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# WELL-POSEDNESS OF NONLINEAR PARABOLIC PROBLEMS WITH NONLINEAR WENTZELL BOUNDARY CONDITIONS

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**Abstract.** Of concern is the nonlinear parabolic problem with nonlinear dynamic boundary conditions

$$u_t + \operatorname{div}(F(u)) = \operatorname{div}(\mathcal{A}\nabla u), \qquad u(0,x) = f(x),$$
$$u_t + \beta\partial_\nu^{\mathcal{A}} u + \gamma(x,u) - q\beta\Delta_{\mathrm{LB}} u = 0,$$

for $x \in \Omega \subset \mathbb{R}^N$ and $t \geq 0$; the last equation holds on the boundary $\partial\Omega$. Here $\mathcal{A} = \{a_{ij}(x)\}_{ij}$ is a real, Hermitian, uniformly positive definite $N \times N$ matrix; $F \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ is Lipschitz continuous; $\beta \in C(\partial\Omega)$, with $\beta > 0$; $\gamma : \partial\Omega \times \mathbb{R} \to \mathbb{R}$; $q \geq 0$; and $\partial_\nu^{\mathcal{A}} u$ is the conormal derivative of $u$ with respect to $A$; everything is sufficiently regular. Here we prove the well-posedness of the problem. Moreover, we prove explicit stability estimates of the solution $u$ with respect to the coefficients $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, and the initial condition $f$. Our estimates cover the singular case of a problem with $q = 0$ which is approximated by problems with positive $q$.

## 1. Introduction

This paper deals with real solutions of the following nonlinear parabolic problem with nonlinear Wentzell (or dynamic) boundary conditions:

$$\begin{cases} u_t + \operatorname{div}(F(u)) = \operatorname{div}(\mathcal{A}\nabla u), & \text{in } (0,\infty)\times\Omega, \\ u(0,\cdot) = f, & \text{in } \Omega, \\ u_t + \beta\partial_\nu^{\mathcal{A}} u + \gamma(x,u) - q\beta\Delta_{\mathrm{LB}} u = 0, & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{1.1}$$

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We extend the existence and stability results obtained in [5, 6, 16], where the authors considered the problem corresponding to the choice $F \equiv 0$.

Such an initial boundary value problem can model a diffusion process, for example, the heat equation with a heat source on the boundary. When the heat source on the boundary depends nonlinearly on the heat flow across and the temperature on the boundary and the heat can transfer along the boundary, we obtain a boundary condition of the form in (1.1) (see [17] for a derivation of such boundary conditions). Many authors have considered parabolic problems with dynamic (and the related general Wentzell) boundary conditions (cf. [5, 6, 10, 11, 12, 13, 15, 16, 23, 24, 25, 26]).

In this paper we assume

*i)* $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded open set with $C^2$ boundary;

*ii)* $\mathcal{A} = \{a_{ij}\}_{ij} \in C^1(\overline{\Omega}; \mathbb{R}^{N\times N})$ is symmetric and uniformly elliptic; in particular, there exist two constants $\alpha_1 \geq \alpha_0 > 0$ such that

$$\alpha_0|\xi|^2 \leq \langle \mathcal{A}(x)\xi, \xi\rangle \leq \alpha_1|\xi|^2,$$

for each $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$;

*iii)* $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ is a Lipschitz-continuous vector field, and we set $L = \|F'\|_{L^\infty(\mathbb{R}^N)}$;

*iv)* $\beta \in C(\partial\Omega)$ and $\beta_1 \geq \beta(x) \geq \beta_0 > 0$, for some constants $\beta_1$, $\beta_0$, and every $x \in \partial\Omega$;

*v)* $0 \leq q < \infty$ is a given constant;

*vi)* $\Delta_{\mathrm{LB}}$ is the Laplace-Beltrami operator on $\partial\Omega$;

*vii)* $\partial_\nu^{\mathcal{A}}$ is the conormal derivative with respect to $\partial\Omega$, namely

$$\partial_\nu^{\mathcal{A}} u = \langle \mathcal{A}\nabla u, \nu\rangle = \sum_{ij} a_{ij}\partial_{x_i} u \nu_j,$$

where $\nu$ is the unit outer normal at $x \in \partial\Omega$;

*viii)* $f \in H^1(\Omega)$; concerning the trace of $f$ on $\partial\Omega$, when $q > 0$ we require that $\nabla_\tau f \in L^2(\partial\Omega)$, where $\nabla_\tau$ is the tangential gradient;

*ix)* $\gamma\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$, $\gamma(\cdot, 0) = 0$, for each $x \in \partial\Omega$ the function $\gamma(x, \cdot)$ is nondecreasing, and

$$|\gamma(x, u)| \leq \Gamma(|u|^p + 1) \tag{1.2}$$

for some positive constants $\Gamma$ and $p$ such that

$$\begin{cases} p \leq \frac{N}{N-2}, & \text{if } N \geq 3,\\ p < \infty, & \text{otherwise.}\end{cases} \tag{1.3}$$

All of the above coefficients are assumed to be real valued.

As a consequence of $ix)$ we have that

$$u \geq 0 \text{ implies } \gamma(x,u) \geq 0, \qquad u \leq 0 \text{ implies } \gamma(x,u) \leq 0, \tag{1.4}$$

and in particular

$$0 \leq u\gamma(x,u) \leq \Gamma(|u|^{p+1} + |u|), \tag{1.5}$$

$$0 \leq (u-v)(\gamma(x,u) - \gamma(x,v)), \tag{1.6}$$

$$0 \leq \int_0^u \gamma(x,\xi)d\xi \leq \Gamma\Big(\frac{|u|^{p+1}}{p+1} + |u|\Big). \tag{1.7}$$

The Sobolev embedding theorem and (1.3) imply[1]

$$H^1(\partial\Omega) \subset L^{p+1}(\partial\Omega), \quad H^1([0,t]\times\partial\Omega) \subset L^{2p}([0,t]\times\partial\Omega), \quad \text{for every } t > 0. \tag{1.8}$$

Therefore, due to (1.5) and (1.7),

$$g \in H^1(\partial\Omega) \quad \text{implies} \quad \gamma(\cdot,g)^2,\, g\gamma(\cdot,g),\, \int_0^g \gamma(\cdot,\xi)d\xi \in L^1(\partial\Omega). \tag{1.9}$$

Moreover, we observe that $\frac{N}{N-2} \geq 1$, for each $N \geq 3$; therefore our assumptions cover also the linear case treated in [5]. Finally, we point out that $\gamma$ is allowed to be discontinuous. In this case $\gamma(x,\cdot)$ determines a unique maximal monotone graph in the plane $\mathbb{R}^2$, which we may also denote by $\gamma(x,\cdot)$. Thus in the boundary condition (1.1) one may replace the inequality $(=)$ by the containment symbol $\ni$.

We show well-posedness and explicit stability estimates for (1.1). At first we prove the existence of a unique solution $u$ to (1.1) via a fixed-point argument. We continue by showing some regularity results for the solution of (1.1). Then we investigate the stability with respect to the coefficients $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, and the initial condition $f$, and we provide explicit estimates.

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[1] Here and in the rest of the paper, for the sake of notational simplicity, we write $L^{2p}$ even in the case $p < \frac{1}{2}$. We use the following identification:

$$L^{2p} = \begin{cases} L^{2p} & \text{if } p \geq \frac{1}{2}, \\ L^1 & \text{if } 0 < p < \frac{1}{2}. \end{cases}$$

Suppose we have a sequence of such problems, written in abbreviated form as

$$\begin{cases} u_{n,t} + \operatorname{div}(F_n(u_n)) = \operatorname{div}(\mathcal{A}_n \nabla u_n), & \text{in } (0,\infty)\times\Omega, \\ u_n(0,\cdot) = f_n, & \text{in } \Omega, \\ u_{n,t} + \beta_n \partial_\nu^{\mathcal{A}_n} u_n + \gamma_n(x,u_n) - q_n\beta_n\Delta_{\mathrm{LB}} u_n = 0, & \text{on } (0,\infty)\times\partial\Omega, \end{cases}$$

for $n = 1, 2, \ldots$, with hypotheses $i), \ldots, ix)$ holding for all $n \geq 1$. If $\beta_n \to \beta_0$, $\gamma_n \to \gamma_0$, $\mathcal{A}_n \to \mathcal{A}_0$, $F_n \to F_0$, and $f_n \to f_0$ uniformly, and if $q_n \to q_0$, then $u_n(t,\cdot) \to u_0(t,\cdot)$ (in various norms), uniformly for $t$ in bounded intervals of $[0,\infty)$. This is shown in [4] when $F_0 \equiv 0$, using the Neveu-Trotter-Kato approximation theorem for operator semigroups (cf. [14, 18, 20]). While this theorem gives $\|u_n(t,\cdot) - u_0(t,\cdot)\| \to 0$ for suitable norms, it does not give (except in special cases not including ours) a rate of convergence, which is coded in an inequality of the form

$$\|u_n(t,\cdot) - u_0(t,\cdot)\| \leq \omega_n(t),$$

where $\omega_n$ is an explicitly constructed function that goes to zero as $n \to \infty$ (e.g., $\omega_n(t) = K(T)n^{-\delta(t)}$ for positive constants $K(t)$ and $\delta(t)$, and for $0 \leq t \leq T < \infty$).

