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ON THE WELL POSEDNESS OF A CLASS OF PDES INCLUDING POROUS MEDIUM AND CHEMOTAXIS EFFECT

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Abstract. In this article we deal with a class of degenerate parabolic systems which exhibit two phenomena: porous medium effects and chemotaxis ones. Such classes of equations arise in the modelling (on the mesoscale) of biomass spreading mechanisms via chemotaxis. Under certain "balance conditions" on the order of the porous medium degeneracy and the growth of the chemotactic functions, we prove well posedness.

1. Introduction

We will consider the following model:

$$M_t = \nabla\cdot(M^\alpha \nabla M) - \nabla\cdot(M^\gamma \nabla\rho) + f(M,\rho) \qquad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

$$\rho_t = \triangle\rho - g(M,\rho) \qquad \text{in } \Omega\times(0,\infty), \tag{1.2}$$

$$M = 0, \quad \rho = 1 \qquad \text{in } \partial\Omega\times(0,\infty), \tag{1.3}$$

$$M(\cdot,0) = M_0, \quad \rho(\cdot,0) = \rho_0 \qquad \text{in } \Omega, \tag{1.4}$$

where α and γ are given constants, satisfying $0 \le \alpha \le 2(\gamma-1)$ and $\alpha \ge \gamma+1$. Moreover, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain ($N = 1,2,3$), and $\rho_0, M_0 \ge 0$. We assume that the functions f and g satisfy

$$-f_1 M^2 \le f(M,\rho) \le f_2 M - f_3 M^2 \quad \text{for } M \ge 0,\ \rho \ge 0 \tag{1.5}$$

$$g(M,\rho) = g_0(\rho)M + g_1\rho \quad \text{for } M \ge 0,\ \rho \ge 0 \tag{1.6}$$

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$$0 \le g_0(\rho) \le g_2 \quad \text{for } \rho \ge 0 \tag{1.7}$$
$$|g_0'(\rho)| \le g_3 \quad \text{for } \rho \ge 0,\ g_0(0) = 0 \tag{1.8}$$
$$\tilde{f}(M,\rho) = f(M^{\frac{2}{\alpha+2}}, \rho) - f_4 M^{\frac{2}{\alpha+2}} \text{ is a } C^1\text{- function,} \tag{1.9}$$

where $f_1, f_2, f_3, f_4, g_0, g_1, g_2, g_3$ are positive constants.

Functions f and g satisfying the conditions (1.5) - (1.9) are for example the following.

Example 1.

$$f(M,\rho) = M - \frac{M^{\frac{\alpha}{2}+3}}{1+M^{\frac{\alpha}{2}+1}} \ \text{or}\ M - \frac{M^{\frac{\alpha}{2}+3}}{1+M^{\frac{\alpha}{2}+1}}(\rho+1),$$
$$g(M,\rho) = \frac{\rho}{\rho+1}M + \rho.$$

In the present paper, we treat weak solutions of the system (1.1)-(1.4). The definition is as follows.

Definition 1.1. *For $T > 0$, $\alpha > 1$ and $\gamma > 1$, a pair of non-negative functions (M,ρ) defined in $\Omega \times [0,T)$ is said to be a weak solution of* (1.1)-(1.4) *for M_0, $\rho_0 \in C^1(\overline{\Omega})$, if*

(i) *$M \in L^\infty(0,T;L^2(\Omega))$, $M^\alpha \in L^2(0,T;H^1(\Omega))$, and $M_t \in C_w([0,T];H^{-1}(\Omega))$,*

(ii) *$\rho \in C([0,T];H^1(\Omega))$,*

(iii) *(M,ρ) satisfies equations* (1.1)-(1.4) *in the following sense:*

$$\int_\Omega (M_t - \nabla\cdot(M^\alpha \nabla M) + \nabla\cdot(M^\gamma \nabla\rho) - f(M,\rho))\,\varphi dx = 0$$

for any $\varphi \in \mathcal{D}(\Omega)$, and

$$\begin{aligned}(\rho(x,t)-1) &= \int_\Omega G(x,y,t)(\rho_0(y)-1)dy \\ &- \int_0^t\int_\Omega G(x,y,t-s)g(M(y,s),\rho(y,s))dyds\end{aligned}$$

for any $(x,t) \in \Omega \times (0,T)$, where G is a heat kernel in Ω with the homogeneous Dirichlet boundary condition.

This system of partial differential equations models, for example, an abstract population, described by its population density M, which grows independent of a substrate with concentration ρ. The substrate is degraded by the population as well as by an abiotic decay. The spatial movement of the population is caused by two different effects. Firstly, the model includes a density dependent diffusion term. This nonlinear diffusion effect becomes stronger the more the population is locally concentrated, following a power law as in the case of the porous medium equation. Secondly, the population moves towards regions with increased substrate availability, i.e., follows the chemical signal ρ. This effect is also controlled by the population density and its intensity increases as the local population density grows. Both effects of population mobility vanish for vanishing populations and increase for increasing populations, each obeying a power law. Thus, the model degenerates for $M = 0$.

The study of this equation is motivated by some previous work of the first author (see [3]). Indeed, in [3], we dealt with biomass spatial spreading mechanisms via nonlinear diffusion (modelling in meso-scale), which led to a reaction-diffusion equation comprising simultaneously two kinds of degeneracy: porous medium and fast diffusion. More precisely, in [3] we studied the existence, uniqueness, the dependence on boundary conditions of global existence in time of the solution, and the existence of a global attractor for the associated semigroup. It is worth noting that numerical simulation of this equation leads to the mushroom-shaped patterns observed in the experimental study (see [1]).

The structure of the global attractor which captures all the dynamics (long time behavior of solutions) was given in [2], which shows, in particular, the dominant role of the order of the porous medium degeneracy over fast diffusion for the spatial spreading mechanism of the biomass. Therefore we simplify the nonlinear diffusion term and include chemotaxis (see also [4] and the references therein).

The main aim of the present study is to prove the well posedness of (1.1)–(1.4). We emphasize that the analysis of chemotaxis equations even without degeneracy, that is, $\alpha = 0$, is quite difficult, see [6, 7, 8, 9, 10] and the references therein, and in proving the well posedness in our degenerate case, we meet significant difficulties. To overcome these difficulties we give so-called "balance" conditions between the order of porous-medium degeneracy and the growth order of the chemotaxis function.

The main result of this paper is Theorem 3.1 which states that if the functions f and g satisfy assumptions (1.5)–(1.9) and the given constants α and γ satisfy $0 \leq \alpha \leq 2(\gamma - 1)$ and $\alpha \geq \gamma + 1$, then the initial boundary-value problem (1.1)–(1.4) has at most one solution.

