Advances in Differential Equations Volume 16, Numbers 9-10 (2011), 867–894

REGULARITY OF A VERY WEAK SOLUTION FOR PARABOLIC EQUATIONS AND APPLICATIONS

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(Submitted by: Roger Temam)

Dedicated to Professor J.I. Díaz on the occasion of his 60^{th} birthday

Abstract. In this paper we study the regularity of the so-called very weak solution for a parabolic equation. This unique solution is only integrable over the parabolic cylinder. The initial data and the right-hand side of the linear parabolic equation are functions integrable with respect to the weight function which corresponds to the distance function. In particular, we prove some global regularity of the space-gradient in Lorentz spaces. The regularity with respect to the time derivative is obtained under the condition that the linear operator is time independent and self-adjoint via m-accretive theory.

1. Introduction

In a recent study [10, 11] we looked at the question of the regularity of the gradient of the so-called weak solution in the sense of H. Brezis for the elliptic linear equation $Lu = f$ in an open bounded smooth set Ω with the Dirichlet boundary $u = 0$ on $\partial\Omega$, where f is a function integrable with respect to the weight function $\delta(x) = \text{distance}\,(x; \partial\Omega)$; that is, $f\delta \in L^1(\Omega)$. H. Brezis [4] has shown the existence of a unique solution u of the following problem:

$$(B) \qquad \begin{cases} u \in L^1(\Omega), \\ \displaystyle\int_\Omega u L^*\varphi\, dx = \int_\Omega f\varphi\, dx, \ \forall\, \varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega. \end{cases}$$

Here L^* is the adjoint operator of L.

We have shown [10] that the solution u belongs to the Lorentz space $L^{N',\infty}(\Omega)$ with $N' = \frac{N}{N-1}$ (in $L^\infty(\Omega)$ if $N = 1$). Moreover, for $f \geqslant 0$,

Accepted for publication: April 2011.
AMS Subject Classifications: 35K10, 35D30, 35K67; 35K65.

$u \in W_0^{1,q}(\Omega)$ for some $q > 1$ if and only if $f \in L^1(\Omega, \delta^\alpha)$ for some $0 \leqslant \alpha < 1$. The precise regularity for a signed f is obtained in Lorentz spaces:

$$|\nabla u| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega) \text{ if } f \in L^1(\Omega, \delta^\alpha).$$

One of the advantages of the Brezis weak formulation is to derive results when one cannot expect a classical solution, for instance, if we consider the following problem:

Let $a \in L^\infty(\Omega)$, $0 < a_0 \leqslant a(x)$ *almost everywhere in* Ω *and let* $\gamma \geqslant 1$ (*for simplicity*). *Then there exists a function* $u \in W_0^{1,1}(\Omega)$ *with* $|\nabla u| \in L^{N(\alpha),\infty}(\Omega)$, $u > 0$ *in* Ω *such that* $-\Delta u(x) = \frac{a(x)}{u(x)^\gamma}$ *in* Ω, *with*

$$\begin{cases} N(\alpha) = \frac{N}{N-1+\alpha}, \ \forall\, \alpha \in [\frac{\gamma-1}{\gamma+1}, 1) & \text{if } \gamma > 1, \\ N(\alpha) = \frac{N}{N-1} & \text{if } \gamma = 1. \end{cases}$$

Such results are direct consequences of the above regularity and extend the result of Crandall-Rabinowitz-Tartar [8]. The proof relies on the fact that

$$au^{-\gamma} \in L^1(\Omega, \delta^\alpha), \text{ for suitable } \alpha \in [0, 1).$$

The same techniques help to improve the results obtained by Diaz-Morel-Oswald [9]. We shall consider an application analogous to [9] at the end of this paper. A more general development treating the above elliptic singular equations is given in [12].

The aim of this paper is to give similar regularity results for the following parabolic problem:

For $T > 0$, $Q_T = \Omega \times (0, T) \subset \mathbb{R}^{N+1}$, $f\delta \in L^1(Q_T)$, $u_0 \in L^1(\Omega, \delta)$, $u \in L^1(Q_T)$ *such that*

$$(B_T) \begin{cases} \displaystyle\int_{Q_T} \Big(-\frac{\partial \varphi}{\partial t} + L^*(t)\varphi \Big) u \, dx \, dt = \int_{Q_T} f\varphi \, dx \, dt + \int_\Omega u_0 \varphi(x, 0) \, dx, \\ \forall\, \varphi \in W_\infty^{2,1}(Q_T), \ \varphi(x, T) = 0, \ \forall\, x \in \overline{\Omega}, \\ \text{and } \varphi(x, t) = 0, \ x \in \partial\Omega, \ t \in [0, T]. \end{cases}$$

Here $W_\infty^{2,1}(Q_T) = \{v \in L^\infty(Q_T) : D_t^r D_x^s v \in L^\infty(Q_T), \ 2r + s \leqslant 2\}$, D_t or D_x denotes the derivative respectively on time variable or space variables. $L^*(t)$ will be the adjoint of the operator

$$L(t)u = -\sum_{i,j=1}^N \partial_{x_i}(a_{ij}(x,t)\partial_{x_j} u) + \sum_{i=1}^N b^i(x,t)\partial_{x_i} u + c_0(x,t)u.$$

In particular, we shall show the following result:

Theorem 1. *The unique weak solution of u of (B_T) satisfies*

$$|u|_{L^{q',\infty}(Q_T)} \leqslant c\big(|f|_{L^1(Q_T,\delta)} + |u_0|_{L^1(\Omega,\delta)}\big),$$

for $1 \leqslant q' < \frac{N+2}{N+1}$*, where* c *is a constant depending only on* T, Ω, *and* N.

We shall give some conditions on f and u_0 in order to derive the regularity $u \in L^{N',\infty}(Q_T)$.

If $\alpha \in [0,1)$, $p' = \frac{N}{N-1+\alpha}$, $u_0 \in L^{p',\infty}(\Omega)$, and $f \in L^{p'}(0,T;L^1(\Omega,\delta^\alpha))$, then

$$|\nabla_x u| \in L^{p',\infty}(Q_T).$$

If L is a self-adjoint operator which does not depend on the time t and $b_i = 0$ (for simplicity) then we can use the m-accretive theory to derive regularity results. In particular,

let φ_1 *be the first eigenfunction of* L,

$$L\varphi_1 = \lambda\varphi_1,\ \varphi_1 \in H_0^1(\Omega)\cap W^{2,p}(\Omega),\ p > N;$$

for $\alpha \in [0,1]$, *we denote* $D(L) = \big\{u \in W_0^{1,1}(\Omega) : Lu \in L^1(\Omega,\varphi_1^\alpha)\big\}$, *with*

$$\begin{array}{rcl} L : D(L) \subset L^1(\Omega,\varphi_1^\alpha) & \to & L^1(\Omega,\varphi_1^\alpha), \\ u & \mapsto & Lu = f, \end{array}$$

if and only if

$$\int_\Omega uL\varphi\,dx = \int_\Omega f\varphi\,dx\ \forall\,\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega.$$

Then L *is an* m*-accretive operator in* $L^1(\Omega,\varphi_1^\alpha)$.

As far as we are aware, H. Brézis and W. Strauss [3] were the first to prove the m-accretive property for the L^1-data semilinear equation, which corresponds to $\alpha = 0$ in our case. Weak solutions for parabolic equations analogous to (B_T)-formulations are considered in [1, 7] but only existence and uniqueness results are given under different frameworks; in particular, they use the Laplacian operator. In [13], the authors prove the existence of a C_o-contraction semigroup $e^{t\Delta}$ on $L^1(\Omega,\varphi_1)$. They show that the semigroup $e^{t\Delta}$ maps $L^q(\Omega,\varphi_1)$ into itself. Moreover, they show a regularizing effect; that is, for $t > 0$ the solution $u(t) = e^{t\Delta}u_0$ belongs to $L^\infty(\Omega)$ whenever u_0 is in $L^1(\Omega,\varphi_1)$. No regularity is presented in the aforementioned studies for the space-gradient nor the time derivative. They essentially studied the effect of the initial data and the nonlinearity of the type $|u|^{p-1}u$ on the right-hand side of the equation. Their method is not based on m-accretive theory as ours is here. We should also notice that $L^q(\Omega,\varphi_1)$ is strictly contained in

$L^1(\Omega, \varphi_1^{\frac{1}{q}})$ and our results on L imply in particular that $-L$ is the generator of a semigroup of contraction in $L^1(\Omega, \varphi_1^\alpha)$ (see [5]). We shall end the paper by presenting an application of the above theorems. Many more applications will be given in a later study (work in progress).

2. Notation and preliminary results

We shall consider Ω an open bounded set of $\mathbb{R}^N$, of class $C^{2,1}$, and $T > 0$. We set $Q_T = \Omega \times (0, T)$, and consider $a_{ij} \in C^{0,1}(\overline{Q_T})$, $b^i \in C^{0,1}(\overline{Q_T})$, $c_0 \in L^\infty(Q_T)$, $c_0 \geqslant 0$, $\forall\, \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$, $B = (b^1, \ldots, b^N)$

$$\sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j \geqslant \alpha_0|\xi|^2,\ \alpha_0 > 0,\ c_0(x,t) - \sum_{i=1}^N \partial_{x_i} b^i(x,t) \geqslant 0 \text{ a.e. } Q_T. \tag{$H1$}$$

We associate the operator $L(t)$ to those functions. Let $L^*(t)$ be its adjoint:

$$L(t)u = -\sum_{i,j=1}^N \partial_{x_i}\big(a_{ij}(x,t)\partial_{x_j}u\big) + \sum_{i=1}^N b^i(x,t)\partial_{x_i}u + c_0(x,t)u, \tag{2.1}$$

and

$$L^*(t)\varphi = -\sum_{i,j=1}^N \partial_{x_j}\big(a_{ij}(x,t)\partial_{x_i}\varphi\big) - \sum_{i=1}^N \partial_{x_i}(b^i\varphi) + c_0\varphi, \tag{2.2}$$

where $\partial_{x_j} = \frac{\partial}{\partial x_j}$ denotes the partial derivative with respect to the variable x_j. We will adopt also some notation used in [14].

If $L(t)$ is independent of t we set $L(t) = L$.

If $\varphi : Q_T \to \mathbb{R}$, then for t fixed, $\varphi(t) : \Omega \to \mathbb{R}$ is defined by $\varphi(t)(x) = \varphi(x,t)$.

