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DOUBLY DEGENERATE PARABOLIC EQUATIONS WITH VARIABLE NONLINEARITY I: EXISTENCE OF BOUNDED STRONG SOLUTIONS

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Abstract. We study the homogeneous Dirichlet problem for the anisotropic parabolic equation with double degeneracy

$$\frac{d}{dt}(|v|^{m(x,t)}\,\mathrm{sign}\; v)=\sum_{i=1}^{n}D_i(a_i(x,t)|D_iv|^{p_i(x,t)-2}D_iv)+b(x,t)|v|^{\sigma(x,t)-2}v+g(x,t).$$

The exponents of nonlinearity $m(x,t)>0$, $p_i(x,t)>1$, and $\sigma(x,t)>1$ are given bounded continuous functions. It is proved that the problem has a bounded solution in a variable-exponent Sobolev space. The main existence result is local in time and is established under minimal restrictions on the low-order terms. It is shown that under further restrictions on b and $\sigma(x,t)$ the constructed solution can be continued to the arbitrary time interval. The energy estimates are derived.

1. Introduction

The paper addresses the questions of existence of solutions of the Dirichlet problem for the doubly nonlinear anisotropic parabolic equation with variable nonlinearity

$$\begin{cases} \dfrac{d}{dt}(|v|^{m(z)}\,\mathrm{sign}\; v) & \\ \quad =\displaystyle\sum_{i=1}^{n}D_i\big(a_i(z)|D_iv|^{p_i(z)-2}D_iv\big)+b(z)|v|^{\sigma(z)-2}v+g & \text{in } Q_T,\\ v=0 & \text{on } \Gamma_T,\\ v(x,0)=v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, and $\Gamma_T = \partial\Omega \times (0,T)$ is the lateral boundary of the cylinder Q_T. The exponents of nonlinearity $m(z)$, $p_i(z)$, and $\sigma(z)$ are given functions of their arguments. Throughout the paper we use the notation $z = (x,t) \in Q_T$. The total derivative with respect to t is calculated according to the rule

$$\frac{d}{dt}\psi(v,m) = \psi_v(v,m)\, v_t + \psi_m(v,m)\, m_t.$$

Equations of the type (1.1) with constant exponents m and p_i arise in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, or glaciology [3, 19, 20]. The questions of existence, uniqueness, and propagation of solutions to doubly degenerate equations of the types

$$\begin{aligned} &(a) \qquad u_t = \operatorname{div}\left(a\,|u|^{\alpha}|\nabla u|^{p-2}\nabla u\right) + f(x,t,u), \\ &(b) \qquad \partial_t\left(|u|^{\beta-1}u\right) = \operatorname{div}\left(a\,|\nabla u|^{p-2}\nabla u\right) + g(x,t,u) \end{aligned} \tag{1.2}$$

with constant exponents of nonlinearity have been studied by many authors. We refer here to papers [14, 15, 16, 19, 24, 27, 29] for the results concerning the boundary-value problems in cylinder domains and to [17, 18, 23] for a discussion of the Cauchy problem. Stationary solutions of the anisotropic version of equation (1.2) (a) were studied in [2, 5].

Equation (1.1) with variable exponents of nonlinearity m and p_i has been studied up to now under special assumptions on m and p_i . The cases $p_i \equiv 2$ or $m \equiv 1$ were studied in [4, 6, 7, 22]. In the former case (1.1) generalizes the porous medium equation; in the latter case it transforms into the anisotropic evolutional $p(x,t)$-Laplace equation. The questions of existence and uniqueness of solutions of problem (1.1) with double nonlinearity have been studied in [1, 12, 13] in the special cases when either the diffusion was isotropic but m might depend on t, or in the case of anisotropic diffusion and $m \equiv m(x)$. The properties of localization and blow-up of solutions of problem (1.1) with $m = 1$ were studied in [8, 9, 10, 11]. As distinguished from these works, we are specially interested now in the situation when both features are presented in the same equation and do not allow one to directly apply the already known results.

The main result of this work is the theorem of existence of bounded strong solutions of problem (1.1). These solutions belong to a variable-exponent Orlicz–Sobolev space prompted by the equation (see Section 2 for the rigorous definitions) and possess the additional regularity property

$|v|^{m(z)-1}v_t^2 \in L^1(Q_T)$. As a byproduct of the existence theorem we obtain a series of energy estimates for the constructed strong solution. These estimates are interesting in themselves: they provide information on the regularity of the constructed solution and turn out to be very useful in the study of the localization and blow-up properties of solutions to problem (1.1). A study of these properties, based on the energy estimates derived in the present paper, will be given in a forthcoming publication.

To give a rigorous formulation of the results, we have to introduce the function spaces the solutions of problem (1.1) belong to.

2. The function spaces

In this section we collect some known facts from the theory of Sobolev spaces with variable exponent. A rigorous and detailed exposition of this theory, as well as an exhaustive review of the existing bibliographic sources, can be found in the monograph [21].

2.1. **Orlicz–Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$: definitions and basic properties.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$. Assume that $p(x)$ is continuous with the logarithmic module of continuity: $\forall\, z,\zeta \in \Omega$, $|z-\zeta|<1$,

$$\sum_{i=1}^{n} |p_i(z)-p_i(\zeta)| \le \omega(|z-\zeta|), \qquad \overline{\lim_{\tau\to 0^+}}\, \omega(\tau)\ln\frac{1}{\tau} = C < +\infty. \tag{2.1}$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm (the Luxemburg norm)

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : A_{p(\cdot)}(f/\lambda) \le 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,p(\cdot)}(\Omega)$ with $p(x) \in [p^-,p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega}. \end{cases} \tag{2.2}$$

Let us indicate the basic properties of the spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ used in the rest of this paper.

• The space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive, provided that $p(x) \in C^0(\overline{\Omega})$.

• If condition (2.1) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$, which can be defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm (2.2). The density of smooth functions in the space $W_0^{1,p(\cdot)}(\Omega)$ is crucial for further proceeding. The condition of log-continuity of $p(x)$ is the best known and the most frequently used sufficient condition for the density of C_0^∞ in $W_0^{1,p(x)}(\Omega)$; see [21, 26, 33]. Although this condition is not necessary and can be replaced by other conditions (see [21, Chapter 9] for a discussion of this question) we keep it throughout the paper for the sake of simplicity of presentation.

• It follows directly from the definition of the norm that

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \le A_{p(\cdot)}(f) \le \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right). \tag{2.3}$$

• Hölder's inequality. For all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1,\infty), \quad p'(x) = \frac{p(x)}{p(x)-1}, \quad p^- = \inf_\Omega p(x), \quad (p')^- = \inf_\Omega p'(x),$$

the following inequality holds:

$$\int_\Omega |f\,g|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)} \le 2\,\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)}. \tag{2.4}$$

2.2. **Spaces $L^{p(\cdot,\cdot)}(Q_T)$ and anisotropic space $\mathbf{W}(Q_T)$.** Let $m(z) > 0$ and $p_i(z) > 1$, $i = 1,\dots,n$, be given functions. We assume that $m(z), p_i(z) \in C^0(\overline{Q}_T)$ and that the $p_i(z)$ satisfy the log-continuity condition in the cylinder Q_T:

$$\begin{gathered}\forall\, z,\zeta \in Q_T,\ z = (x,t),\ \zeta = (y,\tau),\ |z-\zeta|^2 = |x-y|^2 + (t-\tau)^2 < 1,\\ \sum_{i=1}^n |p_i(z) - p_i(\zeta)| \le \omega(|z-\zeta|), \qquad \overline{\lim_{\tau\to 0^+}}\,\omega(\tau)\ln\frac{1}{\tau} = C.\end{gathered} \tag{2.5}$$

Throughout the rest of the text we use the notation

$$\begin{aligned} &p_i^+(t) = \sup_\Omega p_i(x,t), && p_i^-(t) = \inf_\Omega p_i(x,t),\\ &p_i^+ = \sup_{Q_T} p_i(z), && p_i^- = \inf_{Q_T} p_i(z),\\ &p^+ = \max\{p_i^+,\ i = 1,\dots,n\}, && p^- = \min\{p_i^-,\ i = 1,\dots,n\},\\ &m^+ = \sup_{Q_T} m(z), && m^- = \inf_{Q_T} m(z), \end{aligned}$$

$$m^{+}(t)=\sup_{\Omega} m(x,t), \qquad m^{-}(t)=\inf_{\Omega} m(x,t).$$

For every fixed $t\in[0,T]$ we introduce the Banach space

$$\begin{gathered}\mathbf{V}_t(\Omega)=\Big\{u(x)\in L^{m(\cdot,t)+1}(\Omega)\cap W_0^{1,1}(\Omega):|D_iu(x)|^{p_i(x,t)}\in L^1(\Omega)\Big\},\\ \|u\|_{\mathbf{V}_t(\Omega)}=\|u\|_{m(\cdot,t)+1,\Omega}+\sum_i\|D_iu\|_{p_i(\cdot,t),\Omega},\end{gathered}$$

and denote by $\mathbf{V}'_t(\Omega)$ its dual. By $\mathbf{W}(Q_T)$ we denote the Banach space

$$\begin{aligned}\mathbf{W}(Q_T)=\Big\{&|u|^{m(z)+1}\in L^1(Q_T):|D_iu|^{p_i(z)}\in L^1(Q_T),\\ &u(\cdot,t)\in\mathbf{V}_t(\Omega)\text{ for a.e. }t\in(0,T)\Big\},\\ \|u\|_{\mathbf{W}(Q_T)}=&\sum_i\|D_iu\|_{p_i(\cdot,\cdot),Q_T}+\|u\|_{m(\cdot,\cdot)+1,Q_T}.\end{aligned}$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$w\in\mathbf{W}'(Q_T)\iff$$

$$\left\{\begin{array}{l}\exists\, w=(w_0,w_1,\ldots,w_n),\quad w_0\in L^{(m(\cdot)+1)'}(Q_T),\quad w_i\in L^{p'_i(\cdot)}(Q_T),\\ \forall\phi\in\mathbf{W}(Q_T)\quad\langle\langle w,\phi\rangle\rangle=\displaystyle\int_{Q_T}\Big(w_0\phi+\sum_i w_iD_i\phi\Big)\,dz.\end{array}\right.$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)}=\sup\left\{\langle\langle v,\phi\rangle\rangle|\,\phi\in\mathbf{W}(Q_T),\ \|\phi\|_{\mathbf{W}(Q_T)}\le 1\right\}.$$

Let

$$\mathbf{V}_+(\Omega)=\Big\{u(x):u\in L^{m^++1}(\Omega)\cap W_0^{1,1}(\Omega),\ |\nabla u|\in L^{p^+}(\Omega)\Big\}.$$

Since $\mathbf{V}_+(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\psi_k\}\subset\mathbf{V}_+(\Omega)$.

