Advances in Differential Equations Volume 16, Numbers 9-10 (2011), 801–837

EXISTENCE AND MULTIPLICITY RESULTS FOR EQUATIONS WITH NEARLY CRITICAL GROWTH

Francesca Gladiali
Università di Sassari,via Piandanna 4, 07100 Sassari, Italy

Massimo Grossi
Dipartimento di Matematica, Università di Roma "La Sapienza"
P.le A. Moro 2 - 00185 Rome, Italy

(Submitted by: Norman Dancer)

Abstract. We consider the problem

$$\begin{cases} -\Delta u = K(x)u^{p_\epsilon} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \end{cases}$$

where $p = \frac{n+2}{n-2}$, $p_\epsilon = p - \epsilon$, $n \geq 3$, $\epsilon > 0$ and $K(x) > 0$ in $\mathbb{R}^n$. We prove an existence and multiplicity result for single peaked solutions of our problem concentrating at a fixed critical point of $K(x)$ and some other related results.

1. Introduction and statement of the main results

This paper deals with the problem

$$\begin{cases} -\Delta u = K(x)u^{p_\epsilon} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $p = \frac{n+2}{n-2}$, $p_\epsilon = p - \epsilon$, $n \geq 3$, $\epsilon > 0$ and $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{1,2} = \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{\frac{1}{2}}$.

The function $K \in C^1(\mathbb{R}^n)$ satisfies the following assumptions:

$$\tfrac{1}{C} \leq K(x) \leq C; \quad |\nabla K(x)| \leq C, \tag{1.2}$$

for some positive constant C.

The main motivation in studying (1.1) arises from the problem of finding a metric conformal to the standard one of $\mathbb{R}^n$ such that $K(x)$ is the scalar curvature of the new metric.

Accepted for publication: February 2011.
AMS Subject Classifications: 35A01, 35A02.

It is well known that, if $\epsilon = 0$ and $K(x) = n(n-2)$, problem (1.1) has only the two-parameter family of solutions

$$U_{\delta,y}(x) = \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-y|^2)^{\frac{n-2}{2}}}, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $\delta > 0$ and $y \in \mathbb{R}^n$. Hereafter we denote by $U(x) = U_{1,0}(x)$.

In this paper we are interested in single peak solutions u_ϵ concentrating at some point; i.e.,

$$u_\epsilon(x) = U_{\delta_\epsilon, y_\epsilon}(x) + \phi_\epsilon(x), \tag{1.4}$$

where $\delta_\epsilon \to 0$, $y_\epsilon \to y_0 \in \mathbb{R}^n$, $\phi_\epsilon(x) \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

Problem (1.1) has been studied by many authors. Indeed there exist many existence or nonexistence results of solutions of (1.1) depending on the shape of $K(x)$ and the exponent p_ϵ (see for example [9], [11] and [12] and the reference therein).

Let us recall that when $K(x)$ is constant there is no solution to (1.1) (see [3]). The same happens if $(\nabla K(x), x - x_0) \geq 0$ for some $x_0 \in \mathbb{R}^n$ ([10]). Another nonexistence result can be found in [9] where it was showed that if $K(x)$ grows like or faster than $|x|^{2-(n-2)\epsilon}$ then equation (1.1) has no positive solutions.

Concerning the existence results we mention the one in [12] where the authors showed the existence of solutions of (1.1) that concentrate at a nondegenerate critical point y_0 of $K(x)$ satisfying the crucial condition

$$\Delta K(y_0) < 0. \tag{1.5}$$

Here the authors looked for solutions with a suitable decay at infinity, i.e., solutions that belong to a suitable Sobolev space. In their setting, if $K(x)$ is bounded, they had to impose the condition $n > 6$. The proof of this result uses the finite-dimensional reduction method, a tool widely used in this kind of problem.

In [11] S. Yan improved this result by removing the condition $n > 6$ but still assuming (1.5). Note that in [11] the supercritical case $\epsilon < 0$ was also considered.

Finally, we mention the paper [8] which is concerned with the problem

$$\begin{cases} -\Delta u + V(x)u = n(n-2))u^{p_\epsilon} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \end{cases} \tag{1.6}$$

Here the authors obtain existence and multiplicity results assuming that $V(x)$ has just stable (not necessarily nondegenerate) critical points.

In this paper we focus on the multiplicity of single peak solutions at a suitable critical point of the function $K(x)$. To do this we use some ideas of [5] where the nonlinear Schrodinger equation was considered. This method is well suited to our problem and allows us to establish some existence and multiplicity results to (1.1).

One of the main results will be the existence, under suitable assumptions on $K(x)$, of several single peak solutions concentrating at the same point y_0. Before stating the main theorem we have to introduce notation and some hypotheses on the function $K(x)$.

Let y_0 be a fixed critical point of $K(x)$. We make the following assumptions on $K(x)$: there exist functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $R_i : B(0,r) \to \mathbb{R}$ and real constants α_i, β_i with $\alpha_i \geq 1$ for $i = 1,..,n$ such that

i) $\frac{\partial K}{\partial x_i}(x + y_0) = h_i(x) + R_i(x)$ in $B(0,r)$,
ii) $|R_i(x)| \leq C|x|^{\beta_i}$ for $x \in B(0,r)$ and $\beta_i > \alpha_i$,
iii) $h_i(tx) = t^{\alpha_i} h_i(x)$ for any $x \in \mathbb{R}^n$ and $t > 0$,
iv) $(h_1(x),..,h_n(x)) = (0,..,0)$ if and only if $x = 0$.

Now let us introduce the following vector field:

$$\mathcal{L}_{y_0}(y) := \Big(\int_{\mathbb{R}^n} h_i(x+y) U^{p+1}(x) dx \Big)_{i=1,\ldots,n}. \tag{1.7}$$

Note that the definition of $\mathcal{L}$ is well posed if $\alpha_i < n$. Then we set

$$\mathcal{Z}_{y_0} := \{ y \in \mathbb{R}^n : y \text{ is a stable zero of } \mathcal{L}_{y_0} \} \tag{1.8}$$

(see Section 5 for the definition of stable zero) and

$$\alpha = \min\{\alpha_i \,|\, i = 1, \ldots, n\}. \tag{1.9}$$

Our first result is the existence of several single peak solutions concentrating at the point y_0. This is linked to the number of elements of the set $\mathcal{Z}_{y_0}$.

Theorem 1.1. *Let $K(x)$ satisfy* (1.2). *Let y_0 be a critical point of $K(x)$ and suppose that $K(x)$ satisfies hypotheses* i)-iv) *in a neighborhood of y_0 with*

$$1 \leq \alpha_i \leq n-1 \quad \text{for } i = 1, \ldots, n, \tag{1.10}$$

and

$$\sum_{\alpha_j = \alpha} \int_{\mathbb{R}^n} h_j(x + \tilde{y}) x_j U^{p+1}(x) dx < 0 \tag{1.11}$$

for all $\tilde{y} \in \mathcal{Z}_{y_0}$. Then there exists ϵ_0 such that for any $\epsilon \in (0, \epsilon_0)$ the number of single peak solutions of (1.1) blowing up and concentrating at y_0 is greater than or equal to $\#\mathcal{Z}_{y_0}$.

We observe that if y_0 is a nondegenerate critical point of K then $\mathcal{Z}_{y_0} = \{0\}$ and (1.11) becomes (1.5) (see (8.1) and (8.2) in Proposition 8.1). One of the crucial tools of the proof of the previous result is the splitting of the peak point y_ϵ of the solution as

$$y_\epsilon = y_0 + \epsilon^{\frac{1}{\alpha+1}} y + o(\epsilon^{\frac{1}{\alpha+1}})$$

where y belongs to $\mathcal{Z}_{y_0}$. This decomposition enables us to handle functions K whose gradient at y_0 is flat of order α.

If all zeroes of $\mathcal{L}_{y_0}$ belong to $\mathcal{Z}_{y_0}$ we have that the previous theorem is optimal.

Theorem 1.2. *Let us suppose that* 0 *is a regular value for* $\mathcal{L}_{y_0}$ *and suppose that* $K(x)$ *satisfies* (1.2) *and hypotheses* i)-iv) *in a neighborhood of* y_0*. If* (1.10) *and* (1.11) *hold and* $\#\mathcal{Z}_{y_0} < \infty$*, then the number of single peak solutions of* (1.1) *blowing up at* y_0 *is equal to* $\#\mathcal{Z}_{y_0}$*.*

Let us recall that 0 is a regular value for $\mathcal{L}_{y_0}$ if $Jac\,(\mathcal{L}_{y_0}(y)) \neq 0$ for any $y \in \mathbb{R}^n$ such that $\mathcal{L}_{y_0}(y) = 0$.

Since it is known that any single peak solution concentrates at a critical point of K, the previous result classifies the number of single peak solutions to (1.1) (under the assumption that 0 is a regular value for $\mathcal{L}_{y_0}$).

We end with a nonexistence result of single peak solutions. It shows that the assumptions of the previous theorem are "almost" sharp.

Theorem 1.3. *Let* $y_0 \in \mathbb{R}^n$ *be a critical point of* $K(x)$*. Assume* $K(x)$ *satisfies* (1.2) *and hypotheses* i)-iv) *in a neighborhood of* y_0*, with* $1 \leq \alpha_i < n-1$ *for* $i = 1, \dots, n$*. If either*

$$\mathcal{L}_{y_0}(y) \neq 0 \quad \textit{for any } y \in \mathbb{R}^n, \tag{1.12}$$

or

$$\exists \tilde{y} \in \mathbb{R}^n \textit{ such that } \mathcal{L}_{y_0}(\tilde{y}) = 0 \textit{ and } \sum_{\alpha_j = \alpha} \int_{\mathbb{R}^n} h_j(x+\tilde{y}) x_j U^{p+1}(x)dx \geq 0, \tag{1.13}$$

then there is no single peaked solution to (1.1) *blowing up at* y_0*.*

Let us observe that if $K(x)$ is a homogenous polynomial of degree $\alpha+1$ then the assumption that $\tilde{y}$ belongs to $\mathcal{Z}_{y_0}$ becomes

$$\int_{\mathbb{R}^n} \nabla K(x+\tilde{y}) U^{p+1}(x)dx = 0$$

and

$$\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n} h_j(x+\tilde{y})x_jU^{p+1}(x)dx \geq 0$$

becomes (by Euler's theorem for homogeneous functions)

$$\int_{\mathbb{R}^n} K(x+\tilde{y})U^{p+1}(x)dx \geq 0.$$

In this setting Chen and Lin proved that there is no single peak solution to (1.1)(see [1]).

The paper is organized as follows. In Section 2, we report some known facts and we prove some technical estimates; in Section 3, we perform the finite-dimensional reduction while, in Section 4, we prove a useful estimate on the decay of the solutions. In Section 5, we find some necessary conditions for the existence of solutions and we prove Theorem 1.1. In Section 6, we prove the asymptotic behavior of the maximum points of solutions of (1.1) and prove Theorem 1.2, while, in Section 7, we prove Theorem 1.3. Finally, in Section 8, we give some examples where Theorem 1.1-1.3 apply.

2. Preliminaries and notation

We rewrite problem (1.1) in an equivalent way in order to use the Liapunov-Schmidt reduction method. The first step is to scale (1.1). If u_ϵ is a solution of (1.1) we let

$$u^{\epsilon}_{\delta,y} = \delta^{\frac{2}{p_\epsilon-1}}u_\epsilon(\delta x+y) \tag{2.1}$$

with $\delta>0$ and $y\in\mathbb{R}^n$. This function satisfies

$$\begin{cases} -\Delta u^{\epsilon}_{\delta,y}(x) = K(\delta x+y)\big(u^{\epsilon}_{\delta,y}(x)\big)^{p_\epsilon} & \text{in } \mathbb{R}^n \\ u^{\epsilon}_{\delta,y} > 0 \\ u^{\epsilon}_{\delta,y}\in\mathcal{D}^{1,2}(\mathbb{R}^n). \end{cases} \tag{2.2}$$

Of course (1.1) and (2.2) are equivalent.

Let us introduce the operator $i^*: L^{\frac{2n}{n+2}}(\mathbb{R}^n)\to\mathcal{D}^{1,2}(\mathbb{R}^n)$, the adjoint of the embedding $i:\mathcal{D}^{1,2}(\mathbb{R}^n)\to L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. The operator i^* is defined as follows:

$$i^*(u)=v \Leftrightarrow (v,\phi)_{1,2}=\int_{\mathbb{R}^n} u(x)\phi(x)dx \quad \forall\phi\in\mathcal{D}^{1,2}(\mathbb{R}^n).$$

As observed in [7] i^* is continuous; i.e., there exists $C>0$ such that

$$\|i^*(u)\|_{1,2}\leq C\|u\|_{\frac{2n}{n+2}} \qquad \forall u\in L^{\frac{2n}{n+2}}(\mathbb{R}^n). \tag{2.3}$$

We let $f_\epsilon(s) = (s^+)^{p_\epsilon}$ where s^+ denotes the positive part of s and $f_0 = (s^+)^{\frac{n+2}{n-2}}$. So we can rewrite problem (2.2) in this way:

$$u = i^* \left(K(\delta x + y)(u^+)^{p_\epsilon} \right), \tag{2.4}$$

where $u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ for some $s > 1$. Here we point out that if u solves (2.4) then $u > 0$ by the strong maximum principle.

Now we consider the operator $L : \mathcal{D}^{1,2}(\mathbb{R}^n) \to \mathcal{D}^{1,2}(\mathbb{R}^n)$ defined by

$$L(u) = u - i^* \left(n(n-2) U^{\frac{4}{n-2}} u \right).$$

It is known that L is self-adjoint and is a compact perturbation of the identity. Moreover, $Ker L = span\,\{\psi_0, \psi_1, \ldots, \psi_n\}$, where

$$\psi_0(x) = x \cdot \nabla U(x) + \frac{n-2}{2} U(x) = \frac{n-2}{2} \frac{1 - |x|^2}{(1 + |x|^2)^{\frac{n}{2}}}, \tag{2.5}$$

$$\psi_i(x) = \frac{\partial U}{\partial x_i}(x) = -(n-2) \frac{x_i}{(1 + |x|^2)^{\frac{n}{2}}}, \quad i = 1, \ldots, n. \tag{2.6}$$

Let $X = L^s(\mathbb{R}^n) \cap \mathcal{D}^{1,2}(\mathbb{R}^n)$ with the norm $\|u\|_X = \max\{\|u\|_s, \|u\|_{1,2}\}$, for some $s > 1$.