Such an estimate is based upon a comparison of two equations like (1.1) but with different choices of $\mathcal{A}$, $F$, $\beta$, $\gamma$, and $q$. To this end we will compare (1.1) with the additional boundary value problem

$$\begin{cases} v_t + \operatorname{div}(\widetilde{F}(v)) = \operatorname{div}(\widetilde{\mathcal{A}} \nabla v), & \text{in } (0,\infty)\times\Omega, \\ v(0,\cdot) = \widetilde{f}, & \text{in } \Omega, \\ v_t + \widetilde{\beta} \partial_\nu^{\widetilde{\mathcal{A}}} v + \widetilde{\gamma}(x,v) - \widetilde{q}\widetilde{\beta}\Delta_{\mathrm{LB}} v = 0, & \text{on } (0,\infty)\times\partial\Omega, \end{cases}$$

where the assumptions on $\widetilde{\mathcal{A}}$, $\widetilde{F}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{q}$, $\widetilde{f}$ and $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, $f$ are the same. We estimate the difference $u - v$ under all the possible choices of the coefficients $q$ and $\widetilde{q}$, namely, $[q \neq 0,\ \widetilde{q} \neq 0]$, $[q = \widetilde{q} = 0]$, and $[q \neq 0,\ \widetilde{q} = 0]$.

Our main result is the following.

**Theorem 1.1.** *Let $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, and $f$ satisfy the assumptions $i), \ldots, ix)$. The boundary value problem* (1.1) *admits a unique solution $u : [0,\infty)\times\overline{\Omega} \to \mathbb{R}$ such that for every $T > 0$*

$$\begin{aligned} u \in & L^\infty\Big((0,T); L^2(\Omega) \oplus L^2\Big(\partial\Omega, \frac{d\sigma}{\beta}\Big)\Big)\cap \\ & \cap \Big(H^1((0,T)\times\Omega) \oplus H^1((0,T)\times\partial\Omega)\Big) \cap L^2((0,T); H^2(\Omega)); \end{aligned}$$

*moreover, if* $q = 0$,

$$\partial_\nu^{\mathcal{A}} u \in L^2\Big((0,T) \times \partial\Omega, dt \times \frac{d\sigma}{\beta}\Big).$$

*In addition we assume that also* $\widetilde{\mathcal{A}}$, $\widetilde{F}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{q}$, *and* $\widetilde{f}$ *satisfy the assumptions* $i), \dots, ix)$ *and* $u$ *is the unique solution of* (1.1) *and* $v$ *is the unique one of* (1.1) *obtained by replacing the coefficients* $\mathcal{A}$, $F$, $\beta$, $\gamma$, *and* $q$ *by the coefficients* $\widetilde{\mathcal{A}}$, $\widetilde{F}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, *and* $\widetilde{q}$ *and with the initial condition* $\widetilde{f}$. *Then the following estimate holds:*

$$\begin{aligned}
&\|u(t,\cdot) - v(t,\cdot)\|^2_{L^2(\Omega)} + \|u(t,\cdot) - v(t,\cdot)\|^2_{L^2_\beta(\partial\Omega)} \\
&+ e^{\lambda t} \int_0^t e^{-\lambda s} \Big(\alpha_0 \|\nabla u(s,\cdot) - \nabla v(s,\cdot)\|_{L^2(\Omega)} \\
&\qquad +2q \|\nabla_\tau u(s,\cdot) - \nabla_\tau v(s,\cdot)\|^2_{L^2(\partial\Omega)}\Big)\, ds \\
&\quad \le e^{\lambda t} \Big( \Big\|f - \widetilde{f}\Big\|^2_{L^2(\Omega)} + \Big\|f - \widetilde{f}\Big\|^2_{L^2_\beta(\partial\Omega)} \Big) \\
&+ \Big[ \Big\|F - \widetilde{F}_{L^\infty(\mathbb{R}^N)}\Big\| + \sup_{x\partial\Omega,\, \xi\in\mathbb{R}} \Big| \frac{\gamma(x,\xi) - \widetilde{\gamma}(x,\xi)}{|\xi|^p + 1} \Big|^2 \\
&\qquad + \Big\|\mathcal{A} - \widetilde{\mathcal{A}}\Big\|^2_{L^\infty(\Omega)} + \Big\|\beta - \widetilde{\beta}\Big\|^2_{L^\infty(\partial\Omega)} + |q - \widetilde{q}| \Big] \times \\
&\quad \times K_0 e^{\lambda t} \Big( \Big[ M(\widetilde{f}, \widetilde{\beta}, \widetilde{q}) + M(f,\beta,q) + M(\widetilde{f}, \widetilde{\beta}, \widetilde{q})^p \Big] (1+t)^{\max\{1,p\}} \\
&\qquad + t|\partial\Omega| + t|\Omega| \big),
\end{aligned}$$

*for each* $t \ge 0$ *and some positive constants* $K_0$ *and* $\lambda$, *where* $|\partial\Omega|$ *is the* $(N-1)$*-dimensional Lebesgue surface measure of* $\partial\Omega$ *and*

$$L^2_\beta(\partial\Omega) = L^2\Big(\partial\Omega, \frac{d\sigma}{\beta}\Big), \tag{1.10}$$

$$\begin{aligned}
M(f,\beta,q) = {}& \|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)} \\
& + \|\nabla f\|^2_{L^2(\Omega)} + \|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)} + q \|\nabla_\tau f\|^2_{L^2(\partial\Omega)}.
\end{aligned} \tag{1.11}$$

We recall that analogous stability estimates were proved in [7, 8] for the Cauchy problem for a second- and a fourth-order equation of parabolic type, respectively.

The statement and the proof of Theorem 1.1 are split in those of Theorems 2.1, 3.1, and 4.1.

The paper is organized as follows. In Section 2 we prove the existence and uniqueness of solutions to (1.1), in Section 3 we prove some regularity result on those solutions, and in Section 4 we prove our stability estimate.

## 2. Existence and uniqueness

In this section we prove the first part of Theorem 1.1.

**Theorem 2.1.** *The initial boundary value problem* (1.1) *admits a unique solution* $u \in \mathcal{H}_{q,\infty}$, *where*

$$\mathcal{H}_{q,T} := \begin{cases} L^\infty\left((0,T); L^2(\Omega) \oplus L^2\left(\partial\Omega, \frac{d\sigma}{\beta}\right)\right) \cap \\ \quad \cap L^2((0,T); H^1(\Omega) \oplus H^1(\partial\Omega)), & \text{if } q \neq 0, \\ L^\infty\left((0,T); L^2(\Omega) \oplus L^2\left(\partial\Omega, \frac{d\sigma}{\beta}\right)\right) \cap \\ \quad \cap L^2((0,T); H^1(\Omega)), & \text{if } q = 0, \end{cases}$$

*and* $\mathcal{H}_{q,\infty} := \bigcup_{T>0} \mathcal{H}_{q,T}$.

We begin by proving the following local existence and uniqueness result.

**Lemma 2.1** (**Local Existence and Uniqueness.**). *Let* $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, *and* $f$ *satisfy the assumptions* $i), \dots, ix)$. *There exists a time* $T_0 > 0$ *such that the boundary value problem* (1.1) *admits a unique solution* $u \in \mathcal{H}_{q,T_0}$.

We use a fixed-point argument based on the contraction-mapping principle.

For every $u \in \mathcal{H}_{q,\infty}$, let $v = \mathcal{T}(u)$ be the unique solution of

$$\begin{cases} v_t + \operatorname{div}(F(u)) = \operatorname{div}(\mathcal{A}\nabla v), & \text{in } (0,\infty) \times \Omega, \\ v(0,\cdot) = f, & \text{in } \Omega, \\ v_t + \beta\partial_\nu^{\mathcal{A}} v + \gamma(x,v) - q\beta\Delta_{\mathrm{LB}} v = 0, & \text{on } (0,\infty) \times \partial\Omega. \end{cases} \tag{2.1}$$

See [16]. Clearly, $u$ solves (1.1) if and only if $u = \mathcal{T}(u)$. To prove this we need further analysis.

