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# A PRIORI BOUNDS FOR GEVREY–SOBOLEV NORMS OF SPACE-PERIODIC THREE-DIMENSIONAL SOLUTIONS TO EQUATIONS OF HYDRODYNAMIC TYPE

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**Abstract.** We present a technique for derivation of a priori bounds for Gevrey–Sobolev norms of space-periodic three-dimensional solutions to evolutionary partial differential equations of hydrodynamic type. It involves a transformation of the flow velocity in the Fourier space, which introduces a feedback between the index of the norm and the norm of the transformed solution, and results in emergence of a mildly dissipative term. We illustrate the technique, using it to derive finite-time bounds for Gevrey–Sobolev norms of solutions to the Euler and inviscid Burgers equations, and global-in-time bounds for the Voigt-type regularizations of the Euler and Navier–Stokes equation (assuming that the respective norm of the initial condition is bounded). The boundedness of the norms implies analyticity of the solutions in space.

## 1. Introduction

We suggest a new approach to derivation of bounds for analytic Gevrey–Sobolev norms (see their definition (2.6) below) of solutions to evolutionary partial differential equations of hydrodynamic type, which exploits a feedback between the norm of the suitably transformed solution and the index of the norm.

Whether three-dimensional solutions to the Euler equation remain smooth forever is an open question. If the initial flow velocity belongs to the Hölder class $C^{1+\alpha}$ ($\alpha > 0$ is arbitrary), the solution remains in it at least for a finite time [4]. Moreover, if initially the flow is analytic in space, the solution is guaranteed to be analytic in a finite spatio-temporal region [1, 2, 3, 5, 6];

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the loss of analyticity does not occur before the solution ceases to be continuously differentiable [2]. These demonstrations involve construction of bounds for iterated approximations of vorticity followed along characteristics, the knowledge of the Green's function in the integral representation of the flow in the terms of vorticity, and application of methods of the theory of functions of complex variables. It is yet unknown, whether a singularity can develop at finite times.

Foias and Temam [11] proved that viscosity can enhance the smoothness of space-periodic solutions to the Navier–Stokes equation (see also [10]): If initially the flow velocity belongs to the Sobolev space $H_1(T^3)$ (i.e., the space of space-periodic functions, whose derivatives of order 1 are square-integrable), then at any small time $t > 0$ the flow has a bounded Gevrey–Sobolev norm. Their method was used subsequently to demonstrate that a similar result holds for space-periodic solutions to various modifications of the Navier–Stokes equation [16, 31]. The method relies on the presence of the viscous term in the equation, and hence is not directly applicable to the Euler equation.

We overcome the difficulty by performing a suitable transformation of the Fourier coefficients of the flow, which introduces a dependence of the index of a Gevrey–Sobolev norm on the norm of the transformed solution. The transformation results in emergence of a weakly dissipative operator in the modified equation and it becomes possible to proceed following the method of [11] (time dependence of the index giving rise to a weakly dissipative operator was employed in [22]). In Section 2 we use this technique to give a simple proof of the boundedness of a Gevrey–Sobolev norm of a space-periodic three-dimensional solution to the Euler equation on a finite time interval. In Section 3 the same technique is applied to the Burgers equation.

The three-dimensional Navier–Stokes equation can be regularized by introduction of suitable nonlinear [18, 19, 20] or linear terms, such as hyperviscosity [23, 24], or the time derivative of the Laplacian of the flow (proposed by O.A. Ladyzhenskaya at the International Mathematical Congress in 1966) which can be regarded as a non-standard "dynamic" viscosity. Boundary value problems for the regularized equation of this type, known as the Navier–Stokes–Voigt equation and the Oskolkov equation in the Russian literature, were studied in [25, 26, 27, 8]. They describe the motion of non-Newtonian visco-elastic fluids with memory (characterized by an exponential decay of the rate of deformation under a constant stress) arising for a class of linear integro-differential rheology equations [28, 29, 30]. While in the Voigt regularization of the Navier–Stokes equation dynamic viscosity is positive, its negative values may be also physically sound [27]. Regularization by progressive damping of high-wavenumber harmonics in some instances of occurrence of the flow velocity in the equation gives rise to the so-called Camassa–Holm or Navier–Stokes–$\alpha$ equation [12, 13, 16] supposedly describing Lagrangian-averaged flows [15]; however, an additional term must be introduced for this interpretation to be accurate [32, 33].

Bounds for Gevrey–Sobolev norms were derived in [21] for solutions to the so-called Voigt regularization of the Euler equation (this type of regularization was suggested for the Euler equation in [8]) by the method, initially applied in [22] to the "lake equation" (i.e., a modified two-dimensional Euler equation). We consider a Voigt-type regularization of the three-dimensional Euler equation in Section 4 and show that a milder regularization suffices to establish the boundedness of Gevrey–Sobolev norms of solutions. The boundedness of the norms is demonstrated for solutions to the Voigt-type regularization of the Navier–Stokes equation in Section 5; the presence of diffusion enables us to further weaken the regularizing term. Constructions of Sections 4 and 5 illustrate another aspect of our approach: dependence of the index of a Gevrey–Sobolev norm on the norm of the transformed solution allows us to decrease the order of nonlinearity in the energy-type inequality and thus to obtain global (in time) bounds for the norm.

In this paper, we consider zero-mean (note that the means of solutions are conserved in time) space-periodic three-dimensional solutions, assuming for simplicity that the elementary periodicity cell is the cube $T^3 = [0, 2\pi]^3$. It is not difficult to generalize our analysis to encompass the case of arbitrary periods along the Cartesian axes. All the equations mentioned above normally involve a force $\mathbf{f}$. Following a long-established tradition, for the sake of simplicity we consider only the illustrative equations for $\mathbf{f} = 0$. Reinstating the forcing in the analysis is also straightforward (provided the force has the right analyticity properties).

Let us also remark on the accurate procedure for derivation of a priori bounds. The bounds are derived below for Fourier–Galerkin truncations of the solutions. We do this by tacitly using the ODE's governing the evolution of the Fourier coefficients of the truncated solutions. The bounds involve norms of the initial conditions for the truncations. Assuming in their place (in general, larger) norms of non-truncated initial conditions, we obtain bounds for the truncated solutions, which are uniform in the number of harmonics retained in the Fourier–Galerkin truncation. By the standard arguments, these bounds remain valid for the norms of the non-truncated solutions.

## 2. The three-dimensional Euler equation

In this section we establish bounds for Gevrey–Sobolev norms of solutions to the Euler equation on a finite time interval $[0, t_*)$, assuming that the initial condition has a finite Gevrey–Sobolev norm.

The motion of a perfect (inviscid) fluid with velocity $\mathbf{v}$ under the action of a body force $\mathbf{f}$ is governed by the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} + \nabla p. \tag{2.1}$$

Incompressibility of fluid implies the solenoidality of the flow:

$$\nabla \cdot \mathbf{v} = 0. \tag{2.2}$$

The initial (at $t = 0$) distribution of the flow velocity $\mathbf{v}^{(\mathrm{in})}$ is prescribed.

We expand a zero-mean space-periodic flow in $T^3$ into the Fourier series:

$$\mathbf{v} = \sum_{\mathbf{n} \neq 0} \widehat{\mathbf{v}}_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}. \tag{2.3}$$

The flow is real, as long as

$$\widehat{\mathbf{v}}_{\mathbf{n}} = \overline{\widehat{\mathbf{v}}_{-\mathbf{n}}}, \tag{2.4}$$

and solenoidality (2.2) implies

$$\widehat{\mathbf{v}}_{\mathbf{n}} \cdot \mathbf{n} = 0. \tag{2.5}$$

The norm $\| \cdot \|_q$ of a zero-mean space-periodic vector or scalar field $\mathbf{v}$ in the Sobolev space $H_q(T^3)$ is defined in the terms of its Fourier coefficients by the relation

$$\|\mathbf{v}\|_q^2 \equiv \sum_{\mathbf{n} \neq 0} |\widehat{\mathbf{v}}_{\mathbf{n}}|^2 |\mathbf{n}|^{2q},$$

where $| \cdot |$ denotes the Euclidean norm of a three-dimensional vector. For any $\sigma > 0$ we define the Gevrey–Sobolev norm[1]

$$\left\| \mathbf{w} \right\|_{\sigma,q}^2 \equiv \sum_{\mathbf{n} \neq 0} |\widehat{\mathbf{w}}_{\mathbf{n}}|^2 \mathrm{e}^{2\sigma|\mathbf{n}|} |\mathbf{n}|^{2q}. \tag{2.6}$$

Here the first index $\sigma$ is a lower estimate for the radius of the region of analyticity of $\mathbf{w}$ around the real axis in the complex space. For $\sigma = 0$, (2.6)