Remark 2.1. Let $s > \frac{n}{n-2}$. If $u \in L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^{\frac{ns}{n+2s}}(\mathbb{R}^n)$, then $i^*(u) \in L^s(\mathbb{R}^n)$ and

$$\|i^*(u)\|_s \le C(n,s) \|u\|_{\frac{ns}{n+2s}}. \tag{2.7}$$

We quote the following result.

Lemma 2.2. *Let $s > \frac{n}{n-2}$. If $u \in X$, then $L(u) \in X$. Moreover, $L_{|X} : X \to X$ is a continuous function; i.e., there exists $C > 0$ such that*

$$\|L_{|X}\|_X \le C \|u\|_X \quad \forall u \in X. \tag{2.8}$$

Finally, $Ker L_{|X} = Ker L$.

Proof. See Lemmas 2.10 and 2.11 in [7]. □

Let $W = (Ker L)^\perp \cap L^s(\mathbb{R}^n) = \{u \in X : (u, \phi)_{1,2} = 0, \forall \phi \in Ker L\}$. We now consider the projection $\Pi : X \to W$. Recall the following result.

Lemma 2.3. *Let $s > \frac{n}{n+2}$ and $\tilde{L} : W \to W$ defined by $\tilde{L}(u) = \Pi \left[L_{|X}(u) \right]$. Then $\tilde{L}$ is invertible and $\tilde{L}^{-1}$ is a continuous operator; i.e., there exists $C > 0$ such that*

$$\|\tilde{L}^{-1}(u)\|_X \le C \|u\|_X \quad \forall u \in W. \tag{2.9}$$

Proof. See Lemma 2.13 in [7]. □

Now we recall the following useful estimates.

Lemma 2.4. *There exist $\epsilon_0 > 0$ and $C > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ and for all $\phi_1, \phi_2 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ and for all $s > 1$ we have*

$$\|U^{p_\epsilon} - U^p + \epsilon\,(\log U)\,U^p\|_{\frac{2n}{n+2}} \le C\epsilon^2 \tag{2.10}$$

$$\|U^{p_\epsilon} - U^p\|_{\frac{2n}{n+2}} \le C\epsilon\,, \|U^{p_\epsilon} - U^p\|_{\frac{ns}{n+2s}} \le C\epsilon \tag{2.11}$$

$$\|p_\epsilon U^{p_\epsilon - 1} - pU^{p-1}\|_{\frac{n}{2}} \le C\epsilon \tag{2.12}$$

$$\begin{aligned}&\left|\left[(U+\phi_1)^+\right]^{p_\epsilon} - \left[(U+\phi_2)^+\right]^{p_\epsilon} - p_\epsilon\left[(U+\phi_2)^+\right]^{p_\epsilon-1}(\phi_1-\phi_2)\right| \\ &\qquad \le C|\phi_1 - \phi_2|^{p_\epsilon}\end{aligned} \tag{2.13}$$

$$\left|\left[(U+\phi_1)^+\right]^{p_\epsilon-1} - \left[(U+\phi_2)^+\right]^{p_\epsilon-1}\right| \le C|\phi_1-\phi_2|^{p_\epsilon-1}. \tag{2.14}$$

Proof. See [7] Lemma 2.20 and Remark 2.21, 2.22. □

Lemma 2.5. *Let $K(x)$ satisfy* (1.2) *and $s > \frac{n}{n-1}$. Then for all $y \in \mathbb{R}^n$*

$$\|\left(K(\delta x + y) - K(y)\right)U^p\|_{\frac{2n}{n+2}} \le C\delta \tag{2.15}$$

$$\|\left(K(\delta x + y) - K(y)\right)U^p\|_{\frac{ns}{n+2s}} \le C\delta \tag{2.16}$$

$$\|\left(K(\delta x + y) - K(y)\right)U^{p-1}\|_{\frac{n}{2}} \le C\delta. \tag{2.17}$$

Proof. Let us prove (2.15). For some $t = t(x,y) \in (0,1)$ we have

$$\begin{aligned}&\|\left(K(\delta x + y) - K(y)\right)U^p\|_{\frac{2n}{n+2}} \\ &= \Big\{\int_{\mathbb{R}^n}\left|\left(K(\delta x + y) - K(y)\right)U^{\frac{n+2}{n-2}}(x)\right|^{\frac{2n}{n+2}} dx\Big\}^{\frac{n+2}{2n}} \\ &= \delta\Big\{\int_{\mathbb{R}^n}\Big|\int_0^1 \nabla K(\delta t x + y)\cdot x\ dt\ U^{\frac{n+2}{n-2}}(x)\Big|^{\frac{2n}{n+2}} dx\Big\}^{\frac{n+2}{2n}} \\ &\le C\delta\Big\{\int_{\mathbb{R}^n}\left[|x|U^{\frac{n+2}{n-2}}(x)\right]^{\frac{2n}{n+2}} dx\Big\}^{\frac{n+2}{2n}} \le C\delta,\end{aligned}$$

where in the last line we used the fact that $\|\nabla K\|_\infty \le C$. Arguing in the same way we get

$$\begin{aligned}&\|\left(K(\delta x + y) - K(y)\right)U^p\|_{\frac{ns}{n+2s}} \\ &= \Big\{\int_{\mathbb{R}^n}\left|\left(K(\delta x + y) - K(y)\right)U^{\frac{n+2}{n-2}}(x)\right|^{\frac{ns}{n+2s}} dx\Big\}^{\frac{n+2s}{ns}}\end{aligned}$$

$$\leq \delta\Big\{ \int_{\mathbb{R}^n} \Big(|x| U^{\frac{n+2}{n-2}}(x)\Big)^{\frac{ns}{n+2s}} dx\Big\}^{\frac{n+2s}{ns}} \leq C\delta,$$

and

$$\begin{aligned}&\|\left(K(\delta x+y)-K(y)\right)U^{p-1}(x)\|_{\frac{n}{2}}\\ &=\Big\{\int_{\mathbb{R}^n}\Big|\left(K(\delta x+y)-K(y)\right)U^{\frac{4}{n-2}}(x)\Big|^{\frac{n}{2}}dx\Big\}^{\frac{2}{n}}\leq CVD\delta.\end{aligned}$$

3. The finite-dimensional reduction

We are looking for solutions to (2.2) of the type

$$u^\epsilon_{\delta,y} = U + \phi^\epsilon_{\delta,y} \text{ and } \phi^\epsilon_{\delta,y} \to 0 \in \mathcal{D}^{1,2}(\mathbb{R}^n) \text{ as } \epsilon \to 0, \text{ and } \delta \to 0.$$

This turns the problem (2.2) into a finite-dimensional one.

The first step is to show the existence of the function $\phi^\epsilon_{\delta,y}$.

Proposition 3.1. *Let $\gamma \in (0,1)$ be fixed and $\frac{n}{n-2} < s < \frac{2n}{n-2}$. Then there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for all $\epsilon \in (0,\epsilon_0)$, $\delta \in (0,\delta_0)$ and $y \in \mathbb{R}^n$, there exists a unique $\phi^\epsilon_{\delta,y} \in W$ such that*

$$\|\phi^\epsilon_{\delta,y}\|_X \leq (\epsilon+\delta)^\gamma \tag{3.1}$$

$$\Pi\left\{U + \phi^\epsilon_{\delta,y} - i^*\left(K(\delta x+y)\left((U+\phi^\epsilon_{\delta,y})^+\right)^{p_\epsilon}\right)\right\} = 0 \tag{3.2}$$

(*see Section* 2 *for the definition of* W).

Proof. With no loss of generality we can assume that $K(y) = n(n-2)$. Set $\phi = \phi^\epsilon_{\delta,y}$. We first observe that ϕ solves (3.2) if and only if ϕ is a fixed point for the operator $T^\epsilon_{\delta,y} : W \to W$ defined by

$$T^\epsilon_{\delta,y}(\phi) = \tilde{L}^{-1}\left\{\Pi\left[i^*\left(K(\delta x+y)((U+\phi)^+)^{p_\epsilon} - K(y)U^p - K(y)pU^{p-1}\phi\right)\right]\right\}. \tag{3.3}$$

Step A. We begin by proving that the inequality $\|\phi\|_X \leq (\epsilon+\delta)^\gamma$ implies $\|T^\epsilon_{\delta,y}(\phi)\|_X \leq (\epsilon+\delta)^\gamma$. Using (3.3) and (2.9) we have

$$\begin{aligned}\|T^\epsilon_{\delta,y}(\phi)\|_X &\leq C\|i^*\left[K(\delta x+y)\left((U+\phi)^+\right)^{p_\epsilon} - K(y)U^p - K(y)pU^{p-1}\phi\right]\|_X\\ &\leq C\left\{\|i^*\left[K(\delta x+y)\left(\left((U+\phi)^+\right)^{p_\epsilon} - U^{p_\epsilon} - p_\epsilon U^{p_\epsilon-1}\phi\right)\right]\|_X\right.\\ &+\|i^*\left[K(\delta x+y)\left(U^{p_\epsilon}-U^p\right)\right]\|_X\\ &+\|i^*\left[\left(K(\delta x+y)-K(y)\right)U^p\right]\|_X\\ &+\|i^*\left[K(\delta x+y)\left(p_\epsilon U^{p_\epsilon-1}\phi - pU^{p-1}\phi\right)\right]\|_X\\ &+\left.\|i^*\left[\left(K(\delta x+y)-K(y)\right)pU^{p-1}\phi\right]\|_X\right\}\end{aligned}$$

$$= C\left(A_1 + A_2 + A_3 + A_4 + A_5\right). \tag{3.4}$$

By the boundedness of $K(x)$ in L^∞, we get

$$\begin{aligned}
&\|i^*\left(K(\delta x+y)\left(\left((U+\phi)^+\right)^{p_\epsilon} - U^{p_\epsilon} - p_\epsilon U^{p_\epsilon-1}\phi\right)\right)\|_{1,2} \quad (\text{ by (2.3)})\\
&\le C\|\left(\left((U+\phi)^+\right)^{p_\epsilon} - U^{p_\epsilon} - p_\epsilon U^{p_\epsilon-1}\phi\right)\|_{\frac{2n}{n+2}} \ (\text{ by (2.13)})\\
&\le C\|\phi^{p_\epsilon}\|_{\frac{2n}{n+2}} \le C\|\phi\|_X^{p_\epsilon}
\end{aligned} \tag{3.5}$$

$\Big($by interpolation since $\dfrac{2n}{n+2}p_\epsilon \in (s, \dfrac{2n}{n-2})$ if ϵ is sufficiently small$\Big)$.

In the same way, using (2.7) and (2.13) we have

$$\begin{aligned}
&\|i^*\left(K(\delta x+y)\left(\left((U+\phi)^+\right)^{p_\epsilon} - U^{p_\epsilon} - p_\epsilon U^{p_\epsilon-1}\phi\right)\right)\|_{s} \\
&\le \|\left(\left((U+\phi)^+\right)^{p_\epsilon} - U^{p_\epsilon} - p_\epsilon U^{p_\epsilon-1}\phi\right)\|_{\frac{ns}{n+2s}} \le C\|\phi^{p_\epsilon}\|_{\frac{ns}{n+2s}} \le C\|\phi\|_X^{p_\epsilon}.
\end{aligned} \tag{3.6}$$

Hence from (3.5) and (3.6) we can write

$$A_1 = \|i^*\left(K(\delta x+y)\left(\left((U+\phi)^+\right)^{p_\epsilon} - U^{p_\epsilon} - p_\epsilon U^{p_\epsilon-1}\phi\right)\right)\|_X \le C\|\phi\|_X^{p_\epsilon}. \tag{3.7}$$

Using (2.3), (2.7) and (2.11) we get

$$A_2 = \|i^*\left(K(\delta x+y)\left(U^{p_\epsilon} - U^p\right)\right)\|_X \le C\|U^{p_\epsilon} - U^p\|_X \le C\epsilon. \tag{3.8}$$

Using (2.3) and (2.15) we easily have

$$\begin{aligned}
&\|i^*\left[\left(K(\delta x+y) - K(y)\right)U^p\right]\|_{1,2}\\
&\le C\|\left(K(\delta x+y) - K(y)\right)U^p\|_{\frac{2n}{n+2}} \le C\delta
\end{aligned} \tag{3.9}$$

while from (2.7) and (2.16) it follows that

$$\begin{aligned}
&\|i^*\left(\left(K(\delta x+y) - K(y)\right)U^p\right)\|_{s}\\
&\le C\|\left(K(\delta x+y) - K(y)\right)U^p\|_{\frac{ns}{n+2s}} \le C\delta.
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10)

$$A_3 = \|i^*\left(\left(K(\delta x+y) - K(y)\right)U^p\right)\|_X \le C\delta. \tag{3.11}$$

Using (2.3), (2.12) and interpolating we have

$$\begin{aligned}
&\|i^*\left(K(\delta x+y)\left(p_\epsilon U^{p_\epsilon-1}\phi - pU^{p-1}\phi\right)\right)\|_{1,2}\\
&\le C\|K(\delta x+y)\left(p_\epsilon U^{p_\epsilon-1}\phi - pU^{p-1}\phi\right)\|_{\frac{2n}{n+2}}\\
&\le C\|p_\epsilon U^{p_\epsilon-1} - pU^{p-1}\|_{\frac{n}{2}}\|\phi\|_{\frac{2n}{n-2}} \le C\epsilon\|\phi\|_X.
\end{aligned} \tag{3.12}$$

In the same way from (2.7), (2.12) we get that

$$\|i^*\left(K(\delta x+y)\left(p_\epsilon U^{p_\epsilon-1}\phi-pU^{p-1}\phi\right)\right)\|_s\le C\epsilon\|\phi\|_X. \tag{3.13}$$

Estimates (3.12) and (3.13) imply

$$A_4=\|i^*\left(K(\delta x+y)\left(p_\epsilon U^{p_\epsilon-1}\phi-pU^{p-1}\phi\right)\right)\|_X\le C\epsilon\|\phi\|_X. \tag{3.14}$$

From (2.3) and (2.17) we have

$$\begin{aligned}&\|i^*\left(\left(K(\delta x+y)-K(y)\right)U^{p-1}\phi\right)\|_{1,2}\\ \le\ & C\|\left(K(\delta x+y)-K(y)\right)U^{p-1}\phi\|_{\frac{2n}{n+2}}\\ \le\ & C\|\left(K(\delta x+y)-K(y)\right)U^{p-1}\|_{\frac{n}{2}}\|\phi\|_{\frac{2n}{n-2}}\le C\delta\|\phi\|_X\end{aligned} \tag{3.15}$$

while from (2.7) and (2.17) it follows that

$$\begin{aligned}&\|i^*\left(\left(K(\delta x+y)-K(y)\right)U^{p-1}\phi\right)\|_{s}\\ \le\ & C\|\left(K(\delta x+y)-K(y)\right)U^{p-1}\phi\|_{\frac{ns}{n+2s}}\\ \le\ & C\|\left(K(\delta x+y)-K(y)\right)U^{p-1}\|_{\frac{n}{2}}\|\phi\|_{s}\le C\delta\|\phi\|_X.\end{aligned} \tag{3.16}$$

From (3.15) and (3.16) then

$$A_5=\|i^*\left(\left(K(\delta x+y)-K(y)\right)U^{p-1}\phi\right)\|_X\le C\delta\|\phi\|_X. \tag{3.17}$$

Putting together (3.7), (3.8), (3.11), (3.14) and (3.17) we finally get

$$\begin{aligned}\|T^\epsilon_{\delta,y}(\phi)\|_X\ &\le\ C\left(\|\phi\|_X^{p_\epsilon}+\epsilon+\delta+\epsilon\|\phi\|_X+\delta\|\phi\|_X\right) \qquad (3.18)\\ &\le\ C\left(\epsilon+\delta\right)^\gamma\left(\left(\epsilon+\delta\right)^{\gamma(p_\epsilon-1)}+\left(\epsilon+\delta\right)^{1-\gamma}+\epsilon+\delta\right)\le(\epsilon+\delta)^\gamma\end{aligned}$$

if ϵ and δ are small enough.