Proposition 2.1 ([7]). *Let $p_i(z)$ satisfy condition* (2.1) *in Q_T. Then the set $\{\psi_k\}$ is dense in $\mathbf{V}_t(\Omega)$ for every $t\in[0,T]$.*

Proposition 2.2 ([7]). *For every $u\in\mathbf{W}(Q_T)$ there is a sequence $\{d_k(t)\}$, $d_k(t)\in C^1[0,T]$, such that $\sum_{k=1}^{s}d_k(t)\psi_k(x)\to u$ in the norm of $\mathbf{W}(Q_T)$.*

We will also use the embedding theorem for the functions in the anisotropic Sobolev spaces.

Lemma 2.1 ([30]). *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain, $\Omega = \{x \in \mathbb{R}^n : x_i \in (0,\lambda)\}$, $\lambda > 0$. Assume that $p_i = const > 1$. Then*

$$\|u\|_{r^*,\Omega} \le C(a,n)\Big(\prod_{i=1}^{n} \|D_i u\|_{p_i,\Omega}\Big)^{\frac{1}{n}}$$

with

$$r^* = \begin{cases} \dfrac{np}{n-p} & \text{if } p < n, \\ \text{any number from } [1,\infty) & \text{if } p \ge n, \end{cases} \qquad \frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}.$$

For the functions with zero traces on $\partial\Omega$ the condition of rectangularity of Ω can be omitted because such functions can be continued by zero to a rectangular domain containing Ω. For the functions defined in a cylinder Q_T, we will apply this lemma for fixed t with the exponents $p_i^-(t)$ and the corresponding critical exponent $r^*(t)$.

Throughout the rest of the paper we use the convention of denoting by C generic constants which can be explicitly calculated through the known parameters, but whose exact values are unimportant.

3. Assumptions and main result

It is convenient to introduce the functions

$$\Phi(v,m) = m(z)\int_0^v |s|^{m(z)-1}\,ds \equiv |v|^{m(z)}\,\text{sign}\ v,$$

$$f(z,v) = b(z)|v|^{\sigma(z)-2}v + g(z)$$

and to write problem (1.1) in the form

$$\begin{cases} \partial_t\Phi(v,m) = \displaystyle\sum_{i=1}^{n} D_i\left(a_i(z)|D_i v|^{p_i(z)-2}D_i v\right) + f(z,v) \quad \text{in } Q_T, \\ v(x,0) = v_0(x) \text{ in } \Omega, \\ v = 0 \text{ on } \Gamma_T. \end{cases} \tag{3.1}$$

The solution of problem (3.1) is understood in the following way.

Definition 3.1. *A function $v(z)$ is called a* strong solution *of problem* (3.1) *if*

(1) $v \in L^\infty(Q_T) \cap \mathbf{W}(Q_T)$, $\Phi'_v(v,m)v_t^2 \in L^1(Q_T)$, *and* $|D_i v|^{p_i(z)} \in L^\infty(0,T;L^1(\Omega))$;

(2) *for every test-function* $\phi \in \mathbf{W}(Q_T)$

$$\int_{Q_T} \left[\phi \frac{d}{dt}\Phi(v,m) + \sum_{i=1}^{n} a_i |D_i v|^{p_i(z)-2} D_i v \cdot D_i \phi\right] dz = \int_{Q_T} f(z,v)\phi\, dz; \quad (3.2)$$

(3) *for every* $\psi(x) \in C_0^\infty(\Omega)$

$$\int_\Omega \psi(x)(\Phi(v(z),m(z)) - \Phi(v_0,m_0))\, dx \to 0 \quad \text{as } t \to 0,\ m_0 = m(x,0). \quad (3.3)$$

Let us assume that the coefficients of equation (3.1) and the exponents of nonlinearity are subject to the following conditions:

$$\begin{cases} m(z), \sigma(z), p_i(z) \in C^0(\overline{Q}_T), \\ p_i(z) \text{ satisfy the log-continuity condition (2.5)}, \\ \text{there exist finite positive constants } a^\pm,\ b^\pm\ p^\pm,\ m^\pm, \text{ and } \sigma^\pm \text{ such that} \\ 0 < a^- \le a_i(z) \le a^+, \quad -b^- \le b(z) \le b^+, \\ 1 < p^- \le p_i(z) \le p^+, \quad 1 < \sigma^- \le \sigma(z) \le \sigma^+, \quad 0 < m^- \le m(z) \le m^+, \end{cases} \quad (3.4)$$

$$m_t \in L^\infty(Q_T), \quad \exists\, \mu = const > 0 : 0 \le m_t \le \mu \text{ a.e. in } Q_T, \quad (3.5)$$

$$\begin{cases} 0 \le -p_{it}(z) \le C_{pi}, \quad 0 \le\ b(z)\sigma_t(z) \le C_\sigma, \\ 0 \le -a_{it}(z) \le C_a, \quad 0 \le b_t(z) \le C_b, \end{cases} \quad (3.6)$$

with nonnegative constants C_a, C_b, C_σ, and C_{pi}.

The main result of this work is given in the following theorem.

Theorem 3.1. *Let the exponents* $m(z)$, $\sigma(z)$, *and* $p_i(z)$ *satisfy conditions* (3.4), (3.5), *and* (3.6). *Let us assume that there exists a constant* $\alpha > 0$ *such that if* $p^-(t) < n$ *for some* $t \in [0,T]$, *then*

$$m^+(t) + 1 + \alpha < \frac{np^-(t)}{n - p^-(t)}, \qquad \text{where} \quad \frac{1}{p^-(t)} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i^-(t)}. \quad (3.7)$$

Assume that the $m(z)$ *satisfy the oscillation condition*

$$\frac{m^+ - 1}{2} < \min\left\{m^-, \left(1 - \frac{1}{p^-}\right)(m^- + 1)\right\} \text{ if } m^+ > 1 \text{ and } \sum_{i=1}^{n} \frac{1}{p_i^-} > 1. \quad (3.8)$$

Then for every $v_0 \in L^\infty(\Omega) \cap \mathbf{V}_0(\Omega)$ *and every* $g \in L^\infty(Q_T)$ *with* $|g_t(z)|^{1+\frac{1}{m(z)}} \in L^1(Q_T)$ *problem* (3.1) *has at least one strong solution in a cylinder* Q_{T^*},

$$T^* = \sup\{\theta : \|v(t)\|_{\infty,\Omega} < \infty,\ \forall\, t \in (0,\theta)\}.$$

Moreover, for small τ the solution satisfies the estimate

$$\|v(t)\|_{\infty,\Omega} \leq (1 + \|v_0\|_{\infty,\Omega})\, \mathrm{e}^{A\,t}, \quad t \in [0,\tau], \tag{3.9}$$

with a constant A independent of v. If either $\sup_{Q_T}(\sigma(z) - m(z)) \leq 1$, or $b^+ \leq 0$, then estimate (3.9) remains true for all $T > 0$, the constructed solution exists globally in time, and $T^* = T$. In the case $m^- \geq 1$ the solution satisfies the inclusion $\Phi_v(v,m)v_t \in L^2(Q_{T_*})$.

Let us illustrate the assertions of Theorem 3.1 by the example of the model equation with constant coefficients and one space variable:

$$\frac{d}{dt}\left(|v|^{m(z)-1}v\right) = \left(|v_x|^{p(z)-2}v_x\right)_x. \tag{3.10}$$

- $m > 0$ and $p > 1$ are constants. In this simplest case the conditions of the theorem reduce to the conditions on the regularity of the initial datum.
- $m \equiv m(x) > 0$ and $p \equiv p(x) > 1$. A strong solution exists if m and p are log-continuous in Ω.
- $m \equiv m(x) > 0$ and $p \equiv p(x,t) > 1$. It is required that, in addition to the above conditions, $p_t \in [0, C_p]$ with a positive constant C_p.
- $m \equiv m(x,t) > 0$ and $p \equiv p(x,t) > 1$. We have to add to the above conditions the restriction $m_t(x,t) \in [0,\mu]$.
- For $n > 1$ conditions (3.7) and (3.8) confine the assertion of Theorem 3.1 to the equations with slow diffusion. Nonetheless, in this case equation (3.10) still admits weak solutions [12].

The same is true for equation (3.10) with the lower-order term

$$\frac{d}{dt}\left(|v|^{m(z)-1}v\right) = \left(|v_x|^{p(z)-2}v_x\right)_x + b\,|v|^{\sigma(z)-2}v + g(z), \quad b = const,$$

but the results are confined to a small time interval and the possibility of continuation of the solution to the arbitrary interval $(0,T)$ depends on the relations between $m(z)$, $p(z)$, and $\sigma(z)$, and on the signs of $\sigma_t(z)$ and b. The solution exists globally in time if either $\sup_{Q_T}(\sigma(z) - m(z)) < 1$, or $b = const < 0$. Both conditions are well-known for the equations with constant exponents of nonlinearity—see, e.g., [3, 9, 31, 32].