[1] Functions, whose Gevrey–Sobolev norms are finite, are analytic (see [22]), and hence the norm (2.6) might be called the analytic Gevrey–Sobolev norm. We could extend our analysis to non-analytic Gevrey classes of index $\alpha$, $0 < \alpha < 1$, whose norms involve the exponents $\exp(2\sigma|\mathbf{n}|^\alpha)$ instead of $\exp(2\sigma|\mathbf{n}|)$ in (2.6). Under the transformation of equations that we employ a weakly dissipative operator emerges, whose symbol would then grow as $|\mathbf{n}|^\alpha$. Since the operator is the strongest for $\alpha = 1$, we consider only this case.

defines the norm in $H_q(T^3)$. Let $|\cdot|_q$ denote the norm in the Lebesgue space $L_q(T^3)$. By the Sobolev embedding theorem [7, 35], for any positive $q < 3/2$ there exists a constant $C_q$ such that for any function $f \in H_q(T^3)$

$$|f|_{6/(3-2q)} \leq C_q \|f\|_q. \tag{2.7}$$

Let $\mathcal{P}_\mathbf{n}$ denote the linear operator of projection of a three-dimensional vector on the plane normal to $\mathbf{n} \neq 0$:

$$\mathcal{P}_\mathbf{n}\widehat{\mathbf{v}}_\mathbf{m} \equiv \widehat{\mathbf{v}}_\mathbf{m} - \frac{\widehat{\mathbf{v}}_\mathbf{m} \cdot \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}.$$

The evolution of Fourier coefficients of the flow is governed by equations

$$\frac{d\widehat{\mathbf{v}}_\mathbf{n}}{dt} + \mathrm{i}\sum_\mathbf{k}(\widehat{\mathbf{v}}_\mathbf{k} \cdot (\mathbf{n} - \mathbf{k}))\,\mathcal{P}_\mathbf{n}\widehat{\mathbf{v}}_{\mathbf{n}-\mathbf{k}} = 0 \tag{2.8}$$

obtained by substitution of the series (2.3) (and $\mathbf{f} = 0$) into the Euler equation (2.1).

Assuming $[\![\mathbf{v}^{(\mathrm{in})}]\!]_{\sigma,\,s+3/2} < \infty$ for some positive $\sigma$ and $s \leq 1/2$, we consider a transformation

$$\widehat{\mathbf{v}}_\mathbf{n}(t) = \widehat{\mathbf{w}}_\mathbf{n}(t) \exp\left(-\beta|\mathbf{n}|\|\mathbf{w}(\mathbf{x}, t)\|^{-\varepsilon}_{s+3/2}\right), \tag{2.9}$$

$$\mathbf{w}(\mathbf{x}, t) = \sum_\mathbf{n} \widehat{\mathbf{w}}_\mathbf{n}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}},$$

where $\beta$ and $\varepsilon < 2$ are positive constants. The transformation involves solving the system of nonlinear equations (2.9) in $\widehat{\mathbf{w}}_\mathbf{n}(t)$; the solution takes the form

$$\widehat{\mathbf{w}}_\mathbf{n}(t) = \widehat{\mathbf{v}}_\mathbf{n}(t) \exp(\psi(t)|\mathbf{n}|),$$

where the quantity $\psi(t) \geq 0$ satisfies the equation

$$\psi(t)\,[\![\mathbf{v}(\mathbf{x}, t)]\!]^{\varepsilon}_{\psi(t),\,s+3/2} = \beta.$$

It has a unique solution for any $t \geq 0$, because the left-hand side is a continuous monotonically increasing function of $\psi$ (as discussed in the introduction, the a priori estimates are derived for Fourier–Galerkin truncations of the flow, whereby the sum (2.3) is assumed to involve a finite number of terms). If $\beta < \sigma[\![\mathbf{v}^{(\mathrm{in})}]\!]^{\varepsilon}_{\sigma,\,s+3/2}$, then $\|\mathbf{w}(\mathbf{x}, t)\|_{s+3/2}$ is bounded at $t = 0$ (uniformly over the number of terms in the truncated sum (2.3)). Our goal is to show that $\mathbf{w}$ is bounded in $H_{s+3/2}(T^3)$ on some finite-length time interval $[0, t_*)$.

Substitution (2.9) transforms (2.8) into

$$\frac{d\widehat{\mathbf{w}}_\mathbf{n}}{dt} + \beta\varepsilon|\mathbf{n}|\,\|\mathbf{w}(\mathbf{x}, t)\|^{-1-\varepsilon}_{s+3/2}\,\widehat{\mathbf{w}}_\mathbf{n}\,\frac{d}{dt}\|\mathbf{w}(\mathbf{x}, t)\|_{s+3/2} \tag{2.10}$$

$$= -\mathrm{i}\sum_{\mathbf{k}}(\widehat{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))\,\mathcal{P}_{\mathbf{n}}\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\exp\Big(\beta\|\mathbf{w}(\mathbf{x},t)\|^{-\varepsilon}_{s+3/2}\,(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)\Big).$$

Scalar multiplying this equation by $|\mathbf{n}|^{2+2s}\,\overline{\widehat{\mathbf{w}}_{\mathbf{n}}}$ and considering the real part of the sum over $\mathbf{n}$, we find

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{w}(\mathbf{x},t)\|^2_{1+s}+\frac{\beta\varepsilon}{2-\varepsilon}\|\mathbf{w}(\mathbf{x},t)\|^{2-\varepsilon}_{s+3/2}\right)\\ &=\mathrm{Im}\sum_{\mathbf{k},\mathbf{n}}(\widehat{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))(\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\cdot\overline{\widehat{\mathbf{w}}_{\mathbf{n}}})|\mathbf{n}|^{2+2s} \qquad (2.11)\\ &\quad\times\exp\Big(\beta\|\mathbf{w}(\mathbf{x},t)\|^{-\varepsilon}_{s+3/2}\,(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)\Big).\end{aligned}$$

We call (2.11) the energy balance equation. In construction of estimates for the sum arising from the nonlinear term we follow, with some variations, the approach [9]. By the inequality for sides of a triangle, the exponential function in the right-hand side of (2.11) does not exceed 1. We will show now that the absolute value of the right-hand side of (2.11) is bounded by $(D_s/2)\|\mathbf{w}\|^3_{s+3/2}$, where the constant $D_s$ is independent of $\mathbf{w}$ ($D_s\to\infty$ for $s\to 0$).

By virtue of the inequality $|\mathbf{n}|^{s+1/2}\le|\mathbf{n}-\mathbf{k}|^{s+1/2}+|\mathbf{k}|^{s+1/2}$, valid for $0<s\le 1/2$, the right-hand side of (2.11) does not exceed

$$\begin{aligned}&\sum_{\mathbf{n},\mathbf{k}}\Big(|\widehat{\mathbf{w}}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}|^{s+3/2}|\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}|+|\mathbf{k}|^{s+1/2}|\widehat{\mathbf{w}}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}||\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}|\Big)|\mathbf{n}|^{s+3/2}|\widehat{\mathbf{w}}_{\mathbf{n}}|\\ &=(2\pi)^{-3}\int_{T^3}\big(f_0(\mathbf{x})f_{s+3/2}(\mathbf{x})+f_{s+1/2}(\mathbf{x})f_1(\mathbf{x})\big)\,f_{s+3/2}(-\mathbf{x})\,d\mathbf{x}, \qquad (2.12)\end{aligned}$$

where scalar functions $f_q$ are defined as the Fourier series

$$f_q(\mathbf{x},t)\equiv\sum_{\mathbf{n}}|\widehat{\mathbf{w}}_{\mathbf{n}}(t)||\mathbf{n}|^q\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}. \qquad (2.13)$$

By the Cauchy–Buniakowski–Schwarz inequality, for any $f=\sum_{\mathbf{n}}\widehat{f}_{\mathbf{n}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}$,

$$|f|\le\sum_{\mathbf{n}}|\mathbf{n}|^{s+3/2}|\widehat{f}_{\mathbf{n}}||\mathbf{n}|^{-(s+3/2)}\le\|f\|_{s+3/2}\Big(\sum_{\mathbf{n}}|\mathbf{n}|^{-3-2s}\Big)^{1/2}.$$

