

MORSE INDEX ESTIMATES FOR QUASILINEAR EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. This work deals with Morse index estimates for a solution $u \in H_1^p(M)$ of the quasilinear elliptic equation $-\operatorname{div}_g((\alpha + |\nabla u|_g^2)^{(p-2)/2} \nabla u) = h(x, u)$, where (M, g) is a compact, Riemannian manifold, $0 < \alpha$, $2 \leq p < n$. The nonlinear right-hand side $h(x, s)$ is allowed to be subcritical or critical.

1. INTRODUCTION

In recent years there was an increasing interest in nonlinear differential problems on manifolds. Several questions give rise to sophisticated analysis where the geometry of the manifold plays a central role. We refer the reader to [13] for an overview on the topic. In this work we consider (M, g) a compact, connected, orientable, boundaryless Riemannian manifold of class C^∞ with Riemannian metric g and $\dim(M) = n \geq 3$. By Nash's theorem [18], M can be isometrically embedded in \mathbb{R}^N , with $N \geq 2n$, as a regular sub-manifold. Let $H_1^p(M)$ be the Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm $\|u\|_{1,p} = \|\nabla u\|_p + \|u\|_p$, where $\|\cdot\|_p$ is the usual norm of $L^p(M)$ (see Section 2). We are concerned with Morse

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index estimates for the solutions of the following quasilinear problem:

$$\begin{cases} -\operatorname{div}_g \left((\alpha + |\nabla u|_g^2)^{(p-2)/2} \nabla u \right) = h(x, u) \\ u \in H_1^p(M) \end{cases} \quad (1.1)$$

where $0 < \alpha$, $2 \leq p < n$. Here $h : M \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the assumption

- (h) for any $s \in \mathbb{R}$, $h(\cdot, s)$ is continuous on M , $h(x, \cdot)$ is C^1 on \mathbb{R} and $\forall(x, s) \in M \times \mathbb{R}$, $|D_s h(x, s)| \leq c_1 |s|^{p^*-2} + c_2$, with c_1, c_2 positive constants, $p^* = pn/(n-p)$.

The above differential problem involves the p -Laplacian operator on (M, g) which is defined as a natural extension of the Laplace-Beltrami operator corresponding to $p = 2$ and arises in several physical contexts, for instance fluid dynamics, nonlinear elasticity and glaciology (see [1, 10]).

In a Euclidean context, some results in [5, 6] have been obtained concerning the computation of critical groups for solutions of quasilinear equations with Dirichlet boundary-value conditions with nonlinear autonomous right-hand side via differential notions like the Morse index (see also [7, 8] for applications). The main difficulty is the fact that the variational setting is Banach (not Hilbert), so that every solution of problem (1.1) is degenerate in the classical sense given in a Hilbert space and the classical Morse lemma does not hold. Moreover one lacks Fredholm properties of the second derivatives of the Euler functional associated to (1.1), and also generalized versions of the Morse lemma, due to Gromoll and Meyer, fail (see [4]).

As far as we know, the critical group estimates for solutions of quasilinear problems on a Riemannian manifold (M, g) are an open problem (for $p = 2$ we refer to Theorem 4.1 in [4]). Regularity results, developed by Druet in [11], have furnished us with the starting point to perform a Morse lemma for quasilinear equations on manifolds. The first step is to introduce a weighted Hilbert space, depending on the solution, in which the variational setting $H_1^p(M)$ can be embedded. In this way the second derivatives of the Euler functional associated to problem (1.1) can be extended to a Fredholm operator, which provides a natural splitting of the space with respect to its spectral decomposition in $L^2(M)$. By means of *a priori* C^1 -bounds of solutions along a suitable direction, one can regain a finite-dimensional reduction and the critical group estimates for each solution to (1.1) via differential notions.

The main results of this work are Theorems 2.1 and 2.2, which cover nonlinear non-autonomous right-hand sides having subcritical or critical growths. We emphasize that when $h(x, \cdot)$ has a critical growth, problem (1.1) is strictly related to the study of generalized scalar curvature equations

(see [9]). In such a critical case, in order to perform reduction arguments, we prove local weakly semicontinuous properties of the Euler functional associated to (1.1), by means of a generalization of the Lindqvist inequality [16]). Moreover a local *Palais-Smale* condition is established for each level in *small balls* of $H_1^p(M)$.

Finally, we mention the papers [2, 21] where multiplicity results for semi-linear elliptic equations on a Riemannian manifold are obtained via Morse techniques.

2. DEFINITIONS, PRELIMINARIES, AND MAIN RESULTS

Any solution $u \in H_1^p(M)$ corresponds to a critical point of the Euler functional $J_\alpha : H_1^p(M) \rightarrow \mathbb{R}$, defined by

$$J_\alpha(u) = \int_M \frac{1}{p} (\alpha + |\nabla u(x)|_g^2)^{\frac{p}{2}} d\mu_g - \int_M H(x, u(x)) d\mu_g \tag{2.1}$$

where $H(x, s) = \int_0^s h(x, t) dt$.

In what follows, the Morse index of J_α in u , denoted by $m(J_\alpha, u)$, is defined as the supremum of the dimensions of the subspaces of $H_1^p(M)$ on which $J_\alpha''(u)$ is negative definite. Moreover, the large Morse index of J_α in u , denoted by $m^*(J_\alpha, u)$, is defined as the sum of $m(J_\alpha, u)$ and the dimension of the kernel of $J_\alpha''(u)$.

We need to introduce the following definitions (cf. [4]). Let \mathbb{K} be a field. We call

$$C_q(J_\alpha, u) = H^q(J_\alpha^c \cap U, (J_\alpha^c \setminus \{u\}) \cap U) \tag{2.2}$$

the q -th critical group of J_α at u , $q = 0, 1, 2, \dots$, where U is a neighborhood of u , $J_\alpha(u) = c$, $J_\alpha^c = \{v \in H_1^p(M) : J_\alpha(v) \leq c\}$ and $H^q(A, B)$ stands for the q -th Alexander-Spanier cohomology group of the pair (A, B) with coefficients in \mathbb{K} (cf. [4]).

Now we state the main results of this work.

Theorem 2.1. *Let u_0 be a critical point of the functional J_α such that $J_\alpha''(u_0)$ is injective. Then $m(J_\alpha, u_0)$ is finite and*

$$\begin{aligned} C_q(J_\alpha, u_0) &\cong \mathbb{K}, & \text{if } q = m(J_\alpha, u_0), \\ C_q(J_\alpha, u_0) &= \{0\}, & \text{if } q \neq m(J_\alpha, u_0). \end{aligned}$$

This theorem extends a classical result in Hilbert spaces for nondegenerate critical points (cf. Theorem 4.1 in [4]), showing that the critical groups of J_α in u_0 depend only on its Morse index. We conclude with a quantitative result for a possibly degenerate critical point.

Theorem 2.2. *Let u_0 be an isolated critical point of the functional J_α . Then $m(J_\alpha, u_0)$ and $m^*(J_\alpha, u_0)$ are finite and*

$$C_q(J_\alpha, u_0) = \{0\}$$

for any $q \leq m(J_\alpha, u_0) - 1$ and $q \geq m^*(J_\alpha, u_0) + 1$.