As a byproduct of the existence theorem we derive the energy relations for the constructed solution, which are useful in the study of extinction and blow-up properties of strong solutions. These energy relations are given in the final section in Theorems 9.1 and 9.2.

The paper is organized as follows. In Section 4, we introduce the regularized problems. Regularization of the term $\Phi(v,m)$ allows us to avoid double degeneracy of the equation; the term $f(z,v)$ is regularized in such a way that its growth at infinity is controlled by $\Phi(v,m)$. The solution of the regularized problem is obtained as the limit of the sequence of the finite-dimensional Galerkin's approximations. We show that this sequence exists on an arbitrary time interval. In Section 5, we derive uniform energy estimates for the finite-dimensional approximations, which allow us to prove in Section 6 that the regularized problems have weak solutions on an arbitrary time interval. Section 7 is devoted to the proof of the maximum principle for the constructed solutions. It is shown that the solutions of the regularized problems are uniformly bounded on a small time interval. Using this estimate we choose the regularization parameter in the term f in such a way that it becomes a dummy and can be omitted. The proof of Theorem 3.1 is given in Section 8. Finally, in Section 9, we derive the energy estimates for the constructed solution.

4. Regularized problem

Following [7, 13] we construct a solution of problem (3.1) as the limit of the sequence of solutions of the regularized problems

$$\begin{cases} \dfrac{d}{dt}\Phi_\epsilon(v,m) = \operatorname{div}\mathcal{F}(\nabla v, z) + f_K(v,z) \quad \text{in } Q_T, \\ v(x,0) = v_0(x) \text{ in } \Omega, \quad v=0 \text{ on } \Gamma_T, \end{cases} \tag{4.1}$$

where $\mathcal{F}(\nabla v, z) = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$, $\mathcal{F}_i(\nabla v, z) = a_i(z)|D_i v|^{p_i(z)-2}D_i v$, and

$$\Phi_\epsilon(v,m) = \int_0^{v(z)} A_\epsilon(s, m(z))\,ds, \quad A_\epsilon(s,m) = \epsilon(1 + l|s|^{l-1}) + m(z)|s|^{m(z)-1},$$

with the constant exponent $l = \max\{1, m^+\}$ and a given parameter $\epsilon > 0$. For every $K > 1$ the function f_K is defined by the equality

$$f_K(w,z) = g(z) + b(z) \times \begin{cases} (\min\{|w|, K\})^{\sigma(z)-m(z)-1}\ |w|^{m(z)-1}w & \text{if } \sigma(z) \geq m(z)+1, \\ |w|^{\sigma(z)-2}w & \text{otherwise.} \end{cases} \tag{4.2}$$

Theorem 4.1. *Let $v_0 \in L^{l+1}(\Omega) \cap \mathbf{V}_0(\Omega)$. For every $\epsilon > 0$ and $K > 1$ problem* (4.1) *has at least one solution v such that*

(1) $v \in \mathbf{W}(Q_T) \cap L^{l+1}(Q_T)$,
(2) $\dfrac{d}{dt}\Phi_\epsilon(v,m) \in L^{1+\frac{1}{l}}(Q_T) \cap \mathbf{W}'(Q_T)$,

(3) *for every test-function* $\phi \in L^{l+1}(Q_T) \cap \mathbf{W}(Q_T)$

$$\int_{Q_T} \left[\phi \, \frac{d}{dt} \Phi_\epsilon(v,m) + \mathcal{F}(\nabla v, z) \cdot \nabla \phi\right] dz = \int_{Q_T} f_K(v,z)\, \phi \, dz, \tag{4.3}$$

(4) *for every* $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega \phi(x)(\Phi_\epsilon(v,m) - \Phi_\epsilon(v_0(x), m_0(x)))\, dx \to 0 \qquad \textit{as } t \to 0, \quad m_0 = m(x,0).$$

4.1. **Galerkin's approximations.** Let us take a linearly independent system $\{\psi_i\}_{i=1}^\infty$ which forms the orthogonal basis of $L^2(\Omega)$ and is dense in $\mathbf{W}(Q_T)$. A solution of the regularized problem (4.1) is constructed as the limit of the sequence of finite-dimensional Galerkin's approximations

$$v^{(k)} = \sum_{i=1}^k c_i(t)\psi_i(x) \in \mathcal{P}_k \equiv \operatorname{span}\{\psi_1, \ldots, \psi_k\}.$$

The coefficients $c_i(t)$ are defined from the system of equations

$$\left(\frac{d}{dt}\Phi_\epsilon(v^{(k)}, m), \psi_j\right)_{2,\Omega} + \left(\mathcal{F}(\nabla v^{(k)}, z), \nabla \psi_j\right)_{2,\Omega} = \left(f_K(v^{(k)}, z), \psi_j\right)_{2,\Omega}, \tag{4.4}$$

$j = \overline{1,k}$. The equality

$$\frac{d}{dt}\Phi_\epsilon(v,m) = (\Phi_\epsilon)'_v v_t + (\Phi_\epsilon)'_m m_t = A_\epsilon(v,m)\, \partial_t v + \partial_t m(z)\, v\, |v|^{m-1} \ln |v| \tag{4.5}$$

allows one to rewrite system (4.4) in the equivalent form

$$\sum_{i=1}^k B_{ij}(\vec{c}(t), t) c_i'(t) = F_j(\vec{c}(t)), \quad c_j(0) = (v_0(x), \psi_j(x))_{2,\Omega}, \quad j = \overline{1,k}, \tag{4.6}$$

with continuous functions

$$\begin{aligned} F_j = &- \Big(\mathcal{F}(\nabla v^{(k)}, z),\, \nabla \psi_j\Big)_{2,\Omega} \\ &+ \Big(f_K(v^{(k)}, z) - \partial_t m(z)\, v^{(k)}\, |v^{(k)}|^{m(z)-1} \ln |v^{(k)}|, \psi_j\Big)_{2,\Omega}. \end{aligned} \tag{4.7}$$

The entries of the matrix B have the form

$$B_{ij}(\vec{c}(t), t) = \Big(A_\epsilon(v^{(k)}, m)\, \psi_i(x),\, \psi_j(x)\Big)_{2,\Omega}, \quad i,j = \overline{1,k}.$$

The determinant of B is the Gram determinant of the linearly independent system $\left\{A_\epsilon^{\frac{1}{2}}(v^{(k)}, m)\, \psi_i(x)\right\}_{i=1}^k$. Since $A_\epsilon \geq \epsilon > 0$, one can solve system (4.6) with respect to the derivatives $c_j'(t)$ and write the system in the normal form.

By Peano's theorem, for every $k \in \mathbb{N}$ there exists at least one solution of system (4.6) on an interval $(0, T_k)$.

4.2. **A priori estimates.** Let us introduce the function

$$\Psi_\epsilon(v,m) = \int_0^v s\, A_\epsilon(s,m)\, ds = \frac{\epsilon}{2} v^2 + \frac{\epsilon\, l}{l+1} |v|^{l+1} + \frac{m}{m+1} |v|^{m+1}$$

and notice that

$$v \frac{d}{dt} \Phi_\epsilon(v,m) = \frac{d}{dt} \Psi_\epsilon(v,m) - \frac{\partial_t m}{(m+1)^2}\, [1-(m+1)\, \ln|v|]\, |v|^{m+1}.$$

Lemma 4.1. *Let* $\Psi_\epsilon(v_0^{(k)}, m_0) \in L^1(\Omega)$ *and* $g \in L^{(m(\cdot)+1)'}(Q_T)$. *Assume that* $0 \le \partial_t m \le \mu$ *almost everywhere in* Q_T *with a positive constant* μ. *Then each of the functions* $v^{(k)}$ *can be continued from the cylinder* Q_{T_k} *to* Q_T. *The continued functions satisfy the estimates*

$$\sup_{(0,T)} \int_\Omega \Psi_\epsilon(v^{(k)}, m)\, dx + a^- \sum_{i=1}^n \int_{Q_T} |D_i v^{(k)}|^{p_i(z)}\, dz \le M + \int_\Omega \Psi_\epsilon(v_0^{(k)}, m_0)\, dx \tag{4.8}$$

with a constant $M = M(\mu, m^\pm, \sigma^\pm, b^+, K, T)$ *that is independent of* k *and* ϵ.

Proof. For the sake of simplicity of notation, we denote by v the finite-dimensional Galerkin's approximation $v^{(k)}$ of the solution of problem (4.1) and drop the subindex ϵ. Multiplying each of the equations (4.4) by $c_j(t)$, integrating over the interval $(0,\tau) \subset (0, T_k)$, summing up in $j = \overline{1,k}$, and using the definition of f_K, we obtain the inequality

$$\int_\Omega \Psi(v,m)\, dx \Big|_{t=0}^{t=\tau} + a^- \sum_{i=1}^n \int_{Q_\tau} |D_i v|^{p_i(z)}\, dz \tag{4.9}$$

$$\le j_0(\tau) + \int_{Q_\tau} \frac{\partial_t m}{(m+1)^2} [1-(m+1)\ln|v|] |v|^{m+1}\, dz + \int_{Q_\tau} g v\, dz \equiv \sum_{i=0}^3 j_i(\tau),$$

where $j_0(\tau) = j_0^{(1)}(\tau) + j_0^{(2)}(\tau)$,

$$j_0^{(1)}(\tau) = \int_{Q_\tau \cap \{\sigma \ge m+1\}} b(z)\, (\min\{|v|, K\})^{\sigma-m-1}\, |v|^{m+1}\, dz,$$

$$j_0^{(2)}(\tau) = \int_{Q_\tau \cap \{\sigma < m+1\}} b(z) |v|^\sigma\, dz, \qquad j_1(\tau) = \int_{Q_\tau} \frac{\partial_t m}{(m+1)^2}\, |v|^{m+1}\, dz,$$

$$j_2(\tau) = -\int_{Q_\tau} \frac{\partial_t m}{m+1}\, \ln|v|\, |v|^{m+1}\, dz, \qquad j_3(\tau) = \int_{Q_\tau} g v\, dz.$$