The second factor in the left-hand side of this inequality, which we denote by $c_s$, is finite for any $s>0$ (it tends to infinity, when $s\to 0$). Consequently, using Hölder's inequality, the Sobolev embedding theorem (see (2.7)) and Parseval's identity, we find a bound for the right-hand side of (2.12):

$$(2\pi)^{-3}\Big(|f_{s+3/2}|_2\max_{T^3}|f_0|+|f_{s+1/2}|_6|f_1|_3\Big)|f_{s+3/2}|_2$$

$$\leq (2\pi)^{-3}\left(c_s\|f_0\|^2_{s+3/2} + C_1 C_{1/2}\|f_0\|_{s+3/2}\|f_0\|_{3/2}\right)\|f_0\|_{s+3/2} \leq \frac{D_s}{2}\|\mathbf{w}\|^3_{s+3/2}$$

for $D_s \equiv (c_s + C_1 C_{1/2})/(4\pi^3)$. Thus, we obtain from (2.11)

$$\frac{d}{dt}\left(\|\mathbf{w}(\mathbf{x},t)\|^2_{1+s} + A\|\mathbf{w}(\mathbf{x},t)\|^{2-\varepsilon}_{s+3/2}\right) \leq D_s\|\mathbf{w}(\mathbf{x},t)\|^3_{s+3/2},$$

where

$$A \equiv 2\beta\varepsilon/(2-\varepsilon). \tag{2.14}$$

Hence, for $\xi \equiv \|\mathbf{w}(\mathbf{x},t)\|^2_{1+s} + A\|\mathbf{w}(\mathbf{x},t)\|^{2-\varepsilon}_{s+3/2}$,

$$\frac{d\xi}{dt} \leq D_s\, A^{-3/(2-\varepsilon)}\,\xi^{3/(2-\varepsilon)} \quad \Rightarrow \quad -\frac{d}{dt}\xi^{-\theta} \leq D_s\theta A^{-3/(2-\varepsilon)},$$

where $\theta \equiv (1+\varepsilon)/(2-\varepsilon)$. Consequently, the bound

$$\xi \leq \left((\xi|_{t=0})^{-\theta} - D_s\theta A^{-3/(2-\varepsilon)}\, t\right)^{-1/\theta} \tag{2.15}$$

holds for

$$t < t_* \equiv (D_s\theta)^{-1}A^{3/(2-\varepsilon)}(\xi|_{t=0})^{-\theta}. \tag{2.16}$$

For such $t$,

$$\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2} \leq \varphi(t) \equiv \left((A/\xi|_{t=0})^{\theta} - D_s\theta t/A\right)^{-1/(1+\varepsilon)}. \tag{2.17}$$

Assuming in

$$\xi|_{t=0} = \|\mathbf{w}(\mathbf{x},0)\|^2_{1+s} + A\|\mathbf{w}(\mathbf{x},0)\|^{2-\varepsilon}_{s+3/2} \tag{2.18}$$

the norms of the non-truncated vector field $\mathbf{w}(\mathbf{x},0)$, we obtain bounds for $\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2}$ and $t_*$ that are uniform in the number of harmonics in the Fourier–Galerkin truncation of a solution. Inequality (2.17) and transformation (2.9) imply

$$\|\!|\mathbf{v}(\mathbf{x},t)|\!\|_{\beta\varphi^{-\varepsilon},\,s+3/2} \leq \|\!|\mathbf{v}(\mathbf{x},t)|\!\|_{\beta\|\mathbf{w}(\mathbf{x},t)\|^{-\varepsilon}_{s+3/2},\,s+3/2} = \|\mathbf{w}(\mathbf{x},t)\|_{s+3/2} \leq \varphi(t).$$

We have therefore proved

**Theorem 1.** *Let initial condition* $\mathbf{v}^{(\mathrm{in})}$ *of a solution to the force-free Euler equation have a finite Gevrey–Sobolev norm* $\|\!|\mathbf{v}^{(\mathrm{in})}|\!\|_{\sigma,\,s+3/2}$, *where* $0 < s \leq 1/2$. *For* $0 \leq t < t_*$ *the solution satisfies the bound*

$$\|\!|\mathbf{v}(\mathbf{x},t)|\!\|_{\beta\varphi(t)^{-\varepsilon},\,s+3/2} \leq \varphi(t), \tag{2.19}$$

*where* $0 < \beta < \sigma\|\!|\mathbf{v}^{(\mathrm{in})}|\!\|^{\varepsilon}_{\sigma,\,s+3/2}$, $0 < \varepsilon < 2$, $t_*$, *and* $\varphi(t)$ *are defined by relations* (2.16)–(2.18), *and* $\mathbf{w}(\mathbf{x},0)$ *is the result of application of transformation* (2.9) *to the initial condition* $\mathbf{v}^{(\mathrm{in})}$.

## 3. The inviscid Burgers equation

In this section we show that our technique gives the same bound for the Gevrey–Sobolev norms of solutions to the inviscid Burgers equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}.$$

The solenoidality of solutions is not required any more. For the sake of simplicity, we again consider only zero-mean space-periodic solutions for the force-free case $\mathbf{f} = 0$, with the elementary periodicity cell being the cube $T^3$.

A solution is expanded into the Fourier series (2.3); the evolution of Fourier coefficients is now governed by equations

$$\frac{d\widehat{\mathbf{v}}_{\mathbf{n}}}{dt} + \mathrm{i} \sum_{\mathbf{k}} (\widehat{\mathbf{v}}_{\mathbf{k}} \cdot (\mathbf{n} - \mathbf{k}))\widehat{\mathbf{v}}_{\mathbf{n}-\mathbf{k}} = 0. \tag{3.1}$$

After transformation (2.9) is applied, (3.1) becomes

$$\begin{aligned} &\frac{d\widehat{\mathbf{w}}_{\mathbf{n}}}{dt} + \beta\varepsilon|\mathbf{n}|\,\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2}^{-1-\varepsilon}\,\widehat{\mathbf{w}}_{\mathbf{n}}\,\frac{d}{dt}\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2} \qquad (3.2)\\ &= -\mathrm{i}\sum_{\mathbf{k}} (\widehat{\mathbf{w}}_{\mathbf{k}} \cdot (\mathbf{n}-\mathbf{k}))\,\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}} \exp\left(\beta\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2}^{-\varepsilon}\,(|\mathbf{n}| - |\mathbf{k}| - |\mathbf{n}-\mathbf{k}|)\right). \end{aligned}$$

Scalar multiplying this equation by $|\mathbf{n}|^{2+2s}\,\overline{\widehat{\mathbf{w}}_{\mathbf{n}}}$ and considering the real part of the sum over $\mathbf{n}$, we again obtain the energy balance equation (2.11). Since the orthogonality (2.5) was not used to derive from (2.11) the bound (2.17) yielding (2.19), the same derivation applies for the force-free inviscid Burgers equation, implying that the bounds (2.17) and (2.19) hold true for its solutions as well. In other words, Theorem 1 applies literally to solutions to the force-free inviscid Burgers equation.

## 4. Voigt-type regularization of the Euler equation

In this section we present another illustration of our technique, demonstrating global (in time) boundedness of Gevrey–Sobolev norms of solutions to the Voigt-type regularization of the Euler equation. The name "Voigt regularization" was proposed in [21] for $s = 1$ (see (4.1) below). The boundedness was proved *ibid.* for $s = 1$ by the method [22]. Our technique seems simpler in application. We show that a milder regularization, than considered in [21], suffices to guarantee global regularity of solutions.

We consider three-dimensional solenoidal space-periodic zero-mean solutions (2.3) to the equation

$$\alpha^2 \frac{\partial}{\partial t}(-\nabla^2)^s \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} + \nabla p \tag{4.1}$$

for $s > 0$. For $s = 1$ this is the Voigt-regularized Euler equation, studied in [21]. For $\mathbf{f} = 0$, in the Fourier space it takes the form

$$(1+\alpha^2|\mathbf{n}|^{2s})\frac{d\widehat{\mathbf{v}}_\mathbf{n}}{dt} + \mathrm{i}\sum_\mathbf{k}(\widehat{\mathbf{v}}_\mathbf{k}\cdot(\mathbf{n}-\mathbf{k}))\,\mathcal{P}_\mathbf{n}\widehat{\mathbf{v}}_{\mathbf{n}-\mathbf{k}} = 0. \tag{4.2}$$

We assume $\|\mathbf{v}^{(\mathrm{in})}\|_{\sigma,\,s+1/2} < \infty$ for a $\sigma > 0$ and make the transformation

$$\widehat{\mathbf{v}}_\mathbf{n}(t) = \widehat{\mathbf{w}}_\mathbf{n}(t)\exp\left(-\beta|\mathbf{n}|\,|||\mathbf{w}(\mathbf{x},t)|||^{-\varepsilon}\right), \tag{4.3}$$

$$\mathbf{w}(\mathbf{x},t) = \sum_\mathbf{n}\widehat{\mathbf{w}}_\mathbf{n}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}},$$

where $\beta$ and $\varepsilon < 2$ are positive constants and

$$|||\mathbf{w}|||^2 \equiv \sum_\mathbf{n}(1+\alpha^2|\mathbf{n}|^{2s})|\mathbf{n}||\widehat{\mathbf{w}}_\mathbf{n}|^2 \tag{4.4}$$

defines a norm equivalent to $\|\cdot\|_{s+1/2}$. Choosing

$$0 < \beta < \sigma\left(\|\mathbf{v}^{(\mathrm{in})}\|^2_{\sigma,1/2} + \alpha^2\|\mathbf{v}^{(\mathrm{in})}\|^2_{\sigma,\,s+1/2}\right)^{\varepsilon/2}, \tag{4.5}$$

we ensure that $|||\mathbf{w}(\mathbf{x},0)|||$ is bounded uniformly over the number of Fourier harmonics preserved in Fourier–Galerkin truncations of the initial condition $\mathbf{v}^{(\mathrm{in})}$. Our goal is to derive bounds for $|||\mathbf{w}(\mathbf{x},t)|||$.