Step B. Here we want to show that

$$T^\epsilon_{\delta,y}:\{\phi\in X:\|\phi\|_X\le(\epsilon+\delta)^\gamma\}\to\{\phi\in X:\|\phi\|_X\le(\epsilon+\delta)^\gamma\}$$

is a contraction map. Arguing as in the previous step we have

$$\begin{aligned}&\|T^\epsilon_{\delta,y}(\phi_1)-T^\epsilon_{\delta,y}(\phi_2)\|_X\\ &\le C\|i^*(K(\delta x+y)[((U+\phi_1)^+)^{p_\epsilon}-((U+\phi_2)^+)^{p_\epsilon}-K(y)U^{p-1}(\phi_1-\phi_2)]\|_X\\ &\le C\Big\{\Big\|i^*\left(K(\delta x+y)\left(\left((U+\phi_1)^+\right)^{p_\epsilon}-\left((U+\phi_2)^+\right)^{p_\epsilon}\right.\right.\\ &\quad-\ \left.\left.p_\epsilon\left((U+\phi_2)^+\right)^{p_\epsilon-1}(\phi_1-\phi_2)\right)\right)\Big\|_X \qquad (3.19)\\ &+C\|i^*\left(K(\delta x+y)\left(p_\epsilon\left((U+\phi_2)^+\right)^{p_\epsilon-1}-p_\epsilon U^{p_\epsilon-1}\right)(\phi_1-\phi_2)\right)\|_X\\ &+C\|i^*\left(K(\delta x+y)\left(p_\epsilon U^{p_\epsilon-1}-pU^{p-1}\right)(\phi_1-\phi_2)\right)\|_X\end{aligned}$$

$$+ C \left\| i^* \left(\left(K(\delta x + y) - K(y) \right) U^{p-1} \left(\phi_1 - \phi_2 \right) \right) \right\|_X \Big\} .$$

Estimating each term in (3.19) as was done in the previous step we get

$$\begin{aligned} &\|T^{\epsilon}_{\delta,y}(\phi_1) - T^{\epsilon}_{\delta,y}(\phi_2)\|_X \\ &\le C \left\{ \|\phi_1 - \phi_2\|_X^{p_\epsilon} + \|\phi_2\|_X^{p_\epsilon - 1} \|\phi_1 - \phi_2\|_X + \epsilon \|\phi_1 - \phi_2\|_X + \delta \|\phi_1 - \phi_2\|_X \right\} \\ &\qquad \text{(choosing } \epsilon \text{ and } \delta \text{ small enough)} \\ &\le C_0 \|\phi_1 - \phi_2\|_X \end{aligned} \tag{3.20}$$

for some constant $C_0 < 1$. This show that $T^{\epsilon}_{\delta,y}$ is a contraction mapping from $\{\phi \in X : \|\phi\|_X \le (\epsilon + \delta)^\gamma\}$ into itself and the claim of the proposition follows. □

4. Some useful estimates.

In Section 3, we proved the existence of $\phi^{\epsilon}_{\delta,y} \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ with $\frac{n}{n-2} < s < \frac{2n}{n-2}$ such that $\|\phi^{\epsilon}_{\delta,y}\|_X \le (\epsilon + \delta)^\gamma$ and such that $U + \phi^{\epsilon}_{\delta,y}$ is a solution of the equation

$$\Pi \left\{ U + \phi^{\epsilon}_{\delta,y} - i^* \left(K(\delta x + y) \left((U + \phi^{\epsilon}_{\delta,y})^+ \right)^{p_\epsilon} \right) \right\} = 0. \tag{4.1}$$

From (4.1) we deduce the existence of real numbers $c_i(\epsilon, \delta, y)$ such that $u^{\epsilon}_{\delta,y} = U + \phi^{\epsilon}_{\delta,y}$ is a solution of

$$u^{\epsilon}_{\delta,y} - i^* \left(K(\delta x + y) \left((u^{\epsilon}_{\delta,y})^+ \right)^{p_\epsilon} \right) = \sum_{i=0}^{n} c_i(\epsilon, \delta, y) \psi_i \tag{4.2}$$

where the functions ψ_i are defined in (2.5) and (2.6). In order to find solutions of (2.4) we need to find ϵ, δ_ϵ, y_ϵ such that $c_i(\epsilon, \delta_\epsilon, y_\epsilon) = 0$ for $i = 0, \ldots, n$.

We start by showing the following.

Lemma 4.1. *Let ϵ_k, δ_k, and $y_k \in \mathbb{R}^n$ be sequences such that* $\lim \epsilon_k = 0$, $\lim \delta_k = 0$ *and* $\lim y_k = y$. *Let $u^{\epsilon_k}_{\delta_k,y_k} = U + \phi^{\epsilon_k}_{\delta_k,y_k}$ be a solution to* (4.2) *such that $u^{\epsilon_k}_{\delta_k,y_k} \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Then*

$$c_i(\epsilon_k, \delta_k, y_k) = o(1) \qquad \forall i = 1, \ldots, n, \tag{4.3}$$

where $o(1)$ goes to zero as $k \to \infty$.

Proof. To simplify notation we set $\phi_k = \phi^{\epsilon_k}_{\delta_k,y_k}$ and $p_k = p_{\epsilon_k}$. The proof will be divided into two steps.

Step A. We first show that $c_0(\epsilon_k,\delta_k,y_k)=o(1)$. Taking the scalar product of (4.2) and ψ_0 we have

$$\begin{aligned}&\left(U+\phi_k-i^*\left(K(\delta_k x+y_k)\left((U+\phi_k)^+\right)^{p_k}\right),\psi_0\right)_{1,2}\\ =\ &\sum_{i=0}^n c_i\left(\epsilon_k,\delta_k,y_k\right)\left(\psi_i,\psi_0\right)_{1,2}=c_0\left(\epsilon_k,\delta_k,y_k\right)\|\psi_0\|^2_{1,2}. \qquad (4.4)\end{aligned}$$

Now we estimate the left-hand side of (4.4).

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla U\cdot\nabla\psi_0 dx+\int_{\mathbb{R}^n}\nabla\phi_k\cdot\nabla\psi_0 dx-\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left((U+\phi_k)^+\right)^{p_k}\psi_0 dx\\ &=K(y)\int_{\mathbb{R}^n}U^{\frac{n+2}{n-2}}\psi_0 dx+K(y)\int_{\mathbb{R}^n}U^{\frac{4}{n-2}}\psi_0\phi_k dx\\ &-\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left((U+\phi_k)^+\right)^{p_k}\psi_0 dx\\ &=-\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[\left((U+\phi_k)^+\right)^{p_k}-U^{p_k}-p_kU^{p_k-1}\phi_k\right]\psi_0 dx\\ &+\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[-U^{p_k}+U^p-\epsilon_k\ln U U^p\right]\psi_0 dx\\ &+\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[-p_kU^{p_k-1}+pU^{p-1}\right]\phi_k\psi_0 dx\\ &+\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]U^p\psi_0 dx+\epsilon_k K(y)\int_{\mathbb{R}^n}U^p\ln U\psi_0 dx\\ &+\epsilon_k\int_{\mathbb{R}^n}\left[-K(y)+K(\delta_k x+y_k)\right]U^p\ln U\psi_0 dx\\ &+\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]pU^{p-1}\phi_k\psi_0 dx. \qquad (4.5)\end{aligned}$$

We now estimate each term in (4.5). Using (2.13) and interpolating we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[\left((U+\phi_k)^+\right)^{p_k}-U^{p_k}-p_kU^{p_k-1}\phi_k\right]\psi_0 dx\right|\\ &\le\int_{\mathbb{R}^n}|\phi_k|^{p-\epsilon_k}|\psi_0|dx\le\|\psi_0\|_{\frac{2n}{n-2}}\|\phi_k^{p-\epsilon_k}\|_{\frac{2n}{n+2}}\\ &\le C\|\psi_0\|_{\frac{2n}{n-2}}\|\phi_k\|_X^{p-\epsilon_k}=o(1) \qquad (4.6)\end{aligned}$$

since $\frac{2n}{n+2}(p-\epsilon_k)\in(s,\frac{2n}{n-2})$ for ϵ small enough. Using (2.10) we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} K(\delta_k x+y_k)\left[-U^{p_k}+U^p-\epsilon_k\ln U U^p\right]\psi_0 dx\Big|\\ &\le \|U^{p_k}-U^p+\epsilon_k\ln U U^p\|_{\frac{2n}{n+2}}\|\psi_0\|_{\frac{2n}{n-2}}\le C\epsilon_k^2.\end{aligned}\tag{4.7}$$

From (2.12) we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} K(\delta_k x+y_k)\left[-p_k U^{p_k-1}+pU^{p-1}\right]\phi_k\psi_0 dx\Big|\\ &\le C\|p_{\epsilon_k}U^{p_{\epsilon_k}-1}-pU^{p-1}\|_{\frac{n}{2}}\|\phi_k\|_{\frac{2n}{n-2}}\|\psi_0\|_{\frac{2n}{n-2}}\le C\epsilon_k\|\phi_k\|_X.\end{aligned}\tag{4.8}$$

Reasoning as in the proof of (2.15) one can see that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]U^p\psi_0 dx\Big|\\ &\le \|\left[K(y)-K(\delta_k x+y_k)\right]U^p\|_{\frac{2n}{n+2}}\|\psi_0\|_{\frac{2n}{n-2}}\le C\left(\delta_k+|y_k-y|\right)\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\epsilon_k\Big|\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]U^p\ln U\psi_0 dx\Big|\\ &\le C\epsilon_k\|\ln U U^p\|_{\frac{2n}{n+2}}\|\psi_0\|_{\frac{2n}{n-2}}\le C\epsilon_k.\end{aligned}\tag{4.10}$$

In the same way, from (2.17), we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]U^{p-1}\phi_k\psi_0 dx\Big|\\ \le\;& \|\left[K(y)-K(\delta_k x+y_k)\right]U^{p-1})\|_{\frac{n}{2}}\|\phi_k\|_{\frac{2n}{n-2}}\|\psi_0\|_{\frac{2n}{n-2}}\\ \le\;& C\left(\delta_k+|y-y_k|\right)\|\phi_k\|_X.\end{aligned}\tag{4.11}$$

Finally,

$$\epsilon_k K(y)\Big|\int_{\mathbb{R}^n}U^p\ln U\psi_0 dx\Big|\le C\epsilon_k.\tag{4.12}$$

Using estimates (4.6)-(4.12) we get

$$\left|c_0\left(\epsilon_k,\delta_k y_k\right)\right|\|\psi_0\|_{1,2}^2\le o(1)\tag{4.13}$$

which gives the claim.

Step B. Here we will show that $c_i(\epsilon_k,\delta_k,y_k)=o(1)$ for $i=1,\dots,n$. Taking the scalar product of (4.2) and ψ_i for $i=1,\dots,n$ we have

$$\left(U+\phi_k-i^*\left(K(\delta_k x+y_k)\left((U+\phi_k)^+\right)^{p_k}\right),\psi_i\right)_{1,2}\tag{4.14}$$

$$= \sum_{j=0}^{n} c_j\left(\epsilon_k,\delta_k,y_k\right)\left(\psi_j,\psi_i\right)_{1,2} = c_i\left(\epsilon_k,\delta_k,y_k\right)\|\psi_i\|_{1,2}^2$$

for $i=1,\dots,n$. As before we expand the left-hand side of (4.14),

$$\int_{\mathbb{R}^n}\nabla U\cdot\nabla\psi_i dx + \int_{\mathbb{R}^n}\nabla\phi_k\cdot\nabla\psi_i dx - \int_{\mathbb{R}^n}K(\delta_k x+y_k)\left((U+\phi_k)^+\right)^{p_k}\psi_i dx$$
$$-\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[\left((U+\phi_k)^+\right)^{p_k} - U^{p_k} - p_k U^{p_k-1}\phi_k\right]\psi_i dx$$
$$+\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[-U^{p_k}+U^p-\epsilon_k U^p\ln U\right]\psi_i dx$$
$$+\int_{\mathbb{R}^n}K(\delta_k x+y_k)\left[-p_k U^{p_k-1}+pU^{p-1}\right]\phi_k\psi_i dx$$
$$+\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]U^p\psi_i dx$$
$$+\epsilon_k\int_{\mathbb{R}^n}\left[-K(y)+K(\delta_k x+y_k)\right]U^p\ln U\psi_i dx$$
$$+\int_{\mathbb{R}^n}\left[K(y)-K(\delta_k x+y_k)\right]U^{p-1}\phi_k\psi_i dx.$$

Proceeding in the same way as in the previous step we get the claim. □

Lemma 4.2. *Let $u^\epsilon_{\delta,y}\in X$ be a solution of* (4.2) *such that $u^\epsilon_{\delta,y}\to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. For each compact set $G\subset\mathbb{R}^n$ there exist $C>0$, $\delta_0>0$, $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, $\delta\in(0,\delta_0)$ and $y\in G$*

$$|u^\epsilon_{\delta,y}(x)|\le CU(x) \quad \text{in } \mathbb{R}^n \tag{4.15}$$

$$\left|\frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}(x)\right|\le C|\nabla U(x)| \quad \text{in } \mathbb{R}^n. \tag{4.16}$$