We recall some definitions and results about compact connected Riemannian manifolds of class C^∞ (see for example [13]).

We denote by $B(0, R)$ the ball in \mathbb{R}^n of center 0 and radius R and by $B_g(x, R)$ the ball in M of center x and radius R .

On the tangent bundle TM of M the exponential map $\exp : TM \rightarrow M$ is defined. This map has the following properties:

- (i) \exp is of class C^∞ ;
- (ii) there exists a constant $R > 0$ such that

$$\exp_x |_{B(0,R)} : B(0, R) \rightarrow B_g(x, R)$$

is a diffeomorphism for all $x \in M$.

It is possible to choose an atlas \mathcal{C} on M , whose charts are given by the exponential map (normal coordinates). We denote by $\{\psi_C\}_{C \in \mathcal{C}}$ a partition of unity subordinate to the atlas \mathcal{C} . Let g_{x_0} be the Riemannian metric in the normal coordinates of the map \exp_{x_0} .

For any $u \in H_1^p(M)$ we have

$$\|\nabla u\|_p^p = \int_M |\nabla u(x)|_g^p d\mu_g = \sum_{C \in \mathcal{C}} \int_C \psi_C(x) |\nabla u(x)|_g^p d\mu_g.$$

In particular, if u has support inside one chart $C = B_g(x_0, R)$, then setting $\bar{u}(z) = u(\exp_{x_0}(z))$ we have

$$\|\nabla u\|_p^p = \int_{B(0,R)} \sum_{i,j=1}^n \left(g_{x_0}^{ij}(z) \frac{\partial \bar{u}(z)}{\partial z_i} \frac{\partial \bar{u}(z)}{\partial z_j} \right)^{\frac{p}{2}} |g_{x_0}(z)|^{\frac{1}{2}} dz,$$

where $|g_{x_0}(z)| = \det(g_{x_0,ij}(z))$ and $(g_{x_0}^{ij}(z))$ is the inverse matrix of $g_{x_0}(z)$. In particular we have $g_{x_0}(0) = \text{Id}$.

A similar relation holds for the integration of $(\alpha + |\nabla u(x)|^2)^{\frac{p}{2}}$.

We also recall that, for any $u \in H_1^p(M)$,

$$\|u\|_{1,p} = \left(\int_M |\nabla u(x)|_g^p d\mu_g \right)^{\frac{1}{p}} + \left(\int_M |u(x)|^p d\mu_g \right)^{\frac{1}{p}}.$$

Moreover, by Theorem 4.1 in [13], there exists $A > 0$ such that for any $u \in H_1^p(M)$

$$\left(\int_M |u|^{p^*} d\mu_g\right)^{1/p^*} \leq A \left(\int_M |\nabla u|_g^p d\mu_g\right)^{1/p} + Vol_{(M,g)}^{-1/n} \left(\int_M |u|^p d\mu_g\right)^{1/p}. \tag{2.3}$$

3. PROPERTIES OF J_α

For any $\alpha \in [0, 1]$, the functional J_α is C^2 and for any $u, v, w \in H_1^p(M)$ we have

$$\begin{aligned} \langle J'_\alpha(u), v \rangle &= \int_M (\alpha + |\nabla u(x)|_g^2)^{\frac{p-2}{2}} (\nabla u(x)|\nabla v(x))_g d\mu_g \\ &\quad - \int_M h(x, u(x))v(x)d\mu_g; \\ \langle J''_\alpha(u)v, w \rangle &= \int_M (\alpha + |\nabla u(x)|_g^2)^{\frac{p-2}{2}} (\nabla v(x)|\nabla w(x))_g d\mu_g \\ &\quad + (p-2) \int_M (\alpha + |\nabla u(x)|_g^2)^{\frac{p-4}{2}} (\nabla u(x)|\nabla v(x))_g (\nabla u(x)|\nabla w(x))_g d\mu_g \\ &\quad - \int_M D_s h(x, u(x))v(x)w(x)d\mu_g. \end{aligned}$$

We now prove that, in spite of the possible critical growth of h , J_α is locally weakly lower semicontinuous.

In the next preliminary lemma, if B is an $n \times n$ real, symmetric and positive definite matrix, let us denote by $(\cdot|\cdot)_B$ the inner product in \mathbb{R}^n defined by

$$(x|y)_B = x^\top B y = \sum_{i,j=1}^n b_{ij} x_i y_j \quad \forall x, y \in \mathbb{R}^n.$$

We begin by proving the following inequality (the case $\alpha = 0$ and $B = I$ has been proved by Lindqvist in Lemma 4.2 of [16]).

Lemma 3.1. *Let B be an $n \times n$ real, symmetric and positive definite matrix, $(\cdot|\cdot)_B$ and $|\cdot|_B$ the corresponding inner product and norm in \mathbb{R}^n . For any $\alpha \geq 0, p \geq 2$ and $x, y \in \mathbb{R}^n$*

$$(\alpha + |y|_B^2)^{p/2} \geq (\alpha + |x|_B^2)^{p/2} + p(\alpha + |x|_B^2)^{(p-2)/2} (x|y-x)_B + \frac{|y-x|_B^p}{2^{p-1}-1}.$$

Proof. Fix $x, y \in \mathbb{R}^n, x \neq y$. We define the function $f(t) = (\alpha + |x + t(y-x)|_B^2)^{p/2}, t \geq 0$. Since the map f is strictly convex, we have $f(1) >$

$f(0) + f'(0)$; that is,

$$(\alpha + |y|_B^2)^{p/2} > (\alpha + |x|_B^2)^{p/2} + p(\alpha + |x|_B^2)^{(p-2)/2}(x|y - x)_B. \tag{3.1}$$

Replacing y by $\frac{x+y}{2}$ in (3.1), we have

$$\left(\alpha + \left|\frac{x+y}{2}\right|_B^2\right)^{p/2} > (\alpha + |x|_B^2)^{p/2} + \frac{p}{2}(\alpha + |x|_B^2)^{(p-2)/2}(x|y - x)_B. \tag{3.2}$$

Now we derive a generalized Clarkson’s inequality (see, for instance, Theorem IV.10 [3]) when $\alpha = 0$). Recalling that $a^{p/2} + b^{p/2} \leq (a + b)^{p/2}$ for any $a, b \geq 0$, we have

$$\begin{aligned} \left(\alpha + \left|\frac{x+y}{2}\right|_B^2\right)^{p/2} + \left(\left|\frac{x-y}{2}\right|_B^2\right)^{p/2} &\leq \left(\alpha + \left|\frac{x+y}{2}\right|_B^2 + \left|\frac{x-y}{2}\right|_B^2\right)^{p/2} \\ &= (\alpha + \frac{1}{2}(|x|_B^2 + |y|_B^2))^{p/2} \leq \frac{1}{2}((\alpha + |x|_B^2)^{p/2} + (\alpha + |y|_B^2)^{p/2}) \end{aligned} \tag{3.3}$$

where the last inequality follows from the fact that $t \in [0, +\infty[\mapsto (\alpha + t)^{p/2}$ is convex.