Substitution (4.3) transforms (4.2) into

$$(1+\alpha^2|\mathbf{n}|^{2s})\left(\frac{d\widehat{\mathbf{w}}_\mathbf{n}}{dt} + \beta\varepsilon|\mathbf{n}|\,|||\mathbf{w}|||^{-1-\varepsilon}\,\widehat{\mathbf{w}}_\mathbf{n}\frac{d}{dt}|||\mathbf{w}|||\right)$$
$$= -\mathrm{i}\sum_\mathbf{k}(\widehat{\mathbf{w}}_\mathbf{k}\cdot(\mathbf{n}-\mathbf{k}))\,\mathcal{P}_\mathbf{n}\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\exp\left(\beta|||\mathbf{w}|||^{-\varepsilon}\left(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|\right)\right).$$

The real part of the sum over $\mathbf{n}$ of these equations scalar multiplied by $\overline{\widehat{\mathbf{w}}_\mathbf{n}}$ reduces to

$$\frac{d}{dt}\left(\|\mathbf{w}\|_0^2 + \alpha^2\|\mathbf{w}\|_s^2 + A|||\mathbf{w}|||^{2-\varepsilon}\right)$$
$$= 2\mathrm{Im}\sum_{\mathbf{k},\mathbf{n}}(\widehat{\mathbf{w}}_\mathbf{k}\cdot(\mathbf{n}-\mathbf{k}))(\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\cdot\overline{\widehat{\mathbf{w}}_\mathbf{n}})\exp\left(\beta|||\mathbf{w}|||^{-\varepsilon}\left(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|\right)\right)$$
$$= \mathrm{Im}\sum_{\mathbf{k},\mathbf{n}}(\widehat{\mathbf{w}}_\mathbf{k}\cdot(\mathbf{n}-\mathbf{k}))(\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\cdot\overline{\widehat{\mathbf{w}}_\mathbf{n}})\left(\exp\left(\beta|||\mathbf{w}|||^{-\varepsilon}\left(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|\right)\right)\right.$$

$$- \exp\left(\beta|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon}\left(|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|-|\mathbf{n}|\right)\right)\right) \tag{4.6}$$

(above, in one of the sums in $\mathbf{k}$ and $\mathbf{n}$ we have changed the index of summation $\mathbf{n}\to\mathbf{k}-\mathbf{n}$ and relied on the fact that $\mathbf{w}$ is real-valued and solenoidal, which stems from (2.5) and (2.4)).

Note that $|e^{\alpha'}-e^{\alpha''}|\le|\alpha'-\alpha''|$ for any negative $\alpha'$ and $\alpha''$. Consequently, the absolute value of the right-hand side of (4.6) does not exceed

$$\sum_{\mathbf{n},\mathbf{k}}|\widehat{\mathbf{w}}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}||\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}||\widehat{\mathbf{w}}_{\mathbf{n}}|\,2\beta\,|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon}\,||\mathbf{n}|-|\mathbf{n}-\mathbf{k}||$$
$$\le\frac{\beta|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon}}{4\pi^3}\int_{T^3}f_0(-\mathbf{x})f_1^2(\mathbf{x})\,d\mathbf{x}$$

(here and in (4.6) we use the notation $A$ and $f_q$ (2.13) introduced in Section 2)

$$\le\frac{\beta|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon}}{4\pi^3}|f_0|_{9/(2-3\zeta)}|f_1|^2_{18/(7+3\zeta)} \tag{4.7}$$

(by Hölder's inequality for a sufficiently small $\zeta>0$). By the Sobolev embedding theorem

$$|f_0|_{9/(2-3\zeta)}\le C_{5/6+\zeta}\|f_0\|_{5/6+\zeta},\qquad |f_1|_{18/(7+3\zeta)}\le C_{1/3-\zeta/2}\|f_1\|_{1/3-\zeta/2}$$

(see (2.7)). By virtue of (2.13) and by Hölder's inequality, (4.7) is bounded by

$$\frac{\beta C_{5/6+\zeta}C^2_{1/3-\zeta/2}|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon}}{4\pi^3}\|\mathbf{w}\|_{5/6+\zeta}\|\mathbf{w}\|^2_{4/3-\zeta/2}$$
$$\le\frac{\beta C_{5/6+\zeta}C^2_{1/3-\zeta/2}|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon}}{4\pi^3}\|\mathbf{w}\|^{1+6\zeta}_{5/6+\zeta}\|\mathbf{w}\|^{2-6\zeta}_{4/3+\zeta}.$$

Hence, for $s\ge5/6+\zeta$ and $\varepsilon=2-6\zeta$, provided $0<\zeta<1/3$, we obtain from the energy balance equation (4.6)

$$\frac{d}{dt}\left(\|\mathbf{w}\|_0^2+\alpha^2\|\mathbf{w}\|_s^2+A|\!|\!|\mathbf{w}|\!|\!|^{6\zeta}\right)\le\frac{\beta C_{5/6+\zeta}C^2_{1/3-\zeta/2}}{4\pi^3\alpha^\varepsilon}\|\mathbf{w}\|_s^{1+6\zeta}.$$

This implies

$$\frac{d\xi}{dt}\le D\,\xi^{1/2+3\zeta}, \tag{4.8}$$

where it is denoted

$$D\equiv\beta C_{5/6+\zeta}C^2_{1/3-\zeta/2}(\pi\alpha)^{-3}/4,\quad \xi\equiv\|\mathbf{w}\|_0^2+\alpha^2\|\mathbf{w}\|_s^2+A|\!|\!|\mathbf{w}|\!|\!|^{6\zeta}. \tag{4.9}$$

For $0 < \zeta < 1/6$, we obtain from (4.8) a global polynomial bound

$$\xi \leq \Big( (\xi|_{t=0})^{1/2-3\zeta} + (1/2 - 3\zeta)\, Dt \Big)^{(1/2-3\zeta)^{-1}} \equiv \varphi(t;\zeta), \tag{4.10}$$

and for $\zeta = 1/6$ an exponential one

$$\xi \leq \xi|_{t=0}\, \mathrm{e}^{Dt} \equiv \varphi(t; 1/6). \tag{4.11}$$

For $\zeta > 1/6$, (4.8) yields bounds for $\xi$, which are finite-time and thus not of interest. In view of (4.3), (4.10), and (4.11), the solution to equation (4.1) satisfies the inequalities

$$\begin{aligned} \|\!|\mathbf{w}|\!\| &\leq (\varphi/A)^{1/(6\zeta)} \qquad \Rightarrow \\ \big\| \mathbf{v}(\mathbf{x},t) \big\|_{\beta(\varphi(t;\zeta)/A)^{1-1/(3\zeta)},\,1/2} &\leq \big\| \mathbf{v}(\mathbf{x},t) \big\|_{\beta\|\!|\mathbf{w}(\mathbf{x},t)|\!\|^{-\varepsilon},\,1/2} \\ &= \|\mathbf{w}\|_{1/2} \leq \|\!|\mathbf{w}|\!\| \leq (\varphi(t;\zeta)/A)^{1/(6\zeta)}. \end{aligned}$$

Note that the first index of the norm in the left-hand side of this inequality is strictly positive for any $t > 0$.

We have proved the following statement:

**Theorem 2.** *Suppose* $0 < \zeta \leq 1/6$ *and* $s \geq 5/6 + \zeta$. *Let the norm* $\big\|\mathbf{v}^{(\mathrm{in})}\big\|_{\sigma,\, s+1/2}$ *of initial condition* $\mathbf{v}^{(\mathrm{in})}$ *of a solution to the Voigt-type regularization of the force-free three-dimensional Euler equation* (4.1) *be finite for some* $\sigma > 0$. *Then the bound*

$$\big\| \mathbf{v}(\mathbf{x},t) \big\|_{\beta(\varphi(t;\zeta)/A)^{1-1/(3\zeta)},\,1/2} \leq (\varphi(t;\zeta)/A)^{1/(6\zeta)}$$

*holds true for the solution at any time* $t > 0$. *Here* $A$ *is defined by* (2.14), $\varepsilon = 2 - 6\zeta$ *and* $\beta$ *satisfies inequality* (4.5); $\varphi$ *is defined by* (4.10) *and* (4.11), *where* $\xi|_{t=0}$ *is obtained by application of formula* (4.9) *to* $\mathbf{w}(\mathbf{x}, 0)$, *which is the result of application of transformation* (2.9) *to the non-truncated initial condition* $\mathbf{v}^{(\mathrm{in})}$.