The paper is organized as follows. In Section 2 we obtain several a priori estimates for the solutions of (1.1)-(1.4), which in turn lead to L^∞-bounds for the biomass component. Section 3 is devoted to the uniqueness of solutions. In the appendix we present (for the convenience of the reader) some standard ideas we used proving the well posedness of solutions.

2. Global existence and boundedness

In this section we obtain several a priori estimates for the solutions of (1.1)-(1.4), which in turn lead to L^∞-bounds both for $M(t,x)$ and $\rho(t,x)$. Existence of a solution of (1.1)-(1.4) will be obtained by using standard approximation procedures given in the Appendix. To this end, we multiply (1.1) by $\left(\frac{1}{\alpha+1-\gamma}M^{\alpha+1-\gamma}-\rho\right)$ and integrate over Ω. Then, we obtain

$$\mathrm{I} := \int_\Omega M_t\Big(\frac{M^{\alpha+1-\gamma}}{\alpha+1-\gamma}-\rho\Big)dx = \mathrm{II}+\mathrm{III}, \tag{2.1}$$

where by II and III we denote the integrals

$$\begin{aligned} \mathrm{II} &:= \int_\Omega \Big\{\nabla\cdot M^\gamma \nabla\Big(\frac{M^{\alpha+1-\gamma}}{\alpha+1-\gamma}-\rho\Big)\Big\}\cdot\Big(\frac{M^{\alpha+1-\gamma}}{\alpha+1-\gamma}-\rho\Big)dx \\ \mathrm{III} &:= \int_\Omega f(M,\rho)\Big(\frac{M^{\alpha+1-\gamma}}{\alpha+1-\gamma}-\rho\Big)dx \end{aligned}$$

respectively. Note that

$$\begin{aligned} \mathrm{I} &= \frac{1}{(\alpha+2-\gamma)(\alpha+1-\gamma)}\frac{d}{dt}\int_\Omega M^{\alpha+2-\gamma}dx - \frac{d}{dt}\int_\Omega M\rho dx + \int_\Omega M\rho_t dx \\ &= \mathrm{IV}+\mathrm{V}, \end{aligned} \tag{2.2}$$

where by IV and V we denote the integrals

$$\begin{aligned} \mathrm{IV} &:= \frac{d}{dt}\int_\Omega \Big(\frac{1}{(\alpha+2-\gamma)(\alpha+1-\gamma)}M^{\alpha+2-\gamma}-M\rho\Big)dx \\ \mathrm{V} &:= \int_\Omega M\rho_t dx. \end{aligned}$$

Multiplying equation (1.2) by ρ_t and using $\rho_t|_{\partial\Omega}=0$ we get

$$\int_\Omega |\rho_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\rho|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 dx \le \int_\Omega \{g_2M+(g_1+1)|\rho|\}|\rho_t|dx.$$

Hence,

$$\frac{1}{2}\int_\Omega |\rho_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla\rho|^2+\rho^2)dx \le g_2^2\int_\Omega M^2dx + (g_1+1)^2|\Omega|\,\|\rho\|_{L^\infty}^2. \tag{2.3}$$

From (1.5) it follows that

$$f_2 M - f_3 M^2 \le F_2 - F_3 M^2$$

for $M \ge 0$, where F_2 and F_3 are positive constants. Hence,

$$\begin{aligned}
\mathrm{III} \le& \int_\Omega (F_2 - F_3 M^2) \frac{M^{\alpha+1-\gamma}}{\alpha+1-\gamma} dx + f_1 \int_\Omega M^2 \rho dx \\
\le& -\frac{F_3}{2(\alpha+1-\gamma)} \int_\Omega M^{\alpha+3-\gamma} dx + C(\|\rho\|_\infty^3 + 1).
\end{aligned}$$

In fact, since we assume that $\alpha \ge \gamma + 1$, we have

$$\begin{aligned}
\int_\Omega M^2 \rho dx \le& \Big(\int_\Omega M^3 dx\Big)^{\frac{2}{3}} \Big(\int_\Omega \rho^3 dx\Big)^{\frac{1}{3}} \le \frac{2}{3} \int_\Omega M^3 dx + \frac{1}{3} \|\rho\|_\infty^3 \\
\le& \frac{F_3}{4(\alpha+1-\gamma)} \int_\Omega M^{\alpha+3-\gamma} dx + \frac{1}{3} \|\rho\|_\infty^3 + C, \\
\int_\Omega M^{\alpha+1-\gamma} dx \le& \Big(\int_\Omega M^{\alpha+3-\gamma} dx\Big)^{\frac{\alpha+1-\gamma}{\alpha+3-\gamma}} \Big(\int_\Omega 1 dx\Big)^{\frac{2}{\alpha+3-\gamma}} \\
\le& \frac{F_3}{4(\alpha+1-\gamma)} \int_\Omega M^{\alpha+3-\gamma} dx + C. \qquad (2.4)
\end{aligned}$$

Using the comparison theorem and (1.6) and (1.7) we obtain

$$0 \le \rho \le \max(\|\rho_0\|_\infty, 1). \qquad (2.5)$$

By (2.1), (2.2) and (2.4), we have

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \Big(\frac{1}{(\alpha+2-\gamma)(\alpha+1-\gamma)} M^{\alpha+2-\gamma} - M\rho\Big) dx \\
&\le -\int_\Omega M^\gamma \Big|\nabla\Big(\frac{M^{\alpha+1-\gamma}}{\alpha+1-\gamma} - \rho\Big)\Big|^2 dx - \int_\Omega M \rho_t dx \\
&\quad - \frac{F_3}{2(\alpha+1-\gamma)} \int_\Omega M^{\alpha+3-\gamma} dx + C.
\end{aligned}$$

For any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$M^{\alpha+2-\gamma} \le \varepsilon M^{\alpha+3-\gamma} + C\varepsilon \quad \text{ for } M \ge 0.$$

Combining these inequalities with (2.3), we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \Big(\frac{M^{\alpha+2-\gamma}}{(\alpha+2-\gamma)(\alpha+1-\gamma)} - M\rho\Big)\, dx \\
&\le (\tfrac{1}{2} + g_2^2) \int_\Omega M^2 dx - \frac{F_3}{2(\alpha+1-\gamma)} \int_\Omega M^{\alpha+2-\gamma} dx + C, \qquad (2.6)
\end{aligned}$$

where C is a positive constant independent of t. (C may change from one occurrence to another). It follows from (2.5) that

$$\begin{aligned}\int_\Omega M\rho dx &\leq \max(\|\rho_0\|_\infty, 1)\int_\Omega M dx \\ &\leq \frac{1}{2(\alpha+2-\gamma)(\alpha+1-\gamma)}\int_\Omega M^{\alpha+2-\gamma}dx + C.\end{aligned} \tag{2.7}$$