Now putting together (3.2), (3.3), we derive

$$\begin{aligned} (\alpha + |y|_B^2)^{p/2} &> -(\alpha + |x|_B^2)^{p/2} + 2\left(\alpha + \left|\frac{x+y}{2}\right|_B^2\right)^{p/2} + 2\left|\frac{x-y}{2}\right|_B^p \\ &\geq \left(\alpha + |x|_B^2\right)^{p/2} + p(\alpha + |x|_B^2)^{(p-2)/2}(x|y - x)_B + 2\left|\frac{y-x}{2}\right|_B^p. \end{aligned} \tag{3.4}$$

This is the desired inequality with the constant 2^{1-p} in place of $\frac{1}{2^{p-1}-1}$.

Replacing y by $\frac{x+y}{2}$ in (3.4) and combining again with (3.3), we get the constant improved to $2^{1-p} + 4^{1-p}$. By iteration we finally find the constant $2^{1-p} + 4^{1-p} + 8^{1-p} + \dots = \frac{1}{2^{p-1}-1}$. \square

Proposition 3.2. *There exists $\bar{R} > 0$ such that, for any fixed $\alpha \geq 0$ and any $u \in H_1^p(M)$, the functional J_α is weakly lower semicontinuous in $\bar{B}_{\bar{R}}(u) = \{v \in H_1^p(M) : \|v - u\|_{1,p} \leq \bar{R}\}$.*

Proof. Let $\bar{R} > 0$, $u \in H_1^p(M)$. Let u_k be a sequence in $H_1^p(M)$ such that $\|u_k - u\|_{1,p} \leq \bar{R}$. Assume that u_k weakly converges to \bar{u} .

We will show that $J_\alpha(\bar{u}) \leq \liminf_k J_\alpha(u_k)$, if \bar{R} is chosen small enough.

Since $u_k \rightarrow \bar{u}$ weakly in $H_1^p(M)$, then $u_k \rightarrow \bar{u}$ strongly in $L^r(M)$ with $r < p^*$ and $u_k(x) \rightarrow \bar{u}(x)$ almost everywhere in M .

By Lemma 3.1, we have

$$\int_M (\alpha + |\nabla u_k|_g^2)^{p/2} d\mu_g - \int_M (\alpha + |\nabla u|_g^2)^{p/2} d\mu_g \tag{3.5}$$

$$\begin{aligned} &\geq p \int_M (\alpha + |\nabla u|_g^2)^{(p-2)/2} (\nabla u |\nabla u_k - \nabla u)_g d\mu_g \\ &+ \frac{1}{2^{p-1} - 1} \int_M |\nabla u_k - \nabla u|_g^p d\mu_g. \end{aligned}$$

Moreover, since $u_k \rightarrow \bar{u}$ strongly in $L^r(M)$ with $r < p^*$ and $u_k(x) \rightarrow \bar{u}(x)$ almost everywhere in M , we have

$$\int_M H(x, u_k) d\mu_g - \int_M H(x, u_k - \bar{u}) d\mu_g = \int_M H(x, \bar{u}) d\mu_g + o(1). \tag{3.6}$$

Indeed, using Vitali's theorem, it follows that

$$\begin{aligned} &\int_M H(x, u_k) d\mu_g - \int_M H(x, u_k - \bar{u}) d\mu_g \tag{3.7} \\ &= \int_M \int_0^1 \frac{d}{dt} H(x, u_k + (t - 1)\bar{u}) dt d\mu_g \\ &= \int_0^1 \int_M h(x, u_k + (t - 1)\bar{u}) \bar{u} d\mu_g dt \end{aligned}$$

tends to

$$\int_0^1 \int_M h(x, t\bar{u}) \bar{u} d\mu_g dt = \int_M \int_0^1 \frac{d}{dt} H(x, t\bar{u}) dt d\mu_g = \int_M H(x, \bar{u}) d\mu_g. \tag{3.8}$$

By (3.5) and (3.6) we have

$$\begin{aligned} J_\alpha(u_k) - J_\alpha(\bar{u}) &\geq \frac{1}{p} \int_M |\nabla u|_g^{p-2} (\nabla u |(\nabla u_k - \nabla u))_g d\mu_g \tag{3.9} \\ &+ \frac{1}{(2^{p-1} - 1)p} \int_M |\nabla u_k - \nabla u|_g^p - \int_M H(x, u_k - \bar{u}) d\mu_g + o(1) \\ &\geq \frac{1}{(2^{p-1} - 1)p} \int_M |\nabla u_k - \nabla u|_g^p - C \int_M |u_k - \bar{u}|^{p^*} d\mu_g + o(1) \\ &\geq \|u_k - \bar{u}\|_{1,p}^p \left(\frac{1}{(2^{p-1} - 1)p} - C K^p \|u_k - \bar{u}\|_{1,p}^{p^*-p} \right) + o(1), \end{aligned}$$

where $K = \max\{A, Vol_{(M,g)}^{-1/n}\}$ (see (2.3)). By choosing $\bar{R} > 0$ small enough, we derive that $\|u_k - \bar{u}\|_{1,p}^{p^*-p} < 1/(C K^p)$, which implies

$$\liminf_{k \rightarrow +\infty} J_\alpha(u_k) \geq J_\alpha(u). \tag{□}$$

We now prove that a local Palais-Smale condition holds for J_α .

Proposition 3.3. *There exists $R > 0$ such that, for any fixed $\alpha \geq 0$ and any $u \in H_1^p(M)$, the functional J_α satisfies the (P.S.) condition on $\overline{B}_R(u) = \{v \in H_1^p(M) : \|v - u\|_{1,p} \leq R\}$.*

Proof. Let $R > 0$. For convenience, for fixed $\alpha \geq 0$, we denote $J_\alpha = J$. Let $(u_m) \subset \overline{B}_R(u)$ be a sequence such that $J'(u_m) \rightarrow 0$; then (u_m) is bounded, thus converges to some $\bar{u} \in \overline{B}_R(u)$, weakly in $H_1^p(M)$ and strongly in each $L^r(M)$, with $r < p^*$. Moreover, arguing as in Lemma 3.1 in [19], one can prove that $(\alpha + |\nabla u_m|^2)^{\frac{p-2}{2}} \nabla u_m$ converges to $(\alpha + |\nabla \bar{u}|^2)^{\frac{p-2}{2}} \nabla \bar{u}$ weakly in the $(L^{p/(p-1)})^N$ norm and almost everywhere in M . Therefore, for any $z \in H_1^p(M)$,

$$\langle J'(\bar{u}), z \rangle = \lim_{m \rightarrow \infty} \langle J'(u_m), z \rangle = 0$$

so that \bar{u} is a critical point and, in particular,

$$\langle J'(u_m), u_m \rangle - \langle J'(\bar{u}), \bar{u} \rangle = o(1). \tag{3.10}$$

We can show that

$$\begin{aligned} & \int_M (\alpha + |\nabla u_m - \nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla u_m - \nabla \bar{u}|_g^2 d\mu_g \tag{3.11} \\ &= \int_M (\alpha + |\nabla u_m|_g^2)^{\frac{p-2}{2}} |\nabla u_m|_g^2 d\mu_g - \int_M (\alpha + |\nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla \bar{u}|_g^2 d\mu_g + o(1). \end{aligned}$$