To the best of our knowledge, the regularized equation (4.1) with the fractional power of the Laplacian has not yet been considered in Sobolev spaces in literature. It is easy to show incrementally that for $s \geq 5/6$ the solution of (4.1) belongs to Sobolev spaces of arbitrarily high indices (provided the initial data and the forcing $\mathbf{f}$ are sufficiently regular). Such a proof can be started by multiplication of (4.1) by $\mathbf{v}$ and deriving a bound for $\|\mathbf{v}\|_{5/6}$. Scalar multiplying now (4.1) by $(-\nabla^2)^{5/3-s}\mathbf{v}$, using the inequality $|\mathbf{v}|_{9/2} \leq C_{5/6}\|\mathbf{v}\|_{5/6}$ to bound the integral arising from the nonlinear term, it easy to show that $\|\mathbf{v}\|_{5/3}$ is bounded at any $t > 0$. Higher-index Sobolev norms can be subsequently bounded similarly.

## 5. Voigt-type regularization of the Navier–Stokes equation

In this section we prove global (in time) boundedness of Gevrey–Sobolev norms of three-dimensional solenoidal space-periodic zero-mean solutions (2.3) to the Voigt-type regularization of the Navier–Stokes equation

$$\alpha^2 \frac{\partial}{\partial t}(-\nabla^2)^s \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\nabla^2\mathbf{v} = \mathbf{f} + \nabla p, \tag{5.1}$$

where $s > 0$. The Voigt regularization ($s = 1$) was previously investigated in [25, 36]; boundedness of Gevrey–Sobolev norms of solutions for $s = 1$ was demonstrated in [16] by a method different from the one used here. We show here that a milder regularization than the one considered in the previous section for the Euler equation suffices to guarantee global regularity of solutions. In the first two subsections we derive the bounds separately for $s > 1/2$, where the smoothing term is excessively strong, and for the limit value $s = 1/2$—constructions in the two cases are somewhat different. In the last subsection we show that the method [11] can be used for $s \le 1/2$ to establish instantaneous development of analyticity of solutions for mildly regular initial conditions.

5.1. **Excessive damping:** $s > 1/2$**.** Assuming $\left\|\mathbf{v}^{(\mathrm{in})}\right\|_{\sigma,\,s+1/2} < \infty$ for a positive $\sigma$, we make transformation (4.3) for $1 < \varepsilon < 2$, where the norm $\|\!|\cdot|\!\|$, equivalent to $\|\cdot\|_{s+1/2}$, is defined by (4.4); $\beta$ is supposed to satisfy inequality (4.5), implying $\|\!|\mathbf{w}(\mathbf{x},0)|\!\| < \infty$. We need to derive a bound for $\|\!|\mathbf{w}(\mathbf{x},t)|\!\|$. Substituting (2.3) and (4.3), we transform (5.1) (for $\mathbf{f} = 0$) into the system of equations

$$\begin{aligned}&(1+\alpha^2|\mathbf{n}|^{2s})\left(\frac{d\widehat{\mathbf{w}}_{\mathbf{n}}}{dt} + \beta\varepsilon|\mathbf{n}|\,\|\!|\mathbf{w}|\!\|^{-1-\varepsilon}\,\widehat{\mathbf{w}}_{\mathbf{n}}\,\frac{d}{dt}\|\!|\mathbf{w}|\!\|\right) + \nu|\mathbf{n}|^2\widehat{\mathbf{w}}_{\mathbf{n}}\\ &= -\mathrm{i}\sum_{\mathbf{k}}(\widehat{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))\,\mathcal{P}_{\mathbf{n}}\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\exp\left(\beta\|\!|\mathbf{w}|\!\|^{-\varepsilon}\,(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)\right).\end{aligned}$$

The real part of the sum of these equations scalar multiplied by $\overline{\widehat{\mathbf{w}}_{\mathbf{n}}}$ reduces to

$$\begin{aligned}&\frac{d}{dt}\left(\|\mathbf{w}\|_0^2 + \alpha^2\|\mathbf{w}\|_s^2 + A\|\!|\mathbf{w}|\!\|^{2-\varepsilon}\right) + 2\nu\|\mathbf{w}\|_1^2\\ &= \mathrm{Im}\sum_{\mathbf{k},\mathbf{n}}(\widehat{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))(\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\cdot\overline{\widehat{\mathbf{w}}_{\mathbf{n}}})\left(\exp\left(\beta\|\!|\mathbf{w}|\!\|^{-\varepsilon}\,(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)\right)\right.\\ &\left.-\exp\left(\beta\|\!|\mathbf{w}|\!\|^{-\varepsilon}\,(|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|-|\mathbf{n}|)\right)\right)\end{aligned} \tag{5.2}$$

(this identity differs from (4.6) only by the presence of the diffusion-related term $2\nu\|\mathbf{w}\|_1^2$).

Note that $|e^{\alpha'} - e^{\alpha''}| \le |\alpha' - \alpha''|^{1/\varepsilon}$ for all $\alpha' \le 0$ and $\alpha'' \le 0$ (since $\varepsilon > 1$). Consequently, the right-hand side of (5.2) does not exceed

$$\begin{aligned}
&\sum_{\mathbf{n},\mathbf{k}} |\widehat{\mathbf{w}}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}||\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}||\widehat{\mathbf{w}}_{\mathbf{n}}| \left(2\beta|\!|\!|\mathbf{w}|\!|\!|^{-\varepsilon} \,||\mathbf{n}| - |\mathbf{n}-\mathbf{k}||\right)^{1/\varepsilon} \\
&\le \frac{(2\beta)^{1/\varepsilon}}{(2\pi)^3|\!|\!|\mathbf{w}|\!|\!|} \int_{T^3} f_0(-\mathbf{x}) f_{1/\varepsilon}(\mathbf{x}) f_1(\mathbf{x})\, d\mathbf{x} \\
&\le \frac{(2\beta)^{1/\varepsilon}}{(2\pi)^3|\!|\!|\mathbf{w}|\!|\!|} |f_0|_6 |f_{1/\varepsilon}|_{3/(s+1/2)} |f_1|_{3/(2-s)}
\end{aligned}$$

(by Hölder's inequality and definition (2.13) of the functions $f_q$)

$$\le \frac{(2\beta)^{1/\varepsilon}}{(2\pi)^3|\!|\!|\mathbf{w}|\!|\!|} C_1\|f_0\|_1 C_{1-s}\|f_{1/\varepsilon}\|_{1-s} C_{s-1/2}\|f_1\|_{s-1/2} \qquad (5.3)$$

(by the Sobolev embedding theorem; see (2.7)). We have assumed here $1/2 < s < 1$, and we further demand $2 > \varepsilon > 1/s$. Then, by Hölder's inequality and definition (2.13) of the functions $f_q$, (5.3) is bounded by

$$\frac{C_1 C_{1-s} C_{s-1/2}(2\beta)^{1/\varepsilon}}{(2\pi)^3|\!|\!|\mathbf{w}|\!|\!|} \|\mathbf{w}\|_1 \|\mathbf{w}\|_{1-s+1/\varepsilon} \|\mathbf{w}\|_{s+1/2} \le D_{s,\varepsilon}\|\mathbf{w}\|_1^{2-\kappa}\|\mathbf{w}\|_s^{\kappa},$$

where we have denoted

$$D_{s,\varepsilon} \equiv C_1 C_{1-s} C_{s-1/2}(2\beta)^{1/\varepsilon}(2\pi)^{-3}/\alpha, \qquad \kappa \equiv \min(1, (s-1/\varepsilon)/(1-s)).$$

We therefore obtain from (5.2), by Young's inequality,

$$\frac{d}{dt}\left(\|\mathbf{w}\|_0^2 + \alpha^2\|\mathbf{w}\|_s^2 + A|\!|\!|\mathbf{w}|\!|\!|^{2-\varepsilon}\right) \le D'_{s,\varepsilon}\alpha^2\|\mathbf{w}\|_s^2,$$

where

$$D'_{s,\varepsilon} \equiv \frac{D_{s,\varepsilon}^{2/\kappa}\kappa}{2\alpha^2}\left(\frac{2-\kappa}{4\nu}\right)^{(2-\kappa)/\kappa}.$$

Therefore,

$$\begin{aligned}
&\|\mathbf{w}\|_0^2 + \alpha^2\|\mathbf{w}\|_s^2 + A|\!|\!|\mathbf{w}|\!|\!|^{2-\varepsilon} \\
&\le \left(\|\mathbf{w}\|_0^2 + \alpha^2\|\mathbf{w}\|_s^2 + A|\!|\!|\mathbf{w}|\!|\!|^{2-\varepsilon}\right)\Big|_{t=0} \mathrm{e}^{D'_{s,\varepsilon}t} \equiv \varphi(t).
\end{aligned} \qquad (5.4)$$

Since $|\!|\!|\mathbf{w}|\!|\!| \le (\varphi/A)^{1/(2-\varepsilon)}$, transformation (4.3) implies

$$\begin{aligned}
\|\mathbf{v}(\mathbf{x},t)\|_{\beta(\varphi(t)/A)^{-\varepsilon/(2-\varepsilon)},\,1/2} &\le \|\mathbf{v}(\mathbf{x},t)\|_{\beta|\!|\!|\mathbf{w}(\mathbf{x},t)|\!|\!|^{-\varepsilon},\,1/2} \\
&= \|\mathbf{w}\|_{1/2} \le |\!|\!|\mathbf{w}|\!|\!| \le (\varphi(t)/A)^{1/(2-\varepsilon)}.
\end{aligned}$$

The first index of the Gevrey–Sobolev norm in the left-hand side of this inequality is strictly positive for any $t > 0$.

