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CONVERGENCE TO SELF-SIMILAR SOLUTIONS FOR A PARABOLIC-ELLIPTIC SYSTEM OF DRIFT-DIFFUSION TYPE IN $\mathbb{R}^2$

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Abstract. We consider the Cauchy problem for a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$, modeling chemotaxis and self-attracting particles, with L^1-initial data. Under the assumption that the total mass of nonnegative initial data is less than 8π, by using similarity arguments, it is shown that the nonnegative solution converges to a radially symmetric self-similar solution at rate $o(t^{-1+1/p})$ in the L^p-norm ($1 \le p \le \infty$) as time goes to infinity.

1. Introduction

In this paper we are concerned with the large-time behavior of nonnegative solutions of the Cauchy problem for the following nonlinear equation:

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.1}$$

$$-\Delta \psi = u, \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.2}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^2, \tag{1.3}$$

where ψ is defined by

$$\psi(t,x) := (N * u)(t,x) = \int_{\mathbb{R}^2} N(x-y)\, u(t,y)\, dy$$

and $N(x)$ is the logarithmic potential in $\mathbb{R}^2$, namely

$$N(x) = \frac{1}{2\pi} \log \frac{1}{|x|}.$$

This system is a simplified version of a chemotaxis system obtained from the original Keller-Segel model [25] (see also Childress-Percus [13]), and also a model of self-attracting particles in $\mathbb{R}^2$ (see [7, 45]).

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In the subcritical case $\int_{\mathbb{R}^2} u_0\,dx < 8\pi$ for the nonnegative initial data $u_0 \in L^1(\mathbb{R}^2)$, the global existence of nonnegative solutions to the Cauchy problem (1.1)–(1.3) have been studied in [10] under the assumption

$$u_0 \log u_0,\ \ u_0|x|^2 \in L^1(\mathbb{R}^2), \tag{1.4}$$

and in [34] under $u_0 \log(1 + |x|) \in L^1(\mathbb{R}^2)$. On the other hand, in the supercritical case $\int_{\mathbb{R}^2} u_0\,dx > 8\pi$, the nonnegative solutions with initial data of finite second moment blow up in finite time (see [7, 10, 26]). The critical case $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$ has been studied in [6, 42] for radially symmetric solutions, and without symmetry assumptions, in [9] for the initial data of finite second moment and in [8, 38] for the initial data of infinite second moment. For related results for chemotaxis models, see [16, 19, 21, 22, 31, 35, 36, 43], and for models of self-attracting particles, see [4, 5], for example. We also refer to [20, 44] in which we can find related results for chemotaxis models.

Since $\nabla\psi$ may be rewritten as

$$\nabla\psi(t,x) = (\nabla N * u)(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\,u(t,y)\,dy,$$

the Cauchy problem (1.1)–(1.3) leads to

$$\partial_t u = \Delta u - \nabla\cdot(u(\nabla N * u)), \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.5}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^2. \tag{1.6}$$

Only under the assumption

$$u_0 \ge 0 \ \text{ on } \ \mathbb{R}^2, \quad u_0 \in L^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} u_0\,dx < 8\pi, \tag{1.7}$$

the Cauchy problem (1.5)–(1.6) has a unique nonnegative mild solution u globally in time, and the L^p-norms of $u(t)$ decay to zero with the exponents $t^{-1+1/p}$ for every $1 < p \le \infty$ as time goes to infinity (see [32]).

In the study of the large-time behavior of nonnegative global solutions, radially symmetric self-similar solutions play an important role. Equation (1.5) has a scaling invariant property such that for a solution u of (1.5), the function u_λ for $\lambda > 0$ defined by

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad t > 0,\ x \in \mathbb{R}^2$$

is also a solution of (1.5). If $u_\lambda = u$ for all $\lambda > 0$, the solution u is called a self-similar solution. Given $\hat{M} > 0$, consider a radially symmetric self-similar solution $U_{\hat{M}}$ of (1.5) such that

$$U_{\hat{M}}(t,x) = \frac{1}{t}\Phi_{\hat{M}}\Big(\frac{|x|}{\sqrt{t}}\Big), \quad \int_{\mathbb{R}^2} U_{\hat{M}}(t,x)\,dx = \hat{M}, \tag{1.8}$$

where $\Phi_{\hat{M}}$ is nonnegative, integrable, and bounded on $[0,\infty)$. The existence of such a radially symmetric self-similar solution has been studied in [3] by ODE methods and in [39] by PDE methods, and uniqueness has been studied in [6]. The existence result reads as follows: For every $\hat{M} \in (0, 8\pi)$, there exists uniquely a radially symmetric self-similar solution $U_{\hat{M}}$ satisfying (1.8), and if $U_{\hat{M}}$ exists, then $\hat{M} \in (0, 8\pi)$. As for nonnegative self-similar solutions of (1.5) without symmetry assumptions, in [39] it was proved that if $V = V(t,x)$ is a nonnegative self-similar solution of (1.5) satisfying $V(t,x) = \frac{1}{t}\Psi\big(\frac{x}{\sqrt{t}}\big)$, where Ψ is nonnegative and belongs to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then Ψ is radially symmetric about a point $x_0 \in \mathbb{R}^2$. Hence, if $\hat{M} := \int_{\mathbb{R}^2} V(t,x)\,dx < 8\pi$, then $V(t,x) = U_{\hat{M}}(t, x - x_0)$ by the uniqueness of nonnegative radially symmetric self-similar solutions satisfying (1.8).

The aim of this paper is to show that under assumption (1.7) on u_0, the nonnegative mild solution u of (1.5)–(1.6) satisfies

$$\lim_{t\to\infty} t^{1-1/p}\|u(t) - U_{\hat{M}}(t)\|_p = 0 \tag{1.9}$$

for every $1 \le p \le \infty$, where $\hat{M} = \int_{\mathbb{R}^2} u_0\,dx$ (see Theorem 4.2). No restrictions on u_0 are assumed except assumption (1.7). The result (1.9) for $p = 1$ was obtained in [10] under the additional assumption (1.4) on u_0, and the proof relies on entropy methods. The entropy method requires

$$u(t)\log u(t),\ |x|^2 u(t) \in L^1(\mathbb{R}^2),$$

which is not expected only under assumption (1.7). For the radial case, the convergence of the mass distribution function $\int_{|x|<r} u(t,x)\,dx$ to self-similarity was obtained in [6] under assumption (1.7) on radially symmetric initial data by using a different method from that in [10] and this paper. We prove (1.9) by showing

$$\lim_{\lambda\to\infty} \|u_\lambda(1) - U_{\hat{M}}(1)\|_p = 0.$$

This rescaling method goes back to Carpio [12], studying the large-time behavior of solutions to the vorticity equations for incompressible viscous fluids, and we also refer to the book of Giga-Giga-Saal [17] for this method and related topics for the vorticity equations.

We remark that the self-similar property is also satisfied for the following chemotaxis system,

$$\partial_t u = \Delta u - \nabla\cdot(u\nabla\psi), \quad \partial_t\psi = \Delta\psi + u, \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.10}$$

and the convergence of solutions to self-similar solutions has been studied in [23, 37] for small initial data by using different methods. For the global existence of nonnegative solutions to the Cauchy problem for (1.10), see [29, 33] for example. We also mention that the large-time behaviors for the Cauchy problem related to Keller-Segel systems have been studied in [28, 40, 46].

This paper is organized as follows. In Section 2 we mention known results on the global existence of nonnegative solutions to (1.5)–(1.6). In Section 3 we give the estimates on the derivatives of nonnegative solutions. Section 4 is devoted to the proof of (1.9)

2. Global solutions with subcritical L^1-initial data

Throughout this paper, we use the following notation: $L^p(\mathbb{R}^d)$ is the Lebesgue space on $\mathbb{R}^d$ with the usual norm $\|\cdot\|_{L^p}$ for $1 \le p \le \infty$. In the case $d = 2$, for simplicity, we denote $L^p(\mathbb{R}^2)$ and $\|\cdot\|_{L^p}$ by L^p and $\|\cdot\|_p$, respectively. For $Q \subset \mathbb{R}^d$ and a Banach space X, we denote the set of all continuous functions from Q to X by $C(Q;X)$ and the set of all bounded continuous functions by $BC(Q;X)$. If $X = \mathbb{R}$, then we denote $C(Q;\mathbb{R})$ and $BC(Q;\mathbb{R})$ by $C(Q)$ and $BC(Q)$, respectively. Denote by $\mathbb{Z}_+$ the set of all nonnegative integers. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}_+^d$, put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}$, $\partial_j = \frac{\partial}{\partial x_j}$. For $m \in \mathbb{N}$ and $1 \le p \le \infty$, we denote by ∂_x^m any partial derivative of order m with respect to the space variables and put

$$\|\partial_x^m f\|_{L^p} = \sum_{|\alpha|=m} \|\partial_x^\alpha f\|_{L^p}.$$

For a function $f = f(t,x)$, $(t,x) \in (a,b) \times \Omega$, where $-\infty \le a < b \le \infty$, $\Omega \subset \mathbb{R}^d$, we denote by $f(t) : \Omega \to \mathbb{R}$ for $t \in (a,b)$ the function $f(t)(x) = f(t,x)$.

We give the definition of mild solutions of the Cauchy problem (1.5)–(1.6).