Indeed, by Vitali's convergence theorem, denoting by $u_{m,t} = u_m + (t - 1)\bar{u}$, we infer that

$$\begin{aligned} & \int_M (\alpha + |\nabla u_m|_g^2)^{\frac{p-2}{2}} |\nabla u_m|_g^2 d\mu_g \\ & - \int_M (\alpha + |\nabla u_m - \nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla u_m - \nabla \bar{u}|_g^2 d\mu_g \\ &= \int_M \int_0^1 \frac{d}{dt} [(\alpha + |\nabla u_m + (t - 1)\nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla u_m + (t - 1)\nabla \bar{u}|_g^2] dt d\mu_g \\ &= \int_0^1 \int_M (p - 2) (\alpha + |\nabla u_{m,t}|_g^2)^{(p-4)/2} |\nabla u_{m,t}|_g^2 (\nabla u_{m,t} | \nabla \bar{u})_g \\ & + 2 (\alpha + |\nabla u_{m,t}|_g^2)^{(p-2)/2} (\nabla u_{m,t} | \nabla \bar{u})_g d\mu_g dt \end{aligned}$$

tends to

$$\begin{aligned} & \int_0^1 \int_M (p - 2) (\alpha + t^2 |\nabla \bar{u}|_g^2)^{(p-4)/2} |\nabla \bar{u}|_g^4 t^3 \\ & + 2 (\alpha + t^2 |\nabla \bar{u}|_g^2)^{(p-2)/2} |\nabla \bar{u}|_g^2 t d\mu_g dt \end{aligned}$$

$$\begin{aligned}
 &= \int_M \int_0^1 \frac{d}{dt} [(\alpha + t^2 |\nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla \bar{u}|_g^2 t^2] dt d\mu_g \\
 &= \int_M (\alpha + |\nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla \bar{u}|_g^2 d\mu_g.
 \end{aligned}$$

In a similar way, using Vitali’s convergence theorem one can prove that

$$\begin{aligned}
 &\int_M h(x, u_m) u_m d\mu_g - \int_M h(x, u_m - \bar{u})(u_m - \bar{u}) d\mu_g \tag{3.12} \\
 &= \int_M \int_0^1 \frac{d}{dt} [h(x, u_m + (t - 1)\bar{u})(u_m + (t - 1)\bar{u})] dt d\mu_g \\
 &= \int_0^1 \int_M D_s h(x, u_m + (t - 1)\bar{u})(u_m + (t - 1)\bar{u}) \bar{u} + h(x, u_m + (t - 1)\bar{u}) \bar{u} d\mu_g
 \end{aligned}$$

which tends to

$$\begin{aligned}
 &\int_0^1 \int_M [D_s h(x, t\bar{u})(t\bar{u}) \bar{u} + h(x, t\bar{u}) \bar{u}] d\mu_g dt \tag{3.13} \\
 &= \int_M \int_0^1 \frac{d}{dt} [h(x, t\bar{u})(t\bar{u})] dt d\mu_g = \int_M h(x, \bar{u}) \bar{u} d\mu_g.
 \end{aligned}$$

Since $|D_s h(x, t)| \leq c_1 |t|^{p^*-2} + c_2$, with c_1 and c_2 positive constants, there exist $C, D > 0$ such that $h(x, t)t \leq C|t|^{p^*} + Dt^2$ for any $t \in \mathbb{R}$. From (3.10), (3.11), (3.12), we deduce

$$\begin{aligned}
 &\int_M |\nabla u_m - \nabla \bar{u}|_g^p d\mu_g - C \int_M |u_m - \bar{u}|^{p^*} d\mu_g \tag{3.14} \\
 &\leq \int_M (\alpha + |\nabla u_m - \nabla \bar{u}|_g^2)^{\frac{p-2}{2}} |\nabla u_m - \nabla \bar{u}|_g^2 d\mu_g \\
 &\quad - \int_M h(x, u_m - \bar{u})(u_m - \bar{u}) d\mu_g + o(1) \\
 &= \langle J'(u_m), u_m \rangle - \langle J'(\bar{u}), \bar{u} \rangle + o(1) = o(1),
 \end{aligned}$$

where C is a positive constant. Denoting

$$a = \limsup_{m \rightarrow +\infty} \|u_m - \bar{u}\|_{1,p}^p \leq 2^p \limsup_{m \rightarrow +\infty} \int_M |\nabla u_m - \nabla \bar{u}|_g^p d\mu_g \tag{3.15}$$

by (3.14), (2.3) we have

$$a \leq 2^p C \limsup_{m \rightarrow +\infty} \int_M |u_m - \bar{u}|^{p^*} d\mu_g \leq 2^p C K^{p^*} a^{p^*/p},$$

where $K = \max\{A, Vol_{(M,g)}^{-1/n}\}$. Therefore, if $a > 0$, this implies

$$a \geq (2^p C K^{p^*})^{-(N-p)/p},$$

hence,

$$\begin{aligned} (2^p C K^{p^*})^{-(N-p)/p} \leq a &\leq 2^p C \limsup_{m \rightarrow +\infty} (\|u_m - u\|_{1,p} + \|u - \bar{u}\|_{1,p})^{p^*} \quad (3.16) \\ &\leq 2^p C (2R)^{p^*}, \end{aligned}$$

where $K > 0$ is a positive constant. If we take $R > 0$ small enough, such that $2^p C (2R)^{p^*} < (2^p C K^{p^*})^{-(N-p)/p}$, we derive a contradiction. Therefore it must be that $a = 0$ and thus u_m strongly converges to \bar{u} in $H_1^p(M)$. \square

4. CRITICAL GROUP ESTIMATES

We recall this regularity result, which can be proved using Moser’s iterative scheme [17] and following the arguments of Druet in [11], taking into account [12] and [20].

Theorem 4.1. *Let (M, g) be a compact Riemannian n -manifold, $n \geq 2$, $1 < p < n$, $\alpha \in [0, 1]$, $C > 0$ and let $h \in C^0(M \times \mathbb{R})$ be such that*

$$\forall (x, r) \in M \times \mathbb{R}, \quad |h(x, r)| \leq C \left(|r|^{p^*-1} + 1 \right).$$

If $u \in H_1^p(M)$ is a solution of $div_g \left((\alpha + |\nabla u|_g^2)^{(p-2)/2} \nabla u \right) + h(x, u) = 0$, then $u \in C^{1,\beta}(M)$, with $\beta \in]0, 1[$. Moreover, if u belongs to a bounded set $A \subset H_1^p(M)$, then $\|u\|_{C^{1,\beta}(M)} \leq K$, where K depends only on n, p, g, A, α, C .