Thus, we have demonstrated

**Theorem 3.** *Suppose* $2 > \varepsilon > 1/s > 1$*. Let the norm* $\|\mathbf{v}^{(\mathrm{in})}\|_{\sigma,\,s+1/2}$ *of initial condition* $\mathbf{v}^{(\mathrm{in})}$ *of a solution to the Voigt-type regularization of the force-free three-dimensional Navier–Stokes equation* (5.1) *be finite for a* $\sigma > 0$*. Then the bound*

$$\|\mathbf{v}(\mathbf{x},t)\|_{\beta(\varphi(t)/A)^{-\varepsilon/(2-\varepsilon)},\,1/2} \leq (\varphi(t)/A)^{1/(2-\varepsilon)}$$

*holds true for the solution at any time* $t > 0$*. Here* $A$ *is defined by* (2.14)*,* $\beta$ *satisfies inequality* (4.5) *and* $\varphi$ *is defined by* (5.4)*, where* $\mathbf{w}(\mathbf{x},0)$ *is obtained from the non-truncated initial condition* $\mathbf{v}^{(\mathrm{in})}$ *by transformation* (2.9)*.*

5.2. **The critical damping:** $s = 1/2$**.** In this subsection we focus on the limit value $s = 1/2$ and show that in this case bounds for the Gevrey–Sobolev norms can be obtained in a similar way. Instead of (4.3), we now make a substitution

$$\widehat{\mathbf{v}}_{\mathbf{n}}(t) = \widehat{\mathbf{w}}_{\mathbf{n}}(t)\exp\left(-\beta|\mathbf{n}|(1+|||\mathbf{w}(\mathbf{x},t)|||)^{-2}\right). \qquad (5.5)$$

$|||\mathbf{w}(\mathbf{x},0)|||$ is bounded uniformly over truncations of $\mathbf{v}^{(\mathrm{in})}$ provided

$$0 < \beta < \sigma\left(1+\sqrt{\|\mathbf{v}^{(\mathrm{in})}\|^2_{\sigma,1/2}+\alpha^2\|\mathbf{v}^{(\mathrm{in})}\|^2_{\sigma,1}}\,\right)^2 \qquad (5.6)$$

(under this condition a transformation of the non-truncated initial condition $\mathbf{v}^{(\mathrm{in})}$ is also well-defined).

As in the previous subsection, we derive from the Voigt-type regularized Navier–Stokes equation (5.1) (for $\mathbf{f} = 0$) a system of equations governing the evolution of Fourier coefficients $\widehat{\mathbf{w}}_{\mathbf{n}}(t)$ of $\mathbf{w}$, and consider the real part of the sum of these equations, scalar multiplied by $\overline{\widehat{\mathbf{w}}_{\mathbf{n}}}$. The energy balance equation, analogous to (5.2), takes now the form

$$\begin{aligned}
&\frac{d}{dt}(\|\mathbf{w}\|_0^2+\alpha^2\|\mathbf{w}\|_{1/2}^2)+4\beta\,\frac{|||\mathbf{w}|||^2}{(1+|||\mathbf{w}|||)^3}\,\frac{d}{dt}|||\mathbf{w}|||+2\nu\|\mathbf{w}\|_1^2\\
&=\mathrm{Im}\sum_{\mathbf{k},\mathbf{n}}(\widehat{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))(\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\cdot\overline{\widehat{\mathbf{w}}_{\mathbf{n}}})\left(\exp\left(\beta(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)/(1+|||\mathbf{w}|||)^2\right)\right.\\
&\left.-\exp\left(\beta(|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|-|\mathbf{n}|)/(1+|||\mathbf{w}|||)^2\right)\right). \qquad (5.7)
\end{aligned}$$

Using the inequality $|e^{\alpha'}-e^{\alpha''}| \leq |\alpha'-\alpha''|^{1/2}$, which holds for any $\alpha' \leq 0$ and $\alpha'' \leq 0$, we obtain from (5.7) the energy inequality

$$\frac{d}{dt}\Big(\|\mathbf{w}\|_0^2+\alpha^2\|\mathbf{w}\|_{1/2}^2+4\beta\Big(\ln(1+\|\!|\mathbf{w}|\!\|)+\frac{4\|\!|\mathbf{w}|\!\|+3}{2(1+\|\!|\mathbf{w}|\!\|)^2}\Big)\Big)+2\nu\|\mathbf{w}\|_1^2$$
$$\le\sqrt{2\beta}\sum_{\mathbf{n},\mathbf{k}}|\widehat{\mathbf{w}}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}||\widehat{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}||\widehat{\mathbf{w}}_{\mathbf{n}}||\mathbf{k}|^{1/2}(1+\|\!|\mathbf{w}|\!\|)^{-1}$$
$$\le\frac{\sqrt{2\beta}}{(2\pi)^3}C_{1/2}C_1\frac{\|\mathbf{w}\|_1^3}{1+\|\!|\mathbf{w}|\!\|}\le\frac{\sqrt{2\beta}C_{1/2}C_1}{(2\pi)^3\alpha^2}\|\mathbf{w}\|_1^2. \tag{5.8}$$

Consequently, if $\beta\le 2\nu^2(2\pi)^6C_{1/2}^{-2}C_1^{-2}\alpha^4$ (in addition to condition (5.6)), then for

$$\xi(t)\equiv\|\mathbf{w}\|_0^2+\alpha^2\|\mathbf{w}\|_{1/2}^2+4\beta\ln(1+\|\!|\mathbf{w}|\!\|) \tag{5.9}$$

we find, integrating (5.8) with respect to time,

$$\xi(t)\le\xi(0)+4\beta\left(-\frac{4\|\!|\mathbf{w}(\mathbf{x},t)|\!\|+3}{2(1+\|\!|\mathbf{w}(\mathbf{x},t)|\!\|)^2}+\frac{4\|\!|\mathbf{w}(\mathbf{x},0)|\!\|+3}{2(1+\|\!|\mathbf{w}(\mathbf{x},0)|\!\|)^2}\right)\le\xi(0)+6\beta.$$

Each of the three terms constituting $\xi(t)$ is positive; hence we deduce from this inequality

$$1+\|\!|\mathbf{w}(\mathbf{x},t)|\!\|\le\exp(\xi(0)/(4\beta)+3/2),\qquad \|\mathbf{w}(\mathbf{x},t)\|_0\le\sqrt{\xi(0)+6\beta}.$$

In view of (5.5), the two inequalities imply

$$\begin{aligned}\big\bracevert\mathbf{v}(\mathbf{x},t)\big\bracevert_{\beta\exp(-\xi(0)/(2\beta)-3),\,0}&\le\big\bracevert\mathbf{v}(\mathbf{x},t)\big\bracevert_{\beta(1+\|\!|\mathbf{w}(\mathbf{x},t)|\!\|)^{-2},\,0}\\&\le\|\mathbf{w}(\mathbf{x},t)\|_0\le\sqrt{\xi(0)+6\beta}.\end{aligned}$$

Thus, in the case of critical damping the following theorem holds:

**Theorem 4.** *Suppose $\sigma>0$ and the norm $\big\bracevert\mathbf{v}^{(\mathrm{in})}\big\bracevert_{\sigma,1}$ of initial condition $\mathbf{v}^{(\mathrm{in})}$ of a solution to the Voigt-type regularization of the force-free three-dimensional Navier–Stokes equation* (5.1) *for $s=1/2$ is finite. Then the bound*

$$\big\bracevert\mathbf{v}(\mathbf{x},t)\big\bracevert_{\beta\exp(-\xi(0)/(2\beta)-3),\,0}\le\sqrt{\xi(0)+6\beta} \tag{5.10}$$

*holds true for the solution at any time $t>0$. Here $\beta$ satisfies inequality* (5.6) *and $\xi(0)$ is determined by application of formula* (5.9) *to $\mathbf{w}(\mathbf{x},0)$, obtained from the non-truncated initial condition $\mathbf{v}^{(\mathrm{in})}$ by transformation* (5.5).