**Lemma 2.2.** *Let* $u \in \mathcal{H}_{q,\infty}$; *we have that* $\mathcal{T}(u) \in \mathcal{H}_{q,\infty}$. *Moreover, the following inequality holds:*

$$\begin{aligned} &\|\mathcal{T}(u)(t,\cdot)\|^2_{L^2(\Omega)} + \|\mathcal{T}(u)(t,\cdot)\|^2_{L^2_\beta(\partial\Omega)} \\ &\quad + 2e^t \int_0^t e^{-s}\Big(\alpha_0 \|\nabla\mathcal{T}(u)(s,\cdot)\|^2_{L^2(\Omega)} + q\,\|\nabla_\tau\mathcal{T}(u)(s,\cdot)\|^2_{L^2(\partial\Omega)}\Big) ds \end{aligned}$$

$$\le e^t\Big(\,\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)}\Big) + L^2 e^t \int_0^t e^{-s}\,\|\nabla u(s,\cdot)\|^2_{L^2(\Omega)}\,ds,$$

*for each* $t \ge 0$.

**Proof.** Set $v = \mathcal{T}(u)$. Using (2.1), (1.5), *ii*), *iii*), and the divergence theorem,

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega \frac{v^2}{2}dx + \int_{\partial\Omega}\frac{v^2}{2}\frac{d\sigma}{\beta}\Big) = \int_\Omega vv_t\,dx + \int_{\partial\Omega} vv_t\frac{d\sigma}{\beta}\\
&= \int_\Omega \operatorname{div}(\mathcal{A}\nabla v)v\,dx - \int_\Omega \operatorname{div}(F(u))v\,dx\\
&\quad - \int_{\partial\Omega} v\partial_\nu^{\mathcal{A}} v\,d\sigma - \int_{\partial\Omega}\frac{\gamma(x,v)v}{\beta}d\sigma + q\int_{\partial\Omega} v\Delta_{\mathrm{LB}}v\,d\sigma\\
&= -\int_\Omega \langle \mathcal{A}\nabla v, \nabla v\rangle dx - \int_\Omega \langle F'(u), \nabla u\rangle v\,dx\\
&\quad + \int_{\partial\Omega} v\partial_\nu^{\mathcal{A}} v\,d\sigma - \int_{\partial\Omega} v\partial_\nu^{\mathcal{A}} v\,d\sigma - \int_{\partial\Omega}\frac{\gamma(x,v)v}{\beta}d\sigma - q\int_{\partial\Omega}|\nabla_\tau v|^2 d\sigma\\
&\le -\alpha_0\int_\Omega |\nabla u|^2 dx + \frac{L^2}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega v^2 dx - q\int_{\partial\Omega}|\nabla_\tau v|^2 d\sigma\\
&\le -\alpha_0\int_\Omega |\nabla u|^2 dx + \frac{L^2}{2}\int_\Omega |\nabla u|^2 dx\\
&\quad + \Big(\int_\Omega \frac{v^2}{2}dx + \int_{\partial\Omega}\frac{v^2}{2}\frac{d\sigma}{\beta}\Big) - q\int_{\partial\Omega}|\nabla_\tau v|^2 d\sigma.
\end{aligned}$$

Therefore, Lemma 2.2 follows from the Gronwall inequality. □

**Lemma 2.3.** *Let* $u, \widetilde{u} \in \mathcal{H}_{q,\infty}$*; the following inequality is satisfied:*

$$\begin{aligned}
&\|\mathcal{T}(u)(t,\cdot) - \mathcal{T}(\widetilde{u})(t,\cdot)\|^2_{L^2(\Omega)} + \|\mathcal{T}(u)(t,\cdot) - \mathcal{T}(\widetilde{u})(t,\cdot)\|^2_{L^2_\beta(\partial\Omega)}\\
&\qquad + e^{\beta_1 t}\int_0^t e^{-\beta_1 s}\Big(\alpha_0\,\|\nabla\mathcal{T}(u)(s,\cdot) - \nabla\mathcal{T}(\widetilde{u})(s,\cdot)\|^2_{L^2(\Omega)}\\
&\qquad\qquad + 2q\,\|\nabla_\tau\mathcal{T}(u)(s,\cdot) - \nabla_\tau\mathcal{T}(\widetilde{u})(s,\cdot)\|^2_{L^2(\partial\Omega)}\Big)ds\\
&\quad \le L^2 e^{\beta_1 t}\int_0^t e^{-\beta_1 s}\Big(\frac{1}{\alpha_0}\,\|u(s,\cdot) - \widetilde{u}(s,\cdot)\|^2_{L^2(\Omega)}\\
&\qquad\qquad + \beta_1\,\|u(s,\cdot) - \widetilde{u}(s,\cdot)\|^2_{L^2_\beta(\partial\Omega)}\Big)ds,
\end{aligned}$$

*for every* $t \ge 0$.

**Proof.** Let us introduce the notation

$$v = \mathcal{T}(u), \qquad \widetilde{v} = \mathcal{T}(\widetilde{u}), \qquad w = v - \widetilde{v}. \tag{2.2}$$

We have that

$$\begin{cases} w_t + \operatorname{div}(F(u) - F(\widetilde{u})) = \operatorname{div}(\mathcal{A}\nabla w), & \text{in } (0,\infty)\times\Omega, \\ w(0,\cdot) = 0, & \text{in } \Omega, \\ w_t + \beta\partial_\nu^{\mathcal{A}} w + \gamma(x,v) - \gamma(x,\widetilde{v}) - q\beta\Delta_{\mathrm{LB}} w = 0, & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{2.3}$$

Using the divergence theorem and (2.3) we have that

$$\begin{aligned} \frac{d}{dt}\Big(\int_\Omega \frac{w^2}{2}dx + \int_{\partial\Omega}\frac{w^2}{2}\frac{d\sigma}{\beta}\Big) &= \int_\Omega ww_t\,dx + \int_{\partial\Omega} ww_t\frac{d\sigma}{\beta} \\ &= \underbrace{\int_\Omega w\operatorname{div}(\mathcal{A}\nabla w)dx - \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\,d\sigma + q\int_{\partial\Omega} w\Delta_{\mathrm{LB}} w\,d\sigma}_{J_1} \\ &\quad \underbrace{-\int_{\partial\Omega} w\left(\gamma(x,v) - \gamma(x,\widetilde{v})\right)\frac{d\sigma}{\beta}}_{J_2}\underbrace{-\int_\Omega w\operatorname{div}(F(u) - F(\widetilde{u}))dx}_{J_3}. \end{aligned} \tag{2.4}$$

The divergence theorem and *ii*) imply

$$\begin{aligned} J_1 &= -\int_\Omega \langle\mathcal{A}\nabla w, \nabla w\rangle dx + \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\,d\sigma \\ &\quad - \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\,d\sigma - q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma \\ &= -\int_\Omega \langle\mathcal{A}\nabla w, \nabla w\rangle dx - q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma \\ &\le -\alpha_0\int_\Omega |\nabla w|^2 dx - q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma. \end{aligned} \tag{2.5}$$

Due to (1.6) and (2.2),

$$J_2 = -\int_{\partial\Omega}(v - \widetilde{v})\left(\gamma(x,v) - \gamma(x,\widetilde{v})\right)\frac{d\sigma}{\beta} \le 0. \tag{2.6}$$

Thanks to *iii*) and *iv*)

$$\begin{aligned} J_3 &= -\int_\Omega w\operatorname{div}(F(u) - F(\widetilde{u}))dx \\ &= \int_\Omega \langle\nabla w, F(u) - F(\widetilde{u})\rangle dx - \int_{\partial\Omega} w\langle F(u) - F(\widetilde{u}), \nu\rangle d\sigma \end{aligned} \tag{2.7}$$

$$\le \frac{\alpha_0}{2}\int_\Omega |\nabla w|^2 dx + \frac{1}{2\alpha_0}\int_\Omega |F(u)-F(\widetilde{u})|^2 dx$$
$$+ \frac{1}{2}\int_{\partial\Omega} w^2 d\sigma + \frac{1}{2}\int_{\partial\Omega} |F(u)-F(\widetilde{u})|^2 d\sigma$$
$$\le \frac{\alpha_0}{2}\int_\Omega |\nabla w|^2 dx + \frac{\beta_1}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta}$$
$$+ \frac{L^2}{2\alpha_0}\int_\Omega |u-\widetilde{u}|^2 dx + \frac{L^2\beta_1}{2}\int_{\partial\Omega} |u-\widetilde{u}|^2 \frac{d\sigma}{\beta}.$$

Using the estimates (2.5), (2.6), and (2.7) in (2.4), we get

$$\frac{d}{dt}\Big(\int_\Omega \frac{w^2}{2} dx + \int_{\partial\Omega} \frac{w^2}{2}\frac{d\sigma}{\beta}\Big) \tag{2.8}$$
$$\le -\frac{\alpha_0}{2}\int_\Omega |\nabla w|^2 dx - q\int_{\partial\Omega} |\nabla_\tau w|^2 d\sigma + \frac{\beta_1}{2}\int_{\partial\Omega} w^2\frac{d\sigma}{\beta}$$
$$+ \frac{L^2}{2\alpha_0}\int_\Omega |u-\widetilde{u}|^2 dx + \frac{L^2\beta_1}{2}\int_{\partial\Omega} |u-\widetilde{u}|^2\frac{d\sigma}{\beta}$$
$$\le -\frac{\alpha_0}{2}\int_\Omega |\nabla w|^2 dx - q\int_{\partial\Omega} |\nabla_\tau w|^2 d\sigma + \beta_1\Big(\int_\Omega \frac{w^2}{2} dx + \int_{\partial\Omega} \frac{w^2}{2}\frac{d\sigma}{\beta}\Big)$$
$$+ \frac{L^2}{2\alpha_0}\int_\Omega |u-\widetilde{u}|^2 dx + \frac{L^2\beta_1}{2}\int_{\partial\Omega} |u-\widetilde{u}|^2\frac{d\sigma}{\beta}.$$

Therefore, the claim follows from the Gronwall's lemma. □

We are now ready for the proof of Lemma 2.1.