We shall denote by $\Omega_T = \Omega$ or Q_T, and if $E \subset \Omega_T$ a Lebesgue-measurable set we denote by $|E|$ its measure and χ_E its characteristic function.

For $u : \Omega_T \to \mathbb{R}$ a measurable function, we recall that the **decreasing rearrangement of** u is given by $\Omega_{T_*} = (0, |\Omega_T|) \to \mathbb{R}$ such that

$$u_*(s) = \inf\Big\{t \in \mathbb{R} : |u > t| \leqslant s\Big\}, \quad s \in \Omega_{T_*}.$$

$$u_*(0) = \operatorname*{ess\,sup}_{\Omega_T} u, \qquad u_*(|\Omega_T|) = \operatorname*{ess\,inf}_{\Omega_T} u.$$

We set

$$u_{**}(s) = \frac{1}{s}\int_0^s |u|_*(t)dt, \quad s \in \Omega_{T_*}.$$

For $1 < p < +\infty$ and $1 \leqslant q < +\infty$, we define the Lorentz space

$$L^{p,q}(\Omega_T) = \Big\{ v : \Omega_T \to \mathbb{R} \text{ measurable} : |v|^q_{p,q} = \int_0^{|\Omega_T|} \Big[t^{\frac{1}{p}} v_{**}(t) \Big]^q \frac{dt}{t} < +\infty \Big\},$$

and

$$L^{p,\infty}(\Omega_T) = \Big\{ v : \Omega_T \to \mathbb{R} : |v|_{p,\infty} = \sup_{s \leqslant |\Omega_T|} \Big[s^{\frac{1}{p}} v_{**}(s) \Big] < +\infty \Big\}.$$

$$W^2(\Omega, |\cdot|_{p,q}) = \Big\{ v \in W^{2,1}(\Omega) : D^2_x v \in L^{p,q}(\Omega) \Big\}.$$

See [16] for more properties and details.

We also use standard notation on parabolic equations; namely, if V is a Banach function endowed with a norm denoted by $\|\cdot\|$, then for $1 \leqslant p < +\infty$

$$L^p(0,T;V) = \Big\{ v : \big(0,T\big) \to V \text{ Bochner measurable} : \int_0^T \|v(t)\|^p dt < +\infty \Big\}.$$

Its norm is denoted by $\|\cdot\|_{L^p(0,T;V)}$,

$$L^\infty(0,T;V) = \Big\{ v \in L^1(0,T;V) : \operatorname*{ess\,sup}_{t \leqslant T} \|v(t)\| < +\infty \Big\}.$$

In particular, $L^p(0,T;L^p(\Omega)) = L^p(Q_T)$, and the norm on this space shall be denoted sometimes $\|\cdot\|_{p,Q_T}$ or $\|\cdot\|_{L^p(Q_T)}$.

Following the notation in [14], we will use

$$V^{2,1}_{q,0}(Q_T) = \Big\{ v \in W^{2,1}_q(Q_T) : v(x,T) = 0,\ x \in \Omega \Big\}.$$

The space $W^{2,1}_q(Q_T)$ is defined as before and $q > 1$.

One can replace the Lebesgue space $L^q(\Omega)$ by Lorentz spaces $L^{p,q}(\Omega)$, and we shall sometimes consider

$$W^{2,1}_{p,q}(Q_T) = \Big\{ v \in L^{p,q}(Q_T) : D^r_t D^s_x \in L^{p,q}(Q_T) : 2r + s \leqslant 2 \Big\}.$$

The following result has been proved by Sun-Sig Byun [6]:

Theorem 2. *Let $1 < p < +\infty$, $W^{-1,p'}(\Omega)$ be the dual space of $W^{1,p}_0(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $F \in L^p(Q_T;\mathbb{R}^N)$, and $A(x,t) = \big(a_{ij}(x,t)\big)_{1 \leqslant i,j \leqslant N}$, the coefficients as given above. Then there exists a unique function $\varphi \in L^p\big(0,T;W^{1,p}_0(\Omega)\big)$ such that*

$$\frac{\partial \varphi}{\partial t} \in L^{p'}\Big(0,T;W^{-1,p'}(\Omega)\Big),$$

$$-\int_0^T \Big\langle \varphi(t), \frac{\partial \psi}{\partial t}(t) \Big\rangle_{W^{1,p}_0(\Omega),W^{-1,p'}(\Omega)} dt + \int_{Q_T} A \nabla \varphi \nabla \psi \, dx\, dt = \int_{Q_T} F \nabla \psi \, dx\, dt,$$

$\forall\, \psi \in L^{p'}\big(0,T;W_0^{1,p'}(\Omega)\big)$ *such that* $\frac{\partial\psi}{\partial t} \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)$, $\psi(x,T)=0$ *for* $x\in\Omega$. *Moreover, there exists a constant* $c(A,p,\Omega)=c>0$, *such that*

$$\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant c\|F\|_{L^p(Q_T)}.$$

As a corollary of the above Theorem 2 one has the following:

Corollary 1. *Let* $2\leqslant p<+\infty$ *and let* $L(t)$ *be the linear operator given in* (2.1). *Then there exists a unique function* $\varphi\in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ *with*

$$\frac{\partial\varphi}{\partial t} \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)+L^p(Q_T),$$

such that $\forall\,\psi \in L^{p'}\big(0,T;W_0^{1,p}(\Omega)\big)$, $\frac{\partial\psi}{\partial t} \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)+L^{p'}(Q_T)$, $\psi(x,T)=0$ *for* $x\in\Omega$, *one has*

$$-\int_0^T \langle\varphi(t),\frac{\partial\psi}{\partial t}\rangle dt+\int_{Q_T} A\nabla\varphi\nabla\psi\,dx\,dt+\int_{Q_T} B\cdot\nabla\varphi\psi\,dx\,dt+\int_{Q_T} c_0\varphi\psi\,dx\,dt$$
$$=\int_{Q_T} F\nabla\psi\,dx\,dt.$$

Moreover, one has a constant $c>0$*:*

$$\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant c\|F\|_{L^p(Q_T)}.$$

Remark 1. In Byun's results ([6]) the assumptions on A are weaker than those considered here. In the above corollary, we can use Byun's assumptions for the coefficients of A.

We shall denote by c various constants depending on the data.

Proof of Corollary 1. Since $p\geqslant 2$, we deduce that there exists a unique $\varphi\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$ such that

$$\begin{cases}\partial_t\varphi-\mathrm{div}\,(A\nabla\varphi)+B\nabla\varphi+c_0\varphi=-\mathrm{div}\,(F) & \text{in } Q_T,\\ \varphi(0)=0,\end{cases}$$

and

$$\|\varphi\|_{L^\infty(0,T;L^2(\Omega))}+\|\varphi\|_{L^2(0,T:H_0^1(\Omega))}\leqslant c\|F\|_{L^2(Q_T)}. \tag{2.3}$$

Setting $q_1=\frac{2}{N}(N+2)$, we deduce from an interpolation inequality (see [14]) that

$$\|\varphi\|_{L^{q_1}(Q_T)}\leqslant c\|F\|_{L^2(Q_T)}. \tag{2.4}$$

For a fixed t, let us introduce the function

$$w(t)\in H^2(\Omega)\cap H_0^1(\Omega): -\Delta w(t)=(c_0-\mathrm{div}\,(B))\varphi(t). \tag{2.5}$$

One has from an elliptic regularity result

$$\|\nabla w(t)\|_{L^{q_1}(\Omega)} \leqslant c\|\varphi(t)\|_{L^{q_1}(\Omega)}, \quad c \text{ is independent of } t. \tag{2.6}$$

Therefore,

$$\|\nabla w\|_{L^{q_1}(Q_T)} \leqslant c\|\varphi\|_{L^{q_1}(Q_T)} \leqslant c\|F\|_{L^2(Q_T)}. \tag{2.7}$$

Thus the equation satisfied by φ can be written as

$$\partial_t\varphi - \operatorname{div}(A\nabla\varphi) = -\operatorname{div}(F + B\varphi - \nabla w).$$

- If $p \leqslant q_1$, then

$$\|F + B\varphi - \nabla w\|_{L^p(Q_T)} \leqslant c\big(\|F\|_{l^p(Q_T)} + \|F\|_{L^2(Q_T)}\big) \leqslant c\|F\|_{L^p(Q_T)},$$

  and from Theorem 2 we conclude

$$\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega)} \leqslant c\|F\|_{L^p(Q_T)}.$$

- If $p > q_1$, then $F + B\varphi - \nabla w \in L^{q_1}(Q_T)^N$. From Theorem 2 one has

$$\|\varphi\|_{L^{q_1}(0,T;W^{1,q_1}(\Omega))} \leqslant c\|F + B\varphi - \nabla w\|_{L^{q_1}(Q_T)}.$$

  Since

$$\|\varphi\|_{L^{q_1}(Q_T)} + \|\nabla w\|_{L^{q_1}(Q_T)} \leqslant c\|F\|_{L^2(Q_T)},$$

  one deduces

$$\|\varphi\|_{L^{q_1}(0,T;W^{1,q_1}(\Omega))} \leqslant c\|F\|_{L^p(Q_T)}. \tag{2.8}$$

From an interpolation result, we deduce using relation (2.8) that

$$\|\varphi\|_{L^{q_2}(Q_T)} \leqslant c\big(\|\varphi\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi\|_{L^{q_1}(0,T;W^{1,q_1}(\Omega))}\big), \tag{2.9}$$

with $q_2 = \frac{q_1}{N}(N+2)$. From relations (2.3), (2.4), (2.8), and (2.9), one has

$$\|\varphi\|_{L^{q_2}(Q_T)} \leqslant c\|F\|_{L^p(Q_T)}. \tag{2.10}$$

We deduce that

$$\|\nabla w\|_{L^{q_2}(Q_T)} \leqslant c\|\varphi\|_{L^{q_2}(Q_T)} \leqslant c\|F\|_{L^p(Q_T)}. \tag{2.11}$$

Thus we have two cases as before:

- If $p \leqslant q_2$, then

$$\|F + B\varphi - \nabla w\|_{L^p(Q_T)} \leqslant c\big(\|F\|_{L^p(Q_T)}\big),$$

  and applying Theorem 2, one has

$$\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant c\|F\|_{L^p(Q_T)}.$$

  The proof is done.