Now we fix a critical point u_0 of J_α . It is standard that u_0 solves the quasi-linear problem (1.1). By Theorem 4.1, $u_0 \in C^{1,\beta}(M)$. As in the Euclidean case [5], we can introduce a Hilbert space, depending on the critical point u_0 , in which $H_1^p(M)$ is embedded, so that a suitable splitting can be obtained. Precisely, let H_0 be the closure of $C_0^\infty(M)$ under the scalar product

$$\begin{aligned} (v, w)_0 &= \int_M \left((\alpha + |\nabla u_0|_g^2)^{(p-2)/2} (\nabla v | \nabla w)_g \right. \\ &\quad \left. + (p-2)(\alpha + |\nabla u_0|_g^2)^{(p-4)/2} (\nabla u_0 | \nabla v)_g (\nabla u_0 | \nabla w)_g \right) d\mu_g + \int_M vw \, d\mu_g. \end{aligned}$$

As $u_0 \in C^{1,\beta}(M)$, the norm $\|\cdot\|_0$ induced by $(\cdot, \cdot)_0$ is equivalent to the usual norm of $H_1^2(M)$. Hence H_0 is isomorphic to $H_1^2(M)$ and the embedding $H_1^p(M) \hookrightarrow H_0$ is continuous.

Denoting by $\langle \cdot, \cdot \rangle : H_0^* \times H_0 \rightarrow \mathbb{R}$ the duality pairing in H_0 , $J_\alpha''(u_0)$ can be extended to the operator $L_0 : H_0 \rightarrow H_0^*$ defined by setting

$$\langle L_0 v, w \rangle = (v, w)_0 - \langle K v, w \rangle$$

where

$$\langle K v, w \rangle = \int_M (D_s h(x, u_0) + 1) v w \, d\mu_g,$$

for any $v, w \in H_0$.

Lemma 4.2. *L_0 is a compact perturbation of the Riesz isomorphism from H_0 to H_0^* . In particular, L_0 is a Fredholm operator with index zero.*

Proof. In order to prove the assertion it is sufficient to show that K is a compact operator from H_0 to H_0^* . Let $\{v_n\}$ be a bounded sequence in H_0 . Then there exists $v \in H_0$ such that, up to a subsequence, $\{v_n\}$ converges weakly to v in H_0 and strongly in $L^2(M)$. There is a constant $C > 0$ such that, for any $w \in H_0$, $\|w\|_0 = 1$, we have

$$\left| \int_M (D_s h(x, u_0) + 1) (v_n - v) w \, d\mu_g \right| \leq C \left(\int_M |v_n - v|^2 \, d\mu_g \right)^{1/2}$$

which tends to zero as $n \rightarrow +\infty$, uniformly with respect to w . This implies that K is a compact operator. \square

Now let us denote by $m(L_0)$ the maximal dimension of a subspace of H_0 on which L_0 is negative definite. Obviously $m(J_\alpha, u_0) \leq m(L_0)$. Furthermore let us denote by $m^*(L_0)$ the sum of $m(L_0)$ and the dimension of the kernel of L_0 . By Lemma 4.2 we conclude that $m(L_0)$ and $m^*(L_0)$ are finite. Since L_0 is a Fredholm operator in H_0 , we can consider the natural splitting

$$H_0 = H^- \oplus H^0 \oplus H^+$$

where H^- , H^0 , H^+ are, respectively, the negative, null and positive spaces, according to the spectral decomposition of L_0 in $L^2(M)$. Therefore one can easily show that there exists a $\gamma_0 > 0$ such that

$$\langle L_0 v, v \rangle \geq \gamma_0 \|v\|_0^2 \quad \forall v \in H^+;$$

moreover $m(L_0)$, $m^*(L_0)$ are respectively the dimensions of H^- and $H^- \oplus H^0$. Since $u_0 \in C^{1,\beta}(M)$, following the same arguments in Lemma 2.2 and Theorem 2.3 in [11], we derive that $H^- \oplus H^0 \subset H_1^p(M) \cap C^1(M)$. Consequently, denoting by $W = H^+ \cap H_1^p(M)$ and $V = H^- \oplus H^0$, we get the splitting $H_1^p(M) = V \oplus W$ and,

$$\langle J_\alpha''(u_0) w, w \rangle \geq \gamma_0 \|w\|_0^2 \quad \forall w \in W, \tag{4.1}$$

so that $m(L_0) = m(J_\alpha, u_0)$ and $m^*(L_0) = m^*(J_\alpha, u_0)$.

The next proposition states a sort of local (P.S.) condition for J_α in the direction of W .

Proposition 4.3. *There exists $R > 0$ such that, for any fixed $\alpha \geq 0$ and $u \in H_p^1(M)$, if $(u_m) \subset \bar{B}_R(u)$ and $\sup_{w \in W \setminus \{0\}} \langle J_\alpha'(u_m), w \rangle / \|w\| \rightarrow 0$, then (u_m) has a convergent subsequence.*

Proof. Reasoning as in the proof of Proposition 3.3, let us fix $R > 0$, $\alpha \geq 0$, and denote $J_\alpha = J$.

With the same arguments, we infer that (u_m) weakly converges to some $\bar{u} \in \bar{B}_R(u)$ and, for any $z \in H_p^1(M)$,

$$\langle J'(u_m), z \rangle \rightarrow \langle J'(\bar{u}), z \rangle. \quad (4.2)$$

Let $\{e_1, \dots, e_{m^*}\}$ be an L^2 -orthogonal basis of V , where $m^* = m^*(J_\alpha, u_0)$. Denoting by $P_V(z) = \sum_{i=1}^{m^*} (\int_M e_i z dx) e_i$, it is clear that $z - P_V(z) \in W$, for any $z \in H_p^1(M)$. Moreover $P_V(u_m)$ strongly converges to $P_V(\bar{u})$. Exploiting (4.2) we get

$$\begin{aligned} \langle J'(u_m), u_m \rangle - \langle J'(\bar{u}), \bar{u} \rangle &= \langle J'(u_m), P_V(u_m) \rangle - \langle J'(\bar{u}), P_V(\bar{u}) \rangle + o(1) \\ &= \langle J'(u_m), P_V(u_m) - P_V(\bar{u}) \rangle + \langle J'(u_m), P_V(\bar{u}) \rangle - \langle J'(\bar{u}), P_V(\bar{u}) \rangle + o(1) \\ &= o(1). \end{aligned} \quad (4.3)$$

Consequently, as (3.11) and (3.12) hold too, we can complete this proof in the same way as the previous one, using (4.3) instead of (3.10). \square

Now we prove that u_0 is a strict minimum pointing in the direction of W .

In order to do that we need some preliminary results. The first one is found in [14].

Lemma 4.4. *Let $I : L^p(M, \mathbb{R}^k) \times L^q(M, \mathbb{R}^m) \rightarrow (-\infty, +\infty]$ be a functional of the form*

$$I(u, v) = \int_M \phi(x, u, v) d\mu_g$$

where $\phi(x, u, v)$ is a nonnegative, continuous function and $\phi(x, u, \cdot)$ is convex. Then I is lower semicontinuous with respect to the strong convergence of the component u in L^p and with respect to the weak convergence of the component v in L^q .