**Proof of Lemma 2.1.** We use a fixed-point argument that is based on the contraction-mapping principle. Let us introduce the set

$$\mathcal{B}_{q,T} = \{u \in \mathcal{H}_{q,T} : \|u - \mathcal{T}(0)\|_{\mathcal{H}_{q,T}} \le 1\},$$

where

$$\|u\|_{\mathcal{H}_{q,T}} = \Big( \|u\|^2_{L^\infty((0,T);L^2(\Omega))} + \|u\|^2_{L^\infty((0,\infty);L^2_\beta(\partial\Omega))} + \alpha_0 \|\nabla u\|^2_{L^2((0,T)\times\Omega)} + 2q\,\|\nabla_\tau u\|^2_{L^2((0,T)\times\partial\Omega)} \Big)^{\frac{1}{2}}.$$

Lemma 2.3 gives

$$\|\mathcal{T}(u) - \mathcal{T}(\widetilde{u})\|^2_{\mathcal{H}_{q,T}} \le L^2\frac{1+\alpha_0\beta_1}{\alpha_0\beta_1}\big(e^{\beta_1 T} - 1\big)\,\|u-\widetilde{u}\|^2_{\mathcal{H}_{q,T}}\,. \tag{2.9}$$

Moreover, if $u \in \mathcal{B}_{q,T}$, thanks to Lemmas 2.2 and 2.3,

$$\begin{aligned} \|\mathcal{T}(u) - \mathcal{T}(0)\|_{\mathcal{H}_{q,T}} &\le L^2 \frac{1+\alpha_0\beta_1}{\alpha_0\beta_1}\left(e^{\beta_1 T}-1\right)\|u\|^2_{\mathcal{H}_{q,T}} \\ &\le 2L^2 \frac{1+\alpha_0\beta_1}{\alpha_0\beta_1}\left(e^{\beta_1 T}-1\right)\left(1+\|\mathcal{T}(0)\|^2_{\mathcal{H}_{q,T}}\right) \\ &\le 2L^2 \frac{1+\alpha_0\beta_1}{\alpha_0\beta_1}\left(e^{\beta_1 T}-1\right)\left(1+e^T\left(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\right)\right). \end{aligned} \tag{2.10}$$

Since

$$2L^2 \frac{1+\alpha_0\beta_1}{\alpha_0\beta_1}\left(e^{\beta_1 T}-1\right)\left(1+e^T\left(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\right)\right) \longrightarrow 0, \quad \text{as } T\to 0,$$

there exists $T_0 > 0$ such that

$$2L^2 \frac{1+\alpha_0\beta_1}{\alpha_0\beta_1}\left(e^{\beta_1 T_0}-1\right)\left(1+e^{T_0}\left(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\right)\right) = \frac{1}{2}. \tag{2.11}$$

Thanks to (2.10), (2.9), and (2.11) we have

$$\mathcal{T}(u) \in \mathcal{B}_{q,T_0}, \qquad \|\mathcal{T}(u) - \mathcal{T}(\widetilde{u})\|_{\mathcal{H}_{q,T_0}} \le \frac{1}{\sqrt{2}}\|u-\widetilde{u}\|_{\mathcal{H}_{q,T_0}},$$

for all $u, \widetilde{u} \in \mathcal{B}_{q,T_0}$. Therefore the contraction-mapping principle gives the existence of a unique fixed point for $\mathcal{T}$ in $\mathcal{B}_{q,T_0}$. Due to the definition of $\mathcal{T}$ this function is the unique solution of (1.1) in the time interval $[0, T_0)$. □

We conclude by proving a global (in time) estimate for the unique local (in time) solution found in Lemma 2.1.

**Lemma 2.4.** *Let $u$ be the solution of* (1.1)*. We have that*

$$\begin{aligned} &\|u(t,\cdot)\|^2_{L^2(\Omega)} + \|u(t,\cdot)\|^2_{L^2_\beta(\partial\Omega)} \\ &\quad + e^{\frac{L^2}{\alpha_0}t}\int_0^t e^{-\frac{L^2}{\alpha_0}s}\left(\alpha_0\|\nabla u(s,\cdot)\|^2_{L^2(\Omega)} + 2q\|\nabla_\tau u(s,\cdot)\|^2_{L^2(\partial\Omega)}\right)ds \\ &\qquad \le e^{\frac{L^2}{\alpha_0}t}\left(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\right) \end{aligned} \tag{2.12}$$

*for each $t \ge 0$. In particular, for every $t > 0$,*

$$\alpha_0\|\nabla u\|^2_{L^2((0,t)\times\Omega)} + 2q\|\nabla_\tau u\|^2_{L^2((0,t)\times\partial\Omega)} \le e^{\frac{L^2}{\alpha_0}t}\left(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\right). \tag{2.13}$$

**Proof.** Using (1.1), (1.4), *ii*), and the divergence theorem,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega \frac{u^2}{2}dx + \int_{\partial\Omega}\frac{u^2}{2}\frac{d\sigma}{\beta}\Big) = \int_\Omega u u_t\,dx + \int_{\partial\Omega} u u_t \frac{d\sigma}{\beta}\\
&= \int_\Omega \operatorname{div}(\mathcal{A}\nabla u)u\,dx - \int_\Omega \operatorname{div}(F(u))u\,dx\\
&\quad - \int_{\partial\Omega} u\partial_\nu^{\mathcal{A}} u\,d\sigma - \int_{\partial\Omega}\frac{\gamma(x,u)u}{\beta}d\sigma + q\int_{\partial\Omega} u\Delta_{\mathrm{LB}}u\,d\sigma\\
&= -\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle dx - \int_\Omega \langle F'(u), \nabla u\rangle u\,dx\\
&\quad + \int_{\partial\Omega} u\partial_\nu^{\mathcal{A}} u\,d\sigma - \int_{\partial\Omega} u\partial_\nu^{\mathcal{A}} u\,d\sigma - \int_{\partial\Omega}\frac{\gamma(x,u)u}{\beta}d\sigma - q\int_{\partial\Omega}|\nabla_\tau u|^2 d\sigma\\
&\le -\frac{\alpha_0}{2}\int_\Omega |\nabla u|^2 dx + \frac{L^2}{2\alpha_0}\int_\Omega u^2 dx - q\int_{\partial\Omega}|\nabla_\tau u|^2 d\sigma\\
&\le -\frac{\alpha_0}{2}\int_\Omega |\nabla u|^2 dx + \frac{L^2}{\alpha_0}\Big(\int_\Omega \frac{u^2}{2}dx + \int_{\partial\Omega}\frac{u^2}{2}\frac{d\sigma}{\beta}\Big) - q\int_{\partial\Omega}|\nabla_\tau u|^2 d\sigma.
\end{aligned}
$$

Therefore, (2.12) is a consequence of the Gronwall inequality.

Let us prove (2.13). Thanks to (2.12), we have

$$
\begin{aligned}
&e^{\frac{L^2}{\alpha_0}t}\left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)}\right)\\
&\qquad \ge e^{\frac{L^2}{\alpha_0}t}\int_0^t e^{-\frac{L^2}{\alpha_0}s}\Big(\alpha_0\,\|\nabla u(s,\cdot)\|^2_{L^2(\Omega)} + 2q\,\|\nabla_\tau u(s,\cdot)\|^2_{L^2(\partial\Omega)}\Big)ds\\
&\qquad \ge e^{\frac{L^2}{\alpha_0}t}\int_0^t e^{-\frac{L^2}{\alpha_0}t}\Big(\alpha_0\,\|\nabla u(s,\cdot)\|^2_{L^2(\Omega)} + 2q\,\|\nabla_\tau u(s,\cdot)\|^2_{L^2(\partial\Omega)}\Big)ds\\
&\qquad = \alpha_0\,\|\nabla u\|^2_{L^2((0,t)\times\Omega)} + 2q\,\|\nabla_\tau u\|^2_{L^2((0,t)\times\partial\Omega)}\,. \qquad \square
\end{aligned}
$$

**Proof of Theorem 2.1.** Thanks to the global (in time) estimate stated in Lemma 2.4, the local (in time) solution found in Lemma 2.1 is indeed global. Moreover, due to the local (in time) uniqueness stated in Lemma 2.1, it is the unique global (in time) solution to (1.1). This proves the claim. □

## 3. Regularity

This section is devoted to the regularity of the solutions of the nonlinear initial boundary value problem (1.1).