- If $p > q_2$, then

$$\|F + B\varphi - \nabla w\|_{L^{q_2}(Q_T)} \leqslant c\|F\|_{L^p(Q_T)},$$

and from Theorem 2, we have

$$\|\nabla\varphi\|_{L^{q_2}(Q_T)} \leqslant c\|F\|_{L^p(Q_T)}.$$

From an interpolation argument

$$\|\varphi\|_{L^{q_3}(Q_T)} \leqslant c\|F\|_{L^p(Q_T)} \text{ with } q_3 = \frac{q_2}{N}(N+2).$$

Then

$$\|B\varphi - \nabla w\|_{L^{q_3}(Q_T)} \leqslant c\|F\|_{L^p(Q_T)}.$$

- If $q \leqslant q_3$, the proof is done.
- Otherwise, we continue as before and we stop when we reach the number

$$q_m : p < q_m, \quad q_m = \frac{q_{m-1}}{N}(N+2), \quad m \geqslant 1. \qquad \square$$

Next, we also recall the following results obtained in [10]:

Theorem 3. *Let* $f \in L^1(\Omega, \delta^\alpha)$, $\alpha \in [0, 1)$. *Then the unique function* $u \in L^1(\Omega)$,

$$\int_\Omega uL^*\varphi\, dx = \int_\Omega f\varphi\, dx \qquad \forall\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega,$$

satisfies $|\nabla u| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$. *Moreover, there exists a constant* $c(\alpha, \Omega) > 0$ *such that*

$$\|\nabla u\|_{L^{\frac{N}{N-1+\alpha},\infty}(\Omega)} \leqslant c(\alpha, \Omega)\|f\|_{L^1(\Omega,\delta^\alpha)},$$

where L^* *is independent of* t *and* $\|f\|_{L^1(\Omega,\delta^\alpha)} = \int_\Omega |f|\delta^\alpha\, dx$.

3. $L^{q,\infty}(Q_T)$-regularity for the weak solution (B_T).

Now we want to prove Theorem 1.

Proof of Theorem 1. Let $f_n = T_n(f) = \min(|f|; n)\text{sign}\,(f)$, with

$$\text{sign}\,(\sigma) = \begin{cases} +1 & \text{if } \sigma > 0, \\ 0 & \text{if } \sigma = 0, \\ -1 & \text{if } \sigma < 0, \end{cases}$$

and $u_{0n} \in C_c^\infty(\Omega) : u_{0n} \to u_0$ in $L^1(\Omega, \delta)$. There exists a unique function $u_n \in W_q^{2,1}(Q_T)$ for all finite $q > 1$ such that

$$(\mathcal{P}_{nL}T) \begin{cases} \frac{\partial u_n}{\partial t} + L(t)u_n = f_n & \text{in } Q_T, \\ u_n = 0 & \text{on } \Omega \times (0, T), \\ u_n(0) = u_{0n}. \end{cases}$$

Let E be a measurable subset of Q_T and for k and n fixed, we consider φ_E^{nk} a solution of

$$\begin{cases} -\partial_t \varphi_E^{nk} + L^*(t)\varphi_E^{nk} = \chi_E \text{sign}\,(u_n - u_k) = g_{n,k}, \\ \varphi_E^{nk} = 0 \text{ on } \partial\Omega \times [0, T), \\ \varphi_E^{nk}(T) = 0, \quad \varphi_E^{nk} \in W_q^{2,1}(Q_T) \quad \forall\, q > 1 \text{ (finite)}. \end{cases}$$

Such a function exists by solving $\widehat{\varphi_E^{nk}}(x, t)$:

$$\begin{cases} \partial_t \widehat{\varphi_E^{nk}} + L^*(t)\widehat{\varphi_E^{nk}} = g_{n,k} \text{ in } Q_T, \\ \widehat{\varphi_E^{nk}} = 0 \text{ on } \sum_T = \partial\Omega \times (0, T), \\ \widehat{\varphi_E^{nk}}(x, 0) = 0,\ x \in \Omega. \end{cases}$$

Then, we can set $\varphi_E^{nk}(x, t) = \widehat{\varphi_E^{nk}}(x, T - t))$.

Making the difference between the equation satisfied by u_n and the one satisfied by u_k, using as a test function φ_E^{nk} we have

$$\int_{Q_T} \left(-\frac{\partial \varphi_E^{nk}}{\partial t} + L^*(t)\varphi_E^{nk} \right) (u_n - u_k)\, dx\, dt$$
$$= \int_{Q_T} (f_n - f_k)\varphi_E^{nk}\, dx\, dt + \int_\Omega (u_{0n} - u_{0k})\varphi_E^{nk}(0)\, dx. \tag{3.1}$$

Then we deduce

$$\int_E |u_n - u_k|\, dx\, dt \leqslant \int_{Q_T} |f_n - f_k|\ |\varphi_E^{nk}|\, dx\, dt + \int_\Omega |u_{0n} - u_{0k}||\varphi_E^{nk}(x, 0)|\, dx. \tag{3.2}$$

By the regularity of φ_E^{nk} (see [14]) there exists a constant $c(q, \Omega) > 0,\ q > N + 2$, such that

$$\sup_{\sigma \leqslant T} \|\nabla \varphi_E^{nk}(\sigma)\| = \sup_{t \leqslant T} \|\nabla \widehat{\varphi_E}(t)\|_{L^\infty(\Omega)} \leqslant c\|\chi_E\|_{L^q(Q_T)} \leqslant c|E|^{\frac{1}{q}}.$$

Therefore,

$$|\varphi_E^{nk}(x, t)| \leqslant c|E|^{\frac{1}{q}} \delta(x), \quad \forall\, x \in \overline{\Omega},\ \forall\, t \in \big[0, T\big],$$

$$|E|^{-\frac{1}{q}}\int_E |u_n-u_k|\,dx\,dt \leqslant c\left[\int_{Q_T} |f_n-f_k|\delta(x)\,dx\,dt+\int_\Omega |u_{0n}-u_{0k}|\delta(x)\,dx\right].$$

From this, one has

$$\sup_{t\leqslant|Q_T|}\left[t^{\frac{1}{q'}}|u_n-u_k|_{**}(t)\right]\leqslant c\left[|f_n-f_k|_{L^1(Q_T,\delta)}+|u_{0n}-u_{0k}|_{L^1(\Omega,\delta)}\right],$$

$+\infty > q > N+2$ thus $1 \leqslant q' < \frac{N+2}{N+1}$.

The uniqueness follows the same argument given in [2]; see also [7] using regularity of parabolic equations. □

4. Gradient regularity for the very weak solution

We shall now investigate the global regularity of the gradient. Next, we want to prove the following result:

Theorem 4. *Let $\alpha \in [0,1)$, $N(\alpha) = \frac{N}{N-1+\alpha}$, $u_0 \in L^{N(\alpha),\infty}(\Omega)$, and $f \in L^{N(\alpha)}\big(0,T;L^1(\Omega,\delta^\alpha)\big)$. Then the unique weak solution u of (B_T) satisfies*

$$|\nabla_x u| \in L^{N(\alpha),\infty}(Q_T).$$

Moreover, there exists a constant $c(\Omega,\alpha,T)>0$ such that

$$\|\nabla_x u\|_{L^{N(\alpha),\infty}(Q_T)} \leqslant c\Big[\|f\|_{L^{N(\alpha)}(0,T;L^1(\Omega,\delta^\alpha))}+\|u_0\|_{L^{N(\alpha),\infty}(\Omega)}\Big].$$

Proof. Let E be a measurable subset of Q_T and

$$H(u_k)=\begin{cases}\dfrac{\nabla u_k}{|\nabla u_k|} & \text{if } \nabla u_k \neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

where $u_k \in W_q^{2,1}(Q_T)$ $\forall q>1$ is the solution associated to the above regular approximate problem $(\mathcal{P}_{kL}T)$.

Let $\varphi_E \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)\cap C(\overline{Q_T})$, $p>N$, with

$$\frac{\partial\varphi_E}{\partial t}\in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big),\quad \frac{1}{p}+\frac{1}{p'}=1,$$

satisfying

$$(\mathcal{P}_{kE})\begin{cases}\partial_t\varphi_E+L^*(t)\varphi_E=-\mathrm{div}\,(\chi_E H(u_k)) & \text{in } Q_T,\\ \varphi_E=0 & \text{on } \sum_T=\partial\Omega\times(0,T),\\ \varphi(x,0)=0 & x\in\Omega.\end{cases}$$

Setting $\widehat{\varphi_E}(x,t) = \varphi_E(x,T-t)$, we then have

$$\int_E |\nabla u_k|\,dx\,dt = \int_0^T \big\langle -\partial_t\widehat{\varphi_E} + L^*(t)\widehat{\varphi_E}, u_k\big\rangle dt, \tag{4.1}$$

where the bracket denotes the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. By an integration by parts, one deduces

$$\begin{aligned}\int_E |\nabla u_k|\,dx\,dt &= \int_{Q_T} \widehat{\varphi_E}\frac{\partial u_k}{\partial t}\,dx\,dt + \int_{Q_T} \widehat{\varphi_E}L(t)u_k\,dx\,dt + \int_\Omega \widehat{\varphi_E}(x,0)u_{0k}\,dx\\ &= \int_{Q_T} f_k\widehat{\varphi_E}\,dx\,dt + \int_\Omega \widehat{\varphi_E}(x,0)u_{0k}(x)\,dx.\end{aligned} \tag{4.2}$$

Let us choose p such that $\alpha = 1-\frac{N}{p}$. Therefore, by the Sobolev embedding, we have that for all $(x,t)\in\overline{Q_T}$

$$|\widehat{\varphi_E}(x,t)| \leqslant c\|\widehat{\varphi_E}(t)\|_{W_0^{1,p}(\Omega)}\cdot\delta^\alpha(x), \tag{4.3}$$

for some constant $c(\Omega,p)$ depending only on p and Ω. Thus relations (4.2) and (4.3) lead to

$$\begin{aligned}\int_E |\nabla u_k|\,dx\,dt &\leqslant c\Big[\int_0^T \|\nabla\widehat{\varphi_E}(t)\|^p_{L^p(\Omega)}dt\Big]^{\frac1p}\\ &\times\Big[\int_0^T\Big(\int_\Omega |f_k(x,t)|\delta^\alpha(x)\,dx\Big)^{p'}dt\Big]^{\frac{1}{p'}} + \|\widehat{\varphi_E}(0)\|_{L^{p,1}(\Omega)}\|u_{0k}\|_{L^{p',\infty}(\Omega)}.\end{aligned} \tag{4.4}$$

Let us show the following lemma:

Lemma 1. *There exists a constant $c(T,\Omega,p) = c > 0$ such that*

$$\|\widehat{\varphi_E}(t)\|_{L^{p,1}(\Omega)} \leqslant c|E|^{\frac1p} \qquad \forall\, t\in[0,T].$$

Proof. Let $m\geqslant 1$, $\Phi(\varphi_E) = |\varphi_E|^{m-1}\varphi_E$, and $\psi_E = \frac{m}{m+1}|\varphi_E|^{m+1}$.