Definition 2.1. *Given $u_0 \in L^1$, a function u on $[0,T) \times \mathbb{R}^2$ is said to be a mild solution of (1.5)–(1.6) on $[0,T)$ if*

(i) $u \in C([0,T); L^1) \cap C((0,T); L^{4/3})$,
(ii) $\sup_{0<t<T} t^{1/4}\|u(t)\|_{4/3} < \infty$,
(iii) *u satisfies the integral equation*

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds, \quad 0 < t < T,$$

where $e^{t\Delta}$ is the heat semigroup defined by

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^2} G(t, x-y) f(y)\,dy, \quad G(t,x) = \frac{1}{4\pi t}\exp(-\frac{|x|^2}{4t}). \qquad (2.1)$$

A function u on $[0,\infty)\times\mathbb{R}^2$ is a global mild solution of (1.5)–(1.6) *with initial data u_0 if u is a mild solution of* (1.5)–(1.6) *on $[0,T)$ for any $0<T<\infty$.*

In order to get local existence, uniqueness, and regularity for the Cauchy problem (1.5)–(1.6) in [32, 34], we used techniques for the vorticity equation in $\mathbb{R}^2$ (see [2, 11, 18, 24], for example), combined with the following estimate of $f(\nabla N * g)$ involved in the nonlinear term of (1.5), namely, for $4/3 \le q < 2$,

$$\|f(\nabla N * g)\|_{2q/(4-q)} \le C_q \|f\|_q \|g\|_q \quad \text{for all} \quad f, g \in L^q, \tag{2.2}$$

where C_q is a positive constant depending only on q. This inequality is deduced from the Hardy-Littlewood-Sobolev inequality in $\mathbb{R}^2$: For $1<q<2$,

$$\|\frac{1}{|x|} * g\|_{2q/(2-q)} \le C_q \|g\|_q \quad \text{for all} \quad g \in L^q,$$

where C_q is a positive constant depending only on q.

To mention local existence, uniqueness, and regularity, following Kato [24], we introduce function spaces. Let $T>0$. For $1 \le p \le \infty$ and $\gamma \ge 0$, define the Banach space $C_{\gamma,T}(L^p)$ with norm $\|\cdot\|_{p,\gamma,T}$ by

$$C_{\gamma,T}(L^p) = \{u : u \in C((0,T);L^p), \ \sup_{0<t<T} t^\gamma \|u(t)\|_p < \infty\},$$
$$\|u\|_{p,\gamma,T} = \sup_{0<t<T} t^\gamma \|u(t)\|_p \quad \text{for } u \in C_{\gamma,T}(L^p).$$

For $\gamma>0$, define $\dot{C}_{\gamma,T}(L^p) = \{u \in C_{\gamma,T}(L^p) : \lim_{t\to 0} t^\gamma \|u(t)\|_p = 0\}$, and for $\gamma = 0$, $\dot{C}_{0,T}(L^p) = BC([0,T);L^p)$. $\dot{C}_{\gamma,T}(L^p)$ is a closed subspace of $C_{\alpha,T}(L^p)$.

Theorem 2.1. *For the initial data $u_0 \in L^1$, there exists $T \in (0,\infty)$ such that the Cauchy problem* (1.5)–(1.6) *has uniquely a mild solution u on $[0,T)$. Moreover, u satisfies the following:*

(i) *$u(t) \to u_0$ in L^1 as $t \to 0$.*
(ii) *For $1 \le q \le \infty$, $u \in \dot{C}_{1-1/q,T}(L^q)$.*
(iii) *For $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^2$, $1<q<\infty$, $\partial_t^\ell \partial_x^\alpha u \in \dot{C}_{1-1/q+|\alpha|/2+\ell,T}(L^q)$.*
(iv) *Let $\ell \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^2$. For $2<q<\infty$ if $|\alpha| = 0$, and for $1<q<\infty$ if $|\alpha| \ge 1$, $\partial_t^\ell \partial_x^\alpha (\nabla N * u) \in \dot{C}_{1/2-1/q+|\alpha|/2+\ell,T}(L^q)$.*
(v) *u is a classical solution of $\partial_t u = \Delta u - \nabla\cdot(u(\nabla N * u))$ in $(0,T)\times\mathbb{R}^2$.*
(vi) *$\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx$ for all $0<t<T$.*
(vii) *If $u_0 \ge 0$ and $u_0 \not\equiv 0$ on $\mathbb{R}^2$, then $u(t,x)>0$ on $(0,T)\times\mathbb{R}^2$.*

For the proof of this theorem, see [32, 34].

Under the additional assumption $u_0 \log(1+|x|) \in L^1$ on the initial data $u_0 \in L^1$, Proposition 2.1 below ensures that $\psi(t) := N * u(t)$ is well-defined in

$L^1_{loc}(\mathbb{R}^2)$ for the mild solution u to (1.5)–(1.6) on $[0,T)$, and we see that (u,ψ) is a solution of the Cauchy problem (1.1)–(1.3) because of $\nabla\psi = \nabla N * u$, $-\Delta\psi = u$. For the proof of Proposition 2.1, see [32].

Proposition 2.1. *Let the initial data $u_0 \in L^1$ satisfy $u_0 \log(1+|x|) \in L^1$. Then the mild solution u to (1.5)–(1.6) on $[0,T)$ satisfies that for every $0 < t < T$,*

$$\int_{|x|\geq 2} |u(t,x)| \log|x| \, dx \leq \int_{\mathbb{R}^2} |u_0(x)| \log(1+|x|) \, dx + C,$$

where $C > 0$ is a constant depending only on $\sup_{0<t<T} (\|u(t)\|_1 + t^{\frac{1}{4}} \|u(t)\|_{4/3})$ and T. Hence, $u(t)\log(1+|x|) \in L^1$ for any $0 < t < T$.

For the nonnegative initial data $u_0 \in L^1$ of finite second moment, the second moment identity is described in the following proposition. For the proof, see [9, 10].

Proposition 2.2. *Let the nonnegative initial data $u_0 \in L^1$ satisfy $|x|^2 u_0 \in L^1$. Then for the nonnegative mild solution u to (1.5)–(1.6) on $[0,T)$, it holds that for every $0 < t < T$,*

$$\int_{\mathbb{R}^2} |x|^2 u(t,x) \, dx = \int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx + 4\hat{M}\Big(1 - \frac{\hat{M}}{8\pi}\Big)t,$$

where $\hat{M} = \int_{\mathbb{R}^2} u_0 \, dx$.

In order to mention the global existence and decay estimates of the nonnegative solution u for the nonnegative initial data $u_0 \in L^1$ with $\hat{M} := \int_{\mathbb{R}^2} u_0 \, dx < 8\pi$, we consider radially symmetric self-similar solutions $U_{\hat{M}}(t,x)$ satisfying (1.8). As mentioned in the Introduction, for every $\hat{M} \in (0, 8\pi)$, there exists uniquely such a radially symmetric self-similar solution $U_{\hat{M}}(t,x)$.

Following [3, 6], we introduce the mass distribution function

$$M(t,s) = \int_{|x|\leq\sqrt{s}} U_{\hat{M}}(t,x) \, dx, \quad t > 0, \ s \geq 0,$$

and see that the function $M(t,s)$ satisfies the following:

$$\begin{cases} \partial_t M = 4\partial_s^2 M + \frac{1}{\pi} M \partial_s M, & t > 0, \ s > 0, \\ M(t,0) = 0, \ M(t,+\infty) = \hat{M}, & t > 0, \\ \lim_{t\to 0} M(t,s) = \hat{M}, & s > 0. \end{cases}$$

Since $M(t,s)$ has the property that for each $\lambda > 0$, $M(\lambda t, \lambda s) = M(t,s)$, $t > 0$, $s \geq 0$, $M(t,s)$ has the form $M(t,s) = m(s/t)$, $t > 0$, $0 \leq s < \infty$ for some function $m(y)$. The nonnegative function $m(y)$ satisfies

$$\begin{cases} 4\dfrac{d^2m}{dy^2}(y) + \dfrac{dm}{dy}(y) + \dfrac{1}{\pi y}m(y)\dfrac{dm}{dy}(y) = 0, \quad y > 0, \\ m(0) = 0, \quad m(+\infty) = \hat{M}, \end{cases}$$

and it was shown in Lemma 4.1 of [6] that

$$\begin{cases} m \in C^1([0,\infty)), \quad \dfrac{dm}{dy}(y) > 0, \quad \dfrac{d^2m}{dy^2}(y) < 0, \quad y > 0, \\ \hat{M}(1 - e^{-y/4}) \leq m(y) \leq \min\left\{4\dfrac{dm}{dy}(0)(1 - e^{-y/4}), \hat{M}\right\}, \quad y > 0, \\ \dfrac{dm}{dy}(y) \leq \dfrac{dm}{dy}(0)e^{-y/4}, \quad y > 0. \end{cases}$$

The relation between $\Phi_{\hat{M}}(y)$ in (1.8) and $m(y)$ is given by

$$\Phi_{\hat{M}}(y) = \pi^{-1} dm/dy(y^2),$$

and hence $U_{\hat{M}}(t)$ is decreasing with respect to $|x|$ and

$$0 < U_{\hat{M}}(t,x) \leq \frac{C}{t}\exp\Big(-\frac{|x|^2}{4t}\Big), \quad t > 0,\ x \in \mathbb{R}^2. \tag{2.3}$$

The following theorem on the global existence and decay estimates of nonnegative mild solutions to (1.5)–(1.6) was proved in [32] by using rearrangement techniques.