**Theorem 3.1.** *Let $u \in \mathcal{H}_{q,\infty}$ be the unique solution of* (1.1). *We have that*

$$\begin{aligned} u_t,\ \operatorname{div}(\mathcal{A}\nabla u) \in & L^2([0,T]\times\Omega), \quad u_t \in L^2_\beta([0,T]\times\partial\Omega),\\ \nabla_\tau u \in & L^2([0,T]\times\partial\Omega), \end{aligned}$$

*for every $T > 0$. In addition, if $q = 0$,*

$$\partial_\nu^{\mathcal{A}} u \in L^2_\beta([0,T]\times\partial\Omega), \qquad T > 0.$$

We split the proof into the following lemmas and corollaries.

**Lemma 3.1.** *The following estimate holds for the solution $u$ of* (1.1):

$$\begin{aligned} &\frac{\|u_t\|^2_{L^2([0,t]\times\Omega)}}{2} + \|u_t\|^2_{L^2_\beta([0,t]\times\partial\Omega)}\\ &\qquad \le \frac{\alpha_1}{2}\|\nabla f\|^2_{L^2(\Omega)} + \frac{L^2 e^{\frac{L^2}{\alpha_0}t}}{2\alpha_0}\Big(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)}\Big)\\ &\qquad\quad + \Gamma\Big(\frac{\|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)}}{p+1} + \|f\|_{L^1_\beta(\partial\Omega)}\Big) + \frac{q}{2}\|\nabla_\tau f\|^2_{L^2(\partial\Omega)} \end{aligned}$$

*for each $t \ge 0$.*

**Proof.** From (1.1), (1.7), (1.8), (1.9), our assumptions, and Lemma 2.4,

$$\begin{aligned} &\int_0^t\int_\Omega u_t^2\,dx\,ds + \int_0^t\int_{\partial\Omega} u_t^2\,ds\,\frac{d\sigma}{\beta}\\ =&\int_0^t\int_\Omega \operatorname{div}(\mathcal{A}\nabla u)u_t\,dx\,ds - \int_0^t\int_\Omega u_t\operatorname{div}(F(u))dx\,ds\\ &-\int_0^t\int_{\partial\Omega} u_t\partial_\nu^{\mathcal{A}}u\,d\sigma\,ds - \int_0^t\int_{\partial\Omega}\frac{\gamma(x,u)}{\beta}u_t\,d\sigma\,ds + q\int_0^t\int_{\partial\Omega}u_t\Delta_{\mathrm{LB}}u\,d\sigma\,ds\\ =&-\int_0^t\int_\Omega\langle\mathcal{A}\nabla u,\nabla u_t\rangle dx\,ds + \int_0^t\int_{\partial\Omega}u_t\partial_\nu^{\mathcal{A}}u\,d\sigma\,ds\\ &-\int_0^t\int_\Omega u_t\langle F'(u),\nabla u\rangle dx\,ds - \int_0^t\int_{\partial\Omega}u_t\partial_\nu^{\mathcal{A}}u\,d\sigma\,ds\\ &-\int_0^t\int_{\partial\Omega}\frac{\gamma(x,u)}{\beta}u_t\,d\sigma\,ds - q\int_0^t\int_{\partial\Omega}\langle\nabla_\tau u,\nabla_\tau u_t\rangle\,d\sigma\,ds\\ \le&-\int_0^t\int_\Omega\langle\mathcal{A}\nabla u,\nabla u_t\rangle dx\,ds + \frac{L^2}{2}\int_0^t\int_\Omega|\nabla u|^2dx\,ds + \frac{1}{2}\int_0^t\int_\Omega u_t^2\,dx\,ds\\ &-\int_0^t\int_{\partial\Omega}\frac{\gamma(x,u)}{\beta}u_t\,d\sigma\,ds - q\int_0^t\int_{\partial\Omega}\langle\nabla_\tau u,\nabla_\tau u_t\rangle d\sigma\,ds \end{aligned}$$

$$
\begin{aligned}
=&-\frac{1}{2}\int_0^t\int_\Omega(\langle\mathcal{A}\nabla u,\nabla u\rangle)_t\,dx\,ds+\frac{L^2}{2}\int_0^t\int_\Omega|\nabla u|^2dx\,ds\\
&+\frac{1}{2}\int_0^t\int_\Omega u_t^2\,dx\,ds-\int_0^t\int_{\partial\Omega}\Big(\int_0^u\frac{\gamma(x,\xi)}{\beta}d\xi\Big)_t d\sigma\,ds\\
&-\frac{q}{2}\int_0^t\int_{\partial\Omega}(|\nabla_\tau u|^2)_t\,d\sigma\,ds\\
=&-\frac{1}{2}\int_\Omega\langle\mathcal{A}\nabla u(t,x),\nabla u(t,x)\rangle dx+\frac{1}{2}\int_\Omega\langle\mathcal{A}\nabla f,\nabla f\rangle dx\\
&+\frac{L^2}{2}\int_0^t\int_\Omega|\nabla u|^2dx\,ds+\frac{1}{2}\int_0^t\int_\Omega u_t^2\,dx\,ds\\
&-\int_{\partial\Omega}\int_0^{u(t,x)}\frac{\gamma(x,\xi)}{\beta}d\xi\,d\sigma+\int_{\partial\Omega}\int_0^f\frac{\gamma(x,\xi)}{\beta}d\xi\,d\sigma\\
&-\frac{q}{2}\int_{\partial\Omega}|\nabla_\tau u(t,x)|^2d\sigma+\frac{q}{2}\int_{\partial\Omega}|\nabla_\tau f|^2d\sigma\\
\le&\frac{1}{2}\int_\Omega\langle\mathcal{A}\nabla f,\nabla f\rangle dx+\frac{L^2}{2}\int_0^t\int_\Omega|\nabla u|^2dx\,ds\\
&+\frac{1}{2}\int_0^t\int_\Omega u_t^2\,dx\,ds+\int_{\partial\Omega}\int_0^f\frac{\gamma(x,\xi)}{\beta}d\xi\,d\sigma+\frac{q}{2}\int_{\partial\Omega}|\nabla_\tau f|^2d\sigma\\
\le&\frac{\alpha_1}{2}\int_\Omega|\nabla f|^2dx+\frac{L^2e^{\frac{L^2}{\alpha_0}t}}{2\alpha_0}\Big(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\Big)+\frac{1}{2}\int_0^t\int_\Omega u_t^2\,dx\,ds\\
&+\Gamma\Big(\frac{\|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)}}{p+1}+\|f\|_{L^1_\beta(\partial\Omega)}\Big)+\frac{q}{2}\int_{\partial\Omega}|\nabla_\tau f|^2d\sigma.
\end{aligned}
$$

The claim is proved. □

**Corollary 3.1.** *Let $u$ be the solution of* (1.1)*. The following estimate holds:*

$$
\begin{aligned}
\|\operatorname{div}(\mathcal{A}\nabla u)\|^2_{L^2([0,t]\times\Omega)}\le 2\alpha_1\|\nabla f\|^2_{L^2(\Omega)}+\frac{4L^2}{\alpha_0}e^{\frac{L^2}{\alpha_0}t}\Big(\|f\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_\beta(\partial\Omega)}\Big)\\
+4\Gamma\Big(\frac{\|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)}}{p+1}+\|f\|_{L^1_\beta(\partial\Omega)}\Big)+2q\|\nabla_\tau f\|^2_{L^2(\partial\Omega)},
\end{aligned}
$$

*for all $t\ge 0$.*

**Proof.** From the first equation in (1.1) and our assumptions,

$$|\operatorname{div}(\mathcal{A}\nabla u)|=|u_t+\operatorname{div}(F(u))|\le|u_t|+L|\nabla u|,$$

so

$$\|\operatorname{div}(\mathcal{A}\nabla u)\|^2_{L^2([0,t]\times\Omega)} \le 2\,\|u_t\|^2_{L^2([0,t]\times\Omega)} + 2L^2\,\|\nabla u\|^2_{L^2([0,t]\times\Omega)}.$$