By our assumption on the coefficients c_0 and b^i we then have

$$c_0\varphi_E\Phi(\varphi_E) - (\operatorname{div} B)\psi_E \geqslant 0 \text{ almost everywhere.} \tag{4.5}$$

Therefore, the first equation of $(\mathcal{P}_{kE})$ leads to

$$\begin{aligned}\big\langle\partial_t\varphi_E, \Phi(\varphi_E)\big\rangle + \int_\Omega A\nabla\varphi_E\nabla\Phi(\varphi_E)\,dx + \int_\Omega (c_0\varphi_E\Phi(\varphi_E) - \operatorname{div} B\psi_E)\,dx\\ = \int_\Omega \chi_E H(u_k)\nabla\Phi(\varphi_E)\,dx \quad \text{in } \mathcal{D}'(0,T).\end{aligned} \tag{4.6}$$

Using an integration by parts and dropping nonnegative terms, we finally derive from relation (4.6) that

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega |\varphi_E|^{m+1}(x,t)\,dx + \alpha_0\int_\Omega |\nabla\varphi_E|^2\Phi'(\varphi_E)\,dx \leqslant \int_\Omega \chi_E(x,t)\Phi'(\varphi_E)|\nabla\varphi_E|\,dx. \tag{4.7}$$

From (4.7) using Young's inequality, one derives

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega |\varphi_E|^{m+1}(x,t)\,dx + \frac{\alpha_0}{2}\int_\Omega |\nabla\varphi_E|^2\Phi'(\varphi_E)\,dx \leqslant c\int_\Omega \chi_E(x,t)\Phi'(\varphi_E)\,dx. \tag{4.8}$$

Setting $Z(t) = \int_\Omega |\varphi_E|^{m+1}(x,t)\,dx$, one has from relation (4.8)

$$\frac{1}{m+1}Z'(t) \leqslant c\left(\int_\Omega \chi_E(x,t)\,dx\right)^{\frac{2}{m+1}} Z(t)^{\frac{m-1}{m+1}}. \tag{4.9}$$

We obtain

$$Z(t)^{\frac{1}{m+1}} \leqslant c\big(\max(1;T)\big)^{\frac{m-1}{2(m+1)}}|E|^{\frac{1}{m+1}}.$$

Thus, for all $m \geqslant 2$, for all $t \in [0,T]$,

$$\|\varphi_E(T-t)\|_{L^m(\Omega)} = \|\widehat{\varphi_E}(t)\|_{L^m(\Omega)} \leqslant c_{0m}|E|^{\frac{1}{m}}. \tag{4.10}$$

By an interpolation argument, one deduces

$$\|\widehat{\varphi_E}(t)\|_{L^{p,1}(\Omega)} \leqslant c_{0p}|E|^{\frac{1}{p}} \quad \forall\, p \geqslant 2. \tag{4.11}$$

From Buyn's theorem (see Corollary 1 of Theorem 2) one has

$$\left(\int_0^T \|\nabla\widehat{\varphi_E}(t)\|^p_{L^p(\Omega)}dt\right)^{\frac{1}{p}} \leqslant c\|\chi_E H(u_k)\|_{L^p(Q_T)} \leqslant c|E|^{\frac{1}{p}}. \tag{4.12}$$

Therefore, relations (4.4) and (4.12) along with Lemma 1 together lead to

$$|E|^{-\frac{1}{p}}\int_E |\nabla u_k|\,dx\,dt \leqslant c\Big[\int_0^T\Big(\int_\Omega |f_k(x,t)|\delta^\alpha(x)\,dx\Big)^{p'}dt\Big]^{\frac{1}{p'}} + c\|u_{0k}\|_{L^{p',\infty}(\Omega)}. \tag{4.13}$$

From relation (4.13), using the Hardy-Littlewood inequality one derives

$$\sup_{s\leqslant|Q_T|}\left[s^{\frac{1}{p'}}|\nabla u_k|_{**}(s)\right] \leqslant c\left[\|f_k\|_{L^{p'}(0,T;L^1(\Omega,\delta^\alpha))} + \|u_{0k}\|_{L^{p',\infty}(\Omega)}\right]. \tag{4.14}$$

This last relation shows that $(\nabla u_k)_{k\geqslant 0}$ is also a Cauchy sequence in $L^{p',\infty}(Q_T)^N$ since u_k tends to u in $L^1(Q_T)$; thus, $|\nabla_x u| \in L^{p',\infty}(Q_T)$. □

5. $L^{N',\infty}(Q_T)$-regularity for the very weak solution

In this section, we shall give some conditions ensuring the global regularity in $L^{N',\infty}(Q_T)$. We start with a maximum principle.

Lemma 2. *If $f \geqslant 0$ and $u_0 \geqslant 0$, then the unique weak solution $u \geqslant 0$.*

Proof. Indeed, using the approximate solution $u_k \in W_q^{2,1}(Q_T)$ $\forall\, q > 1$ satisfying

$$\begin{cases} \dfrac{\partial u_k}{\partial t} + L(t)u_k = f_k \geqslant 0 \quad \text{in } Q_T, \\ u_k(0) = u_{0k} \geqslant 0, \end{cases}$$

one has by usual maximum principle $u_k \geqslant 0$. □

Theorem 5. *Let $f \in L^\infty\big(0,T;L^1(\Omega,\delta)\big)$ and $u_0 \in L^{N',\infty}(\Omega)$. Then the unique solution u of (B_T) satisfies*

$$\begin{cases} u \in L^{N',\infty}(Q_T), \quad N' = \dfrac{N}{N-1} & \text{if } N \geqslant 2, \\ u \in L^\infty(Q_T) & \text{if } N = 1. \end{cases}$$

Moreover, there exists a constant $c > 0$ such that

$$\|u\|_{L^{N',\infty}(Q_T)} \leqslant c\left[\|f\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|u_0\|_{L^{N',\infty}(\Omega)}\right].$$

Proof. Let E be a measurable subset of Q_T and consider for $\varphi_E \geqslant 0$, $\varphi_E \in W_q^{2,1}(Q_T)$, and $q > N+2$

$$\begin{cases} \partial_t\varphi_E + L^*(t)\varphi_E = \chi_E, \\ \varphi_E = 0, & \text{on } \sum_T = \partial\Omega \times (0,T), \\ \varphi_E(0) = 0. \end{cases}$$

Following the argument of Lemma 1, one has

$$\|\varphi_E(t)\|_{L^{N,1}(\Omega)} \leqslant c|E|^{\frac{1}{N}} \quad \forall\, t \in [0,T]. \tag{5.1}$$

We also need the following interpolation lemma result:

Lemma 3. *Let us define*

$$W_{N,1}^{2,1}(Q_T) = \Big\{v \in L^{N,1}(Q_T) : D_t^r D_x^s v \in L^{N,1}(Q_T),\ 2r + s \leqslant 2\Big\}.$$

Then any solution v of

$$(\mathcal{P}_g)\begin{cases}\dfrac{\partial v}{\partial t}+L^*(t)v=g\in L^{N,1}(Q_T),\\ v=0\ \text{on}\ \sum_T=\partial\Omega\times]0,T[,\\ v(0)=0,\end{cases}$$

belongs to $W^{2,1}_{N,1}(Q_T)$ and there exists a constant $c(\Omega,T)=c>0$ such that

$$\|v\|_{W^{2,1}_{N,1}(Q_T)}\leqslant c\|g\|_{L^{N,1}(Q_T)}.$$

In particular,

$$\|v\|_{L^1(0,T;W^2(\Omega;|\cdot|_{N,1}))}\leqslant c\|g\|_{L^{N,1}(Q_T)}.$$

Remark 2. Let us note that $W^{2,1}_{N,1}(Q_T)\hookrightarrow W^{2,1}_q(Q_T)\ \forall q<N$.

Proof of Lemma 3. The proof relies on Marcinkiewicz's interpolation result. Indeed, let v be the unique solution of $(\mathcal{P}_g)$ for $q>1$. Then the mapping

$$\begin{array}{rcl}\Lambda: L^q(Q_T) & \to & W^{2,1}_q(Q_T),\\ g & \mapsto & \Lambda g=v,\end{array}$$

is linear continuous and

$$\|\Lambda g\|_{W^{2,1}_q(Q_T)}\leqslant c_q\|g\|_{L^q(Q_T)}.$$

For r and s integers such that $2r+s\leqslant 2$, we consider $Tg=D^r_tD^s_x\Lambda g$. Then

$$\|Tg\|_{L^q(Q_T)}\leqslant c_q\|g\|_{L^q(Q_T)}. \tag{5.2}$$

By Marcinkiewicz's interpolation lemma,

$$T\text{ maps }L^{N,1}(Q_T)\text{ into }L^{N,1}(Q_T)\text{ continuously}$$

and

$$\|Tg\|_{L^{N,1}(Q_T)}\leqslant c\|g\|_{L^{N,1}(Q_T)}. \tag{5.3}$$

In particular, we have

$$\|v\|_{W^{2,1}_{N,1}(Q_T)}\leqslant c\|g\|_{L^{N,1}(Q_T)}. \tag{5.4}$$

It remains to show the following lemma:

Lemma 4. *Let $v\in L^{N,1}(Q_T)$. Then*

$$\int_0^T\|v(t)\|_{L^{N,1}(\Omega)}dt\leqslant T^{\frac{N-1}{N}}\|v\|_{L^{N,1}(Q_T)}.$$