Let us introduce the function

$$Y(\tau) = \int_{Q_\tau} |v|^{m+1} dz.$$

By the definition $j_0^{(1)}(\tau) \leq b^+ K^{\sigma^+ - m^- - 1} Y(\tau)$, and, by Young's inequality, $j_0^{(2)}(\tau) \leq C\, b^+ \left(1 + Y(\tau)\right)$. The estimate on $j_1(\tau)$ is immediate: $j_1(\tau) \leq \mu Y(\tau)$. To estimate $j_2(\tau)$ we write it in the form

$$\begin{aligned} j_2(\tau) = &-\int_{Q_\tau \cap \{|v|>1\}} \frac{\partial_t m}{1+m} \left(|v|^{m+1} \ln |v|\right) dz \\ &- \int_{Q_\tau \cap \{|v|\leq 1\}} \left(|v|^{\beta} \ln |v|\right) \frac{\partial_t m}{1+m} |v|^{m+1-\beta} dz \end{aligned}$$

with a constant $\beta \in (0, m^- + 1)$. Dropping the first nonpositive term and estimating the second one by Young's inequality, we arrive at the estimate $j_2(\tau) \leq C\ (1 + Y(\tau))$ with a constant $C = C(\mu, m^\pm)$. By Young's inequality

$$j_3(\tau) \leq C\Big(\int_{Q_\tau} |g|^{(m+1)'} dz + Y(\tau)\Big).$$

Plugging these estimates to (4.9) and dropping the nonpositive terms on the right-hand side, we obtain Gronwall's inequality for $Y(\tau)$,

$$Y'(\tau) \leq C(1 + Y(\tau)), \quad Y(0) = 0.$$

It follows that $Y(t) \leq \mathrm{e}^{Ct} - 1$ with a constant C independent of v, and that (4.8) is fulfilled with a constant $M = M(\mu, m^\pm, \sigma^\pm, b^+, K, T)$ independent of k .

The possibility to continue each $v \equiv v^{(k)}$ to the same interval $[0, T]$ follows from (4.8) because the function $v(x, T_k)$ possesses the same properties as $v_0^{(k)}$. □

Corollary 4.1. *The sequence $\{v^{(k)}\}$ contains a subsequence (we assume that it coincides with the whole sequence) such that*

$$\begin{cases} v^{(k)} \to v & \text{weakly in } L^{l+1}(Q_T) \cap \mathbf{W}(Q_T), \\ \Psi_\epsilon(v^{(k)}, m) \to \psi & \star\text{-weakly in } L^\infty(0,T; L^1(\Omega)), \\ \mathcal{F}_i(\nabla v^{(k)}, z) \to A_i & \text{weakly in } L^{p_i'(\cdot)}(Q_T). \end{cases} \tag{4.10}$$

Lemma 4.2. *The sequence $\left\{\frac{d}{dt}\Phi_\epsilon(v^{(k)}, m)\right\}_{k=1}^{\infty}$ is uniformly bounded with respect to k and ϵ in $L^{1+\frac{1}{l}}(Q_T) \cap \mathbf{W}'(Q_T)$.*

Proof. It suffices to show that

$$\Big|\Big(\phi,\, \frac{d}{dt}\Phi_\epsilon(v^{(k)},m)\Big)_{2,Q_T}\Big| \le C \qquad \forall\, \phi \in L^{l+1}(Q_T)\cap \mathbf{W}(Q_T)$$

with $\|\phi\|_{l+1,Q_T} + \|\phi\|_{\mathbf{W}(Q_T)} \le 1$ and a constant C independent of k, ϵ, and ϕ. By (4.4) and (4.8)

$$\Big|(\phi,\, \frac{d}{dt}\Phi_\epsilon(v^{(k)},m))_{2,Q_T}\Big| \le a^+\|v\|_{\mathbf{W}(Q_T)}\|\phi\|_{\mathbf{W}(Q_T)} + \|\phi\|_{m+1,Q_T}\|f_K\|_{\frac{m+1}{m},Q_T},$$

whence the assertion of the lemma. □

Corollary 4.2. *The sequence* $\big\{\frac{d}{dt}\Phi_\epsilon(v^{(k)},m)\big\}_{k=1}^{\infty}$ *contains a subsequence that converges weakly in* $L^{1+\frac{1}{l}}(Q_T)\cap \mathbf{W}'(Q_T)$.

5. The energy inequality

Recall that wherever it doesn't cause confusion $v \equiv v^{(k)}$ stands for the kth Galerkin's approximation of the solution of problem (4.1) with positive ϵ and $K>1$. Multiplying each of equations (4.4) by $c_j'(t)$ and summing up in $j=\overline{1,k}$ we obtain the equality

$$\begin{aligned}\int_\Omega A_\epsilon(v,m)v_t^2\,dx &+ \int_\Omega \mathcal{F}(\nabla v, z)\cdot\nabla v_t\,dx\\ &= \int_\Omega f_K(v,z)\,v_t\,dx - \int_\Omega m_t v|v|^{m-1}\ln|v|v_t\,dx. \qquad (5.1)\end{aligned}$$

Straightforward computations lead to the formulas

$$\begin{aligned}a_i|D_iv|^{p_i-2}D_iv\cdot D_iv_t &= \partial_t\Big(a_i\frac{|D_iv^{(k)}|^{p_i}}{p_i}\Big)\\ &\quad + a_i|D_iv|^{p_i}\Big(\frac{1}{p_i^2}-\frac{\ln|D_iv|}{p_i}\Big)p_{it} - a_{it}\frac{|D_iv|^{p_i}}{p_i},\\ f_K(v,z)\,v_t &= \partial_t(b(z)\,S) + R + g\,v_t\end{aligned}$$

with

$$S=\begin{cases}K^{\sigma-m-1}\dfrac{|v|^{m+1}}{m+1} & \text{if } v^2\ge K^2 \text{ and } \sigma\ge m+1,\\ \dfrac{|v|^\sigma}{\sigma} & \text{if either } v^2<K^2 \text{ and } \sigma\ge m+1, \text{ or if } \sigma<m+1,\end{cases}$$

$$R = \begin{cases} K^{\sigma-m-1}\dfrac{-b_t}{m+1}|v|^{m+1} - \sigma_t \dfrac{b|v|^{m+1}}{m+1}K^{\sigma-m-1}\ln K \\ \qquad\qquad\qquad\qquad \text{if } v^2 \geq K^2 \text{ and } \sigma \geq m+1, \\ -\dfrac{b_t}{\sigma}|v|^{\sigma} + \dfrac{b}{\sigma^2}(1-\sigma \ln |v|)|v|^{\sigma}\sigma_t \\ \qquad\qquad\qquad\qquad \text{if } v^2 < K^2 \text{ and } \sigma \geq m+1, \text{ or if } \sigma < m+1. \end{cases}$$

Let us introduce the energy function

$$\mathcal{E}(t,v) = \int_\Omega \Big(\sum_{i=1}^n a_i \frac{|D_i v|^{p_i}}{p_i} - b\,S\Big)\,dx. \tag{5.2}$$

Combining the above formulas with (5.1) we obtain the equality

$$\frac{d}{dt}\mathcal{E}(t,v) + \int_\Omega A_\varepsilon(v,m)\,v_t^2\,dx = \Lambda(t) + \int_\Omega R\,dx - \int_\Omega P\,dx + \int_\Omega g\,v_t\,dx \tag{5.3}$$

in which

$$P(z) = m_t v|v|^{m-1}\ln|v|v_t$$

$$\Lambda(t) = \int_\Omega \sum_{i=1}^n \Big(- a_i|D_i v|^{p_i}\Big(\frac{1}{p_i^2} - \frac{\ln|D_i v|}{p_i}\Big)p_{it} + a_{it}\frac{|D_i v|^{p_i}}{p_i}\Big)\,dx.$$

Lemma 5.1. *If condition* (3.7) *is fulfilled, then*

$$-\int_\Omega P\,dx \leq \frac{m_t^+(t)}{2\mu}\int_\Omega A_\epsilon(v,m)v_t^2\,dx + C\,m_t^+(t)\Big(1 + \sum_{i=1}^n \int_\Omega |D_i v|^{p_i}\,dx\Big) \tag{5.4}$$

with a constant C independent of v.

