, and since, for  $(\Phi, \Psi) \in G(\lambda, \mu)$ , we have  $Q(\Phi) = \lambda$ ,  $Q(\Psi) = \mu$  and  $E(\Phi, \Psi) = I(\lambda, \mu)$ , we therefore have

$$E(u_n(x, 0), v_n(x, 0)) \rightarrow I(\lambda, \mu), \quad Q(u_n(x, 0)) \rightarrow \lambda, \quad Q(v_n(x, 0)) \rightarrow \mu.$$

We now choose  $\{\alpha_n\}$  and  $\{\beta_n\}$  such that

$$Q(\alpha_n u_n(x, 0)) = \lambda \quad \text{and} \quad Q(\beta_n v_n(x, 0)) = \mu$$

for all  $n$ . Thus  $\alpha_n \rightarrow 1$  and  $\beta_n \rightarrow 1$ . Hence, the sequence  $f_n = \alpha_n u_n(\cdot, t_n)$  and  $g_n = \beta_n v_n(\cdot, t_n)$  satisfy  $Q(f_n) = \lambda$ ,  $Q(g_n) = \mu$ , and

$$\begin{aligned} \lim_{n \rightarrow \infty} E(f_n, g_n) &= \lim_{n \rightarrow \infty} E(u_n(\cdot, t_n), v_n(\cdot, t_n)) \\ &= \lim_{n \rightarrow \infty} E(u_n(x, 0), v_n(x, 0)) = I(\lambda, \mu). \end{aligned}$$

Consequently,  $\{f_n, g_n\}$  is a minimizing sequence for  $I(\lambda, \mu)$ . It follows from Corollary 3.11 that, for all  $n$  large, there exists  $(\Phi_n, \Psi_n) \in G(\lambda, \mu)$  such that  $\|(f_n, g_n) - (\Phi_n, \Psi_n)\|_X < \epsilon/2$ . But then we have

$$\begin{aligned} \epsilon &\leq \|(u_n(\cdot, t_n), v_n(\cdot, t_n)) - (\Phi_n, \Psi_n)\|_X \\ &\leq \|(u_n(\cdot, t_n), v_n(\cdot, t_n)) - (f_n, g_n)\|_X + \|(f_n, g_n) - (\Phi_n, \Psi_n)\|_X \\ &\leq |1 - \alpha_n| \cdot \|u_n(\cdot, t_n)\|_1 + |1 - \beta_n| \cdot \|v_n(\cdot, t_n)\|_1 + \frac{\epsilon}{2}, \end{aligned}$$

and by taking  $n \rightarrow \infty$ , we obtain that  $\epsilon \leq \epsilon/2$ , a contradiction. □

#### 4. ORBITAL STABILITY OF SOLITARY-WAVE SOLUTIONS

In this section, we will show that the set of minimizers  $G(\lambda, \mu)$  contains just a single pair of functions (modulo translations), and that this pair of functions is indeed a solitary-wave solution of (1.2) given by (2.6)-(2.4). Theorem 2.1 then follows directly from this fact and Theorem 3.12.

We start first with the following lemma that relates two functions  $\Phi$  and  $\Psi$  whenever  $(\Phi, \Psi) \in G(\lambda, \mu)$ .

**Lemma 4.1.** *Let  $(\Phi, \Psi) \in G(\lambda, \mu)$ . Then, for any  $x \in \mathbb{R}$ ,*

$$\left(\frac{b-a}{b-c}\right)^{1/4} |\Phi| = \left(\frac{b-c}{b-a}\right)^{1/4} |\Psi|.$$

**Proof.** It follows from Lemma 3.4 that for any  $(\Phi, \Psi) \in G(\lambda, \mu)$

$$I(\lambda, \mu) = E(\Phi, \Psi) \geq E_1(\Phi) + E_2(\Psi) \geq I_1(\lambda) + I_2(\mu) = I(\lambda, \mu).$$