What happens if the initial data $\mathbf{v}^{(\mathrm{in})}$ is non-analytic and thus Theorem 4 is inapplicable? As in the case of the Voigt-type regularization of the three-dimensional Euler equation, it is possible to develop the theory of solutions to equation (5.1) in the Sobolev spaces. In particular, one can show incrementally that for $s\ge 1/2$ the solution of (5.1) belongs to Sobolev spaces of arbitrarily high indices (when the initial data and the forcing $\mathbf{f}$ are sufficiently regular). Since this question is not in the scope of our paper, we only present a brief sketch of derivation of the bounds. Multiplication of (5.1) by $\mathbf{v}$ demonstrates boundedness of $\int_0^t \|\mathbf{v}(\mathbf{x},\tau)\|_1^2\,d\tau$ for any $t > 0$. Scalar multiplication of the equation by $(-\nabla^2)\mathbf{v}$ and the use of the inequality

$$|(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot(-\nabla^2)\mathbf{v}| \le |\mathbf{v}|_6|\nabla\mathbf{v}|_3|\nabla^2\mathbf{v}|_2 \le (C\|\mathbf{v}\|_1^2\|\mathbf{v}\|_{1+s}^2+\nu\|\mathbf{v}\|_2^2)/2$$

yields

$$\frac{d}{dt}(\alpha^2\|\mathbf{v}\|_{1+s}^2+\|\mathbf{v}\|_1^2) \le \|\mathbf{f}\|_{1-s}^2+(1+C\|\mathbf{v}\|_1^2)\|\mathbf{v}\|_{1+s}^2,$$

whereby

$$\begin{aligned}\alpha^2\|\mathbf{v}\|_{1+s}^2+\|\mathbf{v}\|_1^2 &\le (\alpha^2\|\mathbf{v}(\mathbf{x},0)\|_{1+s}^2+\|\mathbf{v}(\mathbf{x},0)\|_1^2)\\ &\quad\times\exp\Big(\alpha^{-2}\int_0^t(1+C\|\mathbf{v}(\mathbf{x},\tau)\|_1^2)d\tau\Big)\\ &+\int_0^t\|\mathbf{f}(\mathbf{x},\tau)\|_{1-s}^2\exp\Big(\alpha^{-2}\int_\tau^t(1+C\|\mathbf{v}(\mathbf{x},\tau')\|_1^2)d\tau'\Big)d\tau.\end{aligned}$$

Bounds for higher-index Sobolev norms can be subsequently derived by a similar procedure.

5.3. **Instantaneous development of analyticity for $s \le 1/2$.** The remark, made in [16], that the Navier–Stokes–Voigt equation ($s = 1$) exhibits damped hyperbolicity, remains valid for the milder regularization for $s > 1/2$ considered in subsection 5.1. In particular, solutions for $s > 1/2$ apparently cannot instantaneously acquire analyticity—at least, the method [11], revealing that solutions to the Navier–Stokes equation are capable of this, is not directly applicable to equation (5.1). To see this, let us transform a solution to (5.1), following [11], using the relations

$$\mathbf{v}(\mathbf{x},t)=\sum_{\mathbf{n}\ne 0}\widetilde{\mathbf{w}}_{\mathbf{n}}(t)\mathrm{e}^{-\beta t|\mathbf{n}|+\mathrm{i}\mathbf{n}\cdot\mathbf{x}},\qquad \mathbf{w}(\mathbf{x},t)=\sum_{\mathbf{n}}\widetilde{\mathbf{w}}_{\mathbf{n}}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}},\qquad (5.11)$$

where $\beta > 0$ is a constant. The evolution of the transformed Fourier coefficients satisfies the equation

$$\begin{aligned}&(1+\alpha^2|\mathbf{n}|^{2s})\Big(\frac{d\widetilde{\mathbf{w}}_{\mathbf{n}}}{dt}-\beta|\mathbf{n}|\widetilde{\mathbf{w}}_{\mathbf{n}}\Big)+\nu|\mathbf{n}|^2\widetilde{\mathbf{w}}_{\mathbf{n}}\\ &=-\mathrm{i}\sum_{\mathbf{k}}(\widetilde{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))\,\mathcal{P}_{\mathbf{n}}\widetilde{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\exp\left(\beta t(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)\right).\end{aligned}$$

Scalar multiplying it by $|\mathbf{n}|^{2\gamma}\,\overline{\widetilde{\mathbf{w}}_{\mathbf{n}}}$ for $\gamma \geq 0$, summing over $\mathbf{n}$ and taking the real part we find

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{w}\|_\gamma^2 + \alpha^2\|\mathbf{w}\|_{\gamma+s}^2) - \beta(\|\mathbf{w}\|_{1/2+\gamma}^2 + \alpha^2\|\mathbf{w}\|_{1/2+\gamma+s}^2) + \nu\|\mathbf{w}\|_{1+\gamma}^2$$

$$= \operatorname{Im}\sum_{\mathbf{k},\mathbf{n}}(\widetilde{\mathbf{w}}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k}))(\widetilde{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}\cdot\overline{\widetilde{\mathbf{w}}_{\mathbf{n}}})|\mathbf{n}|^{2\gamma}\exp\left(\beta t(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)\right). \quad (5.12)$$

Thus, the strength of the regularizing term for $s > 1/2$ gives rise to a problem: viscous dissipation is too weak to control the term $\beta\alpha^2\|\mathbf{w}\|_{1/2+\gamma+s}^2$ appearing in the left-hand side of the energy balance equation (5.12). Still, for $s > 1/2$ finite-time Gevrey class $G_{1/(2-2s)}$ regularity emerges instantaneously, and this can be established by the method [11].

To show that the method [11] works for $0 < s \leq 1/2$, we choose three quantities $\eta_i > 0$ such that $\eta_1+\eta_2+\eta_3 = \nu$ and $1/2 < \gamma \leq 1$. If $s = 1/2$, we also demand

$$\beta \leq \eta_2/\alpha^2 \quad (5.13)$$

in transformation (5.11). By Hölder's and Young's inequalities,

$$\beta\|\mathbf{w}\|_{1/2+\gamma}^2 \leq \eta_1\|\mathbf{w}\|_{1+\gamma}^2 + (4\eta_1)^{-1}\beta^2\|\mathbf{w}\|_\gamma^2, \quad (5.14)$$

$$\beta\alpha^2\|\mathbf{w}\|_{1/2+\gamma+s}^2 \leq \eta_2\|\mathbf{w}\|_{1+\gamma}^2 + Q_1\alpha^2\|\mathbf{w}\|_{\gamma+s}^2, \quad (5.15)$$

where

$$Q_1 \equiv \begin{cases} 0, & \text{if } \beta \leq \eta_2/\alpha^2, \\ \frac{\beta(1-2s)}{2(1-s)}xbeta(\frac{\beta\alpha^2}{2\eta_2(1-s)}xbeta)^{1/(1-2s)}, & \text{otherwise.} \end{cases}$$

Using Hölder's inequality, the Sobolev embedding theorem (2.7) and Young's inequality, we find that the right-hand side of (5.12) is bounded by the sum

$$\sum_{\mathbf{k},\mathbf{n}}|\widetilde{\mathbf{w}}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}||\widetilde{\mathbf{w}}_{\mathbf{n}-\mathbf{k}}||\widetilde{\mathbf{w}}_{\mathbf{n}}||\mathbf{n}|^{2\gamma}$$
$$= (2\pi)^{-3}\int_{T^3}\Big(\sum_{\mathbf{n}}|\widetilde{\mathbf{w}}_{\mathbf{n}}|\,\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}\Big)\Big(\sum_{\mathbf{n}}|\widetilde{\mathbf{w}}_{\mathbf{n}}||\mathbf{n}|\,\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}\Big)\Big(\sum_{\mathbf{n}}|\widetilde{\mathbf{w}}_{\mathbf{n}}||\mathbf{n}|^{2\gamma}\,\mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\mathbf{x}}\Big)\,d\mathbf{x}$$
$$\leq (2\pi)^{-3}|\mathbf{w}|_{6/(3-2\gamma)}\Big|\sum_{\mathbf{n}}|\widetilde{\mathbf{w}}_{\mathbf{n}}||\mathbf{n}|\,\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}\Big|_3\Big|\sum_{\mathbf{n}}|\widetilde{\mathbf{w}}_{\mathbf{n}}||\mathbf{n}|^{2\gamma}\,\mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\mathbf{x}}\Big|_{6/(1+2\gamma)}$$
$$\leq (2\pi)^{-3}C_\gamma\|\mathbf{w}\|_\gamma C_{1/2}\|\mathbf{w}\|_{3/2}C_{1-\gamma}\|\mathbf{w}\|_{1+\gamma}$$
$$\leq (2\pi)^{-3}C_\gamma C_{1/2}C_{1-\gamma}\|\mathbf{w}\|_\gamma^{1/2+\gamma}\|\mathbf{w}\|_{1+\gamma}^{5/2-\gamma}$$
$$\leq (Q_2/2)\|\mathbf{w}\|_\gamma^{2(1+2\gamma)/(2\gamma-1)} + \eta_3\|\mathbf{w}\|_{1+\gamma}^2, \quad (5.16)$$

where we have denoted

$$Q_2 = 2(2\gamma - 1)\Big(\frac{5-2\gamma}{\eta_3}\Big)^{(5-2\gamma)/(2\gamma-1)}\Big(\frac{C_\gamma C_{1/2}C_{1-\gamma}}{4(2\pi)^3}\Big)^{4/(2\gamma-1)}.$$