Lemma 4.5. *For any $K > 0$ there exist $r_0 > 0$ and $C > 0$ such that, for any $z \in C^1(M)$, with $\|z\|_{C^1(M)} \leq K$, $\|z - u_0\| < r_0$, we have*

$$\langle J''_\alpha(z)w, w \rangle \geq C\|w\|_0^2 \quad \forall w \in W. \tag{4.4}$$

Proof. By contradiction, there exist $K > 0$ and two sequences $z_n \in C^1(M)$ and $w_n \in W$ such that $\|w_n\|_0 = 1$, $\|z_n\|_{C^1(M)} \leq K$, $\|z_n - u_0\|_{1,p} \rightarrow 0$ and

$$\langle J''_\alpha(z_n)w_n, w_n \rangle < \frac{1}{n}. \tag{4.5}$$

Firstly, it is immediate to see that, for a suitable positive constant C_0 , we have

$$\langle J''_\alpha(z_n)w_n, w_n \rangle \geq C_0\|w_n\|_0^2 - \int_M (D_s h(x, z_n) + 1)w_n^2 d\mu_g. \tag{4.6}$$

As w_n is bounded in H_0 , it converges to some \bar{w} weakly in H_0 and strongly in $L^2(M)$. Moreover, as $h(x, \cdot)$ is C^1 , we derive

$$\lim_{n \rightarrow \infty} \int_M (D_s h(x, z_n) + 1)w_n^2 d\mu_g = \int_M (D_s h(x, u_0) + 1)\bar{w}^2 d\mu_g. \tag{4.7}$$

We see that $\bar{w} \neq 0$; in fact, if this were not true, by (4.5), (4.6) and (4.7) we should have $C_0 \leq 0$. By Lemma 4.4 we have

$$\begin{aligned} & \int_M (\alpha + |\nabla u_0|_g^2)^{\frac{p-2}{2}} |\nabla \bar{w}|_g^2 + (p-2)(\alpha + |\nabla u_0|_g^2)^{\frac{p-4}{2}} (\nabla u_0 | \nabla \bar{w})_g^2 \\ & \leq \liminf_n \int_M (\alpha + |\nabla z_n|_g^2)^{\frac{p-2}{2}} |\nabla w_n|_g^2 + (p-2)(\alpha + |\nabla z_n|_g^2)^{\frac{p-4}{2}} (\nabla z_n | \nabla w_n)_g^2. \end{aligned} \tag{4.8}$$

So, by (4.1), (4.5) and (4.8) we have $\gamma_0\|\bar{w}\|_0^2 \leq 0$ which is absurd as $\bar{w} \neq 0$. □

Following the same arguments of Lemma 4.5 in [5] and Lemma 4.2 in [6], one can recognize that u_0 is a strict minimum pointing in the direction of W .

Lemma 4.6. *There exist $\delta > 0$ and $\mu' > 0$ such that, for any $w \in W$ with $\|w\| \leq \delta$, we have*

$$J_\alpha(u_0 + w) - J_\alpha(u_0) \geq \mu'\|w\|^p.$$

We can now obtain the following result which is crucial for developing a finite-dimensional reduction.

Proposition 4.7. *There exist $r > 0$ and $\rho \in (0, r)$ such that, for each v in $V \cap \overline{B}_\rho(0)$, there exists one and only one $\bar{w} \in W \cap B_r(0) \cap L^\infty(\Omega)$ such that for any $z \in W \cap \overline{B}_r(0)$ we have*

$$J_\alpha(v + \bar{w} + u_0) \leq J_\alpha(v + z + u_0).$$

Moreover, \bar{w} is the only element of $W \cap \overline{B}_r(0)$ such that

$$\langle J'_\alpha(u_0 + v + \bar{w}), z \rangle = 0 \quad \forall z \in W \tag{4.9}$$

and

$$S = \{v + z + u_0 : v \in V \cap \overline{B}_\rho(0), w \in W \cap \overline{B}_r(0)\} \subset B_{R'}(u_0), \tag{4.10}$$

where $R' = \min\{R, \bar{R}\}$ and R, \bar{R} are respectively defined by Proposition 4.3 and Proposition 3.2.

Proof. Now let $m^* = m^*(J_\alpha, u_0)$, $\{e_1, \dots, e_{m^*}\}$ be an L^2 -orthogonal basis of $V \subset C^1(M)$ and δ be defined by Lemma 4.6. There exists M depending just on δ such that if $z \in B_\delta(u_0)$ is a solution of $\langle J'_\alpha(z), w \rangle = 0$ for any $w \in W$, denoting by $f_z(x) = \sum_{i=1}^{m^*} (\int_M e_i z dx) e_i(x)$, we have $f_z \in C^1(M)$ and $\|f_z\|_{C^1(M)} \leq M$. Thus z solves the equation

$$\operatorname{div}_g \left((\alpha + |\nabla u|_g^2)^{(p-2)/2} \nabla u \right) + h(x, u) + f_z(x) = 0$$

and there exists $D > 0$ such that

$$\forall (x, r) \in M \times \mathbb{R}, \quad |h(x, r) + f_z(x)| \leq D(|r|^{p^*-1} + 1).$$

Thus, by Theorem 4.1, $z \in C^1(M)$ and $\|z\|_{C^1(M)} \leq K$, where $K > 0$. Now by Lemma 4.5 in correspondence with $2K$ there exist $r_0 \in (0, \delta)$ and $C > 0$ such that (4.4) holds.

We fix $0 < r < \min\left\{\frac{R}{2}, \frac{\bar{R}}{2}, \frac{r_0}{3}\right\}$ where R and \bar{R} are introduced respectively in Proposition 4.3 and Proposition 3.2, so that (4.10) holds and J_α is sequentially lower semicontinuous in $\overline{B}_r(u)$ with respect to the weak topology of $H_1^p(M)$ for any $u \in H_1^p(M)$. Therefore, for any fixed $v \in V \cap \overline{B}_r(0)$, there exists a minimum point $\bar{w} \in W \cap \overline{B}_r(0)$ of the function $w \in W \cap \overline{B}_r(0) \mapsto J_\alpha(u_0 + v + w)$.

We shall prove that there exists $0 < \rho < r$ such that for any $v \in V \cap \overline{B}_\rho(0)$

$$\inf\{J_\alpha(u_0 + v + w) : w \in W, \|w\| = r\} > J_\alpha(u_0 + v). \tag{4.11}$$

Arguing by contradiction, we assume that there exist two sequences $\{w_n\}$ in W and $\{v_n\}$ in V such that $\|w_n\| = r$, $\|v_n\| \rightarrow 0$ and

$$J_\alpha(u_0 + v_n + w_n) \leq J_\alpha(u_0 + v_n) + o(1). \tag{4.12}$$

Moreover, $J_\alpha(u_0 + v_n + w_n) - J_\alpha(u_0 + w_n) = \langle J'_\alpha(u_0 + \beta_n v_n + w_n), v_n \rangle$, where $\beta_n \in (0, 1)$, so that

$$J_\alpha(u_0 + v_n + w_n) = J_\alpha(u_0 + w_n) + o(1)$$

which combined with (4.12) and Lemma 4.6 gives the absurd statement

$$\begin{aligned} \mu' r^p &\leq J_\alpha(u_0 + w_n) - J_\alpha(u_0) = J_\alpha(u_0 + v_n + w_n) - J_\alpha(u_0) + o(1) \\ &\leq J_\alpha(u_0 + v_n) - J_\alpha(u_0) + o(1) = o(1). \end{aligned}$$