Hence the claim follows from Lemmas 2.4 and 3.1. □

**Corollary 3.2.** *Let $u$ be the solution of* (1.1). *We have that*

$$\begin{aligned}
&\|u\|^2_{L^2([0,t]\times\partial\Omega)} + \|\nabla_\tau u\|^2_{L^2([0,t]\times\partial\Omega)}\\
&\quad\le C_0\Big[\Big(\,\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)}\Big)\Big(e^{\frac{L^2}{\alpha_0}t}\frac{\alpha_0^2+L^2+4L^4}{\alpha_0L^2} - \frac{\alpha_0}{L^2}\Big)\\
&\quad+ 2\alpha_1\,\|\nabla f\|^2_{L^2(\Omega)} + 4\Gamma\Big(\frac{\|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)}}{p+1} + \|f\|_{L^1_\beta(\partial\Omega)}\Big) + 2q\,\|\nabla_\tau f\|^2_{L^2(\partial\Omega)}\Big],
\end{aligned}$$

*for each $t \ge 0$ and some constant $C_0 > 0$.*

**Proof.** Let $T > 0$. From Lemma 2.4 we have that

$$u \in L^\infty([0,T];L^2(\Omega)), \qquad \nabla u \in L^2([0,T];L^2(\Omega)), \tag{3.1}$$

and from Corollary 3.1,

$$\operatorname{div}(\mathcal{A}\nabla u) \in L^2([0,T];L^2(\Omega)). \tag{3.2}$$

Since $\mathcal{A}$ is uniformly elliptic (see *ii*)), (3.1) and (3.2) give

$$u \in L^\infty([0,T];H^2(\Omega)). \tag{3.3}$$

Therefore passing to the traces,

$$u|_{\partial\Omega} \in L^\infty([0,T];H^{\frac{3}{2}}(\partial\Omega)) \subset L^\infty([0,T];H^1(\partial\Omega)). \tag{3.4}$$

Finally, rewriting (3.4) in estimate form,

$$\begin{aligned}
&\int_0^T \Big(\|u(t,\cdot)\|^2_{L^2(\partial\Omega)} + \|\nabla_\tau u(t,\cdot)\|^2_{L^2(\partial\Omega)}\Big)\,dt\\
&= \int_0^T \|u(t,\cdot)\|^2_{H^1(\partial\Omega)}\,dt \le c_1\int_0^T \|u(t,\cdot)\|^2_{H^{3/2}(\partial\Omega)}\,dt \le c_2\int_0^T \|u(t,\cdot)\|^2_{H^2(\Omega)}\,dt\\
&\le c_3\int_0^T \Big(\|u(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + \|\operatorname{div}(\mathcal{A}\nabla u(t,\cdot))\|^2_{L^2(\Omega)}\Big)\,dt,
\end{aligned}$$

for some constants $c_1, c_2, c_3 > 0$. Lemma 2.4 and Corollary 3.1 give the claim. □

Let us conclude with the special case $q = 0$.

**Lemma 3.2.** *Let $u$ be the solution of* (1.1)*. If $q = 0$,*

$$\begin{aligned}\left\|\partial_\nu^{\mathcal{A}} u\right\|^2_{L^2_\beta([0,t]\times\partial\Omega)} \leq \frac{2}{\beta_0^2}\Big\{ &\frac{\alpha_1}{2}\left\|\nabla f\right\|^2_{L^2(\Omega)} + \frac{L^2 e^{\frac{L^2}{\alpha_0}t}}{2\alpha_0}\left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)}\right) \\ &+ \Gamma\Big(\frac{\|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)}}{p+1} + \|f\|_{L^1_\beta(\partial\Omega)}\Big) + \frac{q}{2}\left\|\nabla_\tau f\right\|_{L^2(\partial\Omega)} \\ &+ 2\Gamma^2 C_1\Big[\left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2_\beta(\partial\Omega)}\right)\left(e^{\frac{L^2}{\alpha_0}t}\frac{2\alpha_0^2 + 2L^2 + 9L^4}{2\alpha_0 L^2} - \frac{\alpha_0}{L^2}\right) \\ &+ \frac{5}{2}\alpha_1\left\|\nabla f\right\|^2_{L^2(\Omega)} + 5\Gamma\Big(\frac{\|f\|^{p+1}_{L^{p+1}_\beta(\partial\Omega)}}{p+1} + \|f\|_{L^1_\beta(\partial\Omega)}\Big)\Big]^p + \frac{2\Gamma^2|\partial\Omega|}{\beta_0}t\Big\},\end{aligned}$$

*for each $t \geq 0$ and some constant $C_1 > 0$.*

**Proof.** If $q = 0$, the second equation in (1.1) gives

$$\frac{\partial_\nu^{\mathcal{A}} u}{\sqrt{\beta}} = -\frac{u_t}{\beta\sqrt{\beta}} - \frac{\gamma(x,u)}{\beta\sqrt{\beta}}.$$

Therefore, (1.2), (1.7), and (1.8) imply

$$\begin{aligned}\left\|\partial_\nu^{\mathcal{A}} u\right\|^2_{L^2_\beta([0,t]\times\partial\Omega)} &\leq \frac{2}{\beta_0^2}\left(\|u_t\|^2_{L^2_\beta([0,t]\times\partial\Omega)} + \|\gamma(\cdot,u)\|^2_{L^2_\beta([0,t]\times\partial\Omega)}\right) \\ &\leq \frac{2}{\beta_0^2}\left(\|u_t\|^2_{L^2_\beta([0,t]\times\partial\Omega)} + \Gamma^2\left\||u|^p + 1\right\|^2_{L^2_\beta([0,t]\times\partial\Omega)}\right) \\ &\leq \frac{2}{\beta_0^2}\left(\|u_t\|^2_{L^2_\beta([0,t]\times\partial\Omega)} + 2\Gamma^2\|u\|^{2p}_{L^{2p}_\beta([0,t]\times\partial\Omega)} + \frac{2\Gamma^2|\partial\Omega|}{\beta_0}t\right) \\ &\leq \frac{2}{\beta_0^2}\left(\|u_t\|^2_{L^2_\beta([0,t]\times\partial\Omega)} + 2\Gamma^2 c_0\|u\|^{2p}_{H^1([0,t]\times\partial\Omega)} + \frac{2\Gamma^2|\partial\Omega|}{\beta_0}t\right),\end{aligned}$$

for some constant $c_0 > 0$. The claim follows from Lemma 3.1 and Corollary 3.2. □

## 4. Stability estimate

Let us conclude our analysis by proving the explicit stability estimate stated in Theorem 1.1.

**Theorem 4.1.** *Let $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, $f$ and $\widetilde{\mathcal{A}}$, $\widetilde{F}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{q}$, $\widetilde{f}$ satisfy the assumptions i), …, ix). Let $u$ and $v$ be the unique solutions of* (1.1) *obtained in correspondence of $\mathcal{A}$, $F$, $\beta$, $\gamma$, $q$, $f$ and $\widetilde{\mathcal{A}}$, $\widetilde{F}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{q}$, $\widetilde{f}$, respectively. The estimate stated in Theorem* 1.1 *holds.*

Following the notation introduced in Theorems 1.1 and 4.1, let $v$ solve

$$\begin{cases} v_t + \operatorname{div}(\widetilde{F}(v)) = \operatorname{div}(\widetilde{\mathcal{A}}\nabla v), & \text{in } (0,\infty)\times\Omega,\\ v(0,\cdot) = \widetilde{f}, & \text{in } \Omega,\\ v_t + \widetilde{\beta}\partial_\nu^{\widetilde{\mathcal{A}}} v + \widetilde{\gamma}(x,v) - \widetilde{q}\widetilde{\beta}\Delta_{\mathrm{LB}} v = 0, & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{4.1}$$

Introducing the notation

$$w(t,x) = u(t,x) - v(t,x), \qquad t \geq 0, \quad x \in \overline{\Omega}, \tag{4.2}$$

we have that

$$\begin{cases} w_t + \operatorname{div}(F(u) - F(v)) + \operatorname{div}(F(v) - \widetilde{F}(v)) & \\ \qquad\qquad = \operatorname{div}(\mathcal{A}\nabla w) + \operatorname{div}((\mathcal{A} - \widetilde{\mathcal{A}})\nabla v), & \text{in } (0,\infty)\times\Omega,\\ w(0,\cdot) = f - \widetilde{f}, & \text{in } \Omega,\\ w_t + \beta\partial_\nu^{\mathcal{A}} w + \gamma(x,u) - \gamma(x,v) - q\beta\Delta_{\mathrm{LB}} w & \\ \qquad\qquad = \widetilde{\beta}\partial_\nu^{\widetilde{\mathcal{A}}} v - \beta\partial_\nu^{\mathcal{A}} v + \widetilde{\gamma}(x,v) - \gamma(x,v) & \\ \qquad\qquad -(\widetilde{q}\widetilde{\beta} - q\beta)\Delta_{\mathrm{LB}} v, & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{4.3}$$