We have

$$\left(\tfrac{1}{2}+g_2^2\right)M^2 \leq \frac{F_3}{4(\alpha+1-\gamma)}M^{\alpha+2-\gamma}+C$$

by $\alpha \geq \gamma+1$. Then, we observe that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (M^{\alpha+2-\gamma}-(\alpha+2-\gamma)(\alpha+1-\gamma)M\rho)dx \\ &\leq -\frac{F_3(\alpha+2-\gamma)}{4}\int_\Omega (M^{\alpha+2-\gamma}-(\alpha+2-\gamma)(\alpha+1-\gamma)M\rho)dx + C,\end{aligned}$$

or

$$\int_\Omega (M^{\alpha+2-\gamma}-(\alpha+2-\gamma)(\alpha+1-\gamma)M\rho)dx \leq C \quad \text{for } t \geq 0.$$

By these and (2.7), we get

$$\int_\Omega M^{\alpha+2-\gamma}dx \leq C \quad \text{for } t\geq 0. \tag{2.8}$$

Let

$$A = -\triangle + 1 \ \text{ in } \Omega \quad \text{with } \cdot\,|_{\partial\Omega} = 0. \tag{2.9}$$

Then

$$\begin{aligned}(\rho-1)_t =& \triangle(\rho-1)-(\rho-1)-\{g_0(\rho)M+g_1\rho-(\rho-1)\},\\ e^t(\rho-1) =& e^{-At}(\rho_0-1)-\int_0^t e^{-A(t-s)}e^s\{g_0(\rho)M+g_1\rho-(\rho-1)\}ds.\end{aligned}$$

Hence, for $0<\varepsilon<1$,

$$\begin{aligned}&e^t\|A^{1-\varepsilon}(\rho-1)\|_{L^{\alpha-\gamma+2}}\\ &\leq \|A^{1-\varepsilon}e^{-At}(\rho_0-1)\|_{L^{\alpha-\gamma+2}}+\int_0^t e^s\|A^{1-\varepsilon}e^{-A(t-s)}g(M,\rho)\|_{L^{\alpha-\gamma+2}}ds.\end{aligned}$$

Consequently,

$$e^t\|A^{1-\varepsilon}(\rho-1)\|_{L^{\alpha-\gamma+2}} \leq \underbrace{C}_{\text{by } \rho_0\text{: smooth}} + \int_0^t e^s\|A^{1-\varepsilon}e^{-A(t-s)}g\|_{L^{\alpha-\gamma+2}}ds$$

$$\leq C + C\int_0^t e^s(t-s)^{-1+\varepsilon}(\|M\|_{L^{\alpha-\gamma+2}} + 1)ds \leq \frac{C}{\varepsilon}(1+e^t).$$

By $\alpha-\gamma+2 \geq 3$ and $D(A^{4/5}) \subset W^{1,6}(\Omega)$, the above inequality with $\varepsilon = 1/5$ leads to

$$\|\nabla(\rho-1)\|_{L^6} \leq C\|A^{4/5}(\rho-1)\|_{L^{\alpha-\gamma+2}} \leq C(\|M\|_{L^{\alpha-\gamma+2}} + 1). \tag{2.10}$$

Multiplying (1.1) by M^δ and integrating over Ω we obtain

$$\begin{aligned}
&\int_\Omega M_t M^\delta dx = \int_\Omega (\triangle M^{\alpha+1})M^\delta dx - \int_\Omega \nabla\cdot(M^\gamma\nabla\rho)M^\delta dx + \int_\Omega f(M,\rho)M^\delta dx \\
&\leq -\delta(\alpha+1)\int_\Omega M^{\alpha+\delta-1}|\nabla M|^2 + \frac{\delta}{\gamma+\delta}\int_\Omega \nabla\rho\cdot\nabla M^{\gamma+\delta}dx + \frac{f_2^2}{4f_3}\int_\Omega M^\delta dx \\
&= -\frac{4\delta(\alpha+1)}{(\alpha+\delta+1)^2}\int_\Omega |\nabla M^{\frac{\alpha+\delta+1}{2}}|^2 dx - \frac{\delta}{\gamma+\delta}\int_\Omega \nabla\rho\cdot\nabla M^{\gamma+\delta}dx + \frac{f_2^2}{4f_3}\int_\Omega M^\delta dx.
\end{aligned} \tag{2.11}$$

Based on the embedding theorem we obtain

$$\begin{aligned}
&\left(\int_\Omega M^{3(\alpha+\delta+1)}dx\right)^{\frac{1}{3}} \leq C_\delta\int_\Omega |\nabla M^{\frac{\alpha+\delta+1}{2}}|^2 dx, \\
&\frac{\delta}{\gamma+\delta}\left|\int_\Omega \nabla\rho\cdot\nabla M^{\gamma+\delta}dx\right| \leq C\|\nabla\rho\|_{L^6}\|M^{(2\gamma+\delta-1-\alpha)/2}\|_{L^3}\|\nabla M^{(\alpha+\delta+1)/2}\|_{L^2} \\
&\quad\leq \varepsilon\|\nabla M^{(\alpha+\delta+1)/2}\|_{L^2}^2 + \varepsilon\|M^{(2\gamma+\delta-1-\alpha)/2}\|_{L^3}^{2\theta} + C_{\varepsilon,\delta} \\
&\quad\leq \varepsilon\|\nabla M^{(\alpha+\delta+1)/2}\|_{L^2}^2 + \varepsilon|\Omega|^{(2\theta-1)/3}\|M^{\alpha+\delta+1}\|_{L^3} + C_{\varepsilon,\delta} \\
&\quad\leq \varepsilon(1+|\Omega|^{(2\theta-1)/3})\|\nabla M^{(\alpha+\delta+1)/2}\|_{L^2}^2 + C_{\varepsilon,\delta}
\end{aligned}$$

for sufficiently small $\varepsilon > 0$ and $\delta > 1$, where

$$\theta = \frac{2(\alpha+\delta+1)}{(2\gamma+\delta-1-\alpha)} \geq \frac{2(\alpha+\delta+1)}{(2\gamma+\delta-3)}.$$