Proof. The proof will be divided into three steps.
Step A. Here we will prove that for all $R>0$ there exist $C(R)>0$, $\delta_0>0$, $\epsilon_0>0$ such that, for any $y\in G$, $\epsilon\in(0,\epsilon_0)$ and $\delta\in(0,\delta_0)$,

$$|u^\epsilon_{\delta,y}(x)|\le C(R) \quad \text{in } B(0,R). \tag{4.17}$$

From (4.2) we have that $u^\epsilon_{\delta,y}$ satisfies the equation

$$-\Delta u^\epsilon_{\delta,y} = K(\delta x+y)\left(\left(u^\epsilon_{\delta,y}\right)^+\right)^{\frac{n+2}{n-2}-\epsilon} - \sum_{i=0}^{n} c_i(\epsilon,\delta,y)\Delta\psi_i \text{ in } \mathbb{R}^n. \tag{4.18}$$

By Lemma 4.1, we have

$$\Big\| \sum_{i=0}^{n} c_i(\epsilon,\delta,y)\Delta\psi_i \Big\|_{L^\infty(\mathbb{R}^n)} = o(1),$$

where $o(1) \to 0$ as ϵ and δ go to 0. Our assumptions imply that

$$\|K(\delta x + y)\left(\left(u_{\delta,y}^{\epsilon}\right)^{+}\right)^{\frac{4}{n-2}-\epsilon}\|_{L^{\frac{n}{2}}(B(0,4R))} \le C.$$

By Lemma 6 of [6], (see also [8]) we derive that

$$\|u_{\delta,y}^{\epsilon}\|_{L^q(B(0,2R))} \le C\|u_{\delta,y}^{\epsilon}\|_{L^{\frac{2n}{n-2}}(B(0,4R))},$$

where $q = \frac{(\frac{2n}{n-2})^2}{2}$. This implies that

$$\|K(\delta x + y)\left(\left(u_{\delta,y}^{\epsilon}\right)^{+}\right)^{\frac{4}{n-2}-\epsilon}\|_{L^{q\frac{n-2}{4-\epsilon(n-2)}}(B(0,2R))} \le C$$

with $q\frac{n-2}{4-\epsilon(n-2)} > \frac{n}{2}$. So, by elliptic regularity (see [4] or [6]), we can derive

$$\sup_{B(0,R)} |u_{\delta,y}^{\epsilon}| \le C \int_{B(0,2R)} \left(1 + \left(u_{\delta,y}^{\epsilon}\right)^2\right) dx \le C.$$

Step B. In this step we will prove that there exist $R > 0$, $C > 0$, $\delta_0 > 0$, $\epsilon_0 > 0$ such that, for all $y \in G$, $\delta \in (0,\delta_0)$, $\epsilon \in (0,\epsilon_0)$,

$$|u_{\delta,y}^{\epsilon}(x)| \le CU(x) \quad \forall x \in \mathbb{R}^n \setminus B(0,R). \tag{4.19}$$

Let $\hat{u}_{\delta,y}^{\epsilon}(x) \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ be the Kelvin transform of $u_{\delta,y}^{\epsilon}$; i.e.,

$$\hat{u}_{\delta,y}^{\epsilon}(x) = \frac{1}{|x|^{n-2}} u_{\delta,y}^{\epsilon}\Big(\frac{x}{|x|^2}\Big) \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

We want to prove that $|\hat{u}_{\delta,y}^{\epsilon}(x)| \le C$ for all $x \in \mathbb{R}^n \setminus \{0\}$ such that $|x| \le R$. It is standard to see that $\hat{u}_{\delta,y}^{\epsilon}(x)$ satisfies

$$\begin{aligned} -\Delta \hat{u}_{\delta,y}^{\epsilon}(x) &= K\Big(\delta\frac{x}{|x|^2} + y\Big)\frac{1}{|x|^{\epsilon(n-2)}}\big(\big(\hat{u}_{\delta,y}^{\epsilon}\big)^{+}\big)^{\frac{4}{n-2}-\epsilon}\hat{u}_{\delta,y}^{\epsilon}(x) \\ &\quad - \sum_{i=0}^{n} \frac{1}{|x|^{n+2}} c_i(\epsilon,\delta,y)\Delta\psi_i\Big(\frac{x}{|x|^2}\Big) \quad \text{in } \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{4.20}$$

We observe that $\frac{1}{|x|^{n+2}}\Delta\psi_i\big(\frac{x}{|x|^2}\big) \in L^\infty(B(0,4R))$ so that the last term in (4.20) converges to zero as δ and ϵ go to zero. Moreover, using the fact that

$\hat{u}^\epsilon_{\delta,y} \to U(x)$ in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$,

$$\Big\|K\Big(\delta\frac{x}{|x|^2}+y\Big)\frac{1}{|x|^{\epsilon(n-2)}}\left(\hat{u}^\epsilon_{\delta,y}\right)^{\frac{4}{n-2}-\epsilon}\Big\|_{L^{\frac{n}{2}}(B(0,4R))} \le C.$$

Proceeding as in the previous step we get

$$\|\hat{u}^\epsilon_{\delta,y}\|_{L^q(B(0,2R))} \le \|\hat{u}^\epsilon_{\delta,y}\|_{L^{\frac{2n}{n-2}}(B(0,4R))} \le C$$

where $q=\frac{(\frac{2n}{n-2})^2}{2}$. This implies that

$$\|K\Big(\delta\frac{x}{|x|^2}+y\Big)\frac{1}{|x|^{\epsilon(n-2)}}\left(\hat{u}^\epsilon_{\delta,y}\right)^{\frac{4}{n-2}-\epsilon}\|_{L^{\frac{n+1}{2}}(B(0,2R))} \le C\|\hat{u}^\epsilon_{\delta,y}\|_{L^q(B(0,2R))}.$$

By elliptic regularity it follows that

$$\|\hat{u}^\epsilon_{\delta,y}(x)\|_{L^\infty(B(0,R))} \le C.$$

Then (4.15) follows by (4.17) and (4.19).

Step C. This step is devoted to showing (4.16). By Green's representation formula we have

$$\frac{\partial u^\epsilon_{\delta,y}}{\partial x_j}(x) = c_n\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n}\frac{K(\delta z+y)(u^\epsilon_{\delta,y}(z))^{p_\epsilon}-\sum_{i=0}^n c_i(\epsilon,\delta,y)\Delta\psi_i(z)}{|x-z|^{n-2}}dz$$
$$= c_n(2-n)\int_{\mathbb{R}^n}\frac{K(\delta z+y)(u^\epsilon_{\delta,y}(z))^{p_\epsilon}-\sum_{i=0}^n c_i(\epsilon,\delta,y)\Delta\psi_i(z)}{|x-z|^n}(x_j-z_j)dz,$$

where $c_n=\frac{1}{n(2-n)\omega_n}$ and ω_n is the area of the unit sphere in $\mathbb{R}^n$. Using estimate (4.15) we have

$$\Big|\frac{\partial u^\epsilon_{\delta,y}}{\partial x_j}(x)\Big| \le C\int_{\mathbb{R}^n}\frac{\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n-2}{2}p_\epsilon}+\sum_{i=0}^n|c_i(\epsilon,\delta,y)|\big|pU^{p-1}(z)\big||\psi_i(z)|}{|x-z|^{n-1}}dz$$
$$\le C\int_{\mathbb{R}^n}\frac{\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n-2}{2}p_\epsilon}+\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n}{2}+2}\left(1+|z|+|z|^2\right)}{|x-z|^{n-1}}dz. \tag{4.21}$$

Now we let $\mathbb{R}^n=A_1\cup A_2$, where $A_1=\{z\in\mathbb{R}^n : |x-z|>\frac{|x|}{2}\}$ and $A_2=\{z\in\mathbb{R}^n : |x-z|\le\frac{|x|}{2}\}$. We can now split (4.21) into two integrals and estimate these two integrals:

$$\int_{A_1}\frac{\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n-2}{2}p_\epsilon}+\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n}{2}+2}\left(1+|z|+|z|^2\right)}{|x-z|^{n-1}}dz$$

$$\leq C|x|^{1-n}\int_{A_1}\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n-2}{2}p_\epsilon}+\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n}{2}+2}\left(1+|z|+|z|^2\right)dz$$
$$\leq C|x|^{1-n} \tag{4.22}$$

and

$$\int_{A_2}\frac{\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n-2}{2}p_\epsilon}+\left(\frac{1}{(1+|z|^2)}\right)^{\frac{n}{2}+2}\left(1+|z|+|z|^2\right)}{|x-z|^{n-1}}dz$$
$$\leq C\int_{A_2}|x-z|^{1-n}\left(|z|^{-n-2+\epsilon(n-2)}+|z|^{-n-4}(1+|z|+|z|^2)\right)dz.$$
$$\leq C\left(|x|^{-n-2+\epsilon(n-2)}+|x|^{-n-4}\left(1+|x|+|x|^2\right)\right)\int_{|x-z|\leq\frac{|x|}{2}}|x-z|^{1-n}dz$$
$$\leq C\left(|x|^{-n-2+\epsilon(n-2)}+|x|^{-n-4}\left(1+|x|+|x|^2\right)\right)\int_0^{\frac{|x|}{2}}\rho^{1-n}\rho^{n-1}d\rho$$
$$\leq C\left(|x|^{-n-1+\epsilon(n-2)}+|x|^{-n-3}\left(1+|x|+|x|^2\right)\right). \tag{4.23}$$

Using (4.22) and (4.23) we get estimate (4.16). Note that the same can be done for the second derivatives. □

Lemma 4.3. *Let $u^\epsilon_{\delta,y}\in X$ be a solution of* (4.2) *such that $u^\epsilon_{\delta,y}\to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Then*

$$\|\left(u^\epsilon_{\delta,y}\right)^{p_\epsilon}-\left(u^\epsilon_{\delta,y}\right)^{p}+\epsilon\left(\ln u^\epsilon_{\delta,y}\right)\left(u^\epsilon_{\delta,y}\right)^{p}\|_{\frac{2n}{n+2}}\leq C\epsilon^2. \tag{4.24}$$

Proof. By the mean-value theorem we get for all $x\in\mathbb{R}^n$

$$(u^\epsilon_{\delta,y}(x))^{p_\epsilon}-(u^\epsilon_{\delta,y}(x))^{p}+\epsilon(\ln u^\epsilon_{\delta,y}(x))(u^\epsilon_{\delta,y}(x))^{p}=\frac{\epsilon^2}{2}\left(\ln u^\epsilon_{\delta,y}(x)\right)^2\left(u^\epsilon_{\delta,y}(x)\right)^{p-\theta_x\epsilon}$$

for some $\theta_x\in[0,1]$. Using estimate (4.15) it follows that

$$\left(\ln u^\epsilon_{\delta,y}(x)\right)^2\left(u^\epsilon_{\delta,y}(x)\right)^{p-\theta_x\epsilon}\in L^t(\mathbb{R}^n)$$

$t>\frac{n}{n+2}$. The claim is proved. □

Lemma 4.4. *Let $u^\epsilon_{\delta,y}$ be a solution of* (4.2) *such that $u^\epsilon_{\delta,y}\to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $K(x)$ be a bounded function; then*

$$\int_{\mathbb{R}^n}\nabla u^\epsilon_{\delta,y}(x)\cdot\nabla\frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}(x)dx=0 \tag{4.25}$$
$$\int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\left(K(\delta x+y)\left(u^\epsilon_{\delta,y}(x)\right)^{p+1}\right)dx=0. \tag{4.26}$$

Proof. From the divergence theorem and estimate (4.16) we get

$$\int_{B(0,R)} \nabla u^{\epsilon}_{\delta,y}(x)\cdot\nabla\frac{\partial u^{\epsilon}_{\delta,y}}{\partial x_i}(x)dx = \int_{B(0,R)} \frac{\partial}{\partial x_i}\Big(\frac{\left|\nabla u^{\epsilon}_{\delta,y}(x)\right|^2}{2}\Big)dx$$
$$= \int_{\partial B(0,R)} \frac{\left|\nabla u^{\epsilon}_{\delta,y}(x)\right|^2}{2}\nu_i d\sigma \le C\Big(\frac{1}{R^{n-1}}\Big)^2 R^{n-1} \le C\frac{1}{R^{n-1}}.$$

Then by the Lebesgue theorem we have

$$\int_{\mathbb{R}^n} \nabla u^{\epsilon}_{\delta,y}(x)\cdot\nabla\frac{\partial u^{\epsilon}_{\delta,y}}{\partial x_i}(x)dx = 0.$$

The estimate (4.26) follows in the same way. □

Lemma 4.5. *Let $u^{\epsilon}_{\delta,y}$ be a solution of* (4.2) *such that $u^{\epsilon}_{\delta,y}\to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ as $\epsilon\to 0$, and let $K(x)$ be a bounded function; then*

$$\int_{\mathbb{R}^n} div\left(x\left|\nabla u^{\epsilon}_{\delta,y}(x)\right|^2\right)dx = 0 \tag{4.27}$$

$$\int_{\mathbb{R}^n} div\left(K(\delta x+y)x\left(u^{\epsilon}_{\delta,y}(x)\right)^{p+1}\right)dx = 0 \tag{4.28}$$

$$\int_{\mathbb{R}^n} div\left(K(\delta x+y)x\left(u^{\epsilon}_{\delta,y}(x)\right)^{p+1}\ln u^{\epsilon}_{\delta,y}(x)\right)dx = 0. \tag{4.29}$$

Proof. The proof follows exactly as in Lemma 4.4. □

5. A lower bound on the number of solutions

In this section we want to prove Theorem 1.1. A crucial assumption in the next results is to write y as $y = y_0 + \delta\tilde{y}$, where y_0 is a critical point of K and $\tilde{y}\in\mathbb{R}^n$ will be chosen later. We start by proving two lemmas.