Proof. Let v be a nonnegative function in $L^{N,1}(Q_T)$. For a fixed $t \in [0,T]$, and $0 < \sigma < |\Omega|$, $v_*(\sigma,t) = \mathrm{Inf}\left\{y \in \mathbb{R} : \mathrm{meas}\{x \in \Omega : v(x,t) > y\} \leqslant \sigma\right\}$ that is the monotone rearrangement of $v(t) : \Omega \to \mathbb{R}$, $v(t)(x) = v(x,t)$. Then, by the definition of $L^{N,1}$ and using the Hardy-Littlewood inequality

$$\begin{aligned}\int_0^T \|v(t)\|_{L^{N,1}(\Omega)} dt &= \int_0^T \int_0^{|\Omega|} \left[\sigma^{\frac{1}{N}} v_*(\sigma,t)\right] \frac{d\sigma}{\sigma} dt \\ &\leqslant \int_{Q_T^*} \left[(\sigma^{\frac{1}{N}-1})_*(v_*(\sigma,t))_*\right](s) ds, \end{aligned} \tag{5.5}$$

where $Q_T^* = (0, |Q_T| = T|\Omega|)$, one has

$$(\sigma^{\frac{1}{N}-1})_*(s) = T^{\frac{N-1}{N}} s^{\frac{1}{N}-1} \text{ for } s \in (0, T|\Omega|). \tag{5.6}$$

If we consider the rearrangement of v over Q_T, then one has, by equimeasurability,

$$v_*(s) = (v_*(\cdot,\cdot))_*(s), \quad s \in (0, T|\Omega|). \tag{5.7}$$

From relations (5.5) to (5.7), one has

$$\int_0^T \|v(t)\|_{L^{N,1}(\Omega)} dt \leqslant T^{\frac{N-1}{N}} \|v\|_{L^{N,1}(Q_T)}. \qquad \square$$

End of proof of Theorem 5. Without loss of generality we may assume that $f \geqslant 0$ and $u_0 \geqslant 0$. Choosing $\widehat{\varphi_E}(x,t) = \varphi_E(x, T-t)$ as a test function, one derives

$$\int_E u \, dx \, dt = \int_{Q_T} f \widehat{\varphi_E} \, dx \, dt + \int_\Omega u_0 \widehat{\varphi_E}(0) \, dx. \tag{5.8}$$

Since $f \in L^\infty(0,T; L^1(\Omega,\delta))$, one has

$$0 \leqslant \int_{Q_T} \widehat{\varphi_E} f \, dx \, dt \leqslant \left(\int_0^T dt \int_\Omega \frac{|\widehat{\varphi_E}(x,t)|}{\delta(x)} \, dx \right) \left(\sup_{t \leqslant T} \int_\Omega f(x,t)\delta(x) \, dx \right).$$

From Sobolev-Poincaré embedding for Lorentz spaces (see [16]) we have

$$0 \leqslant \widehat{\varphi_E}(x,t) \leqslant c \|\nabla \widehat{\varphi_E}(t)\|_{L^\infty(\Omega)} \delta(x) \leqslant c \|\widehat{\varphi_E}(t)\|_{W^2(\Omega, |\cdot|_{N,1})} \cdot \delta(x). \tag{5.9}$$

Therefore, we derive

$$0 \leqslant \int_{Q_T} \widehat{\varphi_E} f \, dx \, dt \leqslant c \|f\|_{L^\infty(0,T;L^1(\Omega,\delta))} \cdot \int_0^T \|\widehat{\varphi_E}(t)\|_{W^2(\Omega,|\cdot|_{N,1})} dt.$$

Using Lemma 4, one has

$$\int_0^T \|\widehat{\varphi_E}(t)\|_{W^2(\Omega,|\cdot|_{N,1})} dt \leqslant c \|\widehat{\varphi_E}\|_{W^{2,1}_{N,1}(Q_T)}. \tag{5.10}$$

From Lemma 3, we have

$$\|\widehat{\varphi_E}\|_{W^{2,1}_{N,1}(Q_T)} \leqslant c\|\chi_E\|_{L^{N,1}(Q_T)} \leqslant c|E|^{\frac{1}{N}}, \tag{5.11}$$

so that

$$\int_{Q_T} \widehat{\varphi_E} f \leqslant c|E|^{\frac{1}{N}}\|f\|_{L^\infty(0,T;L^1(\Omega,\delta))}. \tag{5.12}$$

Following the argument of Lemma 1 (see also relation (5.1)), for the initial data, one has

$$\int_\Omega u_0\widehat{\varphi}_E(0)\,dx \leqslant \|u_0\|_{L^{N',\infty}(\Omega)}\|\widehat{\varphi}_E(0)\|_{L^{N,1}(\Omega)} \leqslant c|E|^{\frac{1}{N}}\|u_0\|_{L^{N',\infty}(\Omega)}. \tag{5.13}$$

Combining relation (5.8) to (5.13), one derives

$$|E|^{-\frac{1}{N}}\int_E u\,dx\,dt \leqslant c\Big[\|f\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|u_0\|_{L^{N',\infty}(\Omega)}\Big].$$

From all these, we have the result. □

Next, we want to investigate the regularity of the time derivative. To do this, we choose to use the m-accretive theory.

6. Regularity of the time derivative via m-accretive theory

In this section, we shall assume that L is independent of the time t and is self-adjoint ($L = L^*$). Moreover, we shall assume that $B = 0 = c_0$.

Let $\varphi_1 \in H^1_0(\Omega)\cap H^2(\Omega)$ be the first eigenfunction of L with $\lambda_1 > 0$ its associated eigenvalue. We recall that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\delta(x) \leqslant \varphi_1(x) \leqslant c_2\delta(x), \quad \forall\, x \in \overline{\Omega}. \tag{6.1}$$

Let $\alpha \in [0,1]$. We shall say that $v \in L^1(\Omega)$ satisfies

$$Lv \in L^1(\Omega,\varphi_1^\alpha) \text{ if and only if there exists } f \in L^1(\Omega,\varphi_1^\alpha),$$

such that

$$\int_\Omega vL\varphi\,dx = \int_\Omega f\varphi\,dx \quad \forall\,\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega.$$

The above function v belongs to $L^{N',\infty}(\Omega)$ (see [10]) so, taking into account this regularity, we may replace the class of test functions by $\forall\varphi \in W^2(\Omega, |\cdot|_{N,1})\cap H^1_0(\Omega)$.

Remark 3. Let us notice that $L^1(\Omega,\varphi_1^\alpha) = L^1(\Omega,\delta^\alpha)$ but we need to specify that in this section we use the following norm: $\|\cdot\|_{L^1(\Omega,\varphi_1^\alpha)}$. For this reason we keep this notation.

We define the domain of L as $D(L) = \big\{v \in L^1(\Omega) : Lv \in L^1(\Omega, \varphi_1^\alpha)\big\}$,

$$L : D(L) \subset L^1(\Omega, \varphi_1^\alpha) \to L^1(\Omega, \varphi_1^\alpha) \text{ and } C_0^2(\overline{\Omega}) \subset D(L).$$

As a consequence of the result given in [10] (see Theorem 3 given above), one has

Proposition 1. *If* $\alpha \in [0, 1)$, *then*

$$D(L) \subset W^1(\Omega, |\cdot|_{N(\alpha),\infty}) \cap W_0^{1,1}(\Omega) \text{ with } N(\alpha) = \frac{N}{N-1+\alpha}.$$

Proposition 2. *If* $\alpha = 1$, *then*

$$D(L) \subset W^{1,q}(\Omega, \varphi_1),\ 1 \leqslant q < \frac{2N}{2N-1}.$$

The main result of this section is

Theorem 6. *L is an m-accretive operator in $L^1(\Omega, \varphi_1^\alpha)$.*

Proof. Let us note that $C_0^2(\overline{\Omega})$ is dense in $L^1(\Omega, \varphi_1)$. We shall distinguish between two cases:

• $\alpha = 1$. Let $\lambda > 0$, $f \in L^1(\Omega, \varphi_1)$ and $u \in D(L)$ the unique solution of $u + \lambda Lu = f$; that is, $\forall\, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_0^1(\Omega)$ one has

$$\int_\Omega f\varphi\, dx = \int_\Omega (\varphi + \lambda L\varphi)u\, dx. \tag{6.2}$$

Since the equation is linear, we may assume without loss of generality that $f \geqslant 0$.

We choose $\varphi = \varphi_1$ in (6.2) and we have

$$\int_\Omega u\varphi_1\, dx = \frac{1}{1+\lambda\lambda_1}\int_\Omega f\varphi_1\, dx \leqslant \int_\Omega f\varphi_1\, dx. \tag{6.3}$$

Thus, we deduce that for $f \in L^1(\Omega, \varphi_1)$

$$\int_\Omega |u|\varphi_1\, dx \leqslant \int_\Omega |f|\varphi_1\, dx. \tag{6.4}$$

This is equivalent to saying

$$\|(I + \lambda L)^{-1} f\|_{L^1(\Omega,\varphi_1)} \leqslant \|f\|_{L^1(\Omega,\varphi_1)}. \tag{6.5}$$

Since $D(L)$ is dense in $L^1(\Omega, \varphi_1)$, with relation (6.5) we deduce that L is an m-accretive operator.

• $\alpha \in [0,1)$. Let $\lambda > 0$, $f \in L^1(\Omega, \varphi_1^\alpha)$ and $u \in D(L)$ the unique solution of $u + \lambda L u = f$. That is,

$$\int_\Omega f\varphi\,dx = \int_\Omega (\varphi + \lambda L\varphi) u\,dx \quad \forall\,\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega. \tag{6.6}$$

We have to show that

$$\int_\Omega |u|\varphi_1^\alpha\,dx \leqslant \int_\Omega |f|\varphi_1^\alpha\,dx.$$

Without loss of generality, we may assume that $f \geqslant 0$. Let us choose the test function

$$\varphi_{1\varepsilon} = (\varphi_1 + \varepsilon)^\alpha - \varepsilon^\alpha, \quad 1 > \varepsilon > 0 \text{ (small parameter)}.$$

Lemma 5. *For all $u \in L^1(\Omega)$, $u \geqslant 0$ we have*

$$\int_\Omega u L\varphi_{1\varepsilon}\,dx \geqslant 0.$$

Proof. Let $u_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, $p > N$, and let $u_k \geqslant 0$ such that $u_k \to u$ almost everywhere and in $L^1(\Omega)$ as $k \to +\infty$. Then,

$$J_{\varepsilon k} = \int_\Omega u_k L\varphi_{1\varepsilon}\,dx = \int_\Omega A\nabla\varphi_{1\varepsilon}\nabla u_k\,dx.$$