Consequently, by (4.11), we have that for any $v \in V \cap \overline{B}_\rho(0)$ the minimum point \bar{w} belongs to $W \cap B_r(0)$ and then solves

$$\langle J'_\alpha(u_0 + v + w), z \rangle = 0 \quad \forall z \in W. \tag{4.13}$$

Therefore, $\bar{w} \in C^1(M)$ and $\|u_0 + v + \bar{w}\|_{C^1(M)} \leq K$. We can also recognize that \bar{w} is unique. By contradiction we suppose that there exist $w_1 \neq w_2 \in W$ which solve (4.13). Then for any $t \in [0, 1]$ we have $\|v + w_1 + t(w_2 - w_1)\|_{1,p} \leq 3r < r_0$, $\|u_0 + v + w_1 + t(w_2 - w_1)\|_{C^1(M)} \leq 2K$ and, applying Lemma 4.5,

$$\begin{aligned} 0 &= \langle J'_\alpha(u_0 + v + w_2) - J'_\alpha(u_0 + v + w_1), w_2 - w_1 \rangle \\ &= \int_0^1 \langle J''_\alpha(u_0 + v + w_1 + t(w_2 - w_1))(w_2 - w_1), w_2 - w_1 \rangle dt > 0. \quad \square \end{aligned}$$

Thus we can introduce the map

$$\Psi : V \cap \overline{B}_\rho(0) \rightarrow W \cap B_r(0), \tag{4.14}$$

where $\Psi(v) = \bar{w}$ is the unique minimum point of the function $w \in W \cap \overline{B}_r(0) \mapsto J_\alpha(u_0 + v + w)$, and the function

$$\Phi : V \cap \overline{B}_\rho(0) \rightarrow \mathbb{R} \tag{4.15}$$

defined by $\Phi(v) = J_\alpha(u_0 + v + \Psi(v))$.

Now we will show that

$$C_j(J_\alpha, u_0) \simeq C_j(\Phi, 0), \tag{4.16}$$

and, from this, we will prove Theorem 2.1 and Theorem 2.2.

We first need the regularity of Ψ introduced in (4.14).

Proposition 4.8. *Ψ is a continuous function.*

Proof. Let $\{v_n\}$ be a convergent sequence in $V \cap \overline{B}_\rho(0)$, and let v be its limit. By Proposition 4.3, $\Psi(v_n) \rightarrow w \in W \cap \overline{B}_r(0)$. Moreover $\langle J'_\alpha(u_0 + v + w), z \rangle = 0$ for any $z \in W$, thus Proposition 4.7 assures us that $w = \Psi(v)$. \square

Moreover, we need the following technical lemma.

Lemma 4.9. *For any $\sigma \in (0, \rho]$, let us introduce the set*

$$M_\sigma = \left\{ u_0 + v + (1-t)\tilde{w} + t\Psi(v) : v \in V \cap \overline{B}_\sigma(0), \tilde{w} \in W \cap \overline{B}_r(0), \right. \\ \left. J_\alpha(u_0 + v + \tilde{w}) \leq c, t \in [0, 1] \right\}. \tag{4.17}$$

There exists $\rho' \in (0, \rho]$ such that, if $u_0 + v + w \in M_{\rho'}$, where $(v, w) \in V \times W$, then we have, for any $z \in W$ with $\|z\| = r$,

$$J_\alpha(u_0 + v + w) < J_\alpha(u_0 + v + z).$$

Proof. Arguing by way of contradiction, let $\{v_n\}$ be in V , $\{w_n\}$ in $W \cap \overline{B}_r(0)$, $\{z_n\}$ in W , $\{t_n\}$ in $[0, 1]$ be sequences such that $v_n \rightarrow 0$, $J_\alpha(u_0 + v_n + w_n) \leq c$, $\|z_n\| = r$ and

$$J_\alpha(u_0 + v_n + (1-t_n)w_n + t_n\Psi(v_n)) \geq J_\alpha(u_0 + v_n + z_n). \tag{4.18}$$

By Lemma 4.6, there exists μ' such that

$$c + \mu' \|w_n\|^p \leq J_\alpha(u_0 + w_n) = J_\alpha(u_0 + v_n + w_n) + o(1) \leq c + o(1),$$

hence, $w_n \rightarrow 0$. Analogously,

$$c + \mu' r^p \leq J_\alpha(u_0 + v_n + z_n) + o(1). \tag{4.19}$$

Consequently, $u_0 + v_n + (1-t_n)w_n + t_n\Psi(v_n) \rightarrow u_0$ and, applying (4.18) and (4.19), $c + \mu' r^p \leq c + o(1)$ which is absurd. \square

First, fixing $\rho' \in (0, \rho]$ such that the previous lemma is satisfied, we introduce the following sets:

$$M = M_{\rho'} \\ \Phi^c = \{v \in V \cap \overline{B}_{\rho'}(0) : J_\alpha(u_0 + v + \Psi(v)) \leq c\} \\ Y = \{u_0 + v + \Psi(v) : v \in V \cap \overline{B}_{\rho'}(0)\} \\ Y^c = \{u_0 + v + \Psi(v) : v \in V \cap \overline{B}_{\rho'}(0), J_\alpha(u_0 + v + \Psi(v)) \leq c\} \\ U = (u_0 + (V \cap \overline{B}_{\rho'}(0)) + (W \cap \overline{B}_r(0))).$$

Since Ψ is a continuous map and $\Psi(0) = 0$, the topological pair $(\Phi^c, \Phi^c \setminus \{0\})$ is homeomorphic to $(Y^c, Y^c \setminus \{u_0\})$. Therefore, according to (2.2), for any integer j we have

$$C_j(\Phi, 0) \cong H^j(Y^c, Y^c \setminus \{u_0\}).$$

(4.16) will be proved if we show that the topological pair $(Y^c, Y^c \setminus \{u_0\})$ is a deformation retract of $((J_\alpha)^c \cap U, (J_\alpha)^c \cap U \setminus \{u_0\})$.

This will be done in Corollary 4.11, whose proof relies on the following result.

Proposition 4.10. *There exists a continuous function $r : M \rightarrow (J_\alpha)^c \cap U$ such that*

(a) for any $z \in M$: $r(z) - z \in W$
 (b) for any $z \in (J_\alpha)^c \cap U$: $r(z) = z$.

Proof. Let us introduce the continuous function $\beta : H_1^p(M) \rightarrow \mathbb{R}$, $\beta(z) = \sup_{w \in W \setminus \{0\}} \frac{\langle J'_\alpha(z), w \rangle}{\|w\|}$ and the set $Z^* = \{z \in H_1^p(M) : \beta(z) \neq 0\}$. From standard arguments concerning the construction of a pseudogradient vector field (see e.g. [4]) we infer that there exists a continuous vector field $X : Z^* \rightarrow W$ such that, for all $z \in Z^*$,

- (1) $\|X(z)\| \leq 2\beta(z)$,
- (2) $\langle J'_\alpha(z), X(z) \rangle \geq \beta^2(z)$.