Lemma 5.1. *Let y_0 be a critical point of $K(x)$. Assume $K(x)$ satisfies hypotheses* (1.2) *and* i)-iv) *in a neighborhood of y_0 with $1\le\alpha_i<n$ for any $i=1,\dots,n$. Assume $u^{\epsilon}_{\delta,y}$ is a solution to* (4.2) *such that $u^{\epsilon}_{\delta,y}\to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ as $\epsilon\to 0$ and $\delta\to 0$. Then, for ϵ small enough and $y=y_0+\delta\tilde{y}$,*

$$\left(u^{\epsilon}_{\delta,y} - i^*\left(K(\delta x+y)f_\epsilon(u^{\epsilon}_{\delta,y})\right), \frac{\partial u^{\epsilon}_{\delta,y}}{\partial x_i}\right)_{1,2} \tag{5.1}$$
$$= \frac{\delta^{\alpha_i+1}}{p+1}\int_{\mathbb{R}^n} h_i(x+\tilde{y})U^{p+1}(x)dx + o\left(\delta^{\alpha_i+1}\right).$$

Proof. We have

$$\left(u^\epsilon_{\delta,y} - i^*\left(K(\delta x+y)f_\epsilon(u^\epsilon_{\delta,y})\right), \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}\right)_{1,2}$$
$$= \int_{\mathbb{R}^n} \nabla u^\epsilon_{\delta,y}(x)\cdot\nabla \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}(x)dx - \int_{\mathbb{R}^n} K(\delta x+y)\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon} \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}(x)dx.$$

Then by (4.25) and (4.26) we have

$$\begin{aligned}&\left(u^\epsilon_{\delta,y} - i^*\left(K(\delta x+y)f_\epsilon(u^\epsilon_{\delta,y})\right), \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}\right)_{1,2}\\ =\ & -\frac{1}{p_\epsilon+1}\int_{\mathbb{R}^n} K(\delta x+y)\frac{\partial}{\partial x_i}\left(\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}\right)dx\\ =\ & \frac{\delta}{p_\epsilon+1}\int_{\mathbb{R}^n} \frac{\partial K}{\partial x_i}(\delta x+y)\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}dx. \end{aligned} \tag{5.2}$$

Now we want to estimate the integral in (5.2) using the properties of K. So

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{\partial K}{\partial x_i}(\delta x+y)\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}dx\\ =\ & \int_{|\delta x+y|>r} \frac{\partial K}{\partial x_i}(\delta x+y)\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}dx\\ +\ & \int_{|\delta x+y|\le r} \frac{\partial K}{\partial x_i}(\delta x+y)\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}dx = I_1+I_2.\end{aligned} \tag{5.3}$$

Using (4.15) and the fact that $\|\nabla K\|_\infty \le C$,

$$|I_1| \le C\int_{\frac{r}{\delta}}^{+\infty}\left(\frac{1}{1+\rho^2}\right)^{\frac{n-2}{2}\left(\frac{2n}{n-2}\right)}\rho^{n-1}d\rho \le C\delta^n. \tag{5.4}$$

Now we let

$$y = y_0 + \delta\tilde{y} \tag{5.5}$$

with $\tilde{y}\in\mathbb{R}^n$. Then using properties i)-iv) of K we have

$$I_2 = \int_{|\delta x+y|\le r}\left(h_i(\delta x+\delta\tilde{y}) + R_i(\delta x+\delta\tilde{y})\right)\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}dx = A_2+B_2 \tag{5.6}$$

and

$$\begin{aligned}&\int_{|\delta x+y|\le r} h_i(\delta x+\delta\tilde{y})(u^\epsilon_{\delta,y}(x))^{p_\epsilon+1}dx = \delta^{\alpha_i}\int_{|\delta x+y|\le r} h_i(x+\tilde{y})\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1}dx\\ &= \delta^{\alpha_i}\int_{\mathbb{R}^n} h_i(x+\tilde{y})U^{p+1}(x)dx + o(\delta^{\alpha_i})\end{aligned} \tag{5.7}$$

since $|h_i(x+\tilde{y})| \le C\left(|x|^{\alpha_i}+1\right)$ and $\alpha_i < n$, while

$$\begin{aligned} |B_2| &= \Big| \int_{|\delta x+y|\le r} R_i(\delta x+\delta\tilde{y}) \left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1} dx \Big| \\ &\le C\delta^{\beta_i} \int_{|\delta x+y|\le r} |x+\tilde{y}|^{\beta_i} \left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon+1} dx \\ &\le C\delta^{\beta_i} + C\delta^n. \end{aligned} \tag{5.8}$$

Then by (5.6), (5.7) and (5.8) we have

$$I_2 = \delta^{\alpha_i} \int_{\mathbb{R}^n} h_i(x+\tilde{y})\,(U(x))^{p+1}\,dx + o(\delta^{\alpha_i}). \tag{5.9}$$

Finally, we get

$$\begin{aligned} &\left(u^\epsilon_{\delta,y} - i^*\left(K(\delta x+y) f_\epsilon(u^\epsilon_{\delta,y})\right), \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i} \right)_{1,2} \qquad (5.10) \\ &= \frac{\delta^{\alpha_i+1}}{p+1} \int_{\mathbb{R}^n} h_i(x+\tilde{y})\,(U(x))^{p+1}\,dx + o(\delta^{\alpha_i+1}), \end{aligned}$$

which gives the claim. □

Lemma 5.2. *Let y_0 be a critical point of $K(x)$. Assume $K(x)$ satisfies hypotheses* (1.2) *and* i)-iv) *in a neighborhood of y_0 with $1 \le \alpha_i < n-1$ for any $i = 1, \dots, n$. Assume $u^\epsilon_{\delta,y}$ is a solution to* (4.2) *such that $u^\epsilon_{\delta,y} \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ as $\epsilon \to 0$ and $\delta \to 0$. Then, for ϵ small enough and $y = y_0 + \delta\tilde{y}$,*

$$\begin{aligned} &\left(u^\epsilon_{\delta,y} - i^*\left(K(\delta x+y) f_\epsilon(u^\epsilon_{\delta,y})\right), x\cdot\nabla u^\epsilon_{\delta,y} + \frac{n-2}{2} u^\epsilon_{\delta,y} \right)_{1,2} \\ =\;& \frac{(n-2)^2}{4n}\epsilon K(y_0) \int_{\mathbb{R}^n} (U(x))^{p+1}\,dx + o(\epsilon) \\ +\;& \sum_{i=1}^n \frac{\delta^{\alpha_i+1}}{p+1} \int_{\mathbb{R}^n} h_i(x+\tilde{y}) x_i\,(U(x))^{p+1}\,dx + o\left(\delta^{\alpha_i+1}\right). \end{aligned} \tag{5.11}$$

Proof. We have

$$\begin{aligned} &\left(u^\epsilon_{\delta,y} - i^*\left(K(\delta x+y) f_\epsilon(u^\epsilon_{\delta,y})\right), x\cdot\nabla u^\epsilon_{\delta,y} + \frac{n-2}{2} u^\epsilon_{\delta,y} \right)_{1,2} \qquad (5.12) \\ &= \int_{\mathbb{R}^n} \nabla u^\epsilon_{\delta,y}(x) \cdot \nabla\left(x\cdot\nabla u^\epsilon_{\delta,y}(x) + \frac{n-2}{2} u^\epsilon_{\delta,y}(x)\right) dx \\ &- \int_{\mathbb{R}^n} K(\delta x+y) \left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p_\epsilon} \left(x\cdot\nabla u^\epsilon_{\delta,y}(x) + \frac{n-2}{2} u^\epsilon_{\delta,y}(x)\right) dx \end{aligned}$$

$$= \frac{1}{2}\int_{\mathbb{R}^n} div\left(x|\nabla u^{\epsilon}_{\delta,y}(x)|^2\right)dx - \int_{\mathbb{R}^n} K(\delta x+y)\left(x\cdot\nabla u^{\epsilon}_{\delta,y}(x)+\frac{n-2}{2}u^{\epsilon}_{\delta,y}(x)\right)$$
$$\times\left[\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p_\epsilon} - \left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p} + \epsilon\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p}\ln u^{\epsilon}_{\delta,y}(x)\right]dx$$

$$-\frac{n-2}{2}\int_{\mathbb{R}^n} K(\delta x+y)\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$-\frac{1}{p+1}\int_{\mathbb{R}^n} div\left[K(\delta x+y)x\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}\right]dx$$
$$+\frac{1}{p+1}\int_{\mathbb{R}^n} div\left(K(\delta x+y)x\right)\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$+\epsilon\frac{n-2}{2}\int_{\mathbb{R}^n} K(\delta x+y)\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}\ln u^{\epsilon}_{\delta,y}(x)dx$$
$$+\frac{\epsilon}{p+1}\int_{\mathbb{R}^n} div\left[K(\delta x+y)x\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}\left(\ln u^{\epsilon}_{\delta,y}(x)-\frac{1}{p+1}\right)\right]dx$$
$$-\frac{\epsilon}{p+1}\int_{\mathbb{R}^n} div\left(K(\delta x+y)x\right)\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}\left(\ln u^{\epsilon}_{\delta,y}(x)-\frac{1}{p+1}\right)dx.$$

Using (4.27), (4.28) and (4.29) we get

$$\left(u^{\epsilon}_{\delta,y}-i^*\left(K(\delta x+y)f_\epsilon(u^{\epsilon}_{\delta,y})\right), x\cdot\nabla u^{\epsilon}_{\delta,y}+\frac{n-2}{2}u^{\epsilon}_{\delta,y}\right)_{1,2}=$$
$$-\int_{\mathbb{R}^n} K(\delta x+y)\left(x\cdot\nabla u^{\epsilon}_{\delta,y}(x)+\frac{n-2}{2}u^{\epsilon}_{\delta,y}(x)\right)\times$$
$$\times\left[\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p_\epsilon} - \left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p} + \epsilon\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p}\ln u^{\epsilon}_{\delta,y}(x)\right]dx$$
$$+\frac{\delta}{p+1}\int_{\mathbb{R}^n}\nabla K(\delta x+y)\cdot x\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$+\epsilon\frac{n}{(p+1)^2}\int_{\mathbb{R}^n} K(\delta x+y)\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$-\epsilon\frac{\delta}{p+1}\int_{\mathbb{R}^n}\nabla K(\delta x+y)\cdot x\left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1}\left(\ln u^{\epsilon}_{\delta,y}(x)-\frac{1}{p+1}\right)dx$$
$$= I_1+I_2+I_3+I_4. \tag{5.13}$$

We can estimate the first integral as follows:

$$\left|I_1\right| \le C\int_{\mathbb{R}^n}\left|\left((u^{\epsilon}_{\delta,y}(x))^+\right)^{p_\epsilon} - \left((u^{\epsilon}_{\delta,y}(x))^+\right)^{p} + \epsilon\left((u^{\epsilon}_{\delta,y}(x))^+\right)^{p}\ln u^{\epsilon}_{\delta,y}(x)\right|$$

$$\times \left| x\cdot\nabla u^\epsilon_{\delta,y}(x)+\frac{n-2}{2}u^\epsilon_{\delta,y}(x)\right| dx \tag{5.14}$$
$$\le \Big\| \left((u^\epsilon_{\delta,y})^+\right)^{p_\epsilon} - \left((u^\epsilon_{\delta,y})^+\right)^{p} + \epsilon\left((u^\epsilon_{\delta,y})^+\right)^{p}$$
$$\times \ln u^\epsilon_{\delta,y}\|_{\frac{2n}{n+2}} \|x\cdot\nabla u^\epsilon_{\delta,y}+\frac{n-2}{2}u^\epsilon_{\delta,y}\Big\|_{\frac{2n}{n-2}} \le C\epsilon^2.$$

In order to estimate I_2 we split the integral I_2 as follows:

$$I_2=\frac{\delta}{p+1}\int_{\mathbb{R}^n}\nabla K(\delta x+y)\cdot x\left((u^\epsilon_{\delta,y}(x))^+\right)^{p+1}dx \tag{5.15}$$
$$=\frac{\delta}{p+1}\int_{|\delta x+y|\ge r}\nabla K(\delta x+y)\cdot x\left((u^\epsilon_{\delta,y}(x))^+\right)^{p+1}dx$$
$$+\frac{\delta}{p+1}\int_{|\delta x+y|\le r}\nabla K(\delta x+y)\cdot x\left((u^\epsilon_{\delta,y}(x))^+\right)^{p+1}dx=\frac{\delta}{p+1}(I_{21}+I_{22}).$$

From (4.15) and the boundedness of $\|\nabla K\|_\infty$ we have

$$|I_{21}|\le C\int_{\frac{r}{\delta}}^{+\infty}\rho\frac{1}{\rho^{2n}}\rho^{n-1}d\rho\le C\delta^{n-1}. \tag{5.16}$$

To estimate the term I_{22} we use our crucial assumption $y=\delta\tilde y+y_0$. Then

$$I_{22}=\int_{|\delta x+y|\le r}\nabla K(\delta x+y)\cdot x\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$=\sum_{i=1}^n\delta^{\alpha_i}\int_{|\delta x+y|\le r}h_i(x+\tilde y)x_i\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$+\sum_{i=1}^n\int_{|\delta x+y|\le r}R_i(\delta x+\delta\tilde y)x_i\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$=\sum_{i=1}^n\delta^{\alpha_i}\int_{\mathbb{R}^n}h_i(x+\tilde y)x_iU^{p+1}(x)dx+o(\delta^{\alpha_i})+A_{22} \tag{5.17}$$

since $\alpha_i<n-1$ and $|h_i(x+\tilde y)|\le C(|x|^{\alpha_i}+1)$, where

$$\left|A_{22}\right|\le C\sum_{i=1}^n\int_{|\delta x+y|\le r}\left|R_i(\delta x+\delta\tilde y)x_i\right|\left(\left(u^\epsilon_{\delta,y}(x)\right)^+\right)^{p+1}dx$$
$$\le C\delta^{\beta_i}\int_{|\delta x+y|\le r}\left|x+\tilde y\right|^{\beta_i}|x|\,(U(x))^{p+1}dx\le C\delta^{\beta_i}+C\delta^{n-1}. \tag{5.18}$$

Putting together (5.15)-(5.18) we get

$$I_2 = \sum_{i=1}^{n} \frac{\delta^{\alpha_i+1}}{p+1} \int_{\mathbb{R}^n} h_i(x+\tilde{y}) x_i \left(U(x)\right)^{p+1} dx + o\left(\delta^{\alpha_i+1}\right) + O\left(\delta^n\right). \quad (5.19)$$

Concerning I_3 we have

$$\begin{aligned} I_3 &= \epsilon \frac{n}{(p+1)^2} \int_{\mathbb{R}^n} K(\delta x + y) \left(\left(u^{\epsilon}_{\delta,y}(x)\right)^+\right)^{p+1} dx \\ &= \epsilon \frac{n}{(p+1)^2} \int_{\mathbb{R}^n} K(y) U^{p+1}(x) dx + o(\epsilon) \end{aligned}$$

and since $y = y_0 + \delta\tilde{y}$

$$I_3 = \epsilon \frac{n}{(p+1)^2} K(y_0) \int_{\mathbb{R}^n} \left(U(x)\right)^{p+1} dx + o(\epsilon). \quad (5.20)$$

Finally

$$|I_4| \leq C\epsilon\delta \int_{\mathbb{R}^n} |x| \left(U(x)\right)^{p+1} \left|\ln U(x)\right| dx \leq C\epsilon\delta. \quad (5.21)$$

Putting together (5.13), (5.14), (5.19), (5.20) and (5.21) we get (5.11). □

Definition 5.3. *Let $\mathcal{L} \in C(\mathbb{R}^n, \mathbb{R}^n)$ be a vector field. We say that $\tilde{y}$ is a stable zero for $\mathcal{L}$ if*

a) $\mathcal{L}(\tilde{y}) = 0$,
b) *$\tilde{y}$ is isolated,*
c) *if $\mathcal{L}_n$ is a sequence of vector fields such that $\|\mathcal{L}_n - \mathcal{L}\|_{C(B(\tilde{y},r))} \to 0$ for some $r > 0$ then there exists $\tilde{y}_n \in B(\tilde{y}, r)$ such that $\mathcal{L}_n(\tilde{y}_n) = 0$ and $\tilde{y}_n \to \tilde{y}$.*

We want to note that if there exists a neighborhood $B(\tilde{y}, r)$ of $\tilde{y}$ such that

$$deg\left(\mathcal{L}, B(\tilde{y}, r), 0\right) \neq 0, \quad (5.22)$$

then $\tilde{y}$ is a stable zero for $\mathcal{L}$.