Proof. Let us fix $t \in [0,T]$. If $m_t^+(t) = 0$, then $P = 0$ and the assertion is obvious. Assume that $m_t^+(t) > 0$. By the Cauchy inequality

$$-\int_\Omega P\,dx \leq \frac{m_t^+(t)}{2\mu}\int_\Omega A_\epsilon(v,m)v_t^2\,dx + \frac{m_t^+(t)}{2\mu}\int_\Omega \frac{|v|^{2m}}{A_{\epsilon,m}}|\ln|v||^2\,dx$$
$$\equiv \frac{m_t^+(t)}{2\mu}\,(I_1 + I_2)\,.$$

According to (3.7) there is a constant $\alpha \in (0,1)$ such that $m^+(t) + \alpha + 1 \leq np^-(t)/(n - p^-(t))$. Since $s^\alpha\,|\ln s|^2 \to 0$ as $s \to 0+$ and $s^{-\alpha}\,|\ln s|^2 \to 0$ as $s \to \infty$,

$$I_2 \leq \int_{\Omega\cap\{|v|>e\}} |v|^{m+1+\alpha}\left(|v|^{-\alpha}|\ln|v||^2\right)dx + \int_{\Omega\cap\{|v|\leq e\}} |v|^{m+1-\alpha}\left(|v|^{\alpha}|\ln|v||^2\right)dx$$

$$\le C\Big(1 + \|v\|^{m^+(t)+\alpha+1}_{m^+(t)+\alpha+1,\Omega}\Big).$$

Let us make use of the known multiplicative inequality

$$\|v(t)\|_{\lambda(t),\Omega} \le C\,\|v\|^{1-s}_{m^-(t)+1,\Omega}\|\nabla v\|^{s}_{p^-(t),\Omega}, \tag{5.5}$$

in which the constant C is independent of v and the exponents have to be chosen in the following way:

$$s = \Big(\frac{1}{m^-(t)+1} - \frac{1}{p^-(t)}\Big)\Big(\frac{1}{m^-(t)+1} - \frac{1}{p^*_-(t)}\Big)^{-1} \in (0,1), \quad \frac{1}{p^*_-(t)} = \frac{1}{p^-(t)} - \frac{1}{n},$$

$$\lambda(t) \in [m^-(t)+1, p^*_-(t)] \text{ if } p^-(t) < n,$$

$$\lambda(t) \text{ is any number from } [m^-(t)+1, \infty) \text{ if } p^-(t) \ge n.$$

Taking in (5.5) $\lambda(t) = m^+(t)+\alpha+1$, and applying estimate (4.8), Young's inequality, and then (2.4), we obtain

$$\|v(t)\|^{m^+(t)+1+\alpha}_{m^+(t)+1+\alpha,\Omega} \le C\Big(\int_\Omega |\nabla v|^{p^-}\,dx\Big)^{\frac{s}{p^-(t)}(m^+(t)+1+\alpha)}$$

$$\le C'\Big(1 + \int_\Omega |\nabla v|^{p^-}\,dx\Big) \le C''\Big(1 + \sum_{i=1}^n \int_\Omega |D_i v|^{p_i}\,dx\Big). \qquad \square$$

Lemma 5.2. *Let the exponents σ, p_i, and m and the coefficients a_i and b satisfy conditions* (3.4)–(3.6) *with nonnegative constants C_a, C_b, C_σ, and C_{pi}. If the conditions of Lemma* 5.1 *are fulfilled, then*

$$\mathcal{E}(t,v) + \frac{1}{2}\int_0^t \int_\Omega A_\epsilon(v,m)\, v_t^2\,dx\,d\tau \le \mathcal{E}(0, v_0^{(k)}) + C^*(1+t) \tag{5.6}$$

with a constant C^* depending on a^+, $p_i^\pm$, $\sigma^\pm$, $m^\pm$, μ, the constants in (3.4), (3.5), *and* (3.6), *the constant M in* (4.8), *and $\|v_0\|_{m_0(\cdot)+1,\Omega}$.*

Proof. Under the assumptions of the lemma,

$$\Lambda(t) \le \sum_{i=1}^n \int_\Omega a_i |D_i v|^{p_i}\Big(\frac{1}{p_i^2} - \frac{\ln|D_i v|}{p_i}\Big)(-p_{it})\,dx.$$

Using (5.4) we rewrite (5.3) in the form

$$\frac{d}{dt}\mathcal{E}(t,v) + \frac{1}{2}\int_\Omega A_\varepsilon(v,m) v_t^2\,dx \le \sum_{i=1}^n J_{p_i} + \int_\Omega R\,dx + \int_\Omega v_t\, g\,dx + C\, m_t^+(t)\Big(1 + \sum_{i=1}^n \int_\Omega |D_i v|^{p_i}\,dx\Big) \tag{5.7}$$

with

$$J_{p_i} = \int_\Omega \frac{a_i |D_i v|^{p_i}}{p_i^2} \left(1 - p_i \ln |D_i v|\right) (-p_{it})\, dx$$

and the constant C from the conditions of Lemma 5.1. Let us introduce the function

$$G(s) = \frac{s^p}{p^2} \left(1 - p \ln\, s\right), \quad s \in I \equiv (0, \mathrm{e}^{1/p}].$$

It is easy to see that $G(0) = G\left(\mathrm{e}^{1/p}\right) = 0$ and $G'(s) = -s^{p-1} \ln\, s$, whence $\max_I G(s) = G(1) = p^{-2}$. Using this observation we have

$$\begin{aligned} J_{p_i} &= \Big(\int_{\Omega \cap (1 \le p_i \ln |D_i v|)} + \int_{\Omega \cap (p_i \ln |D_i v| < 1)} \Big) \frac{a_i |D_i v|^{p_i}}{p_i^2} (1 - p_i \ln |D_i v|)(-p_{it})\, dx \\ &\le \int_{\Omega \cap (p_i \ln |D_i v| < 1)} |p_{it}| \frac{a_i |D_i v|^{p_i}}{p_i^2} \left(1 - p_i \ln |D_i v|\right)\, dx \le \frac{a^+ C_p}{(p^-)^2} \, |\Omega| \,. \end{aligned}$$

Applying (3.6), we estimate R by a constant depending only on $a^\pm$, $p_i^\pm$, $\sigma^\pm$, $m^\pm$, μ, $|\Omega|$, and the constants in (3.6). Integrating (5.7) with respect to t and then applying (4.8) and Young's inequality we obtain

$$\begin{aligned} &\int_0^t \int_\Omega g\, v_t\, dz = \int_\Omega g\, v\, dx \Big|_{\tau=0}^{\tau=t} - \int_0^t \int_\Omega v\, g_t\, dx \\ &\le C \Big(1 + 2 \sup_{(0,T)} \Psi_\epsilon(v,m) + \sup_{(0,T)} \|g\|_{1+\frac{1}{m},\Omega} + \int_{Q_T} |v|^{m+1} dz + \int_{Q_T} |g_t|^{1+\frac{1}{m}}\, dz \Big) \\ &\le C'. \end{aligned}$$

Finally,

$$C \int_0^t m_t^+(t) \Big(1 + \sum_{i=1}^n \int_\Omega |D_i v|^{p_i}\, dx \Big)\, dt \le C\, \mu \Big(t + \sum_{i=1}^n \int_{Q_T} |D_i v|^{p_i}\, dz \Big),$$

and the required estimate follows now from (4.8). □

Corollary 5.1. (1) *If in the conditions of Lemma* 5.2 $m_t = 0$ *and* $g \equiv 0$, *then* (5.6) *takes on the form*

$$\mathcal{E}(t, v^{(k)}) + \frac{1}{2} \int_0^t \int_\Omega A_\varepsilon(v^{(k)}, m) \left(\partial_t v^{(k)} \right)^2 \, dx\, d\tau \le \mathcal{E}(0, v_0^{(k)}) + C^*\, t. \quad (5.8)$$

(2) *If in the conditions of Lemma* 5.2 $g \equiv 0$, *and if the exponents* p_i, σ, *and* m *and the coefficients* a_i *and* b *are independent of* t, *then* (5.8) *holds with* $C^* = 0$ *and transforms into equality.*

(3) *The constant* C^* *in* (5.6) *and* (5.8) *tends to zero when the constants in* (3.5) *and* (3.6) *tend to zero.*

Corollary 5.2. *Under the conditions of Lemma* 5.2,

$$\sup_{(0,T)} \int_\Omega \sum_{i=1}^n |D_i v^{(k)}|^{p_i(z)}\,dx + \int_{Q_T} A_\epsilon(v,m)(v_t^{(k)})^2\,dz \le M_1 \tag{5.9}$$

with a constant M_1 *independent of* k*. The constant* M_1 *may depend on* K *and the constants in conditions* (3.4), (3.5), *and* (3.6).

Proof. In view of (5.8), it suffices to prove that $\int_\Omega b\,S\,dx$ is uniformly bounded in $(0,T)$. The assertion is obvious if $b(z) \le 0$. In case that $b^+ \ge 0$, it is a byproduct of the definition of S and inequality (4.8). □

Lemma 5.3. *Let us assume that condition* (3.8) *is fulfilled. The sequence* $\{v^{(k)}\}$ *contains a subsequence which possesses the properties* (4.10) *and converges almost everywhere in* Q_T.

Proof. Let us consider the sequence $\{w_k\}$ with $w_k = |v^{(k)}|^\alpha \operatorname{sign} v^{(k)}$ and $\alpha = \frac{1}{2}(1+\max\{m^+,1\}) \ge 1$. Let us define $\theta = \max\{m^+,1\} - m(z)$ and claim that

$$\theta \le m^- + 1 \qquad \text{if} \sum_{i=1}^n \frac{1}{p_i^-} > 1.$$

This condition is automatically fulfilled if $m^+ \le 1$; otherwise, it reads $\frac{m^+-1}{2} \le m^-$. Applying Hölder's inequality and using (4.8) and Lemma 2.1 we estimate

$$\int_{Q_T} |w_{k,t}|\,dz \le C\Big(\int_{Q_T} |v^{(k)}|^\theta\,dx\Big)^{\frac{1}{2}}\Big(\int_{Q_T} A_\epsilon(v^{(k)},m)(v_t^{(k)})^2\,dz\Big)^{\frac{1}{2}} \le C''.$$

On the other hand, by virtue of (5.9), for any $r \in (1,p^-)$ and every $t \in (0,T)$

$$\begin{aligned} I_k \equiv \int_\Omega |\nabla w_k|^r\,dx &\le C\Big(\int_\Omega |\nabla v^{(k)}|^{p^-}\,dx\Big)^{\frac{r}{p^-}}\Big(\int_\Omega |v^{(k)}|^\omega\,dz\Big)^{1-\frac{r}{p^-}} \\ &\le C'\Big(\int_\Omega |v^{(k)}|^\omega\,dz\Big)^{1-\frac{r}{p^-}} \quad \text{with } \omega = (\alpha-1)\frac{rp^-}{p^- - r}. \end{aligned}$$

If $\alpha = 1$ (that is, $m^+ \le 1$), we simply take $r = p^-$; in this case the I_k are uniformly bounded. If $\alpha > 1$ (i.e., $m^+ > 1$), we claim that

$$\frac{m^+-1}{2} < \Big(1-\frac{1}{p^-}\Big)(m^-+1) \quad \Rightarrow \quad \frac{m^+-1}{2(m^-+1)} + \frac{1}{p^-} < 1 \qquad \text{if} \sum_{i=1}^n \frac{1}{p_i^-} > 1.$$