Theorem 2.2. *Assume* $\hat{M} := \int_{\mathbb{R}^2} u_0\,dx < 8\pi$ *for the nonnegative initial data* $u_0 \in L^1$. *Then the nonnegative mild solution* u *of* (1.5)–(1.6) *exists globally in time. Moreover it holds that for every* $1 \leq p \leq \infty$,

$$\|u(t)\|_p \leq (\pi t)^{-1+1/p}\|dm/dy\|_{L^p(0,\infty)} \quad \textit{for} \quad t > 0. \tag{2.4}$$

3. ESTIMATES ON DERIVATIVES OF SOLUTIONS

For a nonnegative initial data $u_0 \in L^1$ satisfying $\hat{M} := \int_{\mathbb{R}^2} u_0\,dx < 8\pi$, let u be the nonnegative mild solution of the Cauchy problem (1.5)–(1.6) on $[0,\infty)$ mentioned in Theorem 2.2. In what follows, we denote by $C(*,\dots,*)$ a positive constant depending only on the quantities appearing in the parentheses. Since $\|dm/dy\|_{L^p(0,\infty)}$ in Theorem 2.2 depends only on $\hat{M}$ and p, we may write (2.4) as

$$\sup_{t>0} t^{1-1/p}\|u(t)\|_p \leq C(\hat{M}, p), \quad 1 \leq p \leq \infty. \tag{3.1}$$

For the estimates on the derivatives of u, we have the following.

Theorem 3.1. *Let $1 \le p \le \infty$. Then it holds that for all $\ell, n \in \mathbb{Z}_+$,*

$$\sup_{t>0} t^{1-1/p+\ell/2+n} \|\partial_t^n \partial_x^\ell u(t)\|_p \le C(\hat{M}, p, \ell, n), \tag{3.2}$$

where $\hat{M} = \int_{\mathbb{R}^2} u_0 \, dx$.

In order to prove this theorem, we need several lemmas. First we give the following lemma based on Lemma 4.2 of [24]. For the proof, see [24].

Lemma 3.1. *For any $\delta > 0$, the solution u satisfies the integral equation*

$$t^\delta u(t) = \delta \int_0^t e^{(t-s)\Delta}(s^{\delta-1}u(s))\, ds - \int_0^t \nabla\cdot e^{(t-s)\Delta}(s^\delta u(s)(\nabla N * u)(s))ds, \quad t > 0.$$

The second lemma is about the well-known L^p-L^q estimates for the heat semigroup $e^{t\Delta}$.

Lemma 3.2. *Let $1 \le q \le p \le \infty$, $n \in \mathbb{N}$, and $\alpha \in \mathbb{Z}_+^2$. Then, for all $f \in L^q$,*

$$\|\partial_t^n \partial_x^\alpha e^{t\Delta} f\|_p \le Ct^{-1/q+1/p-|\alpha|/2-n} \|f\|_q,$$

where C is a constant depending only on p, q, n, and α.

Lemma 3.3. *For all $f \in L^1 \cap L^\infty$,*

$$\|\nabla N * f\|_\infty \le \kappa(\|f\|_1 \|f\|_\infty)^{1/2}, \tag{3.3}$$

where $\kappa = (2/\pi)^{1/2}$.

Proof. For every $A > 0$,

$$2\pi|(\nabla N * f)(x)| \le \left(\int_{|x-y|\le A} + \int_{|x-y|>A}\right) \frac{|f(y)|}{|x-y|}\, dy \le 2\pi\|f\|_\infty A + \|f\|_1 A^{-1}.$$

From this we have

$$2\pi|(\nabla N * f)(x)| \le 2(2\pi\|f\|_\infty \|f\|_1)^{1/2},$$

which implies (3.3). □

We introduce the following notation: For $\ell, n \in \mathbb{Z}_+$,

$$\phi_p(t) = \sup_{0<s<t} s^{1-1/p}\|u(s)\|_p,$$

$$\phi_p^{(\ell)}(t) = \sup_{0<s<t} s^{1-1/p+\ell/2}\|\partial_x^\ell u(s)\|_p,$$

$$\phi_p^{(n,\ell)}(t) = \sup_{0<s<t} s^{1-1/p+\ell/2+n}\|\partial_t^n \partial_x^\ell u(s)\|_p.$$

Lemma 3.4. *Let $1 \le p \le \infty$ and $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| = \ell \in \mathbb{Z}_+$. Then, for $0 < s \le t$,*

$$\begin{aligned} & s^{1-1/p+\ell/2+1}\|\partial_x^\alpha \nabla \cdot (u(s)(\nabla N * u)(s))\|_p \\ & \le s^{1-1/p+\ell/2+1}\|\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)\|_p + \phi_p(t)\phi_\infty^{(\ell)}(t) \\ & \quad + \sum_{k=1}^{\ell} C_k \phi_p^{(k)}(t)\{(\phi_1^{(\ell-k+1)}(t)\phi_\infty^{(\ell-k+1)}(t))^{1/2} + \phi_\infty^{(\ell-k)}(t)\}, \end{aligned} \tag{3.4}$$

where C_k are positive constants depending only on ℓ and k.

Proof. We observe that

$$\nabla \cdot (u(\nabla N * u)) = \nabla u \cdot (\nabla N * u) - u^2$$

because of $\nabla \cdot (\nabla N * u) = -u$, and that for $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| = \ell$,

$$\partial_x^\alpha(\nabla u \cdot (\nabla N * u)) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \nabla \partial_x^\beta u \cdot (\nabla N * \partial_x^{\alpha-\beta} u).$$

Applying Lemma 3.3, for $0 < s \le t$, we have

$$\begin{aligned} & \|\partial_x^\alpha(\nabla u(s) \cdot (\nabla N * u))(s)\|_p \le \|\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)\|_p \\ & \quad + \sum_{k=0}^{\ell-1} C_k \|\partial_x^{k+1} u(s)\|_p (\|\partial_x^{\ell-k} u(s)\|_1 \|\partial_x^{\ell-k} u(s)\|_\infty)^{\frac{1}{2}} \\ & \le \|\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)\|_p \\ & \quad + s^{-1+1/p-\ell/2-1} \sum_{k=1}^{\ell} C_{k-1} \phi_p^{(k)}(t)(\phi_1^{(\ell-k+1)}(t)\phi_\infty^{(\ell-k+1)}(t))^{\frac{1}{2}}, \end{aligned}$$

where $C_0 = C_\ell = 1$. Similarly,

$$\begin{aligned} \|\partial_x^\alpha u^2(s)\|_p & \le \sum_{k=0}^{\ell} C_k \|\partial_x^k u(s)\|_p \|\partial_x^{\ell-k} u(s)\|_\infty \\ & \le s^{-1+1/p-\ell/2-1} \sum_{k=0}^{\ell} C_k \phi_p^{(k)}(t)\phi_\infty^{(\ell-k)}(t). \end{aligned}$$

Hence,

$$\begin{aligned} & s^{1-1/p+\ell/2+1}\|\partial_x^\alpha \nabla \cdot (u(s)(\nabla N * u))(s)\|_p \\ & \le s^{1-1/p+\ell/2+1}\|\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)\|_p + \phi_p(t)\phi_\infty^{(\ell)}(t) \end{aligned}$$

$$+\sum_{k=1}^{\ell} C'_k \phi_p^{(k)}(t)\{(\phi_1^{(\ell-k+1)}(t)\phi_\infty^{(\ell-k+1)}(t))^{1/2}+\phi_\infty^{(\ell-k)}(t)\}.$$

Thus (3.4) is deduced. □

Proof of Theorem 3.1. First we prove (3.2) for the case $n=0$ by induction on ℓ, namely, for all $\ell \in \mathbb{Z}_+$,

$$\sup_{t>0} t^{1-1/p+\ell/2}\|\partial_x^\ell u(t)\|_p \le C(\hat{M},p,\ell). \tag{3.5}$$

By virtue of (3.1), (3.5) is true for $\ell = 0$.

Assume that (3.5) is true for all nonnegative integers less than or equal to ℓ. To prove (3.5) for $\ell+1$, we use Lemma 3.1 to get

$$\begin{aligned} t^\delta u(t) &= \delta \int_0^t e^{(t-s)\Delta}(s^{\delta-1}u(s))ds - \int_0^{t(1-\varepsilon)} \nabla\cdot e^{(t-s)\Delta}(s^\delta u(s)(\nabla N * u)(s))\,ds \\ &\quad - \int_{t(1-\varepsilon)}^t \nabla\cdot e^{(t-s)\Delta}(s^\delta u(s)(\nabla N * u)(s))\,ds \\ &= \delta I(t) + II(t) + III(t), \quad t>0, \end{aligned}$$

where $\delta > 0$ and $0<\varepsilon<1$. We take δ such that $\delta > 1+\ell/2$ and fix it.