The existence of X gives a decreasing flow for J_α in the direction of W . In fact the Cauchy problem

$$\dot{\sigma}(t) = -X(\sigma(t)), \quad \sigma(0) = z \tag{P_z}$$

is locally solvable for any $z \in Z^*$ and the function $t \mapsto J_\alpha(\sigma(t))$ is decreasing as

$$\frac{d}{dt} J_\alpha(\sigma(t)) = \langle J'_\alpha(\sigma(t)), \dot{\sigma}(t) \rangle = -\langle J'_\alpha(\sigma(t)), X(\sigma(t)) \rangle < -\beta^2(\sigma(t)). \tag{4.20}$$

Now let z_0 be an element of $M \cap J_\alpha^{-1}[c, +\infty) \setminus Y$, so that the Cauchy problem (P_{z_0}) is locally solvable. We first note that Lemma 4.9 and (4.20) ensure that $\sigma(t, z_0) \in U$, for any t in which the solution $\sigma(t, z_0)$ to (P_{z_0}) is defined.

It can be proved that there exists $T_{z_0} \geq 0$ such that $\sigma(t, z_0)$ is defined at least in $[0, T_{z_0}]$ and $J_\alpha(\sigma(T_{z_0}, z_0)) = c$. In fact, if not, denoting by T the maximal existence interval for the initial data z_0 , we have $\lim_{t \rightarrow T^-} J_\alpha(\sigma(t, z_0)) > c$. By Proposition 4.3 there is $\varepsilon > 0$ such that, for any t , $\beta(\sigma(t, z_0)) > \varepsilon$, hence, by (4.20),

$$c - J_\alpha(z_0) \leq \int_0^t \frac{d}{ds} J_\alpha(\sigma(s, z_0)) ds < -\varepsilon^2 t \tag{4.21}$$

so that $T < \frac{J_\alpha(z_0) - c}{\varepsilon^2}$. Moreover, for any $t_1 < t_2$ we get

$$\begin{aligned} \|\sigma(t_2) - \sigma(t_1)\| &\leq \int_{t_1}^{t_2} \|\dot{\sigma}(t)\| dt \leq 2 \int_{t_1}^{t_2} \beta(\sigma(t)) dt \\ &\leq 2\sqrt{(t_2 - t_1)(J_\alpha(\sigma(t_1)) - J_\alpha(\sigma(t_2)))} \leq 2\sqrt{(t_2 - t_1)(J_\alpha(z_0) - c)}. \end{aligned} \tag{4.22}$$

This implies that $\sigma(T)$ exists and is not a critical point, hence the flow can be extended beyond T , contradicting the maximality.

Now, denoting by A the set of all $(t, z) \in \mathbb{R} \times Z^*$ such that the solution to (P_z) is defined in t , we see that the function $(t, z) \in A \mapsto J_\alpha(\sigma(t, z)) \in \mathbb{R}$

is C^1 and, by (4.20),

$$\frac{\partial}{\partial t} J_\alpha(\sigma(T_{z_0}, z_0)) < -\beta^2(\sigma(T_{z_0}, z_0)) < 0,$$

so continuity of $z \mapsto T_z$ in z_0 is assured by the implicit function theorem.

Hence, we are ready to define the function $r : M \rightarrow (J_\alpha)^c \cap U$ given by

$$r(z) = \begin{cases} z & \text{if } z \in (J_\alpha)^c \\ \sigma(T_z, z) & \text{if } z \notin (J_\alpha)^c \end{cases}$$

which satisfies (a) and (b).

It is clear that r is continuous in the interior of $(J_\alpha)^c$ and, by ODE theory, also in $M \cap J_\alpha^{-1}[c, +\infty) \setminus Y$, thus it remains only to verify continuity of r in $J_\alpha^{-1}\{c\} \cap Y$.

To this aim, let us fix $z_0 \in J_\alpha^{-1}\{c\} \cap Y$ and let $\{z_n\} \subset M$ be a sequence converging to z_0 . If z_n belongs to $(J_\alpha)^c$, then the assertion easily follows, so let us suppose $z_n \notin (J_\alpha)^c$. We will prove now that in this case $\beta(r(z_n)) \rightarrow 0$, so that the assertion follows from Proposition 4.3.

Indeed, arguing by contradiction, suppose that

$$\beta(r(z_n)) \text{ is not infinitesimal}; \quad (4.23)$$

then, there exist $\bar{T} > 0$ and $\varepsilon_0 > 0$ such that

$$\forall n \in \mathbb{N}, \forall t \in [T_{z_n} - \bar{T}, T_{z_n}] \quad \beta(\sigma(t, z_n)) > \varepsilon_0. \quad (4.24)$$

In fact, if (4.24) is not true, there is a subsequence $\{z_{n_k}\}$ such that, for any $k \in \mathbb{N}$,

$$\beta(\sigma(t_k, z_{n_k})) \leq \frac{1}{k}$$

at least for a suitable $t_k \in [T_{z_{n_k}} - \frac{1}{k}, T_{z_{n_k}}]$. Hence, by Proposition 4.3, $\sigma(t_k, z_{n_k}) \rightarrow z_0$, while, by (4.22),

$$\|r(z_{n_k}) - \sigma(t_k, z_{n_k})\| \leq 2\sqrt{1/k \cdot (J_\alpha(z_{n_k}) - c)} \rightarrow 0,$$

so that also $r(z_{n_k}) \rightarrow z_0$, and in particular $\beta(r(z_{n_k})) \rightarrow 0$, in contradiction with (4.23).

Therefore, (4.24) holds and, reasoning as in (4.21),

$$c - J_\alpha(\sigma(T_{z_n} - \bar{T}, z_n)) \leq -\varepsilon_0^2 \bar{T}$$

which gives the absurd statement

$$c + \varepsilon_0^2 \bar{T} \leq J_\alpha(\sigma(T_{z_n} - \bar{T}, z_n)) \leq J_\alpha(z_n) = c + o(1). \quad \square$$

In order to prove (4.16) and finally Theorems 2.1 and 2.2, we need the following step.

Corollary 4.11. $(Y^c, Y^c \setminus \{u_0\})$ is a deformation retract of $((J_\alpha)^c \cap U, (J_\alpha)^c \cap U \setminus \{u_0\})$.

Proof. It immediately follows from the previous proposition by defining

$$\eta : [0, 1] \times (J_\alpha)^c \cap U \rightarrow (J_\alpha)^c \cap U,$$

$$\eta(t, u_0 + v + w) = r(u_0 + v + (1 - t)w + t\Psi(v)). \quad \square$$

Proof of Theorem 2.1. As $J''_\alpha(u_0)$ is injective, we have $H^0 = \{0\}$ and there is a suitable constant $\mu > 0$ such that

$$\langle J''_\alpha(u_0)v, v \rangle \leq -\mu\|v\|^2 \quad \text{for any } v \in V.$$

As a consequence, u_0 is a local isolated maximum of J_α along V , thus 0 is a local isolated maximum of Φ in $V \cap \overline{B}_{\rho'}$ and by (4.16) the assertion follows (see [4, Example 1, page 33]). \square

Proof of Theorem 2.2. By (4.16), it is clear that $C_j(J_\alpha, u_0) = \{0\}$ if $j > \dim V$. Moreover, by Theorem 3.1 proved by Lancelotti in [15], we have $C_j(J_\alpha, u_0) = \{0\}$ for any $j < m(J_\alpha, u_0)$.

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