A straightforward computation leads to

$$\begin{aligned} J_{\varepsilon k} &= \alpha \int_\Omega A\nabla\varphi_1 \nabla u_k (\varphi_1 + \varepsilon)^{\alpha-1}\,dx \\ &= \alpha \int_\Omega A\nabla\varphi_1 \Big[\nabla(\varphi_1+\varepsilon)^{\alpha-1} u_k - (\alpha-1)(\varphi_1+\varepsilon)^{\alpha-2} u_k \nabla\varphi_1\Big]\,dx. \end{aligned} \tag{6.7}$$

Using the equation satisfied by φ_1, we then deduce from (6.7) that

$$\begin{aligned} J_{\varepsilon k} \ \geqslant\ & \alpha\lambda_1 \int_\Omega \varphi_1(\varphi_1+\varepsilon)^{\alpha-1} u_k\,dx \\ & + \alpha(1-\alpha) \int_\Omega A\nabla\varphi_1\nabla\varphi_1 (\varphi_1+\varepsilon)^{\alpha-2} u_k\,dx. \end{aligned} \tag{6.8}$$

By coercivity condition, one has

$$A\nabla\varphi_1\nabla\varphi_1 \geqslant \alpha_0 |\nabla\varphi_1|^2, \quad \alpha_0 > 0. \tag{6.9}$$

Therefore, we have

$$J_{\varepsilon k} \geqslant \alpha\lambda_1 \int_\Omega \varphi_1(\varphi_1+\varepsilon)^{\alpha-1} u_k\,dx + \alpha(1-\alpha)\alpha_0 \int_\Omega |\nabla\varphi_1|^2 (\varphi_1+\varepsilon)^{\alpha-2} u_k\,dx \geqslant 0.$$

We deduce that

$$\int_\Omega uL\varphi_{1\varepsilon}\,dx = \lim_{k\to+\infty}\int_\Omega u_k L\varphi_{1\varepsilon}\,dx \geqslant 0. \tag{6.10}$$

□

End of proof of Theorem 4. From Lemma 5, using $\varphi_{1\varepsilon}$ as a test function in (6.6),

$$0 \leqslant \int_\Omega u\varphi_{1\varepsilon}\,dx \leqslant \int_\Omega (\varphi_{1\varepsilon} + \lambda L\varphi_{1\varepsilon})u\,dx = \int_\Omega f\varphi_{1\varepsilon}\,dx; \tag{6.11}$$

as $\varepsilon \to 0$, one obtains

$$0 \leqslant \int_\Omega u\varphi_1^\alpha\,dx = \lim_{\varepsilon\to 0}\int_\Omega u\varphi_{1\varepsilon}\,dx \leqslant \lim_{\varepsilon\to 0}\int_\Omega f\varphi_{1\varepsilon}\,dx = \int_\Omega f\varphi_1^\alpha\,dx. \tag{6.12}$$

From relation (6.12), we deduce that

$$\int_\Omega |u|\varphi_1^\alpha\,dx \leqslant \int_\Omega |f|\varphi_1^\alpha\,dx, \tag{6.13}$$

which means that

$$\|(I+\lambda L)^{-1}f\|_{L^1(\Omega,\varphi_1^\alpha)} \leqslant \|f\|_{L^1(\Omega,\varphi_1^\alpha)}. \tag{6.14}$$

Since $D(L)$ is dense in $L^1(\Omega,\varphi_1^\alpha)$, relation (6.14) implies that L is an m-accretive operator in $L^1(\Omega,\varphi_1^\alpha)$. □

As a corollary of the above Theorem 6, one has

Corollary 2. *Let $T>0$, $u_0 \in D(L)$, $f \in W^{1,1}\big(0,T;L^1(\Omega,\varphi_1^\alpha)\big)$, and $\alpha \in [0,1]$. Then, there exists a function satisfying*

$$\begin{cases} u \in C\big([0,T];D(L)\big) \cap C^1\big([0,T];L^1(\Omega,\varphi_1^\alpha)\big), \\ \dfrac{du}{dt}(t) + Lu(t) = f(t)\ \forall\, t \in \big[0,T\big], u(0) = u_0. \end{cases}$$

Proof. We may apply standard results on m-accretive operators (see [5, 2, 15]). □

Combining Propositions 1 and 2 and Corollary 2 of Theorem 6, we have

Corollary 3. *Let $u_0 \in D(L)$, $f \in W^{1,1}\big(0,T;L^1(\Omega,\delta^\alpha)\big)$, and $\alpha \in [0,1]$.*

- *If $\alpha \in [0,1)$, then*

$$\frac{d}{dt}\int_\Omega u(x,t)\varphi(x)\,dx + \int_\Omega A(x)\nabla u(x,t)\nabla\varphi(x)\,dx = \int_\Omega f(x,t)\varphi(x)\,dx, \tag{6.15}$$

in $\mathcal{D}'(0,T)$*, for all* $\varphi \in W^{1,\infty}(\Omega)\cap H_0^1(\Omega)$*.*

$$u \in C([0,T]; W^1(\Omega, |\cdot|_{N(\alpha),\infty})),\ N(\alpha) = \frac{N}{N-1+\alpha},$$

$$\frac{\partial u}{\partial t} \in C([0,T]; L^1(\Omega,\delta^\alpha)),\ u(0) = u_0.$$

- *If* $\alpha = 1$*, then relation* (6.15) *holds for all* $\varphi \in C_c^1(\Omega)$*:*

$$u \in C([0,T]; W^{1,1}(\Omega,\delta))$$

$$\frac{\partial u}{\partial t} \in C([0,T]; L^1(\Omega,\delta)),\ u(0) = u_0.$$

Remark 4. In order to estimate of the solution u with respect to the data, we shall use the Duhamel's formula,

$$u(t) = S_L(t)u_0 + \int_0^t S_L(t-s)f(s)ds,\ t \in [0,T],$$

where S_L is the continuous semigroup of contractions on $L^1(\Omega,\varphi_1^\alpha)$ generated by $-L$.

Proposition 3. *Under the same assumptions as for Theorem* 6,

$$\begin{aligned}
\operatorname*{Max}_{t\leqslant T} \|u(t)\|_{L^1(\Omega,\delta^\alpha)} &\leqslant \|u_0\|_{L^1(\Omega,\delta^\alpha)} + \|f\|_{L^1(0,T;L^1(\Omega,\delta^\alpha))},\\
\operatorname*{Max}_{t\leqslant T} \|\frac{du}{dt}(t)\|_{L^1(\Omega,\delta^\alpha)} &\leqslant c\Big[\|f(0)\|_{L^1(\Omega,\delta^\alpha)} + \|Lu_0\|_{L^1(\Omega,\delta^\alpha)}\\
&\qquad + \int_0^T \|f'(s)\|_{L^1(\Omega,\delta^\alpha)}ds\Big].
\end{aligned}$$

Proof. (See [5, 15] for details.) □

To obtain global regularity in space, for the time derivative, we can combine the above results to obtain

Proposition 4. *Assume that* $f \in C^1\big(]0,T]; L^1(\Omega,\delta)\big)$ *with* $f(t) = 0$ *for* $0 \leqslant t < \eta$ *and* Lu_0 *belongs to* $L^{N',\infty}(\Omega)$*. Then* $\frac{\partial u}{\partial t} \in L^{N',\infty}(Q_T)$*, and*

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{L^{N',\infty}(Q_T)} \leqslant c\big[\|f'\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|Lu_0\|_{L^{N',\infty}(\Omega)}\big].$$

Proof. Let $f_k \in C_c^1(]0,T]; C^2(\overline{\Omega}))$ such that $f_k \to f$ in $C^1([0,T]; L^1(\Omega,\delta)$, and $u_{0k} \in W^2(\Omega, |\cdot|_{N',\infty})$ such that $Lu_{0k} \to Lu_0$ in $L^{N',\infty}(\Omega)$.

Let us introduce $(u_k, w_k) \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, the unique solutions of

$$\begin{cases} \dfrac{\partial u_k}{\partial t} + Lu_k = f_k & \text{in } Q_T, \\ u_k(0) = u_{0k}, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w_k + Lw_k = \dfrac{\partial f_k}{\partial t} & \text{in } Q_T, \\ w_k(0) = -Lu_{0k}. \end{cases}$$

Then, one can check that $w_k = \frac{\partial u_k}{\partial t}$ and using Theorem 5 on w_k, one has

$$\|w_k\|_{L^{N',\infty}(Q_T)} \leqslant c\Big[\Big\|\frac{\partial f_k}{\partial t}\Big\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|Lu_{0k}\|_{L^{N',\infty}(\Omega)}\Big]. \tag{6.16}$$

Since the u_k converge in $L^1(Q_T)$ to the unique solution u of (B_T) and relation (6.16) shows us that $(u_k)_{k\geqslant 0}$ is a Cauchy sequence in $L^{N',\infty}(Q_T)$, we deduce the result, i.e.,

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{L^{N',\infty}(Q_T)} \leqslant c\Big[\Big\|\frac{\partial f}{\partial t}\Big\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|Lu_0\|_{L^{N',\infty}(\Omega)}\Big]. \qquad \square$$

A similar argument leads to the following result:

Proposition 5. *Let $\alpha \in [0,1)$, $u_0 \in W^2(\Omega, |\cdot|_{N(\alpha),\infty})$, $\frac{N}{N-1+\alpha}$, and $f \in C_c^1((0,T];\, L^1(\Omega,\delta^\alpha))$. Then the unique weak solution u of (B_T) satisfies*

$$\Big|\nabla_x \frac{\partial u}{\partial t}\Big| \in L^{N(\alpha),\infty}(Q_T),$$

and

$$\Big\|\nabla_x \frac{\partial u}{\partial t}\Big\|_{L^{N(\alpha),\infty}(Q_T)} \leqslant c\Big[\Big\|\frac{\partial f}{\partial t}\Big\|_{L^\infty(0,T;L^1(\Omega,\delta^\alpha))} + \|Lu_0\|_{L^{N(\alpha),\infty}(\Omega)}\Big],$$

for some constant c depending only on Ω, α, and T.