**Proof of Theorem 4.1.** Using the divergence theorem and (4.3) we have that

$$\begin{aligned} \frac{d}{dt}\Big(\int_\Omega \frac{w^2}{2} dx + \int_{\partial\Omega} \frac{w^2}{2}\frac{d\sigma}{\beta}\Big) &= \int_\Omega w w_t\, dx + \int_{\partial\Omega} w w_t \frac{d\sigma}{\beta} \qquad (4.4)\\ &= \int_\Omega w \operatorname{div}(\mathcal{A}\nabla w) dx + \int_\Omega w \operatorname{div}((\mathcal{A} - \widetilde{\mathcal{A}})\nabla v) dx\\ &- \int_\Omega w \operatorname{div}(F(u) - F(v)) dx - \int_\Omega w \operatorname{div}(F(v) - \widetilde{F}(v)) dx\\ &- \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\, d\sigma - \int_{\partial\Omega} w\left(\gamma(x,u) - \gamma(x,v)\right)\frac{d\sigma}{\beta}\\ &+ q\int_{\partial\Omega} w\Delta_{\mathrm{LB}} w\, d\sigma + \int_{\partial\Omega} w\left(\widetilde{\beta}\partial_\nu^{\widetilde{\mathcal{A}}} v - \beta\partial_\nu^{\mathcal{A}} v\right)\frac{d\sigma}{\beta}\\ &+ \int_{\partial\Omega} w\left(\widetilde{\gamma}(x,v) - \gamma(x,v)\right)\frac{d\sigma}{\beta} - \int_{\partial\Omega} (\widetilde{q}\widetilde{\beta} - q\beta) w\Delta_{\mathrm{LB}} v \frac{d\sigma}{\beta}\\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_\Omega w \operatorname{div}(\mathcal{A}\nabla w) dx - \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\, d\sigma + q\int_{\partial\Omega} w\Delta_{\mathrm{LB}} w\, d\sigma,\\ I_2 &= -\int_{\partial\Omega} w\left(\gamma(x,u) - \gamma(x,v)\right)\frac{d\sigma}{\beta}, \end{aligned}$$

$$
\begin{aligned}
I_3 =& \int_{\partial\Omega} w\left(\widetilde{\gamma}(x,v)-\gamma(x,v)\right)\frac{d\sigma}{\beta},\\
I_4 =& \int_{\Omega} w\,\mathrm{div}((\mathcal{A}-\widetilde{\mathcal{A}})\nabla v)dx - \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}-\widetilde{\mathcal{A}}} v\,d\sigma,\\
I_5 =& \int_{\partial\Omega} \frac{\widetilde{\beta}-\beta}{\beta} w\left(\partial_\nu^{\widetilde{\mathcal{A}}} v - \widetilde{q}\Delta_{\mathrm{LB}} v\right)d\sigma,\\
I_6 =& (q-\widetilde{q})\int_{\partial\Omega} w\Delta_{\mathrm{LB}} v\,d\sigma,\\
I_7 =& -\int_{\Omega} w\,\mathrm{div}(F(u)-F(v))dx,\\
I_8 =& \int_{\Omega} w\,\mathrm{div}(F(v)-\widetilde{F}(v))dx.
\end{aligned}
$$

We continue by estimating these eight terms. The divergence theorem and *ii*) imply

$$
\begin{aligned}
I_1 =& -\int_{\Omega}\langle\mathcal{A}\nabla w,\nabla w\rangle dx + \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\,d\sigma - \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}} w\,d\sigma - q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma\\
=& -\int_{\Omega}\langle\mathcal{A}\nabla w,\nabla w\rangle dx - q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma\\
\le& -\alpha_0\int_{\Omega}|\nabla w|^2 dx - q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma. \qquad (4.5)
\end{aligned}
$$

Due to (1.6) and (4.2),

$$
I_2 = -\int_{\partial\Omega}(u-v)\left(\gamma(x,u)-\gamma(x,v)\right)\frac{d\sigma}{\beta} \le 0. \qquad (4.6)
$$

The Hölder inequality and (1.8) give

$$
\begin{aligned}
I_3 \le& \frac{1}{2}\int_{\partial\Omega}\left(\widetilde{\gamma}(x,v)-\gamma(x,v)\right)^2\frac{d\sigma}{\beta} + \int_{\partial\Omega}\frac{w^2}{2}\frac{d\sigma}{\beta} \qquad (4.7)\\
\le& \frac{1}{2}\int_{\partial\Omega}\left(\frac{\widetilde{\gamma}(x,v)-\gamma(x,v)}{|v|^p+1}\right)^2(|v|^p+1)^2\frac{d\sigma}{\beta} + \int_{\partial\Omega}\frac{w^2}{2}\frac{d\sigma}{\beta}\\
\le& \sup_{x\partial\Omega,\,\xi\in\mathbb{R}}\left|\frac{\widetilde{\gamma}(x,\xi)-\gamma(x,\xi)}{|\xi|^p+1}\right|^2\int_{\partial\Omega}\frac{\widetilde{\beta}}{\beta}\left(|v|^{2p}+1\right)\frac{d\sigma}{\widetilde{\beta}} + \int_{\partial\Omega}\frac{w^2}{2}\frac{d\sigma}{\beta}\\
\le& \sup_{x\partial\Omega,\,\xi\in\mathbb{R}}\left|\frac{\widetilde{\gamma}(x,\xi)-\gamma(x,\xi)}{|\xi|^p+1}\right|^2\left(\frac{\beta_1}{\beta_0}\|v(t,\cdot)\|^{2p}_{L^{2p}_{\widetilde{\beta}}(\partial\Omega)} + \frac{|\partial\Omega|}{\beta_0}\right) + \int_{\partial\Omega}\frac{w^2}{2}\frac{d\sigma}{\beta}.
\end{aligned}
$$

Let $\varepsilon > 0$ be a constant that will be specified later. The divergence theorem and the Hölder inequality imply

$$\begin{aligned}
I_4 &= -\int_\Omega \langle(\mathcal{A}-\widetilde{\mathcal{A}})\nabla v, \nabla w\rangle dx + \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}-\widetilde{\mathcal{A}}} v\, d\sigma - \int_{\partial\Omega} w\partial_\nu^{\mathcal{A}-\widetilde{\mathcal{A}}} v\, d\sigma \qquad (4.8)\\
&= -\int_\Omega \langle(\mathcal{A}-\widetilde{\mathcal{A}})\nabla v, \nabla w\rangle dx \le \frac{1}{2\varepsilon}\int_\Omega |(\mathcal{A}-\widetilde{\mathcal{A}})\nabla v|^2 dx + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 dx\\
&\le \frac{\left\|\mathcal{A}-\widetilde{\mathcal{A}}\right\|^2_{L^\infty(\Omega)}}{2\varepsilon}\int_\Omega |\nabla v|^2 dx + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 dx.
\end{aligned}$$

The Hölder inequality, $ix$), (1.8), and (4.1) yield

$$\begin{aligned}
I_5 &= \int_{\partial\Omega} \frac{\beta-\widetilde{\beta}}{\beta\widetilde{\beta}} w\left(v_t + \widetilde{\gamma}(x,v)\right) d\sigma \qquad (4.9)\\
&\le \frac{1}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta} + \int_{\partial\Omega} \frac{|\beta-\widetilde{\beta}|^2}{\beta\widetilde{\beta}}\left(v_t^2 + \widetilde{\gamma}(x,v)^2\right)\frac{d\sigma}{\widetilde{\beta}}\\
&\le \frac{1}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta} + \frac{\left\|\beta-\widetilde{\beta}\right\|^2_{L^\infty(\partial\Omega)}}{\beta_0^2}\int_{\partial\Omega}\left(v_t^2 + 2\Gamma^2(v^{2p}+1)\right)\frac{d\sigma}{\widetilde{\beta}}\\
&\le \frac{1}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta} + \frac{\left\|\beta-\widetilde{\beta}\right\|^2_{L^\infty(\partial\Omega)}}{\beta_0^2} \times\\
&\quad \times \left(\int_{\partial\Omega} v_t^2 \frac{d\sigma}{\widetilde{\beta}} + 2\Gamma^2\left(\|v(t,\cdot)\|^{2p}_{L^{2p}_{\widetilde{\beta}}(\partial\Omega)} + \frac{|\partial\Omega|}{\beta_0}\right)\right).
\end{aligned}$$

Playing again with the Hölder inequality, the divergence theorem, and (4.2),

$$\begin{aligned}
I_6 &= (\widetilde{q}-q)\int_{\partial\Omega} \langle\nabla_\tau w, \nabla_\tau v\rangle d\sigma \qquad (4.10)\\
&= (\widetilde{q}-q)\int_{\partial\Omega}\left(\langle\nabla_\tau u, \nabla_\tau v\rangle - |\nabla_\tau v|^2\right) d\sigma\\
&\le |\widetilde{q}-q|\int_{\partial\Omega}\left(|\nabla_\tau u|\,|\nabla_\tau v| + |\nabla_\tau v|^2\right) d\sigma\\
&\le |\widetilde{q}-q|\left(\frac{3}{2}\int_{\partial\Omega} |\nabla_\tau v|^2 d\sigma + \frac{1}{2}\int_{\partial\Omega} |\nabla_\tau u|^2 d\sigma\right).
\end{aligned}$$