Let $1 \le p \le \infty$. Applying Lemma 3.2, we then have

$$\begin{aligned} \|\partial_x^{\ell+1} I(t)\|_p &\le \int_0^t \|\partial_x e^{(t-s)\Delta}(s^{\delta-1}\partial_x^\ell u(s))\|_p\,ds \\ &\le C_p \int_0^t (t-s)^{-1/2}s^{\delta-1}\|\partial_x^\ell u(s)\|_p\,ds \\ &\le C_p \int_0^t (t-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\,\phi_p^{(\ell)}(t) \\ &= C_p t^{\delta-1+1/p-(\ell+1)/2}\int_0^1 (1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\,\phi_p^{(\ell)}(t). \end{aligned}$$

By Lemma 3.3 and $\phi_1(t) = \hat{M}$, we get

$$\begin{aligned} &\|\partial_x^{\ell+1} II(t)\|_p \le \int_0^{t(1-\varepsilon)} \|\partial_x^{\ell+1}\nabla\cdot e^{(t-s)\Delta}(s^\delta u(s)(\nabla N * u)(s))\|_p\,ds \\ &\le C_p \int_0^{t(1-\varepsilon)} (t-s)^{-(\ell+2)/2}s^\delta \|u(s)(\nabla N * u)(s)\|_p\,ds \\ &\le C_p \int_0^{t(1-\varepsilon)} (t-s)^{-(\ell+2)/2}s^\delta \|u(s)\|_p(\|u(s)\|_1\|u(s)\|_\infty)^{1/2}\,ds \end{aligned}$$

$$\leq C_p t^{\delta-1+1/p-(\ell+1)/2}\int_0^{1-\varepsilon}(1-s)^{-(\ell+2)/2}s^{\delta-1+1/p-1/2}\,ds\,\phi_p(t)(\hat{M}\phi_\infty(t))^{1/2}.$$

Hence, by the induction assumption,

$$t^{1-1/p+(\ell+1)/2}\big(\|\partial_x^{\ell+1}I(t)\|_p+\|\partial_x^{\ell+1}II(t)\|_p\big)\leq C(\hat{M},p,\ell,\varepsilon)t^{\delta}. \tag{3.6}$$

For $1\leq p\leq\infty$, let $1\leq q\leq p$ with $q<\infty$, $1/q-1/p<1/2$. By Lemma 3.4 and the fact that $\phi_1(t)=\hat{M}$, we obtain

$$\begin{aligned}
\|\partial_x^{\ell+1}III(t)\|_p &= \Big\|\int_{t(1-\varepsilon)}^t \partial_x e^{(t-s)\Delta}\partial_x^\ell\nabla\cdot(s^\delta u(s)(\nabla N*u)(s))\,ds\Big\|_p\\
&\leq C_{p,q}\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^\delta\|\partial_x^\ell\nabla\cdot(u(s)(\nabla N*u)(s))\|_q\,ds\\
&\leq C_{p,q}\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^\delta\|\nabla\partial_x^\ell u(s)\cdot(\nabla N*u)(s))\|_q\,ds\\
&\quad + C_{p,q}\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}\,ds\,\phi_q(t)\phi_\infty^{(\ell)}(t)\\
&\quad + C_{p,q}\sum_{k=1}^{\ell}C_k\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}\,ds\\
&\quad \times\phi_q^{(k)}(t)\{(\phi_1^{(\ell-k+1)}(t)\phi_\infty^{(\ell-k+1)}(t))^{1/2}+\phi_\infty^{(\ell-k)}(t)\}.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}\,ds\\
&= t^{\delta-1+1/p-(\ell+1)/2}\int_{1-\varepsilon}^1(1-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}\,ds
\end{aligned}$$

and

$$\int_0^1(1-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}\,ds<\infty$$

due to $1/q-1/p<1/2$ and $\delta>1+\ell/2$, we have

$$\begin{aligned}
&t^{1-1/p+(\ell+1)/2}\|\partial_x^{\ell+1}III(t)\|_p \qquad (3.7)\\
&\leq C_{p,q}t^{1-1/p+(\ell+1)/2}\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^\delta\|\nabla\partial_x^\ell u(s)\cdot(\nabla N*u)(s))\|_q\,ds\\
&\quad + C_{p,q}t^\delta\int_0^1(1-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}\,ds
\end{aligned}$$

$$\times\Big\{\phi_q(t)\phi_\infty^{(\ell)}(t)+\sum_{k=1}^{\ell}C_k\phi_q^{(k)}(t)\Big((\phi_1^{(\ell-k+1)}(t)\phi_\infty^{(\ell-k+1)}(t))^{1/2}+\phi_\infty^{(\ell-k)}(t)\Big)\Big\}.$$

Consider the case $1<p<\infty$. Take $q=p$ in (3.7). Then, by Lemma 3.3, the first term on the right-hand side of (3.7) is estimated as follows:

$$\begin{aligned}
&C_pt^{1-1/p+(\ell+1)/2}\int_{t(1-\varepsilon)}^{t}(t-s)^{-1/2}s^{\delta}\|\nabla\partial_x^{\ell}u(s)\cdot(\nabla N*u)(s))\|_p\,ds\\
&\le C_pt^{1-1/p+(\ell+1)/2}\int_{t(1-\varepsilon)}^{t}(t-s)^{-1/2}s^{\delta}\|\nabla\partial_x^{\ell}u(s)\|_p(\|u(s)\|_1\|u(s)\|_\infty)^{1/2}\,ds\\
&\le C_pt^{\delta}\int_{1-\varepsilon}^{1}(1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\,(\hat{M}\varphi_\infty(t))^{1/2}\phi_p^{(\ell+1)}(t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&t^{1-1/p+(\ell+1)/2}\|\partial_x^{\ell+1}III(t)\|_p \qquad (3.8)\\
&\le C_pt^{\delta}\int_{1-\varepsilon}^{1}(1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\,(\hat{M}\phi_\infty(t))^{1/2}\phi_p^{(\ell+1)}(t)\\
&\quad+C_pt^{\delta}\int_{0}^{1}(1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\\
&\quad\times\Big\{\phi_p(t)\phi_\infty^{(\ell)}(t)+\sum_{k=1}^{\ell}C_k\phi_p^{(k)}(t)\Big((\phi_1^{(\ell-k+1)}(t)\phi_\infty^{(\ell-k+1)}(t))^{1/2}+\phi_\infty^{(\ell-k)}(t)\Big)\Big\}.
\end{aligned}$$

Therefore, by the induction assumption, it follows from (3.6) and (3.8) that

$$\begin{aligned}
&t^{1-1/p+(\ell+1)/2}\|\partial_x^{\ell+1}u(t)\|_p\\
&\le C(\hat{M},p)\int_{1-\varepsilon}^{1}(1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\,\phi_p^{(\ell+1)}(t)+C(\hat{M},p,\ell,\varepsilon).
\end{aligned}$$

From this it follows that

$$\phi_p^{(\ell+1)}(t)\le C(\hat{M},p)\int_{1-\varepsilon}^{1}(1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\,\phi_p^{(\ell+1)}(t)+C(\hat{M},p,\ell,\varepsilon).$$

Taking $0<\varepsilon<1$ such that

$$C(\hat{M},p)\int_{1-\varepsilon}^{1}(1-s)^{-1/2}s^{\delta-2+1/p-\ell/2}\,ds\le\frac{1}{2},$$

we have

$$\phi_p^{(\ell+1)}(t)\le C(\hat{M},p,\ell+1). \qquad (3.9)$$

Consider the case $p = \infty$. Take $p = \infty$ and $2 < q < \infty$ in (3.7) and fix q. Then, using the induction assumption and the fact that $\phi_q^{(\ell+1)}(t) \leq C(\hat{M}, q, \ell+1)$ by (3.9), we estimate the first term on the right-hand side of (3.7) as follows:

$$\begin{aligned}
&t^{1+(\ell+1)/2} \int_{t(1-\varepsilon)}^{t} (t-s)^{-1/q-1/2} s^{\delta} \|\nabla \partial_x^{\ell} u(s) \cdot (\nabla N * u)(s))\|_q \, ds \\
&\leq t^{1+(\ell+1)/2} \int_0^t (t-s)^{-1/q-1/2} s^{\delta} \|\nabla \partial_x^{\ell} u(s)\|_q \|(\nabla N * u)(s)\|_\infty \, ds \\
&\leq t^{1+(\ell+1)/2} \int_0^t (t-s)^{-1/q-1/2} s^{\delta-2+1/q-\ell/2} \, ds \, \phi_q^{(\ell+1)}(t) (\hat{M} \phi_\infty(t))^{1/2} \\
&\leq C(\hat{M}, q, \ell+1) t^{\delta}.
\end{aligned}$$

Hence, by (3.7) with $p = \infty$ and $2 < q < \infty$, the induction assumption gives

$$t^{1+(\ell+1)/2} \|\partial_x^{\ell+1} III(t)\|_\infty \leq C(\hat{M}, q, \ell+1) t^{\delta}.$$

By this estimate and (3.6) for $p = \infty$, we deduce $\phi_\infty^{(\ell+1)}(t) \leq C(\hat{M}, \ell+1)$.

Consider the case $p = 1$. Take $q = 1$ in (3.7). Then the first term on the right-hand side of (3.7) is estimated by using (2.2) with $q = 4/3$:

$$\begin{aligned}
&t^{(\ell+1)/2} \int_{t(1-\varepsilon)}^{t} (t-s)^{-1/2} s^{\delta} \|\nabla \partial_x^{\ell} u(s) \cdot (\nabla N * u)(s))\|_1 \, ds \\
&\leq C t^{(\ell+1)/2} \int_0^t (t-s)^{-1/2} s^{\delta} \|\nabla \partial_x^{\ell} u(s)\|_{4/3} \|u(s)\|_{4/3} \, ds \\
&\leq C t^{(\ell+1)/2} \int_0^t (t-s)^{-1/2} s^{\delta-1/2-(\ell+1)/2} \, ds \, \phi_{4/3}^{(\ell+1)}(t) \phi_{4/3}(t) \\
&\leq C(\hat{M}, \ell+1) t^{\delta}.
\end{aligned}$$

Here we used $\phi_{4/3}^{(\ell+1)}(t) \leq C(\hat{M}, \ell+1)$ by (3.9). Hence, as in the case $p = \infty$, we have $\phi_1^{(\ell+1)}(t) \leq C(\hat{M}, \ell+1)$. Since we establish that (3.5) is true for $\ell+1$, (3.5) is true for all ℓ.