Relations (5.12)–(5.16) imply the energy-type inequality

$$\begin{aligned}&\frac{d}{dt}(\|\mathbf{w}\|_\gamma^2 + \alpha^2\|\mathbf{w}\|_{\gamma+s}^2)\\ &\le (2\eta_1)^{-1}\beta^2\|\mathbf{w}\|_\gamma^2 + 2Q_1\alpha^2\|\mathbf{w}\|_{\gamma+s}^2 + Q_2\|\mathbf{w}\|_\gamma^{2(1+2\gamma)/(2\gamma-1)}.\end{aligned}$$

Therefore,

$$\frac{d\xi}{dt} \le q\xi + Q_2\xi^{(1+2\gamma)/(2\gamma-1)},$$

where

$$\xi(t) \equiv \|\mathbf{w}\|_\gamma^2 + \alpha^2\|\mathbf{w}\|_{1/2+\gamma}^2, \qquad q \equiv \max((2\eta_1)^{-1}\beta^2, 2Q_1). \tag{5.17}$$

Integrating this inequality, we obtain a bound

$$\xi(t) \le \mathrm{e}^{qt}\left((\xi(0))^{-(\gamma-1/2)^{-1}} - (Q_2/q)(\mathrm{e}^{qt(\gamma-1/2)^{-1}} - 1)\right)^{-(\gamma-1/2)} \equiv \varphi(t), \tag{5.18}$$

valid for

$$t < t_* \equiv \frac{2\gamma-1}{2q}\ln\Big(1 + \frac{q}{Q_2(\xi(0))^{1/(\gamma-1/2)}}\Big). \tag{5.19}$$

From transformation (5.11), relation (5.17), and this bound we infer

**Theorem 5.** *Suppose* $0 < s \le 1/2 < \gamma \le 1$ *and the initial condition of a solution to the Voigt-type regularization of the force-free three-dimensional Navier–Stokes equation* (5.1) *belongs to the Sobolev space* $H_{\gamma+1/2}(T^3)$. *Then for* $t < t_*$ *the solution satisfies the bound*

$$\lvert\!\lvert\!\lvert \mathbf{v}(\mathbf{x},t) \rvert\!\rvert\!\rvert_{\beta t,\gamma} \le \sqrt{\varphi(t)}.$$

*Here a positive constant* $\beta$ *satisfies inequality* (5.13) *if* $s = 1/2$, *and is arbitrary otherwise,* $\varphi$ *and* $t_*$ *are defined by formulae* (5.18) *and* (5.19), *respectively, where* $\xi(0)$ *is determined applying* (5.17) *to* $\mathbf{w}(\mathbf{x},0) = \mathbf{v}^{(\mathrm{in})}$.

This shows that analyticity of solutions to the Voigt-type regularization of the Navier–Stokes equation (5.1) for $s = 1/2$ emerges instantaneously, provided the initial conditions are in the Sobolev space $H_{\gamma+1/2}(T^3)$. For small values of the regularizing parameter $\alpha$ the bounds $\varphi(t)$ and $t_*$ are uniform in $\alpha$ (note that $q$ is independent of sufficiently small $\alpha$).

## 6. Concluding remarks

We have explored a new approach to derivation of inequalities for Gevrey–Sobolev norms of solutions to evolutionary partial differential equations, in which a suitable nonlinear transformation of a solution in the Fourier space introduces a feedback between the norm of the transformed solution and the first index of the norm (2.6). Several examples of application of our technique were discussed.

We have proved that if initially a three-dimensional flow is analytic, then analyticity in spatial variables is preserved by the Euler equation on the interval $[0, t_*)$ (2.16)—this is implied by the bound (2.19) for Gevrey–Sobolev norms of a solution. (Alternatively, a bound for a Gevrey–Sobolev norm of the solution can be obtained [34] from finite-time bounds for solutions to the Voigt regularization, which are uniform in the small parameter in the regularizing term ($\alpha$ in (4.1)); such uniform bounds can be derived [34] by application of the method similar to the one used in [21].)

Pivotal to our technique is transformation (2.9), which results in emergence of a new term,

$$\beta\varepsilon|\mathbf{n}|\,\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2}^{-1-\varepsilon}\,\widehat{\mathbf{w}}_{\mathbf{n}}\,\frac{d}{dt}\|\mathbf{w}(\mathbf{x},t)\|_{s+3/2},$$

in the equations (2.10) and (3.2) governing the evolution of a solution (more precisely, of its transformed Fourier coefficients) to the Euler and Burgers equations. This term represents a new pseudodifferential mildly diffusive operator, controlling the norms in $H_{s+3/2}$; it is analogous to the term $-\dot{\tau}(t)\|(-\nabla^2)^{r+1/2}\mathbf{w}\|^2$ appearing in literature on analyticity of solutions to PDE's of hydrodynamic type [17, 22]. (Our assumption $s \leq 1/2$ is technical; similar bounds can be obtained in other Sobolev spaces for larger $s$ in a slightly different way.) The order of this operator cannot be increased without acquiring difficulties in deriving bounds for the exponent in the energy balance equation (2.11). Evidently, by the same construction one can derive an identical bound on the interval $[0, t_*)$ for solutions to the Navier–Stokes equation. This bound is uniform in viscosity and can be employed for a study of convergence of solutions to the Navier–Stokes equation to the solution to the Euler equation when viscosity tends to zero (such a study is beyond the scope of the present paper).

Solutions to the Burgers equation are known to develop shocks at finite time [14] due to intersection of characteristics; this rules out global (in time) analyticity in spatial variables of its solutions. That our bounds for the Gevrey–Sobolev norms of solutions to the Euler and Burgers equations are identical suggests that the bounds are rough. Other nonlinear transformations of solutions can probably yield more accurate bounds. A clear drawback of our technique is associated with its relative simplicity: we cannot use it to demonstrate persistence of analyticity of solutions to the Euler equation, i.e., the fact that the solution is analytic on any interval $0 \leq t < t_c$, where it is continuously differentiable [2] (persistence of non-analytic Gevrey-class regularity of space-periodic solutions to the Euler equation is shown in [17]).

We have also investigated the Voigt-type regularizations (4.1) of the Euler equation, involving the regularizing term $\alpha^2\partial/\partial t\,(-\nabla^2)^s\mathbf{v}$ and proved boundedness of Gevrey–Sobolev norms of solutions to the regularized Euler equation for $s > 5/6$. Thus, a regularization, milder than the one investigated in [21] (for $s = 1$), suffices to guarantee global regularity and analyticity of the solutions. Our technique enables us to decrease the order of nonlinearity of the energy-type inequality for the norm of the transformed solution; as a result, our bounds are global in time and exhibit a polynomial or, at most, exponential growth in time.

Finally, we have explored the Voigt-type regularization (5.1) of the Navier–Stokes equation with the regularizing term for $s > 1/2$, which is milder than the regularization considered in [16] ($s = 1$) and the regularization of the Euler equation ($s > 5/6$) studied here. We have obtained global exponential-in-time bounds for Gevrey–Sobolev norms. For $s = 1/2$, both the method [11] and our technique can be applied. Consequently, if the initial velocity is in the Sobolev space $H_{\gamma+1/2}(T^3)$, then the solution has a bounded Gevrey–Sobolev norm (5.18) on the time interval $[0, t_*)$ defined by (5.19). Furthermore, starting at any point $t_0$, $0 < t_0 < t_*$, we obtain a global (in time) bound (5.10) (performing optimization of the bounds in $t_*$ and other parameters is desirable). Unlike all other bounds for Gevrey–Sobolev bounds that we have derived in this paper, the bound for solutions to the Voigt-type regularization (5.1) of the Navier–Stokes equation for $s = 1/2$ does not deteriorate in time—neither the indices of the norm nor the right-hand side depend on time in inequality (5.10).

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