Proof of Theorem 1.1. We want to prove that any value $\tilde{y}_0 \in \mathcal{Z}_{y_0}$ gives rise to a single peak solution of (1.1) which blows up at y_0. We note that we have already shown, in Section 3, that if ϵ and δ are sufficiently small for each $y \in \mathbb{R}^n$ there exists a function $\phi^{\epsilon}_{\delta,y} \in W$ such that $U + \phi^{\epsilon}_{\delta,y}$ is a solution of (3.2). Here we want to show that there exist d_ϵ such that $\frac{1}{c} < d_\epsilon < c$ and $\tilde{y}_\epsilon$ such that, letting $\delta_\epsilon^{\alpha+1} = \frac{1}{d_\epsilon}\epsilon$ and $y_\epsilon = y_0 + \delta_\epsilon \tilde{y}_\epsilon$, the function

$u^\epsilon_{\delta_\epsilon,y_\epsilon}=U+\phi^\epsilon_{\delta_\epsilon,y_\epsilon}$ satisfies equation (4.2) with coefficients $c_i(\epsilon,\delta_\epsilon,y_\epsilon)=0$ for all $i=0,\dots,n$. To this end we let

$$G^0_\epsilon(d,\tilde y)=\frac{1}{\epsilon}\left(u^\epsilon_{\delta,y}-i^*\left(K(\delta x+y)\left((u^\epsilon_{\delta,y})^+\right)^{p_\epsilon}\right),x\cdot\nabla u^\epsilon_{\delta,y}+\frac{n-2}{2}u^\epsilon_{\delta,y}\right)_{1,2}$$
$$G^i_\epsilon(d,\tilde y)=\frac{1}{\delta^{\alpha_i+1}}\left(u^\epsilon_{\delta,y}-i^*\left(K(\delta x+y)\left((u^\epsilon_{\delta,y})^+\right)^{p_\epsilon}\right),\frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}\right)_{1,2}.$$

Taking the scalar product of equation (4.2) and the function $\frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}$ for $i=1,\dots,n$ or $x\cdot\nabla u^\epsilon_{\delta,y}+\frac{n-2}{2}u^\epsilon_{\delta,y}$, and letting $\delta_\epsilon^{\alpha+1}=\frac{1}{d}\epsilon$ and $y_\epsilon=y_0+\delta_\epsilon\tilde y$, we have

$$\begin{aligned}G^0_\epsilon(d,\tilde y)&=\frac{1}{\epsilon}\left(c_0(\epsilon,d,\tilde y)\|\psi_0\|^2_{1,2}+o(1)\right)\\ G^i_\epsilon(d,\tilde y)&=\frac{1}{\delta^{\alpha_i+1}}\left(c_i(\epsilon,d,\tilde y)\|\psi_i\|^2_{1,2}+o(1)\right).\end{aligned}$$

Finding a solution of (2.2) means finding d_ϵ and $\tilde y_\epsilon$ such that

$$\begin{aligned}G^0_\epsilon(d_\epsilon,\tilde y_\epsilon)&=0\\ G^i_\epsilon(d_\epsilon,\tilde y_\epsilon)&=0\end{aligned}$$

for $i=1,\dots,n$ and for each ϵ in $(0,\epsilon_0)$ for some $\epsilon_0>0$. Using (5.1) and (5.11) we have that

$$G^0_\epsilon(d,\tilde y)=\frac{1}{d}\left(d\frac{(n-2)^2}{4n}K(y_0)\int_{\mathbb{R}^n}U^{p+1}(x)dx+\psi(\tilde y)\right)+o(1) \tag{5.23}$$
$$G^i_\epsilon(d,\tilde y)=\frac{1}{p+1}\int_{\mathbb{R}^n}h_i(x+\tilde y)U^{p+1}(x)dx+o(1),$$

where

$$\psi(\tilde y)=\frac{1}{p+1}\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n}h_j(x+\tilde y)x_jU^{p+1}(x)dx.$$

We observe that the leading term of $G^i_\epsilon(d,\tilde y)$ is independent of d. So, since $\tilde y_0$ is a stable zero to $\mathcal{L}_{y_0}$, there exists $\tilde y_\epsilon\to\tilde y_0$ independent of d such that

$$G^i_\epsilon(d,\tilde y_\epsilon)=0 \text{ for any } d\in\mathbb{R}.$$

Inserting $\tilde y_\epsilon$ into (5.23) we derive that $G^0_\epsilon(d,\tilde y_\epsilon)$ becomes

$$G^0_\epsilon(d,\tilde y_\epsilon)=\frac{1}{d}\left(Ad+\psi(\tilde y_\epsilon)\right)+o(1),$$

where

$$A = \frac{(n-2)^2}{4n} K(y_0) \int_{\mathbb{R}^n} U^{p+1}(x)dx.$$

Hence there exists d_ϵ close to $-\frac{\psi(\tilde{y})}{A}$ such that $G^0_\epsilon(d_\epsilon, \tilde{y}_\epsilon) = 0$ which proves the existence of the solution.

It remains to prove that two different stable zeros of $\mathcal{L}_{y_0}$ give rise to two different solutions of (1.1). So let $\tilde{y}^1_0$ and $\tilde{y}^2_0$ be two different points of $\mathcal{Z}_{y_0}$, and let d^i_ϵ and $\tilde{y}^i_\epsilon$ be the parameter associated with these two points. Let $\left(\delta^i_\epsilon\right)^{\alpha+1} = \frac{1}{d^i_\epsilon}\epsilon$, $y^i_\epsilon = y_0 + \delta^i_\epsilon \tilde{y}^i_\epsilon$ and $u^\epsilon_{\delta^1_\epsilon, y^1_\epsilon}$ $u^\epsilon_{\delta^2_\epsilon, y^2_\epsilon}$ be the two solutions of (2.2) generated by the parameters. For $i = 1, 2$ we let

$$u^\epsilon_i(x) = (\delta^i_\epsilon)^{-\frac{2}{p-\epsilon-1}} u^\epsilon_{\delta^i_\epsilon, y^i_\epsilon}\left(\frac{x - y^i_\epsilon}{\delta^i_\epsilon}\right)$$

be the corresponding solutions of (1.1). Then

$$\left(\delta^1_\epsilon\right)^{\frac{2}{p-\epsilon-1}} u^\epsilon_1(y^1_\epsilon) = u^\epsilon_{\delta^1_\epsilon, y^1_\epsilon}(0) \to U(0)$$
$$\left(\delta^2_\epsilon\right)^{\frac{2}{p-\epsilon-1}} u^\epsilon_2(y^1_\epsilon) = u^\epsilon_{\delta^2_\epsilon, y^2_\epsilon}\left(\frac{\delta^1_\epsilon}{\delta^2_\epsilon}\tilde{y}^1_\epsilon - \tilde{y}^2_\epsilon\right) \to U(c\tilde{y}^1_0 - \tilde{y}^2_0),$$

where

$$c = \lim_{\epsilon\to 0} \frac{\delta^1_\epsilon}{\delta^2_\epsilon} = \lim_{\epsilon\to 0}\left(\frac{d^2_\epsilon}{d^1_\epsilon}\right)^{\frac{1}{\alpha+1}} = \left(\frac{d^2_0}{d^1_0}\right)^{\frac{1}{\alpha+1}}.$$

We have the following alternative:

1) $c = 1$. Then

$$\lim_{\epsilon\to 0} \frac{u^\epsilon_1(y^1_\epsilon)}{u^\epsilon_2(y^1_\epsilon)} = \frac{U(0)}{U(\tilde{y}^1_0 - \tilde{y}^2_0)} \neq 1$$

and the claim follows.

2) $c \neq 1$. There is no loss of generality in assuming $c < 1$. In this case we have

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{u^\epsilon_1(y^1_\epsilon)}{u^\epsilon_2(y^1_\epsilon)} &= \lim_{\epsilon\to 0}\left(\frac{\delta^1_\epsilon}{\delta^2_\epsilon}\right)^{-\frac{2}{p-\epsilon-1}} \frac{u^\epsilon_{\delta^1_\epsilon, y^1_\epsilon}(0)}{u^\epsilon_{\delta^2_\epsilon, y^2_\epsilon}\left(\frac{\delta^1_\epsilon}{\delta^2_\epsilon}\tilde{y}^1_\epsilon - \tilde{y}^2_\epsilon\right)} \\ &= \left(\frac{1}{c}\right)^{\frac{2}{p-1}} \frac{U(0)}{U(c\tilde{y}^1_0 - \tilde{y}^2_0)} = \left(\frac{1}{c}\right)^{\frac{2}{p-1}} \left(1 + |c\tilde{y}^1_0 - \tilde{y}^2_0|^2\right)^{\frac{n-2}{2}} > 1.\end{aligned}$$

This proves that $u^\epsilon_1 \neq u^\epsilon_2$. □

6. A useful estimate

In this section we want to prove that if u_ϵ is a solution of (1.1) that blows-up and concentrate at y_0 and if y_ϵ stands for its peak then

$$\frac{y_0 - y_\epsilon}{\epsilon^{\frac{1}{\alpha+1}}} \to \tilde{y},$$

where $\tilde{y}$ satisfies $\mathcal{L}_{y_0}(\tilde{y}) = 0$. We start by proving the following.

Proposition 6.1. *Let y_0 be a critical point of $K(x)$. Assume $K(x)$ satisfies* (1.2) *and hypotheses* i)-iv) *in a neighborhood of y_0. Let ϵ_n be a sequence that goes to zero and $u_n = u_{\epsilon_n}$ the corresponding single peaked solutions of* (1.1). *If $y_n = y_{\epsilon_n}$ denotes the peak of u_n and if $y_n \to y_0$ as $\epsilon \to 0$, then there exists a positive constant $C > 0$ such that*

$$\left| \frac{y_0 - y_n}{\epsilon_n^{\frac{1}{\alpha+1}}} \right| \le C, \tag{6.1}$$

where α is defined in (1.9) *and depends only on the shape of $K(x)$ in a neighborhood of y_0.*

Proof. We suppose by contradiction that there exists a sequence ϵ_n such that

$$\left| \frac{y_0 - y_n}{\epsilon_n^{\frac{1}{\alpha+1}}} \right| \to \infty. \tag{6.2}$$

We let $v_n(x) = (\epsilon_n^{\frac{1}{\alpha+1}})^{\frac{2}{p-\epsilon_n-1}} u_n(\epsilon_n^{\frac{1}{\alpha+1}} x + y_n)$. Then v_n satisfies the equation

$$-\Delta v_n(x) = K(\epsilon_n^{\frac{1}{\alpha+1}} x + y_n)\,(v_n(x))^{p-\epsilon_n} \qquad \text{in } \mathbb{R}^n. \tag{6.3}$$

Here we note that v_n satisfies the hypotheses of Lemma 4.3 and hence it satisfies $v_n \le CU$. Moreover, $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ where v is a solution of

$$-\Delta v = K(y_0) v^p \qquad \text{in } \mathbb{R}^n$$

and hence $v = \frac{1}{K(y_0)^{\frac{n-2}{4}}} U(x)$. Multiplying (6.3) by $\frac{\partial v_n}{\partial x_i}$ and integrating we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla v_n \cdot \nabla\Big(\frac{\partial v_n}{\partial x_i}\Big) dx - \int_{\mathbb{R}^n} K(\epsilon_n^{\frac{1}{\alpha+1}} x + y_n)\,(v_n(x))^{p-\epsilon_n} \frac{\partial v_n}{\partial x_i} dx \\ &= -\frac{1}{p-\epsilon_n+1} \int_{\mathbb{R}^n} K(\epsilon_n^{\frac{1}{\alpha+1}} x + y_n) \frac{\partial}{\partial x_i} (v_n(x))^{p-\epsilon_n+1}\, dx \\ &\quad (\text{ using } v_n \le CU) \end{aligned} \tag{6.4}$$

$$= \frac{\epsilon_n^{\frac{1}{\alpha+1}}}{p-\epsilon_n+1}\int_{\mathbb{R}^n}\frac{\partial K}{\partial x_i}(\epsilon_n^{\frac{1}{\alpha+1}}x+y_n)\,(v_n(x))^{p-\epsilon_n+1}\,dx. \tag{6.5}$$

Arguing as in the proof of the Lemma 5.1, using properties i)-iv) of $K(x)$ we find

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n}\frac{\partial K}{\partial x_i}(\epsilon_n^{\frac{1}{\alpha+1}}x+y_n)\,(v_n(x))^{p-\epsilon_n+1}\,dx \\ &= \int_{|\epsilon_n^{\frac{1}{\alpha+1}}x+y_n|>r}\frac{\partial K}{\partial x_i}(\epsilon_n^{\frac{1}{\alpha+1}}x+y_n)\,(v_n(x))^{p-\epsilon_n+1}\,dx \\ &+ \int_{|\epsilon_n^{\frac{1}{\alpha+1}}x+y_n|\le r}h_i(\epsilon_n^{\frac{1}{\alpha+1}}x+y_n-y_0)\,(v_n(x))^{p-\epsilon_n+1}\,dx \\ &+ \int_{|\epsilon_n^{\frac{1}{\alpha+1}}x+y_n|\le r}R_i(\epsilon_n^{\frac{1}{\alpha+1}}x+y_n-y_0)\,(v_n(x))^{p-\epsilon_n+1}\,dx \\ &= I_1+I_2+I_3. \end{aligned} \tag{6.6}$$