It remains to prove (3.2) by induction on $n \in \mathbb{N}$:

$$\sup_{t>0} t^{1-1/p+\ell/2+n} \|\partial_t^n \partial_x^{\ell} u(t)\|_p \leq C(\hat{M}, p, \ell, n) \quad \text{for all} \quad \ell \in \mathbb{Z}_+. \tag{3.10}$$

To this aim we observe that by equation (1.5),

$$\partial_t^{n+1} \partial_x^{\alpha} u = \partial_t^n \partial_x^{\alpha} \Delta u - \partial_t^n \partial_x^{\alpha} (\nabla u \cdot (\nabla N * u)) + \partial_t^n \partial_x^{\alpha} u^2. \tag{3.11}$$

From (3.11) for $n = 0$, we deduce (3.10) for $n = 1$ by using the Leibniz formula for the second and third terms on the right-hand side of (3.11) and applying Lemma 3.3 and the induction assumption. Similarly, we can deduce that if (3.10) holds for all natural numbers less than or equal to n, then (3.10) holds for $n+1$. Therefore, we complete the proof of Theorem 3.1.

4. Large-time behavior of solutions

For the nonnegative initial data $u_0 \in L^1$ with $\hat{M} := \int_{\mathbb{R}^2} u_0\,dx < 8\pi$, let u be the nonnegative mild solution of (1.5)–(1.6) on $[0,\infty)$; namely, u satisfies the following:

$$\begin{aligned} & u \in C([0,\infty); L^1) \cap C((0,\infty); L^{4/3}), \\ & \sup_{0<t<T} t^{1/4}\|u(t)\|_{4/3} < \infty \quad \text{for every } T > 0, \\ & u(t) = e^{t\Delta}u_0 - \int_0^t \nabla\cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds, \quad t > 0. \end{aligned} \tag{4.1}$$

By Theorem 2.1, u is smooth on $(0,\infty)\times\mathbb{R}^2$ and a classical solution of (1.5).

For $\lambda > 0$, define $u_{0\lambda}(x)$ and $u_\lambda(t,x)$ by

$$u_{0\lambda}(x) = \lambda^2 u_0(\lambda x), \quad u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad t > 0,\ x \in \mathbb{R}^2.$$

Lemma 4.1. *u_λ satisfies*

$$u_\lambda(t) = e^{t\Delta}u_{0\lambda} - \int_0^t \nabla\cdot e^{(t-s)\Delta}(u_\lambda(s)(\nabla N * u_\lambda)(s))\,ds, \quad t > 0. \tag{4.2}$$

Proof. The integral equation (4.1) is rewritten as

$$\begin{aligned} u(t,x) = & \int_{\mathbb{R}^2} G(t,x-y)u_0(y)\,dy \\ & - \int_0^t ds \int_{\mathbb{R}^2} (\nabla G)(t-s,x-y)\cdot u(s,y)(\nabla N * u)(s,y)\,dy, \end{aligned}$$

where $G(t,x)$ is the heat kernel given in (2.1). Take $(\lambda^2 t, \lambda x)$ as (t,x) in this integral equation. Then, by using $\lambda^2 G(\lambda^2 t, \lambda x) = G(t,x)$ and $\nabla G(t,x) = \lambda^3(\nabla G)(\lambda^2 t, \lambda x)$ and observing $(\nabla N * u)(\lambda^2 t, \lambda x) = \lambda^{-1}(\nabla N * u_\lambda)(t,x)$, direct calculations give (4.2). □

By Lemma 4.1, we have the following.

Proposition 4.1. *For each $\lambda > 0$, u_λ is a nonnegative mild solution of (1.5)–(1.6) on $[0,\infty)$ with the nonnegative initial data $u_{0\lambda}$.*

Since $u_{0\lambda}$ satisfies

$$\int_{\mathbb{R}^2} u_{0\lambda}(x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx = \hat{M} < 8\pi,$$

Theorem 3.1 ensures that for all $1 \le p \le \infty$ and $\ell, n \in \mathbb{Z}_+$,

$$\sup_{t>0} t^{1-1/p+\ell/2+n} \|\partial_t^n \partial_x^\ell u_\lambda(t)\|_p \le C(\hat{M}, p, \ell, n). \tag{4.3}$$

We remark that $C(\hat{M}, p, \ell, n)$ is independent of λ. Therefore, by Ascoli-Arzela's theorem, for any sequence $\{\lambda_j\}_{j=1}^\infty$ satisfying $\lambda_j \nearrow \infty$ as $j \nearrow \infty$, there exist a subsequence of $\{\lambda_j\}_{j=1}^\infty$, denote it by $\{\lambda_j\}_{j=1}^\infty$ again, and a nonnegative function $U \in C^\infty((0,\infty) \times \mathbb{R}^2)$ such that

$$\lim_{j\to\infty} \partial_t^n \partial_x^\alpha u_{\lambda_j} = \partial_t^n \partial_x^\alpha U \quad \text{locally uniformly in} \quad (0,\infty) \times \mathbb{R}^2$$

for all $n \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^2$. Since

$$\int_{\mathbb{R}^2} u_\lambda(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx = \hat{M},$$

by Fatou's lemma,

$$\int_{\mathbb{R}^2} U(t,x)\,dx \le \hat{M} \quad \text{for all} \quad t > 0.$$

Moreover, by (4.3), we see that for all $1 \le p \le \infty$ and $\ell, n \in \mathbb{Z}_+$, U satisfies

$$\sup_{t>0} t^{1-1/p+\ell/2+n} \|\partial_t^n \partial_x^\ell U(t)\|_p \le C(\hat{M}, p, \ell, n). \tag{4.4}$$

As for the nonlinear term $\nabla \cdot (u_{\lambda_j}(\nabla N * u_{\lambda_j}))$, we have the following.

Lemma 4.2. *It holds that*

$$\lim_{j\to\infty} \nabla \cdot (u_{\lambda_j}(\nabla N * u_{\lambda_j})) = \nabla \cdot (U(\nabla N * U)) \ \textit{locally uniformly in} \ (0,\infty) \times \mathbb{R}^2. \tag{4.5}$$

Proof. To prove this lemma, we claim

$$\lim_{j\to\infty} \nabla N * u_{\lambda_j} = \nabla N * U \text{ locally uniformly in } (0,\infty) \times \mathbb{R}^2. \tag{4.6}$$

Once we get this claim, we deduce (4.5) by observing

$$\nabla \cdot (u_{\lambda_j}(\nabla N * u_{\lambda_j})) = \nabla u_{\lambda_j} \cdot (\nabla N * u_{\lambda_j}) - u_{\lambda_j}^2$$

and

$$\nabla \cdot (\nabla N * u_{\lambda_j}) = -u_{\lambda_j}, \ \nabla \cdot (\nabla N * U) = -U.$$

We prove (4.6). For any fixed $R_1 > 0$ we take any R_2 with $R_2 > 2R_1$. Then, for $|x| \le R_1$ and $|y| > R_2$, we have $|x-y| \ge R_2/2$ and

$$|(\nabla N * u_{\lambda_j})(t,x) - (\nabla N * U)(t,x)| \le \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{|x-y|}|u_{\lambda_j}(t,y) - U(t,y)|\,dy$$

$$\le \frac{1}{2\pi}\Big(\int_{|y|\le R_2} + \int_{|y|>R_2}\Big)\frac{1}{|x-y|}|u_{\lambda_j}(t,y) - U(t,y)|\,dy$$

$$\le R_2 \sup_{|y|\le R_2} |u_{\lambda_j}(t,y) - U(t,y)| + \frac{2}{\pi R_2}\hat{M}.$$

From this, for any $0 < t_1 < t_2$ it follows that

$$\limsup_{j\to\infty}\Big(\sup_{\substack{t_1\le t\le t_2\\ |x|\le R_1}} |(\nabla N * u_{\lambda_j})(t,x) - (\nabla N * U)(t,x)|\Big) \le \frac{2}{\pi R_2}\hat{M},$$

and hence, by letting $R_2 \to \infty$, (4.5) is deduced. □

Since u_λ is a classical solution of (1.5), by Lemma 4.2, we see that U is a classical solution of (1.5) on $(0,\infty)\times\mathbb{R}^2$, namely

$$\partial_t U = \Delta U - \nabla\cdot(U(\nabla N * U)) \text{ in } (0,\infty)\times\mathbb{R}^2.$$

We also see that U is a weak solution of (1.5)–(1.6) with initial data $\hat{M}\delta_0(x)$, where $\delta_0(x)$ is the Dirac measure supported at the origin. To say it precisely, we define a weak solution of (1.5)–(1.6) with initial data $M\delta_0(x)$, where $M \in \mathbb{R}$.

Definition 4.1. *A function v on $(0,\infty)\times\mathbb{R}^2$ is said to be a weak solution of* (1.5)–(1.6) *with initial data $M\delta_0(x)$, where $M\in\mathbb{R}$, if*

(i) $v \in C((0,\infty); L^1 \cap L^{4/3})$,
(ii) $\sup_{0<t<1} t^{1/4}\|v(t)\|_{4/3} < \infty$,
(iii) *for any $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^2)$, v satisfies*

$$0 = M\varphi(0,0) + \int_0^\infty\int_{\mathbb{R}^2}(\partial_t\varphi + \Delta\varphi)v\,dx\,dt + \int_0^\infty\int_{\mathbb{R}^2}\nabla\varphi\cdot(v(\nabla N * v))\,dx\,dt.$$

Proposition 4.2. *For the limit function U mentioned above, it holds that*

(i) *U is a classical solution of* (1.5),
(ii) *U is a weak solution of* (1.5)–(1.6) *with initial data $\hat{M}\delta_0(x)$, where $\hat{M} = \int_{\mathbb{R}^2} u_0\,dx$.*

Proof. We only prove (ii) because (i) has already been shown above.