This allows us to choose r in such a way that $\omega \leq m^- + 1$ and leads to a uniform estimate on I_k. By [28, Section 5, Theorem 8] the sequence $\{w_k\}$ is compact in $L^\nu(Q_T)$ with some $\nu \geq 1$ and, therefore, contains a subsequence that converges almost everywhere. □

Lemma 5.4. *For every fixed ϵ the sequence $\{\Psi_\epsilon(v^{(k)}, m)\}$ is precompact in $C^0([0,T]; L^1(\Omega))$.*

Proof. According to (4.8) $\|\Psi_\epsilon(v^{(k)}, m)\|_{1,\Omega}(t)$ are uniformly bounded in $(0,T)$. For every $t_1, t_2 \in [0,T]$, $t_1 \leq t_2$,

$$\int_\Omega |\Psi_\epsilon|_{t=t_2} - \Psi_\epsilon|_{t=t_1}|\, dx = \int_\Omega \Big| \int_{t_1}^{t_2} \partial_t \Big(\int_0^v sA_\epsilon(s,m)\, ds \Big)\, dt \Big|\, dx$$
$$\leq 2 \int_{t_1}^{t_2} \int_\Omega \Big(|v|\, |A_\epsilon(v,m)| |v_t| + \Big| \int_0^v s\, (A_\epsilon)'_m(s,m)\, m_t\, ds \Big| \Big)\, dz \equiv 2(I_1 + I_2).$$

By Hölder's inequality and due to (4.8) and (5.9),

$$|I_1| \leq C \Big(\int_{t_1}^{t_2} \int_\Omega v^2 A_\epsilon(v,m)\, dz \Big)^{\frac{1}{2}} \Big(\int_{Q_T} A_\epsilon(v,m)(v_t)^2\, dz \Big)^{\frac{1}{2}} \leq C' \sqrt{t_2 - t_1}.$$

To estimate I_2 we represent it as the sum of the integrals I_2^+ and I_2^- over the sets where $|v| \geq 1$ or $|v| \leq 1$. Recall that

$$\int_0^v s(A_\epsilon)'_m(s,m)\, ds = m \int_0^v s|s|^{m-1} \ln |s|\, ds = \frac{m}{m+1} \Big(\ln |v| - \frac{1}{m+1} \Big) |v|^{m+1}.$$

Following the proof of Lemma 5.1 and applying Young's inequality, Lemma 2.1, and then (5.9), we find that for small $\alpha = \mathit{const} > 0$

$$|I_2^+| \leq C\, \mu \Big(\sup_{s \geq 1} |s|^{-\alpha} \ln s \Big) \int_{t_1}^{t_2} \Big(1 + \int_\Omega |v|^{m^+(t)+\alpha+1}\, dx \Big) dt \leq \mu\, C''\, (t_2 - t_1).$$

On the other hand, by virtue of (4.8),

$$|I_2^-| \leq C\, \mu \Big(\sup_{0<s\leq 1} s^{\frac{1}{2}} |\ln s| \Big) \int_{t_1}^{t_2} \int_\Omega \big(1 + |v|^{m+1} \big)\, dz \leq \mu\, C'\, (t_2 - t_1).$$

The assertion follows now from the Arzelá–Ascoli theorem. □

6. Proof of Theorem 4.1

We may choose a subsequence of $\{v^{(k)}\}$ in such a way that (4.10) is fulfilled and, by virtue of Lemmas 5.3 and 5.4,

$$\begin{aligned}
&\Phi_\epsilon(v^{(k)},m) \to \Phi_\epsilon(v,m) \text{ a.e. in } Q_T,\\
&f_K(v^{(k)},z) \to f_K(v,z) \text{ a.e. in } Q_T,\\
&\Psi_\epsilon(v^{(k)},m) \to \Psi_\epsilon(v,m) \text{ in } C^0([0,T];L^1(\Omega)).
\end{aligned}$$

By construction, for every $k,s\in\mathbb{N}$ and $\phi^{(s)}\in \operatorname{span}\{\psi_1,\dots,\psi_s\}$,

$$\int_{Q_T}\Big[\phi^{(s)}\,\frac{d}{dt}\Phi_\epsilon(v^{(k)},m)+\sum_{i=1}^n \mathcal{F}_i(\nabla v^{(k)},z)\cdot D_i\phi^{(s)}-\phi^{(s)}\,f_K(v^{(k)},z)\Big]\,dz=0. \tag{6.1}$$

Letting in (6.1) $k\to\infty$ and then $s\to\infty$, we find that $\forall\,\phi\in L^{l+1}(Q_T)\cap \mathbf{W}(Q_T)$,

$$\int_{Q_T}\Big[\phi\,\frac{d}{dt}\Phi_\epsilon(v,m)+\sum_{i=1}^n A_i\cdot D_i\phi-\phi\,f_K(v,z)\Big]\,dz=0. \tag{6.2}$$

Let $\phi(x)\in C_0^\infty(\Omega)$. Fix some $t_1,t_2\in(0,T)$ $(t_1<t_2)$ and test (6.2) with the function

$$\zeta(x,t)=\phi(x)\xi_s(t),\quad \xi_s(t)=\begin{cases}0 & \text{for } t\geq t_2\\ s(t_2-t) & \text{for } t\in(t_2-1/s,t_2),\\ 1 & \text{for } t\in(t_1+1/s,t_2-1/s),\\ s(t-t_1) & t\in(t_1,t_1+1/s),\\ 0 & \text{for } t\leq t_1,\end{cases}$$

for $s\in\mathbb{N}$, which gives

$$\begin{aligned}
\int_0^T\Big(\int_\Omega\zeta\frac{d}{dt}\Phi_\epsilon(v,m)\,dx\Big)dt &= \int_0^T\frac{d}{dt}\Big(\int_\Omega\zeta\Phi_\epsilon(v,m)\,dx\Big)\,dt-\int_{Q_T}\zeta_t\Phi_\epsilon(v,m)\,dz\\
&= s\int_{t_2-\frac{1}{s}}^{t_2}\int_\Omega\Phi_\epsilon(v,m)\phi(x)\,dx\,dt - s\int_{t_1}^{t_1+\frac{1}{s}}\int_\Omega\Phi_\epsilon(v,m)\phi(x)\,dx\,dt\\
&= -\int_{t_1}^{t_2}\int_\Omega\Big[\sum_{i=1}^n A_i\cdot D_i\phi(x)-\phi(x)\,f_K(v,z)\Big]\,dz.
\end{aligned}$$

The inclusion $\Phi_\epsilon(v,m)\in L^{1+\frac{1}{l}}(\Omega)$ for almost every $t\in(0,T)$ allows one to pass to the limit as $s\to\infty$ and apply the Lebesgue differentiation theorem:

for almost every $\tau \in (0,T)$,

$$\lim_{s\to\infty} \frac{1}{1/s} \int_{\tau}^{\tau+1/s} \Big(\int_{\Omega} \Phi_\epsilon(v,m)\phi(x)\,dx \Big) = \int_{\Omega} \Phi_\epsilon(v(x,\tau),m(x,\tau))\phi(x)\,dx.$$

It follows that for almost every $t_1, t_2 \in [0,T]$,

$$\int_{\Omega} \Phi_\epsilon(v,m)\phi(x)\,dx \Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \int_{\Omega} \Big[\sum_{i=1}^{n} A_i \cdot D_i\phi - \phi\, f_K(v,z) \Big]\,dz.$$

By the property of absolute continuity of the integral the right-hand side is continuous, which gives the inclusion $\Phi_\epsilon(v,m) \in C^0_w(0,T; L^{1+\frac{1}{l}}(\Omega))$. It remains to check that $A_i = \mathcal{F}_i(\nabla v, z)$. We prove this equality adapting the monotonicity arguments of [25, Chapter 2]. Choosing the limit function v for the test-function in (6.2), we obtain the equality

$$\int_{Q_T} \Big[v\,\frac{d}{dt}\Phi_\epsilon(v,m) + \sum_{i=1}^{n} A_i \cdot D_i v - v\, f_K(v,z) \Big]\,dz = 0. \tag{6.3}$$

We rely on the following well-known inequality: for $\mu > 1$,

$$\forall\, \xi, \zeta \in \mathbb{R} \qquad (|\xi|^{\mu-2}\xi - |\zeta|^{\mu-2}\zeta)(\xi - \zeta) \ge 0.$$

It follows that for every $\zeta \in \mathbf{W}(Q_T)$ and $k \in \mathbb{N}$,

$$X^{(k)} := \int_{Q_T} \Big(\mathcal{F}(\nabla v^{(k)}, z) - \mathcal{F}(\nabla\zeta, z) \Big) \cdot \nabla(v^{(k)} - \zeta)\,dz \ge 0.$$

Following the derivation of estimate (4.8), for every $k = 1, 2, \ldots$ we obtain the equality

$$\begin{aligned} &\int_{\Omega} \Psi_\epsilon(v^{(k)}, m)\,dx \Big|_{t=0}^{t=T} - \int_{Q_T} \frac{\partial_t m}{(m+1)^2} \Big[1 - (m+1)\, \ln|v^{(k)}| \Big]\, |v^{(k)}|^{m+1}\,dz \\ &+ \int_{Q_T} \Big[\mathcal{F}(\nabla v^{(k)}, z) \cdot \nabla v^{(k)} - v^{(k)}\, f_K(v^{(k)}, z) \Big]\,dz = 0. \end{aligned} \tag{6.4}$$