Combining these and the Hölder inequality and using $\alpha \geq \gamma+1$ gives

$$\frac{d}{dt}\int_\Omega M^{\delta+1}dx + \int_\Omega M^{\delta+1}dx + \frac{1}{C_\delta}\int_\Omega |\nabla M^{(\alpha+\delta+1)/2}|^2 dx \leq C_\delta, \tag{2.12}$$

for some $C_\delta > 1$. Then, we have

$$\int_\Omega M^{\delta+1}dx + \int_0^t \frac{e^{s-t}}{C_\delta}\int_\Omega |\nabla M^{(\alpha+\delta+1)/2}|^2 dxds \leq C_\delta, \text{ for all } \delta \geq 1. \tag{2.13}$$

Taking $\delta = 6$ and using an argument similar to that of (2.10), we get

$$\|A^{4/5}(\rho - 1)\|_{L^7} \le C\Big(\sup_{0\le t}\|M(t)\|_{L^7} + 1\Big) \le C.$$

Since $D(A^{4/5}) \subset C^1(\overline{\Omega})$, it follows that

$$\|\nabla\rho\|_{L^\infty} \le C\Big(\sup_{t\ge 0}\|M(t)\|_{L^7} + 1\Big) \le C. \tag{2.14}$$

Then, we obtain

$$\begin{aligned}\Big|\int_\Omega \nabla\rho\cdot\nabla M^{\gamma+\delta}dx\Big| &\le C\int_\Omega \Big|\nabla M^{\gamma+\delta}\Big|\,dx \qquad (2.15)\\ &\le C\frac{2(\gamma+\delta)}{\gamma+\delta+1}\int_\Omega\Big|\nabla M^{(\gamma+\delta+1)/2}\Big|\,M^{\left(\frac{2(\gamma+\delta)}{\alpha+\delta+1}-1\right)\frac{\alpha+\delta+1}{2}}dx\\ &\le \frac{\delta(\alpha+1)}{(\alpha+\delta+1)^2}\int_\Omega\Big|\nabla M^{(\gamma+\delta+1)/2}\Big|^2dx + C(1+\delta)\Big(\int_\Omega M^{\alpha+\delta-3}dx + 1\Big).\end{aligned}$$

Combining the embedding formula

$$\|u\|_{L^2} \le C\left(\|\nabla u\|_{L^2} + \|u\|_{L^2}\right)^{N/(N+3)}\cdot\|u\|_{L^{6/5}}^{3/(N+3)}$$

with $\|u\|_{L^{6/5}} \le |\Omega|^{1/3}\|u\|_{L^2}$, we get

$$\|u\|_{L^2} \le C\left(\|\nabla u\|_{L^2} + \|u\|_{L^2}\right)^{1/2}\|u\|_{L^{6/5}}^{1/2}.$$

Then, we obtain

$$\begin{aligned}&\int_\Omega M^{\alpha+\delta+1}dx \qquad (2.16)\\ &\le C\Big(\int_\Omega|\nabla M^{(\alpha+\delta+1)/2}|^2dx + \int_\Omega M^{\alpha+\delta+1}dx\Big)^{\frac12}\Big(\int_\Omega M^{3(\alpha+\delta+1)/5}dx\Big)^{\frac56}.\end{aligned}$$

Hence,

$$\begin{aligned}C\int_\Omega M^{\alpha+\delta-3}dx &\le C\Big(\int_\Omega M^{\alpha+\delta+1}dx + 1\Big) \qquad (2.17)\\ &\le \frac{\delta(\alpha+1)}{(\alpha+\delta+1)^2}\int_\Omega|\nabla M^{(\alpha+\delta+1)/2}|^2dx + C(1+\delta)\Big(\int_\Omega M^{3(\alpha+\delta+1)/5}dx\Big)^{\frac53} + C\\ &\le \frac{\delta(\alpha+1)}{(\alpha+\delta+1)^2}\int_\Omega|\nabla M^{(\alpha+\delta+1)/2}|^2dx\\ &\quad + C(1+\delta)\Big(\int_\Omega M^{4(\delta+1)/5}dx\Big)^{\frac54}\Big(\int_\Omega M^{12\alpha/5}dx\Big)^{\frac{5}{12}} + C.\end{aligned}$$

Using an argument similar to the above, we get

$$C\int_\Omega M^\delta dx \le \frac{\delta(\alpha+1)}{(\alpha+\delta+1)^2}\int_\Omega |\nabla M^{(\alpha+\delta+1)/2}|^2 dx + C(1+\delta)\Big(\int_\Omega M^{4(\delta+1)/5}dx\Big)^{5/4}\Big(\int_\Omega M^{12\alpha/5}dx\Big)^{5/12}+C.$$

It follows from (2.12) with $\delta=(12/5)\alpha-1$ that

$$\sup_{0\le t}\int_\Omega (M(x,t))^{12\alpha/5}\,dx<\infty. \tag{2.18}$$

Combining these estimates with (2.15) and (2.11), we have

$$\frac{d}{dt}\int_\Omega M^{\delta+1}dx+\frac{\delta(\delta+1)(\alpha+1)}{(\alpha+\delta+1)^2}\int_\Omega |\nabla M^{(\alpha+\delta+1)/2}|^2 dx \le C(1+\delta^2)\Big(\int_\Omega M^{4(\delta+1)/5}dx+1\Big)^{5/4}.$$

By this and (2.18), we have

$$\int_\Omega M^{\delta+1}dx \le \frac{\delta(\alpha+1)}{(\alpha+\delta+1)^4}\int_\Omega \Big|\nabla M^{(\alpha+\delta+1)/2}\Big|^2 dx + C\Big(\int_\Omega M^{4(\delta+1)/5}dx\Big)^{\frac{5}{4}}+C.$$

This implies

$$\frac{d}{dt}\Big(\int_\Omega M^{\delta+1}dx+1\Big)+\Big(\int_\Omega M^{\delta+1}dx+1\Big) \le C(1+\delta^2)\Big(\int_\Omega M^{4(\delta+1)/5}dx+1\Big)^{5/4},$$

or

$$\sup_{0\le t}\Big(\int_\Omega M^{\delta+1}dx+1\Big) \le \Big(\int_\Omega M_0^{\delta+1}dx+1\Big)+C(1+\delta)^2\sup_{0\le t}\Big(\int_\Omega M^{4(\delta+1)/5}dx+1\Big)^{5/4}.$$