U satisfies (i) and (ii) of Definition 4.1 by virtue of (4.4). To prove (iii) of Definition 4.1, we multiply

$$\partial_t u_{\lambda_j} = \Delta u_{\lambda_j} - \nabla \cdot (u_{\lambda_j}(\nabla N * u_{\lambda_j}))$$

by φ and integrate on $(0,\infty)\times\mathbb{R}^2$. Then by integration by parts, we have

$$\begin{aligned} 0 = \int_{\mathbb{R}^2} \varphi(0,x) u_{0\lambda_j}(x)\,dx + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t\varphi + \Delta\varphi) u_{\lambda_j}\,dx\,dt \\ + \int_0^\infty \int_{\mathbb{R}^2} \nabla\varphi \cdot u_{\lambda_j}(\nabla N * u_{\lambda_j})\,dx\,dt. \end{aligned} \tag{4.7}$$

It is easily obtained that

$$\lim_{j\to\infty} \int_{\mathbb{R}^2} \varphi(0,x) u_{0\lambda_j}(x)\,dx = \varphi(0,0)\int_{\mathbb{R}^2} u_0(x)\,dx = \varphi(0,0)\hat{M}.$$

By Lemma 4.2, for each $t > 0$,

$$\begin{aligned} &\lim_{j\to\infty} \int_{\mathbb{R}^2} \nabla\varphi(t,x)\cdot(u_{\lambda_j}(t,x)(\nabla N * u_{\lambda_j})(t,x))\,dx \\ &= \int_{\mathbb{R}^2} \nabla\varphi(t,x)\cdot(U(t,x)(\nabla N * U)(t,x))\,dx, \end{aligned}$$

and by (2.2) and (4.3),

$$\begin{aligned} &\Big| \int_{\mathbb{R}^2} \nabla\varphi(t,x)\cdot(u_{\lambda_j}(t,x)(\nabla N * u_{\lambda_j})(t,x))\,dx \Big| \\ &\le \|u_{\lambda_j}(t)(\nabla N * u_{\lambda_j})(t)\|_1 \|\nabla\varphi\|_\infty \\ &\le C\|u_{\lambda_j}(t)\|_{4/3}^2 \|\nabla\varphi\|_\infty \le C(\hat{M})\|\nabla\varphi\|_\infty t^{-1/2}. \end{aligned}$$

Hence, by letting $j\to\infty$ in (4.7), the Lebesgue convergence theorem ensures

$$0 = \hat{M}\varphi(0,0) + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t\varphi + \Delta\varphi) U\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla\varphi\cdot(U(\nabla N * U))\,dx\,dt.$$

This implies (iii) of Definition 4.1. □

4.1. **Uniqueness of nonnegative weak solutions with a Dirac measure.** From the definition of weak solutions the following lemma follows.

Lemma 4.3. *Let v be a weak solution of (1.5)–(1.6) with initial data $M\delta_0(x)$, where $M \in \mathbb{R}$. Then for all $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^2)$ and for all $T > 0$,*

$$0 = M\varphi(0,0) - \int_{\mathbb{R}^2} \varphi(T,x) v(T,x)\,dx + \int_0^T \int_{\mathbb{R}^2} (\partial_t\varphi + \Delta\varphi) v\,dx\,dt \tag{4.8}$$

$$+\int_0^T\int_{\mathbb{R}^2}\nabla\varphi\cdot(v(\nabla N*v))\,dx\,dt.$$

Proof. Take any positive number T and fix it. For $0<h<1$, let $\eta_h\in C_0^\infty([0,\infty))$ be such that $0\le\eta_h(t)\le 1$, $\eta_h'(t)\le 0$ $(t\ge 0)$, $\eta_h(t)=1$ $(0\le t\le T)$, and $\eta_h(t)=0$ $(t\ge T+h)$, where $\eta_h'=d\eta_h/dt$. Take $\eta_h\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^2)$ as φ in Definition 4.1. Then

$$\begin{aligned}0&=M\varphi(0,0)+\int_T^{T+h}\eta_h'(t)\Big(\int_{\mathbb{R}^2}\varphi(t)v(t)\,dx\Big)dt\\&\quad+\int_0^{T+h}\eta_h(t)\Big(\int_{\mathbb{R}^2}(\partial_t\varphi(t)+\Delta\varphi(t))v(t)\,dx\Big)dt\\&\quad+\int_0^{T+h}\eta_h(t)\Big(\int_{\mathbb{R}^2}\nabla\varphi(t)\cdot(v(t)(\nabla N*v(t)))\,dx\Big)dt\\&=M\varphi(0,0)+I_h+II_h+III_h.\end{aligned}\tag{4.9}$$

Since $t\mapsto\int_{\mathbb{R}^2}\varphi(t,x)v(t,x)\,dx$ is continuous on $(0,\infty)$ and $\eta_h'\le 0$, we deduce

$$\lim_{h\to 0}I_h=-\int_{\mathbb{R}^2}\varphi(T,x)v(T,x)\,dx.$$

By (2.2) and (i) and (ii) of Definition 4.1, for $0<t<T$, we have

$$\int_{\mathbb{R}^2}|\nabla\varphi(t)\cdot(v(t)(\nabla N*v)(t))|\,dx\le C\|\nabla\varphi(t)\|_\infty\|v(t)\|_{4/3}^2\le Ct^{-1/2},\tag{4.10}$$

where C is a positive constant independent of t. By (4.10), the Lebesgue convergence theorem ensures that

$$\lim_{h\to 0}III_h=\int_0^T\int_{\mathbb{R}^2}\nabla\varphi\cdot(v(\nabla N*v))\,dx\,dt.$$

Similarly,

$$\lim_{h\to 0}II_h=\int_0^T\int_{\mathbb{R}^2}(\partial_t\varphi+\Delta\varphi)v\,dx\,dt.$$

Hence, letting $h\to 0$ in (4.9), we deduce (4.8). □

For later use, we give some properties of nonnegative weak solutions of (1.5)–(1.6) with initial data $M\delta_0(x)$, where $M\ge 0$.

Proposition 4.3. *Let v be a nonnegative weak solution of* (1.5)–(1.6) *with initial data $M\delta_0(x)$, where $M\ge 0$. Then the following hold:*

(i) *For all* $t > 0$,

$$\int_{\mathbb{R}^2} v(t,x)\,dx = M. \tag{4.11}$$

(ii) *For all* $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\lim_{t\to 0}\int_{\mathbb{R}^2} \varphi(x)v(t,x)\,dx = M\varphi(0). \tag{4.12}$$

Proof. Take any $T > 0$ and fix it. Let $\eta \in C_0^\infty([0,\infty))$ be such that $\eta(t) = 1$ for $0 \le t \le T$, and let $\varphi \in C_0^\infty(\mathbb{R}^2)$. Take $\eta(t)\varphi(x)$ as $\varphi(t,x)$ in (4.8). Then

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x)v(T,x)\,dx = M\varphi(0) &+ \int_0^T\int_{\mathbb{R}^2} \Delta\varphi v\,dx\,dt \\ &+ \int_0^T\int_{\mathbb{R}^2} \nabla\varphi\cdot(v(\nabla N * v))\,dx\,dt.\end{aligned} \tag{4.13}$$

By (4.10), we have

$$\int_{\mathbb{R}^2} |\nabla\varphi(x)\cdot(v(t,x)(\nabla N*v)(t,x))|\,dx \le C\|\nabla\varphi\|_\infty\, t^{-1/2},\ \ 0 < t \le T. \tag{4.14}$$

For $R > 1$, let $\varphi_R \in C_0^\infty(\mathbb{R}^2)$ be a cutoff function such that

$$\begin{aligned}&0 \le \varphi_R \le 1,\ \ \varphi_R(x) = 1\ (|x| \le R),\ \varphi_R(x) = 0\ (|x| \ge 2R),\\ &|\nabla\varphi_R(x)| \le \tfrac{C}{R},\quad |\nabla^2\varphi_R(x)| \le \tfrac{C}{R^2},\end{aligned}$$

where C is a positive constant independent of R. Take φ_R as φ in (4.13). Then

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi_R(x)v(T,x)\,dx = M &+ \int_0^T\int_{\mathbb{R}^2} \Delta\varphi_R v\,dx\,dt \\ &+ \int_0^T\int_{\mathbb{R}^2} \nabla\varphi_R\cdot(v(\nabla N * v))\,dx\,dt.\end{aligned} \tag{4.15}$$

By (i) and (ii) of Definition 4.1,

$$\int_{\mathbb{R}^2} |\Delta\varphi_R(x)v(t,x)|\,dx \le \|\Delta\varphi_R\|_4\|v(t)\|_{4/3} \le \frac{C}{R^2}t^{-1/4},\ \ 0 < t < T.$$

Putting $\varphi = \varphi_R$ in (4.14) we have

$$\int_{\mathbb{R}^2} |\nabla\varphi_R(x)\cdot(v(t,x)(\nabla N * v)(t,x))|\,dx \le \frac{C}{R^{3/2}}\,t^{-1/2}.$$

Hence, letting $R \to \infty$ in (4.15), we deduce $\int_{\mathbb{R}^2} v(T,x)\,dx = M$ by the Lebesgue convergence theorem.