The above results lead us to study of the regularizing effect. We begin with the following result:

Proposition 6. *Assume as before that*

$$L = -\mathrm{div}\,(A(x)\nabla\cdot), \quad A(x) = (a_{ij}(x))_{i,j\leqslant N}$$

satisfying $a_{ij} = a_{ji}$ and the above coercivity (see (H1)). Then the unique solution u of (B_T) satisfies $u \in L^\infty(0,T; L^1(\Omega,\delta))$. Moreover, one has a constant $c > 0$ such that

$$\operatorname*{ess\,sup}_{t\leqslant T} \|u(t)\|_{L^1(\Omega,\delta)} \leqslant c\Big[\|u_0\|_{L^1(\Omega,\delta)} + \|f\|_{L^1(Q_T,\delta)}\Big]. \tag{6.17}$$

Proof. For $\varepsilon \in (0,1)$, we define

$$S_\varepsilon(\sigma) = \begin{cases} \dfrac{\sigma}{\varepsilon} & \text{if } |\sigma| \leqslant \varepsilon, \\ \operatorname{sign}(\sigma) & \text{otherwise.} \end{cases}$$

Let φ_1 be the first eigenfunction of L, and $u_n \in W_q^{1,2}(Q_T)$, $q > N$, the regular solution defined as in the proof of Theorem 1. Taking $\varphi_1 S_\varepsilon(u_n)$ as a test function, dropping nonnegative terms, one has

$$\int_\Omega \frac{\partial u_n}{\partial t} S_\varepsilon(u_n)\varphi_1\, dx + \int_\Omega A(x)\nabla u_n \nabla\varphi_1 S_\varepsilon(u_n)\, dx \leqslant \int_\Omega |f_n|\varphi_1\, dx. \tag{6.18}$$

Setting

$$\wedge_\varepsilon(u_n) = \int_0^{u_n} S_\varepsilon(\sigma)d\sigma > 0,$$

the above relation (6.18) gives

$$\frac{d}{dt}\int_\Omega \wedge_\varepsilon(u_n)\varphi_1\, dx + \int_\Omega A(x)\nabla\varphi_1\nabla \wedge_\varepsilon (u_n)\, dx \leqslant \int_\Omega |f_n|\varphi_1\, dx. \tag{6.19}$$

Using the equation satisfied by φ_1, relation (6.19) implies

$$\frac{d}{dt}\int_\Omega \wedge_\varepsilon(u_n)\varphi_1\, dx + \lambda_1 \int_\Omega \wedge_\varepsilon(u_n)\varphi_1\, dx \leqslant \int_\Omega |f_n|\varphi_1\, dx. \tag{6.20}$$

From Gronwall's inequality, after an integration, we let $\varepsilon \to 0$ to derive (6.17) for u_n. Letting $n \to +\infty$ in the resulting inequality, we have the result for u. □

Theorem 7 (**Regularizing effect**). *Let $f \in L^\infty(0,T;L^1(\Omega,\delta))$ and $u_0 \in L^1(\Omega,\delta)$. Then for all $\eta \in (0,\frac{T}{2})$, we have the unique solution u of (B_T) in $L^{N',\infty}(\Omega, \times(\eta,T))$. Moreover, there exists a constant $c_\eta > 0$ such that*

$$\|u\|_{L^{N',\infty}(\Omega,\times(\eta,T))} \leqslant c_\eta\Big[\|f\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|u_0\|_{L^1(\Omega,\delta)}\Big].$$

Proof. Let $\theta \in C^\infty[0,T]$ such that

$$\theta(t) = \begin{cases} 0 & \text{for } 0 < t \leqslant \eta, \\ 1 & \text{if } t \geqslant 2\eta, \\ 0 \leqslant \theta \leqslant 1. \end{cases}$$

Let us consider u_n, the sequence approximating u as in the proof of Theorem 1. Since one has

$$\partial_t u_n + Lu_n = f_n \text{ in } Q_T,$$

we deduce

$$\partial_t(\theta u_n) + L(\theta u_n) = \theta f_n + \theta' u_n, \qquad (\theta u_n)(x,0) = 0,\ x \in \Omega.$$

Applying Theorem 5, we have

$$\|\theta u_n\|_{L^{N',\infty}(Q_T)} \leqslant c\Big[\|\theta f_n + \theta' u_n\|_{L^\infty(0,T;L^1(\Omega,\delta))}\Big]. \tag{6.21}$$

From relation (6.21) we have

$$\|\theta u_n\|_{L^{N',\infty}(Q_T)} \leqslant c_\eta\Big[\|f_n\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|u_n\|_{L^\infty(0,T;L^1(\Omega,\delta))}\Big]. \tag{6.22}$$

By Proposition 6, we also have

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega,\delta))} \leqslant c\Big[\|u_{0n}\|_{L^1(\Omega,\delta)} + \|f_n\|_{L^1(0,T;L^1(\Omega,\delta))}\Big]. \tag{6.23}$$

From these relations, we derive

$$\|\theta u_n\|_{L^{N',\infty}(Q_T)} \leqslant c_\eta\Big[\|f_n\|_{L^\infty(0,T;L^1(\Omega,\delta))} + \|u_{0n}\|_{L^1(\Omega,\delta)}\Big], \tag{6.24}$$

and the result follows. □

We can use the arguments from Theorem 3 to derive

Theorem 8 (**Regularizing effect of the gradient**). *Let $\alpha \in [0,1)$, $N(\alpha) = \frac{N}{N-1+\alpha}$, $u_0 \in L^1(\Omega,\delta^\alpha)$, and $f \in L^{N(\alpha)}(0,T;L^1(\Omega,\delta^\alpha))$. We consider the same hypothesis as in Proposition 6. Then the weak solution u of (B_T) satisfies, $\forall\,\eta \in (0,\frac{T}{2})$,*

$$|\nabla_x u| \in L^{N(\alpha),\infty}(\Omega,\times(\eta,T)).$$

Moreover, one has a constant $c_\eta = c(\Omega,\alpha,\eta,T) > 0$ such that

$$\|\nabla_x u\|_{L^{N(\alpha),\infty}(\Omega,\times(\eta,T))} \leqslant c_\eta\Big[\|f\|_{L^{N(\alpha)}(0,T;L^1(\Omega,\delta^\alpha))} + \|u_0\|_{L^1(\Omega,\delta^\alpha)}\Big].$$

Proof. The proof needs a result similar to Proposition 6.

Proposition 7. *Let $f \in L^1(Q_T,\delta^\alpha)$, $\alpha \in [0,1)$, and $u_0 \in L^1(\Omega,\delta^\alpha)$. Then the unique solution u in (B_T) satisfies $u \in L^\infty(0,T;L^1(\Omega,\delta^\alpha))$. Moreover,*

$$\|u\|_{L^\infty(0,T;L^1(\Omega,\delta^\alpha))} \leqslant c\Big[\|f\|_{L^1(\Omega,\delta^\alpha)} + \|u_0\|_{L^1(Q_T,\delta^\alpha)}\Big],$$

for some constant $c = c(\Omega,T,\alpha) > 0$.

Proof. Let $\varepsilon \in (0,1)$ and φ_1 be the first eigenfunction associated with L. Let us define $\varphi_{1\varepsilon} = (\varphi_1 + \varepsilon)^\alpha - \varepsilon^\alpha$. Considering the approximating sequence $u_n \in W_q^{2,1}(Q_T),\ q > 1$ given in the proof of Theorem 1, we have

$$\int_\Omega \frac{\partial u_n}{\partial t} S_\varepsilon(u_n)\varphi_{1\varepsilon}\,dx + \int_\Omega A(x)\nabla u_n \nabla \varphi_{1n} S_\varepsilon(u_n)\,dx \leqslant \int_\Omega |f_n|\varphi_{1\varepsilon}\,dx, \quad (6.25)$$

where S_ε is as in the proof of Proposition 6. Then, relation (6.25) gives

$$\frac{d}{dt}\int_\Omega \wedge_\varepsilon(u_n)\varphi_{1\varepsilon}\,dx + \alpha\int_\Omega A(x)\nabla \wedge_\varepsilon (u_n)\nabla\varphi_1(\varphi_1+\varepsilon)^{\alpha-1}\,dx \leqslant \int_\Omega |f_n|\varphi_{1\varepsilon}\,dx, \quad (6.26)$$

where

$$\wedge_\varepsilon(u_n) = \int_0^{u_n} S_\varepsilon(\sigma)d\sigma.$$

Writing

$$\nabla\wedge_\varepsilon(u_n)(\varphi_1+\varepsilon)^{\alpha-1} = \nabla(\wedge_\varepsilon(u_n)(\varphi_1+\varepsilon)^{\alpha-1}) + (1-\alpha)(\varphi_1+\varepsilon)^{\alpha-2}\wedge_\varepsilon(u_n)\nabla\varphi_1, \quad (6.27)$$

and using the equation satisfied by φ_1, one obtains from relation (6.26)

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \wedge_\varepsilon(u_n)\varphi_{1\varepsilon}\,dx + \alpha\lambda_1\int_\Omega \varphi_1 \wedge_\varepsilon (u_n)(\varphi_1+\varepsilon)^{\alpha-1} \quad (6.28)\\ &+ \alpha_0\alpha(1-\alpha)\int_\Omega(\varphi_1+\varepsilon)^{\alpha-2}\wedge_\varepsilon (u_n)|\nabla\varphi_1|^2\,dx \leqslant \int_\Omega |f_n|\varphi_{1\varepsilon}\,dx. \end{aligned}$$

Since $\wedge_\varepsilon(u_n) \geqslant 0$, we drop nonnegative terms to obtain the following inequality:

$$\frac{d}{dt}\int_\Omega \wedge_\varepsilon(u_n)\varphi_{1\varepsilon}\,dx \leqslant \int_\Omega |f_n|\varphi_{1\varepsilon}dx. \quad (6.29)$$

From (6.29), we derive after an integration and letting $\varepsilon \to 0$

$$\int_\Omega |u_n(x,t)|\varphi_1^\alpha\,dx \leqslant \int_{Q_T}|f_n|\varphi_1^\alpha\,dx + \int_\Omega |u_{0n}|\varphi_1^\alpha\,dx. \quad (6.30)$$

Letting $n \to +\infty$ and using the fact that

$$c_0\delta(x) \leqslant \varphi_1(x) \leqslant c_1\delta(x),\ c_i > 0,\ x \in \overline{\Omega},$$

we derive the result. □

End of Proof of Theorem 8. Considering the same function θ as in the proof of Theorem 7, we have from Theorem 4

$$\|\theta\nabla_x u\|_{L^{N(\alpha),\infty}(Q_T)} \leqslant c\Big[\|\theta f\|_{L^{N(\alpha)}(0,T;L^1(\Omega,\delta^\alpha))} + \|\theta' u\|_{L^{N(\alpha)}(0,T;L^1,\delta^\alpha)}\Big]. \quad (6.31)$$