We set $\delta=2(5/4)^n-1$,

$$A(n):=\sup_{0\le t}\Big(\int_\Omega M^{2(5/4)^n}dx+1\Big)$$

and $B:=\|M_0\|_{L^\infty}|\Omega|+1$. We can show

$$(a+b)^{5/4}\le \tfrac{5}{4}\big(a^{5/4}+b^{5/4}\big) \quad \text{for } a\ge 0 \quad \text{and} \quad b\ge 0.$$

This inequality entails that for $n = 1, 2, 3, \cdots$

$$(a+b)^{(5/4)^n} \le \left(\tfrac{5}{4}\right)^{1+(\frac{5}{4})+(\frac{5}{4})^2+\cdots+(\frac{5}{4})^{n-1}} \left(a^{(5/4)^n}+b^{(5/4)^n}\right) \text{ for } a \ge 0 \text{ and } b \ge 0.$$

Then we get

$$\begin{aligned}
A(n) &\le B^{2(\frac{5}{4})^n} + C(\tfrac{5}{4})^{2n} A(n-1)^{5/4} \\
&\le B^{2(\frac{5}{4})^n} + C(\tfrac{5}{4})^{2n}\Big(B^{2(\frac{5}{4})^{n-1}} + C(\tfrac{5}{4})^{2(n-1)} A(n-2)^{5/4}\Big)^{5/4} \\
&\le B^{2(\frac{5}{4})^n} + \{[C(\tfrac{5}{4})^{2n}](\tfrac{5}{4})\} B^{2(\frac{5}{4})^n} \\
&\quad + \{[C(\tfrac{5}{4})^{2n}](\tfrac{5}{4})\}[C(\tfrac{5}{4})^{2(n-1)}]^{(\frac{5}{4})} A(n-2)^{(\frac{5}{4})^2} \\
&\le B^{2(\frac{5}{4})^n} + \{[C(\tfrac{5}{4})^{2n}](\tfrac{5}{4})\} B^{2(\frac{5}{4})^n} + \cdots \\
&+ \prod_{k=0}^{n-2} \Big\{ \Big[C(\tfrac{5}{4})^{2(n-k)}\Big]^{(\frac{5}{4})^k} (\tfrac{5}{4})^{1+(\frac{5}{4})+\cdots+(\frac{5}{4})^k} \Big\} B^{2(\frac{5}{4})^n} \\
&+ \prod_{k=0}^{n-2} \Big\{ \Big[C(\tfrac{5}{4})^{2(n-k)}\Big]^{(\frac{5}{4})^k} (\tfrac{5}{4})^{1+(\frac{5}{4})+\cdots+(\frac{5}{4})^k} \Big\} \Big[C(\tfrac{5}{4})^2\Big]^{(\frac{5}{4})^{n-1}} A(0)^{(\frac{5}{4})^n} \\
&\le n B^{2(\frac{5}{4})^n} \Big\{ \prod_{k=0}^{n-2} \Big[C(\tfrac{5}{4})^{2(n-k)}\Big]^{(\frac{5}{4})^k} \Big\} \Big\{ (\tfrac{5}{4})^{\sum_{k=1}^{n-1}(n-k)(\frac{5}{4})^{k-1}} \Big\} \\
&+ A(0)^{(\frac{5}{4})^n} \Big\{ \prod_{k=0}^{n-1} \Big[C(\tfrac{5}{4})^{2(n-k)}\Big]^{(\frac{5}{4})^k} \Big\} \Big\{ (\tfrac{5}{4})^{\sum_{k=1}^{n-1}(n-k)(\frac{5}{4})^{k-1}} \Big\}.
\end{aligned}$$

Consequently,

$$\sup_{0 \le t} \|M(t)\|_{L^{2(5/4)^n}} \le n^{\frac{1}{2}\left(\frac{4}{5}\right)^n} C^{8/5} \left(\tfrac{5}{4}\right)^{2\cdot 3\cdot 4^2/5} B + C^2 \left(\tfrac{5}{4}\right)^{3\cdot 4^2} A(0).$$

As $n \to \infty$, this leads to

$$\sup_{0 \le t} \|M(t)\|_{L^\infty} < \infty.$$

This proves the L^∞-bounds for $M(t,x)$.

To show well posedness it remains to prove the uniqueness of the solutions.

3. Uniqueness

Let $(M_1, \rho_1), (M_2, \rho_2)$ be bounded solutions to (1.1)-(1.4) in $\Omega \times (0, \infty)$ with $\rho_1(0) = \rho_2(0)$, $M_1(0) = M_2(0)$. Then

$$\begin{aligned}(M_1 - M_2)_t =& \frac{1}{\alpha+1} \triangle (M_1^{\alpha+1} - M_2^{\alpha+1}) - \nabla \cdot (M_1^\gamma \nabla \rho_1 - M_2^\gamma \nabla \rho_2) \\ &+ f(M_1, \rho_1) - f(M_2, \rho_2). \end{aligned} \tag{3.1}$$

Multiplying the equation (3.1) by $(-\triangle)^{-1}(M_1 - M_2)$ and integrating over Ω we obtain

$$\begin{aligned}&\int_\Omega (M_1 - M_2)_t (-\triangle)^{-1} (M_1 - M_2) dx \\ &= -\frac{1}{\alpha+1} \int_\Omega (M_1^{\alpha+1} - M_2^{\alpha+1})(M_1 - M_2) dx \\ &\quad + \int_\Omega (M_1^\gamma \nabla \rho_1 - M_2^\gamma \nabla \rho_2)(-\triangle)^{-\frac{1}{2}} (M_1 - M_2) dx \\ &\quad + \int_\Omega (f(M_1, \rho_1) - f(M_2, \rho_2))(-\Delta)^{-1}(M_1 - M_2) dx.\end{aligned}$$

Then, we let

$$\mathrm{VI} := \frac{1}{2} \frac{d}{dt} \int_\Omega |(-\triangle)^{-\frac{1}{2}} (M_1 - M_2)|^2 dx = -\mathrm{VII} + \mathrm{VIII} + \mathrm{IX} + \mathrm{X}, \tag{3.2}$$

where

$$\begin{aligned}\mathrm{VII} &:= \frac{1}{\alpha+1} \int_\Omega (M_1^{\alpha+1} - M_2^{\alpha+1})(M_1 - M_2) dx, \\ \mathrm{VIII} &:= \int_\Omega (M_1^\gamma - M_2^\gamma) \nabla \rho_1 \cdot (-\triangle)^{-\frac{1}{2}} (M_1 - M_2) dx, \\ \mathrm{IX} &:= \int_\Omega M_2^\gamma \nabla (\rho_1 - \rho_2)(-\triangle)^{-\frac{1}{2}} (M_1 - M_2) dx, \\ \mathrm{X} &:= \int_\Omega (f(M_1, \rho_1) - f(M_2, \rho_2))(-\Delta)^{-1} (M_1 - M_2) dx.\end{aligned}$$