(4.12) is obtained by letting $T \to 0$ in (4.13) □

Proposition 4.4. *Let v be a nonnegative weak solution of* (1.5)–(1.6) *with initial data $M\delta_0(x)$, where $M \geq 0$. Then, for all $t > 0$, $|x|^2 v(t) \in L^1$ and*

$$\int_{\mathbb{R}^2} |x|^2 v(t,x)\,dx = 4M\Big(1 - \frac{M}{8\pi}\Big)t. \tag{4.16}$$

Proof. For $R > 1$, let $\varphi_R \in C_0^\infty(\mathbb{R}^2)$ be the same cutoff function as in the proof of Proposition 4.3. Take $\Phi_R = |x|^2\varphi_R$ and t as φ and T in (4.13), respectively:

$$\begin{aligned}\int_{\mathbb{R}^2} \Phi_R(x)v(t,x)\,dx = \int_0^t \int_{\mathbb{R}^2} \Delta\Phi_R(x)v(s,x)\,dx\,ds \\ + \int_0^t \int_{\mathbb{R}^2} v(s,x)(\nabla N * v)(s,x)\cdot\nabla\Phi_R(x)\,dx\,ds.\end{aligned} \tag{4.17}$$

We note

$$\int_{\mathbb{R}^2} v(t,x)\,dx = M$$

by (4.11), and $\|\Delta\Phi_R\|_\infty \leq C$ by the definition of Φ_R, where C is independent of R. Then

$$\int_0^t \int_{\mathbb{R}^2} |v(s,x)\Delta\Phi_R(x)|\,dx\,ds \leq CMt.$$

We observe that the second term on the right-hand side of (4.17) may be rewritten as

$$\begin{aligned}I_R(s) &:= \int_{\mathbb{R}^2} v(s,x)(\nabla N * v)(s,x)\cdot\nabla\Phi_R(x)\,dx \\ &= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v(s,x)v(s,y)\frac{x-y}{|x-y|^2}\cdot\nabla\Phi_R(x)\,dy\,dx \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v(s,x)v(s,y)\frac{(x-y)\cdot(\nabla\Phi_R(x)-\nabla\Phi_R(y))}{|x-y|^2}\,dy\,dx.\end{aligned}$$

Since

$$|\nabla\Phi_R(x) - \nabla\Phi_R(y)| \leq \|\nabla^2\Phi_R\|_\infty|x-y| \leq C|x-y|,$$

where C is a constant independent of R, we have

$$|I_R(s)| \leq C\Big(\int_{\mathbb{R}^2} v(s,x)\,dx\Big)^2 = CM^2,$$

and hence

$$\int_{\mathbb{R}^2} \Phi_R(x)v(t,x)\,dx \leq C(M+M^2)t.$$

Letting $R \to \infty$, by Fatou's lemma we obtain

$$\int_{\mathbb{R}^2} |x|^2 v(t,x)\,dx \le C(M+M^2)t.$$

Noting that as $R \to \infty$,

$$\Delta\Phi_R(x) \to 4, \qquad \frac{(x-y)\cdot(\nabla\Phi_R(x)-\nabla\Phi_R(y))}{|x-y|^2} \to 2,$$

by the Lebesgue convergence theorem we deduce

$$\int_0^t \int_{\mathbb{R}^2} v(s,x)\Delta\Phi_R(x)\,dx\,ds \to 4\int_0^t \int_{\mathbb{R}^2} v(s,x)\,dx\,ds = 4Mt,$$
$$\int_0^t I_R(s)\,ds \to -\frac{1}{2\pi}\int_0^t \Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v(s,x)v(s,y)\,dy\,dx\Big)ds = -\frac{M^2}{2\pi}t,$$

and hence (4.16) by letting $R \to \infty$ in (4.17). □

For a nonnegative weak solution v of (1.5)–(1.6) with initial data $\hat{M}\delta_0(x)$, we show that if $0 < \hat{M} < 8\pi$ then $v = U_{\hat{M}}$, where $U_{\hat{M}}$ is the radially symmetric self-similar solution of (1.5) with

$$\int_{\mathbb{R}^2} U_{\hat{M}}(t,x)\,dx = \hat{M}$$

mentioned in Section 2. In order to show $v = U_{\hat{M}}$, we need a result by Gallagher-Gallay-Lions [15] in which they proved the uniqueness of weak solutions of the vorticity equation in $\mathbb{R}^2$ with a Dirac measure by applying rearrangement techniques. To mention this result (Proposition 4.5 below), we introduce the notion of rearrangements.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function satisfying

$$||f| > \theta| := |\{x \in \mathbb{R}^d : |f(x)| > \theta\}| < \infty \quad \text{for any} \quad \theta > 0,$$

where $|A|$ is the Lebesgue measure of a Lebesgue-measurable set A in $\mathbb{R}^d$. The distribution function μ_f of f is defined by $\mu_f(\theta) = ||f| > \theta|$ $(\theta \ge 0)$, and the decreasing rearrangement f^* of f by

$$f^*(s) = \inf\{\theta \ge 0 : \mu_f(\theta) \le s\} \quad (s \ge 0).$$

The function $f^\sharp(x)$, called the symmetric rearrangement or the Schwarz symmetrization of f, is defined by $f^\sharp(x) = f^*(c_d|x|^d)$, where c_d is the volume of the unit ball in $\mathbb{R}^d$.

Some basic properties about rearrangements are as follows (see [1, 27, 30, 41] for example):

(i) f^* is nonincreasing and right-continuous on $[0,\infty)$.

(ii) If f is continuous on $\mathbb{R}^d$, then f^* and $f^\sharp$ are continuous on $[0,\infty)$ and $\mathbb{R}^d$, respectively.
(iii) If $f:\mathbb{R}^d \to [0,\infty)$ is radially symmetric and nonincreasing with respect to $|x|$, then $f = f^\sharp$.
(iv) For every Borel-measurable function $\Phi : \mathbb{R} \to [0,\infty)$,

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\,dx = \int_{\mathbb{R}^d} \Phi(f^\sharp(x))\,dx = \int_0^\infty \Phi(f^*(s))\,ds.$$

(v) Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be integrable on $\mathbb{R}^d$. If $\int_0^s f^*(\sigma)\,d\sigma \le \int_0^s g^*(\sigma)\,d\sigma$ for all $s > 0$, then

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\,dx \le \int_{\mathbb{R}^d} \Phi(|g(x)|)\,dx$$

for all convex functions $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$.
(vi) (Contraction property) Let $1 \le p \le \infty$. For $f, g \in L^p(\mathbb{R}^d)$,

$$\|f^* - g^*\|_{L^p(0,\infty)} \le \|f - g\|_{L^p(\mathbb{R}^d)}, \quad \|f^\sharp - g^\sharp\|_{L^p(\mathbb{R}^d)} \le \|f - g\|_{L^p(\mathbb{R}^d)}.$$

Let $T > 0$. For a measurable function $f : (0,T) \times \mathbb{R}^d \to \mathbb{R}$, we denote by f^* the decreasing rearrangement of f with respect to the space variable $x \in \mathbb{R}^d$; that is, $f^*(t,s) = f(t)^*(s)$ for $t \in (0,T)$ and $s \ge 0$, where $f(t)^*$ is the decreasing rearrangement of $f(t)$. Define the Schwarz symmetrization $f^\sharp$ of f with respect to the space variable by $f^\sharp(t,x) = f^*(t, c_d|x|^d) = f(t)^*(c_d|x|^d)$.

Proposition 4.5 (Proposition 4.2, [15])**.** *Let $f, g : \mathbb{R}^d \to [0,+\infty)$ be continuous and integrable functions satisfying*
(i) $\int_0^s f^*(\sigma)\,d\sigma \le \int_0^s g^*(\sigma)\,d\sigma$ *for all* $s > 0$*,*
(ii) *g is radially symmetric and nonincreasing with respect to $|x|$,*
(iii) $\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} g(x)\,dx$*,*
(iv) $\int_{\mathbb{R}^d} |x|^d f(x)\,dx = \int_{\mathbb{R}^d} |x|^d g(x)\,dx < \infty$*.*

Then $f = g$.

For the nonnegative mild solution u of (1.5)–(1.6) with nonnegative initial data $u_0 \in L^1$ and the radially symmetric self-similar solution $U_{\hat{M}}$ with $\hat{M} = \int_{\mathbb{R}^2} u_0\,dx$, the following comparison between $\int_0^s u^*(t,\sigma)\,d\sigma$ and $\int_0^s U^*_{\hat{M}}(t,\sigma)\,d\sigma$ was obtained in [32].

Proposition 4.6. *Assume $\hat{M} := \int_{\mathbb{R}^2} u_0\,dx < 8\pi$ for the nonnegative initial data $u_0 \in L^1$. Then for the nonnegative mild solution u of* (1.5)–(1.6) *on* $[0,T)$*, it holds that for each $0 < t < T$,*

$$\int_0^s u^*(t,\sigma)\,d\sigma \le \int_0^s U^*_{\hat{M}}(t,\sigma)\,d\sigma \quad \textit{for all} \;\; s > 0.$$

Applying Proposition 4.6, we have the following.