On the other hand,

$$\begin{aligned} \mathcal{F}(\nabla v^{(k)}, z) \cdot \nabla v^{(k)} &= \mathcal{F}(\nabla v^{(k)}, z) \cdot \nabla(v^{(k)} - \zeta) + \mathcal{F}(\nabla v^{(k)}, z) \cdot \nabla\zeta \\ &= \Big(\mathcal{F}(\nabla v^{(k)}, z) - \mathcal{F}(\nabla\zeta, z) \Big) \cdot \nabla(v^{(k)} - \zeta) \\ &\quad + \mathcal{F}(\nabla\zeta, z) \cdot \nabla(v^{(k)} - \zeta) + \mathcal{F}(\nabla v^{(k)}, z) \cdot \nabla\zeta. \end{aligned}$$

Gathering these two equalities, we obtain

$$0 \le X_i^{(k)} = -\int_{\Omega} \Psi_\epsilon(v^{(k)}, m)\,dx \Big|_{t=0}^{t=T} \tag{6.5}$$

$$+\int_{Q_T} v^{(k)} f_K(v^{(k)},z)\,dz + \int_{Q_\tau} \frac{\partial_t m}{(m+1)^2}\left[1-(m+1)\ln|v^{(k)}|\right]|v^{(k)}|^{m+1}\,dz$$
$$-\int_{Q_T} (\mathcal{F}(\nabla v^{(k)},z)-\mathcal{F}(\nabla\zeta,z))\cdot\nabla\zeta\,dz - \int_{Q_T}\mathcal{F}(\nabla\zeta,z)\cdot\nabla v^{(k)}\,dz.$$

By the property of lower semicontinuity of the norm,

$$-\liminf_{k\to\infty}\int_\Omega \Psi_\epsilon(v^{(k)}(x,T),m(x,T))\,dx \le -\int_\Omega \Psi_\epsilon(v(x,T),m(x,T))\,dx,$$

while

$$\liminf_{k\to\infty}\int_\Omega \Psi_\epsilon(v_0^{(k)},m)\,dx = \int_\Omega \Psi_\epsilon(v_0,m(x,0))\,dx$$

because $v_0\in L^{l+1}(\Omega)$. For every fixed $\epsilon>0$

$$v\in L^{l+1}(Q_T)\cap\mathbf{W}(Q_T), \qquad \frac{d}{dt}\Phi_\epsilon(v,m)\in L^{1+\frac{1}{l}}(Q_T)\cap\mathbf{W}'(Q_T),$$

and for almost every $t_1,t_2\in[0,T]$ the integration-by-parts formula holds:

$$\int_{t_1}^{t_2}\int_\Omega v\frac{d}{dt}\Phi_\epsilon(v,m)\,dz \tag{6.6}$$
$$=\int_\Omega \Psi_\epsilon(v,m)\,dx\Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_\Omega \frac{\partial_t m}{(m+1)^2}\left[1-(m+1)\ln|v|\right]|v|^{m+1}\,dz.$$

According to (6.3) $\Psi_\epsilon(v,m)\in C^0([0,T];L^1(\Omega))$ and (6.6) is true for every $t\in[0,T]$. Letting $k\to\infty$ and plugging (6.6) into (6.3) we arrive at the inequality

$$0\le -\int_\Omega \Psi_\epsilon(v(x,T),m(x,T))\,dx + \int_\Omega \Psi_\epsilon(v_0,m_0)\,dx$$
$$-\int_{Q_T}\Big[\sum_i A_i\cdot D_i v - v\,f_K(v,z)\Big]\,dz$$
$$+\int_{Q_T}\frac{\partial_t m}{(m+1)^2}\left[1-(m+1)\ln|v|\right]|v|^{m+1}\,dz + \sum_i\int_{Q_T} A_i\cdot D_i(v-\zeta)\,dz$$
$$-\int_Q \mathcal{F}(\nabla\zeta,z)\cdot\nabla(v-\zeta)\,dz = \sum_i\int_{Q_T}(A_i-\mathcal{F}_i(\nabla\zeta,z))\,D_i(v-\zeta)\,dz.$$

We may now take $\zeta=v\pm\lambda\eta$ with arbitrary $\lambda>0$ and $\eta\in L^{l+1}(Q_T)\cap\mathbf{W}(Q_T)$. Simplifying and letting $\lambda\to 0$ we obtain the inequalities

$$\pm\sum_i\int_{Q_T}\left[A_i-\mathcal{F}_i(\nabla v,z)\right]\cdot D_i\eta\,dz\ge 0,$$

which are impossible unless $A_i(z) = \mathcal{F}_i(\nabla v, z)$ almost everywhere in Q_T. The proof of Theorem 4.1 is completed.

7. Bounded solutions of problem (4.1)

Let us show that the solutions of problem (4.1) are uniformly bounded on a time interval chosen according to $\|v_0\|_{\infty,\Omega}$. This estimate would mean that for big K the solutions of problem (4.1) are, in fact, independent of K and solve the same problem but with f_K replaced by f.

Theorem 7.1. *Let in the conditions of Theorem 4.1 $v_0 \in L^\infty(\Omega)$. There exist T_0 and $A = const$, which depend on $\|v_0\|_{\infty,\Omega}$, $\|g\|_{\infty,Q_T}$, $m^\pm$, $\sigma^\pm$, and μ, but are independent of ϵ, such that the solutions of problem (4.1) satisfy the estimate*

$$\|v(t)\|_{\infty,\Omega} \le (1 + \|v_0\|_{\infty,\Omega})\, e^{A\,t} \qquad \text{for } t \in [0, T_0]. \tag{7.1}$$

These solutions solve problem (4.1) in Q_{T_K} with $f_K \equiv f$.

The proof relies on the following technical assertion.

Lemma 7.1. *Let $K > 1$, $m > 1$, $l > 1$, and $s \ge 1$ be given constants. For every constant $C_s > 2s + \max\{2; m+1; l+1\}$,*

$$G(v) \equiv C_s \int_0^{\max\{0,v\}} \rho\, A_\epsilon(\rho, m) \min\{K^{2(s-1)}, \rho^{2(s-1)}\}\, d\rho \\ - v \max\{0, v\}\, A_\epsilon(v, m) \min\{K^{2(s-1)}, v^{2(s-1)}\} \ge 0.$$

Proof. Let us write G in the form

$$G(v) = \begin{cases} C_s \displaystyle\int_0^v \rho^{2s-1} A_\epsilon(\rho, m)\, d\rho - A_\epsilon(v, m)\, v^{2s} & \text{for } v \in (0, K), \\ C_s \Big(\displaystyle\int_0^K \rho^{2s-1} A_\epsilon(\rho, m)\, d\rho + \int_K^v \rho^{2s-1} A_\epsilon(s, m)\, ds \Big) & \\ \qquad\qquad -v^2 A_\epsilon(v, m) K^{2(s-1)} & \text{if } v \ge K, \\ 0 & \text{if } v \le 0. \end{cases}$$

A straightforward calculation gives

$$G'(v) = v^{2s-1} \Big(\epsilon\, l\, (C_s - 2s(l-1))\, v^{l-1} + m\, v^{m-1} (C_s - m) + \epsilon\, C_s \Big) \ge 0$$

if $v \in (0, K]$ and

$$G'(v) \ge \epsilon\, v^{2s-1} (C_s - 2) + \epsilon\, l\, v^{2s+l-2} (C_s - (l+1)) + m v^{m+2s-2} (C_s - (m+1)) > 0$$

if $v > K$. The assertion follows because $G(0) = 0$ and the function $G(v)$ is nondecreasing. □

Proof of Theorem 7.1. It is sufficient to show that

$$\|\max\{0,v\}\|_{\infty,\Omega} \le (1+\|\max\{0,v_0\}\|_{\infty,\Omega})\exp(At)$$

with a suitable constant A. Let us fix some $K>1$ and test (6.2) with the function

$$\phi_s(v)=\max\{0,v\}\min\{K^{2(s-1)},\, v^{2(s-1)}\}.$$

It is easy to see that

$$\begin{aligned}
&\phi_s(v)\,\frac{d\Phi_\epsilon(v,m)}{dt}=\phi_s(v)\,A_\epsilon(v,m)v_t+\phi_s(v)m_t|v|^m\ln|v|\\
&=\frac{d}{dt}\Big(\int_0^v \phi_s(\rho)A_\epsilon(\rho,m)\,d\rho\Big)\\
&\quad+m_t\Big(\phi_s(v)\int_0^{\max\{0;v\}}(A_\epsilon)'_m(\rho,m)\,d\rho-\int_0^{\max\{0;v\}}\phi_s(\rho)(A_\epsilon)'_m(\rho,m)\,d\rho\Big).
\end{aligned}$$

Let us consider the function

$$H(v)=\phi_s(v)\int_0^v (A_\epsilon)'_m(\rho,m)\,d\rho-\int_0^v \phi_s(\rho)(A_\epsilon)'_m(\rho,m)\,d\rho.$$

It is continuous for $v\ge 0$, continuously differentiable on $[0,K)\cup(K,\infty)$, $H(0)=0$, and

$$H'(v)=\phi'_s(v)\int_0^v (A_\epsilon)'_m(\rho,m)\,d\rho=v^m\ln v\times\begin{cases}(2s-1)v^{2(s-1)} & \text{if } v\in[0,K),\\ K^{2(s-1)} & \text{if } v>K.\end{cases}$$

It follows that $H(v)$ is monotone decreasing on the interval $(0,1)$ and monotone increasing for $v>1$. $H(v)$ attains its minimal value at the point $v=1$: $H(1)=-\frac{2s-1}{(2s+m-1)^2}$. Since $0\le m_t\le\mu$ almost everywhere in Q_T, we estimate

$$\phi_s(v)\,\frac{d\Phi_\epsilon(v,m)}{dt}\ge\frac{d}{dt}\Big(\int_0^v \phi_s(\rho)A_\epsilon(\rho,m)\,d\rho\Big)-\mu\frac{2s-1}{(2s+m-1)^2}.$$