We first estimate the integral VIII. We have

$$\mathrm{VIII} \leq \varepsilon \int_\Omega (M_1^\gamma - M_2^\gamma)^2 dx + C_\varepsilon \|\nabla \rho_1\|_\infty^2 \int_\Omega |(-\triangle)^{-\frac{1}{2}} (M_1 - M_2)|^2 dx.$$

Combining this with $\|M_1\|_{L^\infty}, \|M_2\|_{L^\infty} \leq C$ and using $\gamma \geq \frac{\alpha}{2} + 1$, we obtain

$$(M_1^\gamma - M_2^\gamma)^2 \leq C \left(\|M_1\|_{L^\infty}^{\gamma - \frac{\alpha}{2} - 1} + \|M_2\|_{L^\infty}^{\gamma - \frac{\alpha}{2} + 1} \right)^2 (M_1^{\frac{\alpha}{2}+1} - M_2^{\frac{\alpha}{2}+1})^2$$

$$\begin{aligned}
&\le \quad C\Big(\int_{M_2}^{M_1}\tau^{\frac{\alpha}{2}}d\tau\Big)^2 \le C\Big|\int_{M_2}^{M_1}\tau^{\alpha}d\tau\Big|\Big|\int_{M_2}^{M_1}d\tau\Big| \\
&= \quad C(M_1^{\alpha+1}-M_2^{\alpha+1})(M_1-M_2). \qquad (3.3)
\end{aligned}$$

Hence, the expression VIII can be estimated by (choosing ϵ sufficiently small)

$$\begin{aligned}
\mathrm{VIII} \le\ & \frac{1}{2}\int_\Omega (M_1^{\alpha+1}-M_2^{\alpha+1})(M_1-M_2)dx \qquad (3.4)\\
&+ C\|\nabla\rho_1\|_\infty^2\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx\\
&\le \frac{1}{2}\int_\Omega (M_1^{\alpha+1}-M_2^{\alpha+1})(M_1-M_2)dx + C\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx.
\end{aligned}$$

Here, we used the estimate (2.14). Next, multiplying the equation (1.2) by $\rho=\rho_1$ and $\rho=\rho_2$, respectively, and subtracting one from the other we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho_1-\rho_2)^2dx + \int_\Omega|\nabla(\rho_1-\rho_2)|^2dx + g_1\int_\Omega(\rho_1-\rho_2)^2dx\\
&= \int_\Omega (g_0(\rho_1)M_1 - g_0(\rho_2)M_2)(\rho_1-\rho_2)dx\\
&= \int_\Omega (g_0(\rho_1)-g_0(\rho_2))(\rho_1-\rho_2)M_1dx + \int_\Omega g_0(\rho_2)(M_1-M_2)(\rho_1-\rho_2)dx\\
&\le \underbrace{g_3\|M_1\|_{L^\infty}\int_\Omega|\rho_1-\rho_2|^2dx}_{\text{by } |g_0'(\rho)|\le g_3} + \frac{1}{4g_2^2}\int_\Omega|\nabla\{g_0(\rho_2)(\rho_1-\rho_2)\}|^2dx\\
&\quad + g_2^2\int_\Omega|(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx.
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\rho_1-\rho_2|^2dx + \frac{1}{2}\int_\Omega|\nabla(\rho_1-\rho_2)|^2dx\\
&\qquad\qquad \le C\int_\Omega|\rho_1-\rho_2|^2dx + C\int_\Omega|(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\frac{d}{dt}\left(e^{-Ct}\int_\Omega|\rho_1-\rho_2|^2dx\right) + \frac{1}{2}e^{-Ct}\int_\Omega|\nabla(\rho_1-\rho_2)|^2dx\\
&\qquad\qquad \le Ce^{-Ct}\int_\Omega|(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx.
\end{aligned}$$

Integrating the last inequality over $[0,t]$, we have

$$
\begin{aligned}
&e^{-Ct}\int_\Omega |\rho_1(t)-\rho_2(t)|^2dx+\int_0^t e^{-Cs}\int_\Omega |\nabla(\rho_1(s)-\rho_2(s))|^2dxds\\
&\le \int_0^t Ce^{-Cs}\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1(s)-M_2(s))|^2dxds+\underbrace{\int_\Omega |\rho_1(0)-\rho_2(0)|^2dx.}_{=0}
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int_0^t\int_\Omega (|\rho_1(s)-\rho_2(s)|^2+|\nabla(\rho_1(s)-\rho_2(s))|^2)dxds\\
&\quad\le Ce^{Ct}\int_0^t\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1(s)-M_2(s))|^2dxds. \qquad (3.5)
\end{aligned}
$$

The boundedness of M_2 guarantees

$$
\begin{aligned}
|\mathrm{IX}| &\le \|M_2\|^\gamma_{L^\infty}\frac{1}{2}\left(\int_\Omega |\nabla(\rho_1-\rho_2)|^2dx+\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx\right)\\
&\le C\left(\int_\Omega |\nabla(\rho_1-\rho_2)|^2dx+\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1-M_2)|^2dx\right).
\end{aligned}
$$

It follows from (1.9) that

$$
\begin{aligned}
&|(f(M_1,\rho_1)-f_4M_1)-(f(M_2,\rho_2)-f_4M_2)|=|\tilde f(M_1^{\frac{\alpha}{2}+1},\rho_1)-\tilde f(M_2^{\frac{\alpha}{2}+1},\rho_2)|\\
&\le C(|M_1^{\frac{\alpha}{2}+1}-M_2^{\frac{\alpha}{2}+1}|+|\rho_1-\rho_2|)
\end{aligned}
$$