As in Lemma 5.1 we get

$$|I_1| \le C\big(\epsilon_n^{\frac{1}{\alpha+1}}\big)^n. \tag{6.7}$$

Moreover, using (6.2) we have (up to a subsequence)

$$\begin{aligned} I_2 &= |y_n-y_0|^{\alpha_i}\int_{|\epsilon_n^{\frac{1}{\alpha+1}}x+y_n|\le r}h_i\Big(\frac{\epsilon_n^{\frac{1}{\alpha+1}}}{|y_n-y_0|}x+\frac{y_n-y_0}{|y_n-y_0|}\Big)\,(v_n(x))^{p-\epsilon_n+1}\,dx \\ &= |y_n-y_0|^{\alpha_i}h_i(z)\int_{\mathbb{R}^n}(U(x))^{p+1}\,dx+o(|y_n-y_0|^{\alpha_i}), \end{aligned} \tag{6.8}$$

where $z\in\partial B(0,1)$ is given by (up to a subsequence)

$$z=\lim_{n\to\infty}\frac{y_n-y_0}{|y_n-y_0|}.$$

Reasoning as before we get

$$|I_3| \le C|y_n-y_0|^{\beta_i}+C\big(\epsilon_n^{\frac{1}{\alpha+1}}\big)^n \tag{6.9}$$

and $\epsilon_n^{\frac{n}{\alpha+1}}=o\left(|y_n-y_0|^{\alpha_i}\right)$ by (6.2). We have shown so far that

$$0=|y_n-y_0|^{\alpha_i}h_i(z)\int_{\mathbb{R}^n}(U(x))^{p+1}\,dx+o\left(|y_n-y_0|^{\alpha_i}\right)$$

and this implies $h_i(z)=0$ which is not possible from the properties iv) on K. □

Proposition 6.2. *Let y_0 be a critical point of $K(x)$. Assume $K(x)$ satisfies* (1.2) *and hypotheses* i)-iv) *in a neighborhood of y_0 with $1 \le \alpha_i < n-1$ for all $i = 1, \dots, n$. Let ϵ_n be a sequence that goes to zero and $u_n = u_{\epsilon_n}$ the corresponding single peaked solutions of* (1.1). *If $y_n = y_{\epsilon_n}$ denotes the peak of u_n and if $y_n \to y_0$ as $\epsilon \to 0$, then, up to a subsequence, we have*

$$y_n = y_0 - \epsilon_n^{\frac{1}{\alpha+1}} \tilde{y} + o(\epsilon_n^{\frac{1}{\alpha+1}}), \tag{6.10}$$

$$\mathcal{L}_{y_0}(\tilde{y}) = 0 \tag{6.11}$$

and

$$\sum_{\alpha_j=\alpha} \frac{1}{p+1} \int_{\mathbb{R}^n} h_j(x+\tilde{y}) x_j U^{p+1}(x) dx = -\frac{n-2}{2} K(y_0) \int_{\mathbb{R}^n} U^{p+1}(x) dx, \tag{6.12}$$

where α is defined in (1.9) *and depends only on the shape of $K(x)$ in a neighborhood of y_0.*

Proof of Proposition 6.2. Let us prove (6.10) and (6.11). As in the proof of Proposition 6.1 we let

$$v_n(x) = \big(\epsilon_n^{\frac{1}{\alpha+1}}\big)^{\frac{2}{p-\epsilon_n-1}} u_n\big(\epsilon_n^{\frac{1}{\alpha+1}} x + y_n\big).$$

Since u_n is a single peaked solution of (1.1), $v_n \to U(x)$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Multiplying (2.2) by $\frac{\partial v_n}{\partial x_i}$ we have (see also (5.2))

$$\begin{aligned} 0 &= \frac{\epsilon_n^{\frac{1}{\alpha+1}}}{p-\epsilon_n+1} \int_{\mathbb{R}^n} \frac{\partial K}{\partial x_i}(\epsilon_n^{\frac{1}{\alpha+1}} x + y_n) \, (v_n(x))^{p-\epsilon_n+1} \, dx \\ &= \frac{\epsilon_n^{\frac{1}{\alpha+1}}}{p-\epsilon_n+1} \left(I_{1,n} + I_{2,n} + I_{3,n}\right), \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} I_{1,n} &= \big(\epsilon_n^{\frac{1}{\alpha+1}}\big)^{\alpha_i} \int_{|\epsilon_n^{\frac{1}{\alpha+1}} x + y_n - y_0| \le r} h_i\Big(x + \frac{y_n - y_0}{\epsilon_n^{\frac{1}{\alpha+1}}}\Big) \, (v_n(x))^{p-\epsilon_n+1} \, dx \\ &= \epsilon_n^{\frac{\alpha_i}{\alpha+1}} \int_{\mathbb{R}^n} h_i(x+\tilde{y}) \, (U(x))^{p+1} \, dx + o(\epsilon_n^{\frac{\alpha_i}{\alpha+1}}) \end{aligned} \tag{6.14}$$

and up to a subsequence

$$\tilde{y} = \lim_{n\to\infty} \frac{y_n - y_0}{\epsilon_n^{\frac{1}{\alpha+1}}}. \tag{6.15}$$

Arguing as in the proof of Lemma 5.1 we get

$$I_{2,n} = O(\epsilon_n^{\frac{\beta_i}{\alpha+1}}) + O(\epsilon_n^{\frac{n}{\alpha+1}}) \tag{6.16}$$

and

$$I_{3,n} = O(\epsilon_n^{\frac{n}{\alpha+1}}). \tag{6.17}$$

Hence from (6.13)-(6.17) we get

$$\int_{\mathbb{R}^n} h_i(x+\tilde{y})U^{p+1}(x)dx = 0$$

and then $\mathcal{L}_{y_0}(\tilde{y}) = 0$.

Let us prove (6.12). Taking the scalar product of (4.2) with $x\cdot\nabla v_n + \frac{n-2}{2}v_n$ we get from (5.11)

$$0 = \sum_{j=1}^{n} \frac{\epsilon_n^{\frac{\alpha_j+1}{\alpha+1}}}{p+1} \int_{\mathbb{R}^n} h_j(x+\tilde{y})x_j\,(U(x))^{p+1}\,dx \tag{6.18}$$

$$+ \frac{n-2}{2}K(y_0)\epsilon_n \int_{\mathbb{R}^n} (U(x))^{p+1}\,dx + o(\epsilon_n)$$

and this proves the claim. □

Proof of Theorem 1.3. This follows from Proposition 6.2. □

7. An exact multiplicity result

Proof of Theorem 1.2. By contradiction let us suppose that $\#\{$ single peak solutions of (1.1) concentrating at $y_0\} > \#\mathcal{Z}_{y_0}$. Since $\#\mathcal{Z}_{y_0} < \infty$ from Proposition 6.2 there exist $\tilde{y} \in \mathcal{Z}_{y_0}$, a sequence $\epsilon_n \to 0$ and two distinct solutions u_n^1 and u_n^2 of (1.1) such that if y_n^1 and y_n^2 denote their peaks we have

$$\lim_{n\to\infty} \frac{y_n^1 - y_0}{\epsilon_n^{\frac{1}{\alpha+1}}} = \lim_{n\to\infty} \frac{y_n^2 - y_0}{\epsilon_n^{\frac{1}{\alpha+1}}} = \tilde{y}.$$

Letting $\delta_n = \epsilon_n^{\frac{1}{\alpha+1}}$ the functions $v_n^i(x) = \delta_n^{\frac{2}{p-\epsilon_n-1}} u_n^i(\delta_n x + y_0)$ satisfy the equation

$$-\Delta v_n^i(x) = K(\delta_n x + y_0)\left(v_n^i(x)\right)^{p-\epsilon_n} \quad \text{in } \mathbb{R}^n \tag{7.1}$$

and $v_n^i(x) \to U(x-\tilde{y})$ uniformly on $\mathbb{R}^n$ for $i = 1, 2$. By Lemma 4.2 we have that $0 < v_n^i(x) \le CU(x-\tilde{y})$ in $\mathbb{R}^n$. Since $u_n^1(x) \neq u_n^2(x)$ we can consider the function

$$v_n(x) = \frac{v_n^1(x) - v_n^2(x)}{\|v_n^1(x) - v_n^2(x)\|_{1,2}} \quad \text{in } \mathbb{R}^n. \tag{7.2}$$

The function v_n satisfies

$$-\Delta v_n(x) = K(\delta_n x + y_0) c_n(x) v_n(x) \quad \text{in } \mathbb{R}^n, \tag{7.3}$$

where

$$c_n(x) = (p - \epsilon_n) \int_0^1 \left[t v_n^1(x) + (1-t) v_n^2(x) \right]^{p-1-\epsilon_n} dt.$$

Here we observe that from (4.15) we have

$$0 \le c_n(x) \le \frac{C}{(1 + |x - \tilde{y}|^2)^2}, \quad \forall x \in \mathbb{R}^n. \tag{7.4}$$

Up to a subsequence we can assume that $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and almost everywhere in $\mathbb{R}^n$ where v satisfies

$$-\Delta v(x) = K(y_0) p U^{p-1}(x - \tilde{y}) v(x) \quad \text{in } \mathbb{R}^n. \tag{7.5}$$

Then there exist real numbers a_i such that

$$v(x) = \sum_{i=0}^{n} a_i \psi_i(x - \tilde{y}) \tag{7.6}$$

with ψ_i as defined in (2.5) and (2.6). Moreover, reasoning exactly as in the proof of Lemma 4.2 one can see that

$$|v_n(x)| \le \frac{C}{1 + |x - \tilde{y}|^{n-2}} \quad \text{in } \mathbb{R}^n. \tag{7.7}$$

Now we multiply the equation (7.1) by $\frac{\partial v_n^i}{\partial x_j}$ and we integrate over $\mathbb{R}^n$. Using (4.26) we get

$$0 = \frac{1}{p - \epsilon_n + 1} \delta_n \int_{\mathbb{R}^n} \frac{\partial K}{\partial x_j}(\delta_n x + y_0) \left(v_n^i(x) \right)^{p-\epsilon_n+1} dx. \tag{7.8}$$

Subtracting equation (7.8) evaluated for $i = 1, 2$ we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \frac{\partial K}{\partial x_j}(\delta_n x + y_0) \left[\left(v_n^1(x) \right)^{p-\epsilon_n+1} - \left(v_n^2(x) \right)^{p-\epsilon_n+1} \right] dx \\ 0 &= \int_{\mathbb{R}^n} \frac{\partial K}{\partial x_j}(\delta_n x + y_0) \tilde{c}_n(x) v_n(x) dx, \end{aligned}$$

where

$$\tilde{c}_n(x) = (p + 1 - \epsilon_n) \int_0^1 \left[t v_n^1(x) + (1-t) v_n^2(x) \right]^{p-\epsilon_n} dt \tag{7.9}$$

and by (4.15)

$$0 < \tilde{c}_n(x) \le C U^p(x - \tilde{y}). \tag{7.10}$$

Using the properties i)-iv) of $K(x)$ in a neighborhood of y_0 we get

$$\begin{aligned} 0 &= \int_{|\delta_n x+y_0|>r} \frac{\partial K}{\partial x_j}(\delta_n x+y_0)\tilde{c}_n(x)v_n(x)dx \\ &+ \int_{|\delta_n x+y_0|\le r} h_j(\delta_n x)\tilde{c}_n(x)v_n(x)dx \\ &+ \int_{|\delta_n x+y_0|\le r} R_j(\delta_n x)\tilde{c}_n(x)v_n(x)dx = I_1+I_2+I_3. \end{aligned} \tag{7.11}$$

By (7.7), (7.10) and (1.2) we get

$$\big|I_1\big| \le C\delta^n. \tag{7.12}$$

Moreover, by (7.10) and $\alpha_j < n$, we have

$$\begin{aligned} I_2 &= \delta_n^{\alpha_j}\int_{|\delta_n x+y_0|\le r} h_j(x)\tilde{c}_n(x)v_n(x)dx \\ &= \delta_n^{\alpha_j}\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde{y})v(x)dx + o\left(\delta_n^{\alpha_j}\right) \end{aligned} \tag{7.13}$$

and

$$\big|I_3\big| \le C\delta_n^{\beta_j}\int_{|\delta_n x+y_0|\le r} |x|^{\beta_j}\tilde{c}_n(x)v_n(x)dx \le C\delta_n^{\beta_j} + C\delta_n^n. \tag{7.14}$$

From (7.11)-(7.14) we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde{y})v(x)dx \qquad (7.15) \\ &= a_0\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde{y})\left[(x-\tilde{y})\cdot\nabla U(x-\tilde{y}) + \frac{n-2}{2}U(x-\tilde{y})\right]dx \\ &+ \sum_{i=1}^n a_i\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde{y})\frac{\partial U}{\partial x_i}(x-\tilde{y})dx \end{aligned}$$

for $j = 1,\dots,n$. On the other hand we have that

$$\begin{aligned} &\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde{y})\left[(x-\tilde{y})\cdot\nabla U(x-\tilde{y}) + \frac{n-2}{2}U(x-\tilde{y})\right]dx \\ &\text{(since } \tilde{y} \text{ is a zero of } \mathcal{L}_{y_0}) \qquad (7.16) \\ &= \sum_{i=1}^n\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde{y})(x_i-(\tilde{y})_i)\frac{\partial U}{\partial x_i}(x-\tilde{y})dx \end{aligned}$$

$$= -\frac{1}{p+1}\int_{\mathbb{R}^n}\sum_{i=1}^n \frac{\partial h_j(x)}{\partial x_i}(x_i-\tilde y_i)U^{p+1}(x-\tilde y)dx$$

(using the Euler theorem for homogeneous functions)

$$= -\frac{\alpha_j}{p+1}\int_{\mathbb{R}^n} h_j(x)U^{p+1}(x-\tilde y)dx + \frac{\tilde y_i}{p+1}\int_{\mathbb{R}^n}\frac{\partial h_j(x)}{\partial x_i}U^{p+1}(x-\tilde y)dx$$
$$= \frac{\tilde y_i}{p+1}\int_{\mathbb{R}^n}\frac{\partial h_j(x)}{\partial x_i}U^{p+1}(x-\tilde y)dx = -\tilde y_i\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde y)\frac{\partial U}{\partial x_i}(x-\tilde y)dx.$$

Hence, (7.15) becomes

$$0 = \sum_{i=1}^n (a_i - a_0\tilde y_i)\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde y)\frac{\partial U}{\partial x_i}(x-\tilde y)dx. \tag{7.17}$$

Recalling the definition of $\mathcal{L}_{y_0}$ we have

$$Jac\,(\mathcal{L}_{y_0}(\tilde y)) = (p+1)\Big(\int_{\mathbb{R}^n} h_j(x)U^p(x-\tilde y)\frac{\partial U}{\partial x_i}(x-\tilde y)dx\Big)_{i,j=1,\dots,n}.$$