Proposition 4.7. *Let v be a nonnegative weak solution of (1.5)–(1.6) with initial data $\hat{M}\delta_0(x)$ and assume $0 < \hat{M} < 8\pi$. Then for each $t > 0$,*

$$\int_0^s v^*(t,\sigma)\,d\sigma \le \int_0^s U^*_{\hat{M}}(t,\sigma)\,d\sigma \quad \textit{for all} \;\; s > 0. \tag{4.18}$$

Proof. Take an arbitrary number $\tau > 0$ and fix it. Define w on $[0,\infty) \times \mathbb{R}^2$ by $w(t,x) = v(t+\tau, x)$. Then $w \in C([0,\infty); L^1 \cap L^{4/3})$, and we see that w satisfies the following: For any $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^2)$,

$$\begin{aligned} 0 = &\int_{\mathbb{R}^2} \varphi(0,x)v(\tau,x)\,dx + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t\varphi + \Delta\varphi)w\,dx\,dt \\ &+ \int_0^\infty \int_{\mathbb{R}^2} \nabla\varphi \cdot f\,dx\,dt, \end{aligned} \tag{4.19}$$

where $f = w(\nabla N * w)$. By (2.2) and (i) and (ii) of Definition 4.1, $f \in C([0,\infty); L^1)$. We claim that for all $t > 0$,

$$w(t) = e^{t\Delta}v(\tau) - \int_0^t \nabla \cdot e^{(t-s)\Delta} f(s)\,ds. \tag{4.20}$$

To prove this claim, we define $\tilde{w}$ on $[0,\infty) \times \mathbb{R}^2$ by the right-hand side of (4.20). We observe that $\tilde{w}$ satisfies (4.19) replacing w by $\tilde{w}$. In fact, for $\{f_n\}_{n=1}^\infty \subset C_0^\infty([0,\infty) \times \mathbb{R}^2)$ satisfying

$$\max_{0 \le t \le T} \|f_n(t) - f(t)\|_1 \to 0 \;\; \text{as} \;\; n \to \infty \;\; \text{for all} \;\; T > 0,$$

define $\tilde{w}_n$ on $[0,\infty) \times \mathbb{R}^2$ by

$$\tilde{w}_n(t) = e^{t\Delta}v(\tau) - \int_0^t \nabla \cdot e^{(t-s)\Delta} f_n(s)\,ds.$$

Then $\tilde{w}_n \in C([0,\infty); L^1 \cap L^{4/3}) \cap C^\infty((0,\infty) \times \mathbb{R}^2))$, and applying L^p-L^1 estimates for $e^{t\Delta}$ yields that for all $1 \le p < 2$,

$$\max_{0 \le t \le T} \|\tilde{w}_n(t) - \tilde{w}(t)\|_p \to 0 \;\; \text{as} \;\; n \to \infty \;\; \text{for all} \;\; T > 0.$$

Since $\tilde{w}_n$ satisfies (4.19) replacing w and f by $\tilde{w}_n$ and f_n, respectively (see Chapter 4 of [17] for example), $\tilde{w}$ satisfies (4.19) replacing w by $\tilde{w}$ by letting $n \to \infty$. Hence, by the uniqueness of weak solutions for the heat equation (see Theorem 4.4.2 of [17] for example), we conclude $w = \tilde{w}$ and hence (4.20).

From (4.20) it follows that w is a nonnegative mild solution of (1.5)–(1.6) with initial data $v(\tau)$. Since $\int_{\mathbb{R}^2} v(\tau)\,dx = \hat{M} < 8\pi$, applying Proposition 4.6 yields that for each $t > 0$,

$$\int_0^s v^*(t+\tau,\sigma)\,d\sigma = \int_0^s w^*(t,\sigma)\,d\sigma \le \int_0^s U^*_{\hat{M}}(t,\sigma)\,d\sigma \quad \text{for all} \quad s > 0. \tag{4.21}$$

We observe $\|v^*(t+\tau) - v^*(t)\|_1 \to 0$ as $\tau \to 0$ by the contraction property of the decreasing rearrangement, and hence, letting $\tau \to 0$ in (4.21), we conclude (4.18). □

By Proposition 4.5, we have the following result on uniqueness.

Theorem 4.1. *Let v be a nonnegative weak solution of* (1.5)–(1.6) *with initial data $\hat{M}\delta_0(x)$. If $0 < \hat{M} < 8\pi$, then $v = U_{\hat{M}}$.*

Proof. To prove this theorem, we apply Proposition 4.5 with $f = v(t)$ and $g = U_{\hat{M}}(t)$. For each $t > 0$, the function $x \mapsto U_{\hat{M}}(t,x)$ is radially symmetric and nonincreasing with respect to $|x|$. Proposition 4.7 and (i) of Proposition 4.3 imply (i) and (iii) of Proposition 4.5, respectively. To prove (iv) of Proposition 4.5, we claim

$$\int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(t,x)\,dx = 4\hat{M}\Big(1 - \frac{\hat{M}}{8\pi}\Big)t \quad \text{for } t > 0. \tag{4.22}$$

In fact, by Proposition 2.2, we see that for any $0 < \varepsilon < t$,

$$\int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(t,x)\,dx = \int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(\varepsilon,x)\,dx + 4\hat{M}\Big(1 - \frac{\hat{M}}{8\pi}\Big)(t-\varepsilon).$$

From this relation, (4.22) is deduced because

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(\varepsilon,x)\,dx = 0$$

by virtue of (2.3). Hence it follows from Proposition 4.4 and (4.22) that

$$\int_{\mathbb{R}^2} |x|^2 v(t,x)\,dx = \int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(t,x)\,dx.$$

Therefore, Proposition 4.5 ensures $v(t) = U_{\hat{M}}(t)$ for each $t > 0$. □

Remark 4.1. To prove uniqueness by our method requires nonnegativity for weak solutions. For small $\hat{M} \in \mathbb{R}$, the uniqueness of weak solutions without nonnegativity is obtained by the same method as in [18, 24].

4.2. **Convergence to a radially symmetric self-similar solution.** We come back to study the convergence of u_{λ_j} to U as $j \to \infty$, where u_{λ_j} and U are the same as before.

Proposition 4.8. *Let $1 \le p \le \infty$. Then $u_{\lambda_j}(t) \to U(t)$ in L^p as $j \to \infty$ for all $t > 0$.*

Proof. For fixed $t > 0$, $u_{\lambda_j}(t,x) \to U(t,x)$ as $j \to \infty$ for all $x \in \mathbb{R}^2$, and $\|u_{\lambda_j}(t)\|_1 = \hat{M} = \|U(t)\|_1$ by (i) of Proposition 4.3. Hence,

$$\lim_{j\to\infty} \|u_{\lambda_j}(t) - U(t)\|_1 = 0.$$

Let $1 < p < \infty$. Then

$$\|u_{\lambda_j}(t) - U(t)\|_p \le \|u_{\lambda_j}(t) - U(t)\|_\infty^{1-1/p} \|u_{\lambda_j}(t) - U(t)\|_1^{1/p},$$

from which together with $t\|u_{\lambda_j}(t) - U(t)\|_\infty \le C(\hat{M})$ by (4.3) and (4.4) it follows that

$$\lim_{j\to\infty} \|u_{\lambda_j}(t) - U(t)\|_p = 0.$$

We consider the case $p = \infty$. Recall the following interpolation inequalities (for example, see Theorem 9.3 of [14]): Let $2 < p < \infty$. Then there is a positive constant C, depending only on p, such that for any $f \in W^{1,p}(\mathbb{R}^2)$,

$$\|f\|_\infty \le C\|\nabla f\|_p^{2/p} \|f\|_p^{1-2/p}.$$

Applying this inequality yields

$$\|u_{\lambda_j}(t) - U(t)\|_\infty \le C\|\nabla(u_{\lambda_j}(t) - U(t))\|_p^{2/p} \|u_{\lambda_j}(t) - U(t)\|_p^{1-2/p}.$$

Since $t^{1-1/p}\|\nabla(u_{\lambda_j}(t) - U(t))\|_p \le C(\hat{M}, p)$ by (4.3) and (4.4), we deduce

$$\lim_{j\to\infty} \|u_{\lambda_j}(t) - U(t)\|_\infty = 0.$$

Therefore, we establish the proof of this proposition. □

We are now in a position to mention our main result.

Theorem 4.2. *Assume $\hat{M} = \int_{\mathbb{R}^2} u_0\, dx < 8\pi$ for the nonnegative initial data $u_0 \in L^1$. Then for the nonnegative global mild solution u of (1.5)–(1.6), it holds that for all $1 \le p \le \infty$,*

$$\lim_{t\to\infty} t^{1-1/p} \|u(t) - U_{\hat{M}}(t)\|_p = 0. \tag{4.23}$$

Proof. By Proposition 4.2, U is a nonnegative weak solution of (1.5)–(1.6) with initial data $\hat{M}\delta_0$, and by Theorem 4.1, $U = U_{\hat{M}}$. Hence, by Proposition 4.8,

$$\lim_{j\to\infty} \|u_{\lambda_j}(t) - U_{\hat{M}}(t)\|_p = 0 \tag{4.24}$$

for all $t > 0$, where $1 \le p \le \infty$. Since for any sequence $\{\lambda_j\}$ satisfying $\lambda_j \nearrow \infty$ $(j \nearrow \infty)$ there exists a subsequence of $\{\lambda_j\}$ for which (4.24) is satisfied, we deduce that for all $t > 0$ and for all $1 \le p \le \infty$,

$$\lim_{\lambda\to\infty} \|u_\lambda(t) - U_{\hat{M}}(t)\|_p = 0.$$

Putting $t = 1$ and then taking $\lambda = t^{1/2}$ yields that

$$\lim_{t\to\infty} t^{1-1/p}\|u(t,\cdot) - \frac{1}{t}U_{\hat{M}}(1, \frac{1}{\sqrt{t}}\cdot)\|_p = 0,$$

which completes the proof of Theorem 4.2 because

$$t^{-1}U_{\hat{M}}(1, xt^{-1/2}) = U_{\hat{M}}(t,x).$$

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