Using these and an argument similar to that of (2.10), for $\varepsilon>0$ we obtain that

$$
\begin{aligned}
|\mathrm{X}| &= |\int_\Omega (f(M_1,\rho_1)-f(M_2,\rho_2))(-\Delta)^{-1}(M_1-M_2)dx|\\
&\le \epsilon\int_\Omega (M_1^{\frac{\alpha}{2}+1}-M_2^{\frac{\alpha}{2}+1})^2dx+C_\epsilon\int_\Omega(\rho_1-\rho_2)^2dx\\
&+C_\epsilon\int_\Omega |(-\Delta)^{-1}(M_1-M_2)|^2dx+f_4\int_\Omega (M_1-M_2)(-\Delta)^{-1}(M_1-M_2)dx\\
&\le \epsilon C\int_\Omega (M_1^{\alpha+1}-M_2^{\alpha+1})(M_1-M_2)dx\\
&+C_\epsilon\int_\Omega(\rho_1-\rho_2)^2dx+C_\epsilon\int_\Omega |(-\Delta)^{-\frac{1}{2}}(M_1-M_2)|^2dx, \qquad (3.6)
\end{aligned}
$$

where C_ε are positive constants depending on ε. Taking ϵ sufficiently small and combining with (3.2) and (3.4) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1 - M_2)|^2 dx \tag{3.7}$$
$$\leq C\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1 - M_2)|^2 dx + C\int_\Omega (\rho_1 - \rho_2)^2 + |\nabla(\rho_1 - \rho_2)|^2 dx.$$

Using this, (3.5) and the Poincaré inequality with $M_1(0) = M_2(0)$, we have

$$\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1(t) - M_2(t))|^2 dx$$
$$\leq C(1 + e^{Ct})\int_0^t \int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1(s) - M_2(s))|^2 dxds. \tag{3.8}$$

We let

$$R(t) := \int_0^t \int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1(s) - M_2(s))|^2 dxds.$$

Then $R(t)$ satisfies

$$R'(t) \leq Ce^{Ct}R(t), \quad \text{or} \quad R(t) \leq e^{e^{Ct}}R(0) = 0.$$

Combining this with (3.7) implies that

$$\int_\Omega |(-\triangle)^{-\frac{1}{2}}(M_1(t) - M_2(t))|^2 dx = 0 \text{ or } M_1(t) = M_2(t).$$

This proves uniqueness.

Summarizing the above, we can state the main result of the paper.

Theorem 3.1. *Let the functions f and g satisfy assumptions* (1.5)–(1.9) *and let the given constants α and γ satisfy $0 \leq \alpha \leq 2(\gamma - 1)$ and $\alpha \geq \gamma + 1$. Then the initial boundary-value problem* (1.1)–(1.4) *has at most one solution.*

4. Appendix (Existence of solution)

A1. In this part of Appendix, that is, in A1, we start with non-degenerate approximations of (1.1)-(1.2), namely:

$$M_{\varepsilon t} = \nabla(M_\varepsilon + \varepsilon)^\alpha \nabla M_\varepsilon - \nabla(M_\varepsilon + \varepsilon)^{\gamma-1} M_\varepsilon \nabla\rho_\varepsilon + f(M_\varepsilon, \rho_\varepsilon) \quad \text{in } \Omega \times (0,T). \tag{4.1}$$
$$\rho_{\varepsilon t} = \triangle\rho_\varepsilon - g(M_\varepsilon, \rho_\varepsilon) \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

where T is a positive constant. By using an argument similar to the one in [11], we get a unique classical non-negative solution to the above system, if

M_0 and ρ_0 are non-negative. We denote this solution by $(M_\varepsilon(t,x), \rho_\varepsilon(t,x))$. In the same manner as in Section 2 one can show that

$$||M_\varepsilon(t,x)||_{L^\infty(Q_T)} + ||\rho_\varepsilon(t,x)||_{L^\infty(Q_T)} + ||\nabla\rho_\varepsilon(t,x)||_{L^\infty(Q_T)} \le C, \qquad (4.3)$$

where the constant C is independent of ε. Using Appendix 2 and Appendix 3 (see below) we show that $(M_\varepsilon(t,x), \rho_\varepsilon(t,x))$ is a solution of (4.1)-(4.2) in the sense of Definition 1.1.

A2. In this section we show that any solution $(M_\varepsilon, \rho_\varepsilon)$ of (4.1), (4.2), (1.3) and (1.4) satisfies

$$\int_0^T \|\rho_{\varepsilon t}(t)\|^\delta_{L^\delta(\Omega)} dt, \quad \int_0^T \|\Delta\rho_\varepsilon(t)\|_{L^\delta(\Omega)} dt, \sup_{0\le t\le T} \|\nabla\rho_\varepsilon(t)\|_{L^\infty(\Omega)} dx \le C \qquad (4.4)$$

for any $\delta > 1$.

Proof. By the maximal regularity and (4.3), we can get the first and second estimates of (4.3) (see [5]). Recall $A = -\triangle + 1$ with $\cdot |_{\partial\Omega} = 0$. Then

$$\begin{aligned}(\rho_\varepsilon - 1)_t =& \triangle(\rho_\varepsilon - 1) - (\rho_\varepsilon - 1) - g_0(\rho_\varepsilon)M_\varepsilon - g_1\rho_\varepsilon + (\rho_\varepsilon - 1)\\ e^t(\rho_\varepsilon - 1) =& e^{-At}(\rho_0 - 1) - \int_0^t e^{-A(t-s)} e^s (g_0(\rho_\varepsilon)M_\varepsilon + g_1\rho_\varepsilon - (\rho_\varepsilon - 1))ds.\end{aligned}$$

Hence, for any $\delta \ge 1$,

$$\begin{aligned}e^t\|A^{3/4}(\rho_\varepsilon - 1)\|_{L^\delta} \le& \|A^{3/4}e^{-At}(\rho_0 - 1)\|_{L^\delta}\\ &+ \int_0^t e^s\|A^{3/4}e^{-A(t-s)}\{g((M_\varepsilon), \rho_\varepsilon) - (\rho_\varepsilon - 1)\}\|_{L^\delta} ds. \qquad (4.5)\end{aligned}$$

Consequently,

$$\begin{aligned}e^t\|A^{3/4}(\rho_\varepsilon - 1)\|_{L^\delta} \le& \underbrace{C}_{\text{by } \rho_0\text{: smooth}} + \int_0^t e^s\|A^{3/4}e^{-A(t-s)}g\|_{L^\delta} ds\\ \le& C + \int_0^t e^s(t-s)^{-3/4}C(\|M_\varepsilon\|_{L^{2\delta}} + 1)ds. \qquad (4.6)\end{aligned}$$

Combining this with the boundedness of M_ε, we have

$$\|A^{3/4}(\rho_\varepsilon - 1)\|_{L^\delta(\Omega)} \le C \qquad (4.7)$$

for any $\delta > 1$. For any sufficiently large $\delta > 1$, this entails

$$\|\nabla\rho_\varepsilon\|_{L^\infty(\Omega)} \le C \text{ for } t \in [0,T]. \qquad (4.8)$$

Then, we obtain (4.4).