Since 0 is a regular value for $\mathcal{L}_{y_0}$, $\mathcal{L}_{y_0}(\tilde y)=0$ implies that the linear system (7.17) admits only the solution

$$a_i = a_0\tilde y_i \quad \text{ for } i=1,\dots,n. \tag{7.18}$$

Our next aim is to show that $a_0=0$. To do this let us multiply equation (7.1) by $x\cdot\nabla v_n^i$; integrating over $\mathbb{R}^n$ we get

$$0 = \left(\frac{n-2}{2} - \frac{n}{p-\epsilon_n+1}\right)\int_{\mathbb{R}^n} K(\delta_n x + y_0)\left(v_n^i(x)\right)^{p-\epsilon_n+1}dx$$
$$- \quad \frac{\delta_n}{p-\epsilon_n+1}\int_{\mathbb{R}^n}\nabla K(\delta_n x+y_0)\cdot x\left(v_n^i(x)\right)^{p-\epsilon_n+1}dx. \tag{7.19}$$

Subtracting (7.19) evaluated for $i=1,2$, we get

$$0 = \left(\frac{n-2}{2} - \frac{n}{p-\epsilon_n+1}\right)\int_{\mathbb{R}^n} K(\delta_n x + y_0)\tilde c_n(x)v_n(x)dx$$
$$- \quad \frac{\delta_n}{p-\epsilon_n+1}\int_{\mathbb{R}^n}\nabla K(\delta_n x+y_0)\cdot x\tilde c_n(x)v_n(x)dx = I_1 + I_2,$$

where $\tilde c_n$ are as defined in (7.11). Let us estimate I_1 using (7.7) and (7.10):

$$I_1 = \left(\frac{n-2}{2} - \frac{n}{p-\epsilon_n+1}\right)\int_{\mathbb{R}^n} K(\delta_n x + y_0)\tilde c_n(x)v_n(x)dx$$
$$= \left(-\frac{(n-2)^2}{4n}\epsilon_n + o(\epsilon_n)\right)K(y_0)\int_{\mathbb{R}^n} U^p(x-\tilde y)v(x)dx = o(\epsilon_n)$$

because a straightforward computation gives that $\int_{\mathbb{R}^n} U^p(x-\tilde{y})v(x)dx = 0$.

Concerning I_2 we get

$$\begin{aligned}
I_2 &= \frac{\delta_n}{p-\epsilon_n+1}\int_{\mathbb{R}^n} \nabla K(\delta_n x + y_0)\cdot x\tilde{c}_n(x)v_n(x)dx \\
&= \frac{\delta_n}{p-\epsilon_n+1}\Big[\int_{|\delta_n x+y_0|>r} \frac{\partial K}{\partial x_j}(\delta_n x + y_0)x_j\tilde{c}_n(x)v_n(x)dx \\
&+ \int_{|\delta_n x+y_0|\le r} h_j(\delta_n x)x_j\tilde{c}_n(x)v_n(x)dx \\
&+ \int_{|\delta_n x+y_0|\le r} R_j(\delta_n x)x_j\tilde{c}_n(x)v_n(x)dx\Big] = J_1 + J_2 + J_3.
\end{aligned}$$

Using (7.7) and (7.10) one gets $J_1 = O\left(\delta^{n-1}\right)$ and $J_3 = O(\delta^{\beta_j} + \delta^{n-1})$. Finally,

$$\begin{aligned}
J_2 &= \frac{\delta_n}{p-\epsilon_n+1}\int_{|\delta_n x+y_0|\le r} h_j(\delta_n x)x_j\tilde{c}_n(x)v_n(x)dx \\
&= \sum_{\alpha_j=\alpha}\frac{\delta_n^{\alpha+1}}{p-\epsilon_n+1}\int_{\mathbb{R}^n} h_j(x)x_jU^p(x-\tilde{y})v(x)dx + o\left(\delta_n^{\alpha+1}\right)
\end{aligned}$$

since $\alpha_j < n-1$. Since $\epsilon = \delta_n^{\alpha+1}$ we deduce that (7.20) becomes

$$\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n} h_j(x)x_jU^p(x-\tilde{y})v(x)dx = 0 \tag{7.20}$$

and then (recalling that $a_i = a_0\tilde{y}_i$)

$$\begin{aligned}
&\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n} h_j(x)x_jU^p(x-\tilde{y})v(x)dx \\
&= a_0\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n} h_j(x)x_jU^p(x-\tilde{y})\Big(\sum_{i=1}^n x_i\frac{\partial U}{\partial x_i}(x-\tilde{y}) + \frac{n-2}{2}U(x-\tilde{y})\Big)dx \\
&= -\frac{a_0}{p+1}\sum_{i=1}^n\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\Big(h_j(x)x_ix_j\Big)U^{p+1}(x-\tilde{y})dx \\
&+ \frac{(n-2)a_0}{2}\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n} h_j(x)x_jU^{p+1}(x-\tilde{y})dx \\
&= -\frac{a_0}{p+1}\sum_{i=1}^n\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n}\Big(h_j(x)x_j + h_j(x)\delta_j^i x_i + \frac{\partial h_j(x)}{\partial x_i}x_ix_j\Big)U^{p+1}(x-\tilde{y})dx
\end{aligned}$$

$$+\frac{(n-2)a_0}{2}\sum_{\alpha_j=\alpha}\int_{\mathbb{R}^n} h_j(x)x_jU^{p+1}(x-\tilde y)dx$$

(using the Euler theorem for homogeneous functions)

$$=a_0\Big(\frac{n-2}{2}-\frac{n+1}{p+1}-\frac{\alpha}{p+1}\Big)\int_{\mathbb{R}^n}\sum_{\alpha_j=\alpha} h_j(x)x_jU^{p+1}(x-\tilde y)dx.$$

Since

$$\Big(\frac{n-2}{2}-\frac{n+1}{p+1}-\frac{\alpha}{p+1}\Big)=-\frac{\alpha+1}{p+1}<0,$$

using (1.11) we deduce that (7.20) becomes

$$0=a_0\Big(\frac{n-2}{2}-\frac{n+1}{p+1}-\frac{\alpha}{p+1}\Big)\int_{\mathbb{R}^n} h_j(x)x_jU^{p+1}(x-\tilde y)dx\neq 0,$$

a contradiction if $a_0\neq 0$. Since $a_0=0$, from (7.18) we get

$$a_i=0\quad\text{ for } i=1,\dots,n.$$

Then, by (7.6), $v(x)\equiv 0$. In order to end the proof we have to show that

$v(x)\equiv 0$ is impossible. Multiplying (7.3) by v_n and integrating we have

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n}\big|\nabla v_n\big|^2=\int_{\mathbb{R}^n} K(\delta x+y_0)c_n(x)v_n^2(x)dx \\ &\le C\Big(\int_{|x|\le R} c_n(x)v_n^2(x)dx+\int_{|x|>R} c_n(x)v_n^2(x)dx\Big) \end{aligned}\tag{7.21}$$

for any $R>0$. By (7.3) we have

$$\int_{|x|>R} c_n(x)v_n^2(x)dx\le\Big(\int_{|x|>R} c_n(x)^{\frac{n}{2}}dx\Big)^{\frac{2}{n}}\Big(\int_{|x|>R} v_n(x)^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}\le\frac{1}{2C}$$

if R is big enough. Moreover, since $v_n\rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $v_n\to 0$ uniformly on compact subsets of $\mathbb{R}^n$, it follows that

$$\int_{|x|\le R} c_n(x)v_n^2(x)dx\le\frac{1}{4C}$$

if ϵ is small enough. Then a contradiction arises from (7.21) and we get the claim. □

8. Examples and final remarks

In this section we consider some particular cases of functions $K(x)$ where our results apply. We always assume that $y_0 = 0$. Let us start by considering the case of a nondegenerate critical point.

Proposition 8.1. *Let us assume that* 0 *is a nondegenerate critical point of* K *satisfying*

$$\Delta K(0) < 0.$$

Then there exists exactly one solution to (1.1) *blowing up at* 0*.*

Proof. By Taylor's formula we have that

$$K(x) = K(0) + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 K}{\partial x_j \partial x_k}(0) x_j x_k + R(x) \tag{8.1}$$

in a neighborhood of 0. So we have

$$h_i(x) = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 K}{\partial x_j \partial x_i}(0) x_j.$$

Hence we deduce that

$$\begin{aligned}\mathcal{L}_0(y) &= \int_{\mathbb{R}^n} h_j(x+y) U^{p+1}(x)dx \\ &= \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 K}{\partial x_j \partial x_i}(0) \int_{\mathbb{R}^n} (x_i + y_i)\, U^{p+1}(x)dx \\ &= \sum_{j=1}^{n} \frac{\partial^2 K}{\partial x_j \partial x_i}(0) y_i \int_{\mathbb{R}^n} U^{p+1}(x)dx.\end{aligned}$$

Since the matrix $\frac{\partial^2 K}{\partial x_j \partial x_i}(0)$ is invertible we get that the linear system

$$\mathcal{L}_0(y) = 0$$

admits only the solution $y = 0$. So

$$\mathcal{Z}_0 = \{0\}. \tag{8.2}$$

Since $\alpha_i = 1$ for any $i = 1, .., n$, by Theorem 1.1 and Theorem 1.2 we get the existence and uniqueness of the solution blowing up at zero. $\square$

Remark 8.2. The same computations of the previous proposition allow us to handle the following case:

$$K(x) = 1 + \sum_{j=1}^{n} c_j x_j^{s_j}$$

where $c_j \in \mathbb{R}$ and $s_j \geq 2$ are even positive integers. We again have that $\mathcal{Z}_0 = \{0\}$ and under the assumption

$$\sum_{\alpha_j=\alpha} c_j s_j \int_{\mathbb{R}^n} x_j^{s_j} U^{p+1}(x)dx < 0$$

we get the existence and uniqueness of the solution blowing up at zero. □

Proposition 8.3. *Let us assume that*

$$K(x) = 1 + \sum_{j=1}^{n} c_j x_j^{s_j}$$

where $c_j \in \mathbb{R}$ and s_j are positive integers. Then if at least one of the integers s_j is odd there is no solution to (1.1) *blowing up at* 0*.*

Proof. Since (1.13) is satisfied by Theorem (1.3) the claim follows. □

In the final proposition we exhibit a function $K(x)$ such that (1.1) admits exactly two single peak solutions concentrating at the origin.

Proposition 8.4. *Let us assume that $K(x) : \mathbb{R}^3 \to \mathbb{R}$ is given by*

$$K(x) = K(x_1, x_2, x_3) = 1 + x_1^3 - x_1 x_2^2 - x_3^2 \quad \text{for } x \in B(0,1).$$

Then there exist exactly two single peak solutions concentrating at the origin.

Proof. Computing the functions $h_i(x)$ for $i = 1, 2, 3$, we deduce that $\mathcal{L}_0(\tilde{y}) = 0$ at the points

$$\tilde{y}_1 = \left(0, -\sqrt{2\frac{\int_{\mathbb{R}^n} x_1^2 U^{p+1}(x)dx}{\int_{\mathbb{R}^n} U^{p+1}(x)dx}}, 0\right), \quad \tilde{y}_2 = \left(0, \sqrt{2\frac{\int_{\mathbb{R}^n} x_1^2 U^{p+1}(x)dx}{\int_{\mathbb{R}^n} U^{p+1}(x)dx}}, 0\right).$$

Note that in this case $\alpha_1 = \alpha_2 = 2$ and $\alpha = \alpha_3 = 1$. Since

$$\sum_{\alpha_j=\alpha} \int_{\mathbb{R}^n} h_j(x + \tilde{y}) x_j U^{p+1}(x) = -2 \int_{\mathbb{R}^n} x_3^2 U^{p+1}(x)dx < 0,$$

Theorem 1.11 applies and then (1.1) admits at least two single peak solutions concentrating at the origin. In order to show that we have exactly two solutions we apply Theorem 1.2. This leads to checking that $\tilde{y}_1$ and $\tilde{y}_2$ are regular values for $\mathcal{L}_0$. Indeed

$$\begin{aligned}\det \operatorname{Jac}\mathcal{L}_0(\tilde{y}_1) &= \det \operatorname{Jac}\mathcal{L}_0(\tilde{y}_2)\\ &= -16\Big(\int_{\mathbb{R}^n} x_1^2 U^{p+1}(x)dx\Big)\Big(\int_{\mathbb{R}^n} U^{p+1}(x)dx\Big)^2 \neq 0,\end{aligned}$$

and this gives the claim. □

Remark 8.5. The same computation of the previous proposition can be performed in any dimension $n \geq 3$. With a little work it is also possible to construct examples of functions $K(x)$ such that (1.1) admits exactly m solutions, for any integer $m \geq 1$.

REFERENCES

[1] Chiun-Chuan Chen and Chang-Shou Lin, *Estimate of the conformal scalar curvature equation via the method of moving planes. II,* J. Differential Geom., 49 (1998), 115–178.

[2] Andreas Floer and Alan Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential,* J. Funct. Anal., 69 (1986), 397–408.

[3] Basilis Gidas and Joel Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations,* Comm. Pure Appl. Math., 34 (1981), 525–598. 261–280.

[4] David Gilbarg and Neil S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin, (1983).

[5] Massimo Grossi, *On the number of single-peak solutions of the nonlinear Schrödinger equation,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 19 (2002), 261–280.

[6] Zheng-Chao Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 8 (1991), 159–174.

[7] Anna Maria Micheletti and Angela Pistoia, *Existence of blowing-up solutions for a slightly subcritical or a slightly supercritical non-linear elliptic equation on* $\mathbb{R}^n$, Nonlinear Anal., 52 (2003), 173–195.

[8] Anna Maria Micheletti, Angela Pistoia, and Daniela Visetti, *On the number of blowing-up solutions to a nonlinear elliptic equation with critical growth,* Rocky Mountain J. Math., 37 (2007), 291–325.

[9] Wei Ming Ni, *On the elliptic equation* $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$*, its generalizations, and applications in geometry,* Indiana Univ. Math. J., 31 (1982), 493–529.

[10] Stanislav Pohozaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$*,* Dokl. Akad. Nauk. SSSR., 165 (1965), 36–39.

[11] Shusen Yan, *High-energy solutions for a nonlinear elliptic problem with slightly supercritical exponent,* Nonlinear Anal., 38 (1999), 527–546.

[12] Xuefeng Wang and Juncheng Wei, *On the equation* $\Delta u + K(x)u^{(n+2)/(n-2)\pm\epsilon^2} = 0$ *in* R^n*,* Rend. Circ. Mat. Palermo, 44 (1995), 365–400.