Using Proposition 7 and relation (6.27), we derive

$$\|\theta\nabla_x u\|_{L^{N(\alpha),\infty}(Q_T)} \leqslant c_\eta\Big[\|f\|_{L^{N(\alpha)}(0,T;L^1(\Omega,\delta^\alpha))} + \|u_0\|_{L^1(\Omega,\delta^\alpha)}\Big]. \tag{6.32}$$

This shows the result. □

7. Application

Assumption (H_a)**.** *Let $a > 0$, $\beta = \frac{2}{1+a}$, $f \in L^\infty(0,T;L^1(\Omega,\delta^\alpha))$, and $f \geqslant 0$ with $\alpha \in [0,1)$ and $u_0 \in L^{N',\infty}(\Omega)$. Let $T_n(f) = f_n = \min(f;n)$, $n \geqslant 1$ and let K be the bounded function*

$$K(x) = \beta(1-\beta)A(x)\nabla\varphi_1(x)\nabla\varphi_1(x), \quad x \in \Omega.$$

We consider a function $h \geqslant 0$ such that there exists $n_0 \geqslant 1$

$$K(x) + h(x,t)\left[\frac{(\varphi_1+1)^\beta}{(\varphi_1+1)^\beta - 1}\right]^a \leqslant T_{n_0}\big(f(x,t)\big) - \lambda_1\beta\varphi_1(x)(\varphi_1(x)+1).$$

Proposition 8. *Let $\underline{w}(x) = (\varphi_1(x)+1)^\beta - 1$, $x \in \Omega$. Then*

$$L\underline{w}(x) + h(x,t)\underline{w}^{-a}(x) \leqslant T_{n_0}(f(x,t)) \ \text{a.e. } (x,t) \in Q_T,$$

when $L = -\mathrm{div}\,(A(x)\nabla\cdot)$.

Proof. A straightforward computation leads to

$$\begin{aligned} L\underline{w}(x) + h\underline{w}^{-a}(x) &= (\varphi_1(x)+1)^{(\beta-2)}\beta(1-\beta)A(x)\nabla\varphi_1(x)\nabla\varphi_1(x) \\ &\quad + \lambda_1\beta\varphi_1(x)(\varphi_1(x)+1)^{\beta-1} + h\underline{w}^{-a}(x), \end{aligned}$$

$$\begin{aligned} &L\underline{w}(x) + h(x,t)\underline{w}^{-a}(x) \\ &= (\varphi_1(x)+1)^{\beta-2}\left[K(x) + \lambda_1\beta\varphi_1(x)(\varphi_1(x)+1) + h(x,t)\frac{(\varphi_1(x)+1)^{2-\beta}}{\underline{w}^a(x)}\right]. \end{aligned}$$

Noticing that $\beta - 2 = -a\beta$, we then derive

$$\begin{aligned} &L\underline{w}(x) + h(x,t)\underline{w}^{-a}(x) \\ &\leqslant K(x) + \lambda_1\beta\varphi_1(x)(\varphi_1(x)+1) + h(x,t)\Big[\frac{(\varphi_1(x)+1)^\beta}{(\varphi_1(x)+1)^\beta - 1}\Big]^a \leqslant T_{n_0}(f(x,t)). \end{aligned}$$

Let $u_0 \in L^{N',\infty}(\Omega)$ be such that $u_0 \geqslant \underline{w}$. □

We consider a sequence $(\varepsilon_n)_{n\geqslant 0}$, $\varepsilon_n > 0$, $\varepsilon_n \xrightarrow[n\to+\infty]{} 0$. We define the sequence $(v_n)_{n\geqslant 0}$ with $v_0 = \underline{w}$ and for $n \geqslant 0$:

$$(\mathcal{D}_{n+1}) \begin{cases} \partial_t v_{n+1} + Lv_{n+1} + h(v_n+\varepsilon_n)^{-a} = f_{n+n_0} \text{ in } Q_T, \\ v_{n+1} = 0 \text{ on } \sum_T = \partial\Omega \times (0,T), \\ v_{n+1}(0) = u_{0,n+1}, \text{ with } u_{0,n+1} \geqslant u_{0,n} \geqslant \underline{w}, \\ u_{0,n} \to u_0 \text{ in } L^1(\Omega,\delta^\alpha), \ u_{0,n} \in L^2(\Omega). \end{cases}$$

Proposition 9. *One has the following relations about* $(v_n)_n$*:*

(1) *For all* $n \geqslant 0$, $v_n \geqslant \underline{w}$.
(2) *The function* v_{n+1} *in* $L^2(0,T;H_0^1(\Omega)\cap C([0,T],L^2(\Omega)))$ *with*

$$\partial_t v_{n+1} \in L^2(0,T;H^{-1}\Omega) \cap L^2(\Omega\times(\eta,T)), \ \forall\, \eta > 0.$$

(3) $v_n \leqslant v_{n+1}$.

Proof. We prove the result by induction. For $n = 0$, we have

$$f_{n_0} - h(v_0+\varepsilon_0)^{-a} \in L^\infty(Q_T).$$

The existence and uniqueness of v_1 is classical. Using Proposition 8 we have

$$h(\underline{w}+\varepsilon_0)^{-a} + L\underline{w} \leqslant f_{n_0} = \partial_t v_1 + Lv_1 + h(\underline{w}+\varepsilon_0)^{-a}$$

$$\partial_t(\underline{w}-v_1) + L(\underline{w}-v_1) \leqslant 0 : v_0 = \underline{w} \leqslant v_1.$$

Assuming that $\underline{w} \leqslant v_{n-1} \leqslant v_n$ for $n \geqslant 1$, we deduce from the fact that

$$f_{n+n_0} - h(v_n+\varepsilon_n)^{-a} \in L^\infty(Q_T), \ u_{0,n} \in L^2(\Omega),$$

that v_{n+1} is the unique solution being in $L^2(0,T;H_0^1(\Omega))\cap C([0,T],L^2(\Omega))$. We also have

$$(\underline{w}+\varepsilon_n)^{-a} + L\underline{w} \leqslant f_{n_0+n} = \partial_t v_{n+1} + Lv_{n+1} + h(v_n+\varepsilon_n)^{-a},$$

$$\partial_t(\underline{w}-v_{n+1}) + L(\underline{w}-v_{n+1}) \leqslant h[(v_n+\varepsilon_n)^{-a} - (\underline{w}+\varepsilon_n)^{-a}] \leqslant 0,$$

since $\underline{w} \leqslant v_n$. By the maximum principle we have $\underline{w} \leqslant v_{n+1}$ (knowing that $\underline{w} \leqslant u_{0,n+1}$).

Since $f_{n+n_0-1} \leqslant f_{n+n_0}$ and $u_{0,n+1} \geqslant u_{0,n}$, we derive that $v_{n+1} \geqslant v_n$. □

Proposition 10. *The sequence* v_n *remains in a bounded set* $L^{N',\infty}(Q_T)$ *and its gradient in a bounded set of* $L^{N(\alpha),\infty}(Q_T)$, $N(\alpha) = \frac{N}{N-1+\alpha}$ *with* $\alpha \in [0,1)$.

Proof. One has

$$0 \leqslant \frac{h}{(v_n + \varepsilon_n)^a} \leqslant \frac{h}{\underline{w}^a} \leqslant n_0$$

(by our assumption). Therefore, $g_n = f_{n+n_0} - h(v_n + \varepsilon_n)^{-a}$ remains in a bounded set of $L^\infty(0,T;L^1(\Omega,\delta^\alpha))$ and we deduce the following from Theorem 4 and Theorem 5:

- v_n remains in a bounded set of $L^{N',\infty}(Q_T)$.
- $|\nabla_x v_n|$ remains in a bounded set of $L^{N(\alpha),\infty}(Q_T)$, $N(\alpha) = \frac{N}{N-1+\alpha}$.

□

Since $\underline{w} \leqslant v_n \leqslant v_{n+1}$, we deduce that there exists u such that

- $u(x,t) = \lim v_n(x,t)$ almost everywhere in Q_T,
- $\|u\|_{L^{N',\infty}(Q_T)} \leqslant c\left[\|f\|_{L^\infty(0,T;L^1(\Omega,\delta)} + \|u_0\|_{L^{N',\infty}(\Omega)} + 1\right]$,
- $\|\nabla_x u\|_{L^{N(\alpha),\infty}(Q_T)} \leqslant c[\|f\|_{L^\infty(0,T;L^1(\Omega,\delta^\alpha)} + \|u_0\|_{L^{N',\infty}(\Omega)} + 1]$.

We may pass to the limit in the system $(\mathcal{D}_{n+1})$; then one deduces the following:

Theorem 9. *Under the assumption (H_a) on $a > 0$, f, h, and u_0, there exists at least one function $u \in L^{N',\infty}(Q_T)$, $|\nabla_x u| \in L^{N(\alpha),\infty}(Q_T)$, $N(\alpha) = \frac{N}{N-1+\alpha}$ with $\alpha \in [0,1)$ such that*

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu + \dfrac{h}{u^a} = f \ \text{ in } Q_T, \\ u = 0 \ \text{ on } \sum_T = \partial\Omega \times (0,T), \\ u(0) = u_0, \end{cases}$$

with $u \geqslant (\varphi_1 + 1)^\beta - 1 = \underline{w}$, $hu^{-a} \in L^\infty(Q_T)$, and $\beta = \frac{2}{1+a}$.

Remark 5. (1) For all $\varphi \in C^2(\overline{\Omega})$,

$$\lim_{n\to+\infty} \int_\Omega h(x,t)(v_n + \varepsilon_n)^{-a}(x,t)\varphi(x)\,dx = \int_\Omega hu^{-a}\varphi\,dx \text{ for a.e. } t.$$

(2) The initial data makes sense. Indeed, according to Theorem 3, we have $L^1(\Omega,\delta^\alpha) \subset W^{-1,r}(\Omega)$ for some $r > 1$; we then have $\partial_t u \in L^1(0,T;W^{-1,q}(\Omega))$ for some $q > 1$. This implies with the regularity of u that it belongs to $C([0,T];W^{-1,q}(\Omega))$. [17]

Acknowledgment. I am grateful to Professor J.I. Díaz for his comments and to point out references [3, 13], Professor M.-Petcu for her remarks to improve the presentation of the previous version of this paper, and Professor Andrew King for correcting the English in the Introduction.

I am grateful to the unknown referee for her/his useful remarks.

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