Thus,

$$\frac{a}{2} |\Phi|_4^4 + \frac{c}{2} |\Psi|_4^4 + b \int_{-\infty}^{\infty} |\Phi|^2 |\Psi|^2 dx = \frac{b^2 - ac}{2(b-c)} |\Phi|_4^4 + \frac{b^2}{2(b-a)} |\Psi|_4^4$$

which implies that

$$2 \int_{-\infty}^{\infty} |\Phi|^2 |\Psi|^2 dx = \frac{b-a}{b-c} |\Phi|_4^4 + \frac{b-c}{b-a} |\Psi|_4^4. \tag{4.1}$$

We can rewrite (4.1) as

$$\int_{-\infty}^{\infty} \left( \sqrt{\frac{b-a}{b-c}} |\Phi|^2 - \sqrt{\frac{b-c}{b-a}} |\Psi|^2 \right)^2 dx = 0,$$

from which the statement of the lemma immediately follows. □

With the above lemma in hand, we proceed to show the following.

**Theorem 4.2.** *For any  $\omega > 0$  fixed,*

$$G(\lambda(\omega), \mu(\omega)) = \left\{ \left( e^{i\alpha} \sqrt{\frac{b-c}{b^2-ac}} \phi(\cdot+y), e^{i\beta} \sqrt{\frac{b-a}{b^2-ac}} \phi(\cdot+y) \right) : \alpha, \beta, y \in \mathbb{R} \right\},$$

where  $\phi(\xi) = \sqrt{2\omega} \operatorname{sech}(\sqrt{\omega}\xi)$ .

**Proof.** It has been established (Corollary 3.5) that for any  $\omega > 0$  fixed and for  $\alpha, \beta, y \in \mathbb{R}$

$$\left( e^{i\alpha} \sqrt{\frac{b-c}{b^2-ac}} \phi(\cdot+y), e^{i\beta} \sqrt{\frac{b-a}{b^2-ac}} \phi(\cdot+y) \right) \in G(\lambda(\omega), \mu(\omega)).$$

Hence the theorem is proved if we can show that any minimizer in  $G(\lambda(\omega), \mu(\omega))$  must be of the form given here. Now, since the constrained minimizer for the variational problem exists, there are Lagrange multipliers  $\omega_1, \omega_2 \in \mathbb{R}$  such that

$$\begin{cases} -\Phi_{xx} + \omega_1 \Phi = a|\Phi|^2 \Phi + b|\Psi|^2 \Phi, \\ -\Psi_{xx} + \omega_2 \Psi = c|\Psi|^2 \Psi + b|\Phi|^2 \Psi. \end{cases} \tag{4.2}$$

Using Lemma 4.1, we can rewrite this system as two uncoupled equations

$$\begin{aligned} -\Phi_{xx} + \omega_1 \Phi &= \frac{b^2-ac}{b-c} |\Phi|^2 \Phi, \\ -\Psi_{xx} + \omega_2 \Psi &= \frac{b^2-ac}{b-a} |\Psi|^2 \Psi. \end{aligned} \tag{4.3}$$

A bootstrap argument shows that any pair of  $L^2$ -distribution solutions of (4.3) must indeed be smooth and an elementary calculation shows then that the only pair of  $L^2$ -solution of (4.3) is given by

$$\Phi(x) = e^{i\alpha} \sqrt{\frac{2\omega_1(b-c)}{b^2-ac}} \operatorname{sech}(\sqrt{\omega_1}x + y_1)$$

and

$$\Psi(x) = e^{i\beta} \sqrt{\frac{2\omega_2(b-a)}{b^2-ac}} \operatorname{sech}(\sqrt{\omega_2}x + y_2)$$

where  $y_1, y_2, \alpha$  and  $\beta \in \mathbb{R}$ . Now recall that, for any  $x \in \mathbb{R}$ , we must have

$$\left(\frac{b-a}{b-c}\right)^{1/4} |\Phi(x)| = \left(\frac{b-c}{b-a}\right)^{1/4} |\Psi(x)|;$$

and

$$|\Phi|_2^2 = \lambda = 4\sqrt{\omega} \frac{b-c}{b^2-ac}; \quad |\Psi|_2^2 = \mu = 4\sqrt{\omega} \frac{b-a}{b^2-ac}.$$

It is easy to see then that  $y_1 = y_2$  and  $\omega = \omega_1 = \omega_2$ . The theorem is thus proved.  $\square$

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