Thanks to *iii*) and *iv*),

$$
\begin{aligned}
I_7 &= -\int_\Omega w \operatorname{div}(F(u)-F(v))dx \qquad (4.11)\\
&= -\int_\Omega \langle \nabla w, F(u)-F(v)\rangle dx - \int_{\partial\Omega} w\langle F(u)-F(v),\nu\rangle d\sigma\\
&\le \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 dx + \frac{1}{2\varepsilon}\int_\Omega |F(u)-F(v)|^2 dx\\
&\quad + \frac{1}{2}\int_{\partial\Omega} w^2 d\sigma + \frac{1}{2}\int_{\partial\Omega} |F(u)-F(v)|^2 d\sigma\\
&\le \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 dx + \frac{L^2}{2\varepsilon}\int_\Omega w^2 dx + \frac{(L^2+1)\beta_1}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
I_8 &= -\int_\Omega w \operatorname{div}(F(v)-\widetilde{F}(v))dx \qquad (4.12)\\
&= -\int_\Omega \langle \nabla w, F(v)-\widetilde{F}(v)\rangle dx - \int_{\partial\Omega} w\langle F(v)-\widetilde{F}(v),\nu\rangle d\sigma\\
&\le \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 dx + \frac{1}{2\varepsilon}\int_\Omega |F(v)-\widetilde{F}(v)|^2 dx\\
&\quad + \frac{1}{2}\int_{\partial\Omega} w^2 d\sigma + \frac{1}{2}\int_{\partial\Omega} |F(v)-\widetilde{F}(v)|^2 d\sigma\\
&\le \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 dx + \frac{\beta_1}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta} + \left\|F-\widetilde{F}\right\|_{L^\infty(\mathbb{R}^N)} \frac{\varepsilon|\partial\Omega|+|\Omega|}{2\varepsilon}.
\end{aligned}
$$

Using the estimates (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) in (4.4), we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega \frac{w^2}{2}dx + \int_{\partial\Omega}\frac{w^2}{2}\frac{d\sigma}{\beta}\Big) \qquad (4.13)\\
&\le \Big(3\frac{\varepsilon}{2}-\alpha_0\Big)\int_\Omega |\nabla w|^2 dx - q\int_{\partial\Omega} |\nabla_\tau w|^2 d\sigma\\
&+ \frac{L^2}{2\varepsilon}\int_\Omega w^2 dx + \frac{(L^2+2)\beta_1+2}{2}\int_{\partial\Omega} w^2 \frac{d\sigma}{\beta}\\
&+ \left\|F-\widetilde{F}\right\|_{L^\infty(\mathbb{R}^N)} \frac{\varepsilon|\partial\Omega|+|\Omega|}{2\varepsilon}\\
&+ \sup_{x\partial\Omega,\,\xi\in\mathbb{R}}\left|\frac{\widetilde{\gamma}(x,\xi)-\gamma(x,\xi)}{|\xi|^p+1}\right|^2 \left(\frac{\beta_1}{\beta_0}\,\|v(t,\cdot)\|^{2p}_{L^{2p}_{\widetilde{\beta}}(\partial\Omega)} + \frac{|\partial\Omega|}{\beta_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\left\|\mathcal{A}-\widetilde{\mathcal{A}}\right\|^2_{L^\infty(\Omega)}}{2\varepsilon}\int_\Omega |\nabla v|^2 dx \\
&+ \frac{\left\|\beta-\widetilde{\beta}\right\|^2_{L^\infty(\partial\Omega)}}{\beta_0^2}\left(\int_{\partial\Omega} v_t^2\frac{d\sigma}{\widetilde{\beta}} + 2\Gamma^2\left(\|v(t,\cdot)\|^{2p}_{L^{2p}_{\widetilde{\beta}}(\partial\Omega)} + \frac{|\partial\Omega|}{\beta_0}\right)\right) \\
&+ |\widetilde{q}-q|\left(\frac{3}{2}\int_{\partial\Omega}|\nabla_\tau v|^2 d\sigma + \frac{1}{2}\int_{\partial\Omega}|\nabla_\tau u|^2 d\sigma\right).
\end{aligned}
$$

We choose $\varepsilon = \frac{\alpha_0}{3}$ and introduce the notation

$$
\begin{aligned}
\lambda =& \frac{3L^2}{\alpha_0} + (L^2+2)\beta_1 + 2, \\
\Lambda =& \sup_{x\partial\Omega,\,\xi\in\mathbb{R}}\left|\frac{\widetilde{\gamma}(x,\xi)-\gamma(x,\xi)}{|\xi|^p+1}\right|^2 \\
&+ \left\|\mathcal{A}-\widetilde{\mathcal{A}}\right\|^2_{L^\infty(\Omega)} + \left\|\beta-\widetilde{\beta}\right\|^2_{L^\infty(\partial\Omega)} + |\widetilde{q}-q| + \left\|F-\widetilde{F}\right\|_{L^\infty(\mathbb{R}^N)}, \\
G(t) =& \|v(t,\cdot)\|^{2p}_{L^{2p}_{\widetilde{\beta}}(\partial\Omega)} + |\partial\Omega| + |\Omega| + \|\nabla v(t,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|v_t(t,\cdot)\|^2_{L^2_{\widetilde{\beta}}(\partial\Omega)} + \|\nabla_\tau v(t,\cdot)\|^2_{L^2(\partial\Omega)} + \|\nabla_\tau u(t,\cdot)\|^2_{L^2(\partial\Omega)}.
\end{aligned}
$$

Therefore, (4.13) says

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega w^2 dx + \int_{\partial\Omega} w^2\frac{d\sigma}{\beta}\right) + \alpha_0\int_\Omega|\nabla w|^2 dx + 2q\int_{\partial\Omega}|\nabla_\tau w|^2 d\sigma \qquad (4.14)\\
&\le \lambda\left(\int_\Omega w^2 dx + \int_{\partial\Omega} w^2\frac{d\sigma}{\beta}\right) + c_1\Lambda G(t),
\end{aligned}
$$

for some constant $c_1 > 0$. The Gronwall lemma gives

$$
\begin{aligned}
&\|w(t,\cdot)\|^2_{L^2(\Omega)} + \|w(t,\cdot)\|^2_{L^2_\beta(\partial\Omega)} \qquad (4.15)\\
&\qquad + e^{\lambda t}\int_0^t e^{-\lambda s}\left(\alpha_0\|\nabla w(s,\cdot)\|_{L^2(\Omega)} + 2q\|\nabla_\tau w(s,\cdot)\|^2_{L^2(\partial\Omega)}\right) ds \\
&\qquad \le e^{\lambda t}\left(\|w(0,\cdot)\|^2_{L^2(\Omega)} + \|w(0,\cdot)\|^2_{L^2_\beta(\partial\Omega)}\right) + c_1\Lambda e^{\lambda t}\int_0^t e^{-\lambda s}G(s)ds.
\end{aligned}
$$

Due to Lemmas 2.4 and 3.1, and Corollary 3.2,

$$
\int_0^t e^{-\lambda s}G(s)ds \le \int_0^t G(s)ds
$$

$$
\begin{aligned}
&= \|v\|^{2p}_{L^{2p}_{\widetilde{\beta}}([0,t]\times\partial\Omega)} + t|\partial\Omega| + t|\Omega| + \|\nabla v\|^2_{L^2([0,t]\times\Omega)} \\
&\quad + \|v_t\|^2_{L^2_{\widetilde{\beta}}([0,t]\times\partial\Omega)} + \|\nabla_\tau v\|^2_{L^2([0,t]\times\partial\Omega)} + \|\nabla_\tau u\|^2_{L^2([0,t]\times\partial\Omega)} \\
&\leq c_2 \|v\|^{2p}_{H^1([0,t]\times\partial\Omega)} + t|\partial\Omega| + t|\Omega| + \|\nabla v\|^2_{L^2([0,t]\times\Omega)} \\
&\quad + \|v_t\|^2_{L^2_{\widetilde{\beta}}([0,t]\times\partial\Omega)} + \|\nabla_\tau v\|^2_{L^2([0,t]\times\partial\Omega)} + \|\nabla_\tau u\|^2_{L^2([0,t]\times\partial\Omega)} \\
&\leq c_3 \left[M(\widetilde{f},\widetilde{\beta},\widetilde{q}) + M(f,\beta,q) + M(\widetilde{f},\widetilde{\beta},\widetilde{q})^p\right](1+t)^{\max\{1,p\}} + t|\partial\Omega|,
\end{aligned}
$$

for some constants $c_2, c_3 > 0$, where $M(\cdot,\cdot,\cdot)$ is defined in (1.11). □

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