A3. In this section we show that any solution $(M_\varepsilon, \rho_\varepsilon)$ of (4.1), (4.2), (1.3) and (1.4) satisfies $M_\varepsilon \in H^1((0,T); H^{-1}(\Omega))$, $\rho_\varepsilon \in C([0,T]; W^{1,\delta}(\Omega))$ and uniformly in ε satisfies

$$\int_0^T \int_\Omega |(M_\varepsilon + \varepsilon)_t^{(\alpha+2)/2}|^2 dxdt \le CT.$$

Proof. We multiply (4.1) by $[(M_\varepsilon + \varepsilon)^{\alpha+1}]_t$ and integrate over Ω. Noting that $M_{\varepsilon t} = 0$ on $\partial\Omega$ and that

$$\begin{aligned}|\nabla\{(M_\varepsilon+\varepsilon)^{\gamma-1}M_\varepsilon\}| &\le (\gamma-1)(M_\varepsilon+\varepsilon)^{\gamma-2}M_\varepsilon|\nabla M_\varepsilon| + (M_\varepsilon+\varepsilon)^{\gamma-1}|\nabla M_\varepsilon| \\ &\le \gamma(M_\varepsilon+\varepsilon)^{\gamma-1}|\nabla M_\varepsilon| = |\nabla(M_\varepsilon+\varepsilon)^\gamma|,\end{aligned}$$

we have

$$\begin{aligned}&(\alpha+1)\left(\frac{\alpha+2}{2}\right)^2 \int_\Omega \left|\left\{(M_\varepsilon+\varepsilon)^{(\alpha+2)/2}\right\}_t\right|^2 dx \\ &\qquad + \frac{1}{2(\alpha+1)}\frac{d}{dt}\int_\Omega |\nabla(M_\varepsilon+\varepsilon)^{\alpha+1}|^2 dx \\ &= -\int_\Omega \left(\nabla\cdot(M_\varepsilon+\varepsilon)^{\gamma-1}M_\varepsilon\nabla\rho_\varepsilon + f(M_\varepsilon,\rho_\varepsilon)\right)(M_\varepsilon^{\alpha+1})_t dx \\ &\le \frac{2\gamma}{2\gamma+\alpha}\frac{2(\alpha+1)}{\alpha+2}\int_\Omega \left|\nabla(M_\varepsilon+\varepsilon)^{(2\gamma+\alpha)/2}\right| |\nabla\rho_\varepsilon| \left|(M_\varepsilon+\varepsilon)_t^{(\alpha+2)/2}\right| dx \\ &\qquad + \frac{2(\alpha+1)}{\alpha+2}\int_\Omega \left\{(M_\varepsilon+\varepsilon)^{(2\gamma+\alpha)/2}|\Delta\rho_\varepsilon|\right\} \left|(M_\varepsilon+\varepsilon)_t^{(\alpha+2)/2}\right| dx \\ &\qquad + \int_\Omega f(M_\varepsilon,\rho_\varepsilon)\left|(M_\varepsilon+\varepsilon)_t^{(\alpha+2)/2}\right| dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\int_0^t \int_\Omega \left|\left\{(M_\varepsilon+\varepsilon)^{(\alpha+2)/2}\right\}_s\right|^2 dx + \int_\Omega |\nabla(M_\varepsilon+\varepsilon)^{\alpha+1}|^2 dx \\ &\le C\|\nabla\rho_\varepsilon\|_{L^\infty}^2 \int_0^t \int_\Omega \left|\nabla(M_\varepsilon+\varepsilon)^{(2\gamma+\alpha)/2}\right|^2 dxds \\ &\quad + C\int_0^t \|M_\varepsilon+\varepsilon\|_{L^\infty(\Omega)}^{(2\gamma+\alpha)}\|\Delta\rho_\varepsilon\|_{L^2}^2 ds + \int_0^t \int_\Omega |f(M_\varepsilon,\rho_\varepsilon)|^2 dxds.\end{aligned}$$

Combining this with (A2) implies that

$$\int_0^t \int_\Omega \left|\left\{(M_\varepsilon+\varepsilon)^{(\alpha+2)/2}\right\}_s\right|^2 dxds \le Ct. \tag{4.9}$$

We can get the same estimate as (2.12) for $M_\varepsilon + \varepsilon$. Then, $(M_\varepsilon + \varepsilon)^{\delta+1} \in C([0,T]; L^2(\Omega))$ and $(M_\varepsilon + \varepsilon)^{(\alpha+\delta+1)/2} \in L^2((0,T); H^1(\Omega))$ for $\delta \geq 1$. Since, for any $\delta \geq 1$, $\|M_\varepsilon + \varepsilon\|_{L^\delta}$ is bounded, then $\rho_\varepsilon \in C([0,T]; W^{1,\delta}(\Omega))$ for any $\delta \geq 1$, by standard arguments. Therefore, $((M_\varepsilon + \varepsilon)^{(\alpha+2)/2}, \rho_\varepsilon)$ is bounded in $H^1(\Omega \times (0,T))$. We deduce from this that there exists a subsequence of $\{((M_\varepsilon)^{(\alpha+2)/2}, \rho_\varepsilon)\}_{\varepsilon>0}$ converging to $(M^{(\alpha+2)/2}, \rho)$ almost everywhere in $\Omega \times (0,T)$ and in $L^2(\Omega \times (0,T))$. Combining this with (4.4) and (4.9), we obtain that for $t \in [0,T]$ the subsequence $\{((M_{\varepsilon(n)})^{(\alpha+2)/2}, \rho_{\varepsilon(n)})\}_{n\geq 1}$ converges to $(M^{(\alpha+2)/2}, \rho)$ in $L^2(\Omega)$, from which and (4.4) it follows that the sequence $\{\rho_{\varepsilon(n)})\}_{n\geq 1}$ converges to ρ in $W^{1,\delta}(\Omega)$ for $\delta \geq 2$. Since $(M_\varepsilon, \rho_\varepsilon)$ is a solution to (4.1) and (4.2), we infer that $\partial M_{\varepsilon(n)}/\partial t$ converges to M_t in $H^{-1}(\Omega)$ for $t \in [0,T]$, which entails $M_\varepsilon \in H^1((0,T); H^{-1}(\Omega))$. Thus we showed that any solution to (4.1)-(4.2), (1.3)-(1.4) satisfies uniformly in ε estimates (4.3), (4.8) and belongs to the class described in Definition 1.1. As a consequence we obtain that $(M, \rho) = \lim_{n\to\infty}(M_{\varepsilon(n)}, \rho_{\varepsilon(n)})$ is a solution of (1.1)-(1.4) in the sense of Definition 1.1.

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