Further, $\mathcal{F}_i(\nabla v)\,D_i\phi_s(v)\ge 0$ almost everywhere in Q_T. Substituting these inequalities in (6.2) we arrive at the following estimate: for every $t,t+h\in(0,T)$,

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h}\frac{d}{dt}\Big(\int_\Omega\int_0^{\max\{0;v\}}\phi_s(\rho)\,A_\epsilon(\rho,m)\,d\rho\,dx\Big)dt-\mu\frac{(2s-1)|\Omega|}{(2s+m^--1)^2}\\
&\qquad\le\frac{b^+}{h}\int_t^{t+h}\int_{\Omega\cap\{\sigma\ge m+1\}}\phi_s(v)\,v^m\,(\min\{v,\,K\})^{\sigma-m-1}\,dz
\end{aligned}\tag{7.2}$$

$$+\frac{b^+}{h}\int_t^{t+h}\int\limits_{\Omega\cap\{\sigma<m+1\}}\phi_s(v)\,v^{\sigma-1}\,dz+\frac{1}{h}\int_t^{t+h}\int\limits_{\Omega}\phi_s(v)\,g(z)\,dz.$$

Let us denote $v_+ = \max\{0, v\}$. Applying Young's inequality and the fact that $m^- v_+^m \leq v_+ A_\epsilon(v_+, m)$, we estimate the integrands of the second and third terms on the right-hand side of (7.2) by means of the inequalities

$$\begin{aligned}
\phi_s(v)v^{\sigma-1} &= \begin{cases} v_+^\sigma v^{2(s-1)} & \text{if } v \leq K \\ v_+^\sigma K^{2(s-1)} & \text{if } v > K > 1 \end{cases}\\
&\leq \begin{cases} \frac{\sigma+1+2(s-1)}{m+1+2(s-1)} v_+^{m+1} v^{2(s-1)} + \frac{m-(\sigma+1)}{m+1+2(s-1)} & \text{if } v < K, \\ v_+^{m+1} K^{2(s-1)} & \text{if } v > K > 1 \end{cases}\\
&\leq \begin{cases} v^m \phi_s(v) + \frac{m-(\sigma+1)}{m+1+2(s-1)} & \text{if } v \leq K, \\ v^m \phi_s(v) & \text{if } v > K > 1, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\phi_s(v) &= \begin{cases} v^{-m}\left(v^m \phi_s(v)\right) & \text{if } v \geq K > 1, \\ v^{2s-1} & \text{if } v < K \end{cases}\\
&\leq \begin{cases} \frac{1}{m^-} v\,\phi_s(v) A_s(v) & \text{if } v \geq K > 1 \\ \frac{2s-1}{2(s-1)+m+1} v^{2(s-1)+m+1} + \frac{m}{2(s-1)+m+1} & \text{if } v < K \end{cases}\\
&\leq \begin{cases} \frac{1}{m^-} v\,\phi_s(v) A_s(v) & \text{if } v \geq K > 1 \\ \frac{1}{m^+}\frac{2s-1}{2(s-1)+m+1} v\,\phi_s(v) A_\epsilon(v) + \frac{m}{2(s-1)+m+1} & \text{if } v < K. \end{cases}
\end{aligned}$$

Let us introduce the function

$$I(t) = \int_\Omega \int_0^{v_+} \rho\, A_\epsilon(\rho, x) \min\{K^{2(s-1)},\, \rho^{2(s-1)}\}\, d\rho\, dx.$$

By virtue of Lemma 7.1 with $C_s = 2s + (l+1)$, inequality (7.2) entails the inequality

$$\begin{aligned}
\frac{1}{h}\int_t^{t+h} \frac{dI(\tau)}{d\tau}\,d\tau &\leq \frac{C_s}{m^-}\Big(\frac{1}{h}\int_t^{t+h}\Big(\|g(\cdot,\tau)\|_{\infty,\Omega} + b^+\Big(1+K^\gamma\Big)\Big) I(\tau)\,d\tau\Big)\\
&+ \frac{m^+|\Omega|}{2(s-1)+m^-+1}\Big(\frac{1}{h}\int_t^{t+h}\|g(\cdot,\tau)\|_{\infty,\Omega}\,d\tau\Big)\\
&+ \mu\frac{(2s-1)|\Omega|}{(2s+m^--1)^2} + b^+|\Omega|\frac{m^+-(\sigma^-+1)}{m^-+2(s-1)},
\end{aligned}$$

$\gamma = \max\{0, \sigma - m - 1\}$. Letting $h \to 0$, we arrive at the differential inequality for $I(t)$: for almost every $t \in (0, T)$,

$$I'(t) \le C_s M_1 \, I(t) + M_2, \qquad I(0) = I_0,$$

with the constants

$$M_1 = \frac{1}{m^-}\left(G + b^+ \left(1 + K^\gamma\right)\right), \qquad G = \operatorname{ess\,sup}_{(0,T)} \|g\|_{\infty,\Omega},$$

$$M_2 = \mu \frac{(2s-1)|\Omega|}{(2s+m^- - 1)^2} + b^+ \, |\Omega| \, \frac{m^+ - (\sigma^- + 1)}{m^- + 2(s-1)} + \frac{m^+ |\Omega| \, G}{2(s-1) + m^- + 1} \sim \frac{1}{s}$$

as $s \to \infty$. The direct integration of this inequality gives

$$I^{\frac{1}{2C_s}}(t) \le \mathrm{e}^{M_1 t} \left(I^{\frac{1}{2C_s}}(0) + (M_2 \, t)^{\frac{1}{2C_s}} \right). \tag{7.3}$$

Let us now pass to the limit as $s \to \infty$. Fix some $t \in (0, T]$, denote

$$\Omega_+ = \{x \in \Omega : \; v(x,t) > K\} \quad \text{and} \quad \Omega_- = \{x \in \Omega : \; v(x,t) \le K\},$$

and represent $I(t)$ as the sum of the integrals $I_\pm(t)$ taken over the sets $\Omega_\pm$. A straightforward calculation shows that

$$I_+(t) \ge K^{2(s-1)} \left(K^{m^-} + \frac{\epsilon}{2} K^2 + \epsilon \, K^l \right) |\Omega_+|,$$

$$I_-(t) \ge \frac{m^-}{2(s-1) + m^+ + 1} \int_{\Omega_-} |v|^{2(s-1)+m+1} \, dx.$$

Since $\|v_0\|_{\infty,\Omega} < K$,

$$I(0) = I_-(0) \quad \text{and} \quad \lim_{s \to \infty} I^{\frac{1}{2C_s}}(0) = \|v_0\|_{\infty,\Omega}.$$

Letting $s \to \infty$ in (7.3) we also obtain the inequalities

$$K \le \lim_{s \to \infty} I_+^{\frac{1}{2C_s}}(t) \le \mathrm{e}^{M_1 t} \left(1 + \|v_0\|_{\infty,\Omega}\right) \quad \text{if } |\Omega_+| \ne 0,$$

$$\|v(t)\|_{\infty,\Omega_-} \le \mathrm{e}^{M_1 t} \left(1 + \|v_0\|_{\infty,\Omega}\right),$$

whence

$$K + \|v(t)\|_{\infty,\Omega_-} \le 2 \, \mathrm{e}^{M_1 t} \left(1 + \|v_0\|_{\infty,\Omega}\right). \tag{7.4}$$

Given $K > 1 + 2\left(1 + \|v_0\|_{\infty,\Omega}\right)$, let us set

$$T_K = \frac{1}{M_1} \ln\left(1 + \frac{1}{1 + 2(1 + \|v_0\|_{\infty,\Omega})}\right). \tag{7.5}$$

It is easy to see that for $t \in [0, T_K]$ inequality (7.4) is impossible unless $|\Omega_+| = 0$, which means that $\Omega = \Omega_-$ for all $t \in [0, T_K]$. Estimate (7.1) follows now from (7.3) with $I(t) \equiv I_-(t)$ as $s \to \infty$. □

Corollary 7.1. *In the special cases when*

$$\text{either} \quad \sup_{Q_T}(\sigma(z) - m(z) - 1) \le 0, \quad \text{or} \quad b^+ = 0,$$

the constant M_1 and, respectively, T_K are independent of K. This means that estimate (7.1) *remains true for every finite T.*

Now we are in position to show that for small T_0 the parameter K is a dummy and can be omitted. Let us consider the problem with the unique regularization parameter ϵ:

$$\begin{cases} \partial_t \Phi_\epsilon(v,x) = \text{div } \mathcal{F}(\nabla v, z) + f(v,z) \quad \text{in } Q_T, \\ v(x,0) = v_0(x) \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_T. \end{cases} \tag{7.6}$$

Theorem 7.2. *Let us assume that the conditions of Theorem* 4.1 *are fulfilled. If, in addition, $v_0 \in L^\infty(\Omega)$, then there exists $T_0 \in (0,T]$ such that in the cylinder Q_{T_0} the solution of problem* (4.1) *solves problem* (7.6) *and satisfies the estimate*

$$\|v(t)\|_{\infty,\Omega} \le (1 + \|v_0\|_{\infty,\Omega})\, \mathrm{e}^{A\,t} \qquad \text{for } t \in [0, T_0] \tag{7.7}$$

with a constant A independent of ϵ.

Proof. It follows from Theorem 7.1 that for every $K > 2(1 + \|v_0\|_{\infty,\Omega})$ the solutions of problem (4.1) are bounded in O_{T_K} independently of ϵ, which means that there is an interval $[0, T_0]$ where $\|v(t)\|_{\infty,\Omega} < K$. Thus, $f_K(v,z) \equiv f(v,z)$ in Q_{T_0} and v is a bounded solution of problem (7.6). □

7.1. **Continuation to the maximal existence interval.** The constructed solution is defined on an interval $(0, T_0)$ with T_0 from Theorem 7.1. Since the function $v(x, T_0)$ possesses the same properties as the initial function v_0, one may take it for the initial datum and repeat all the above arguments to show that the solution of problem (7.6) can be continued to a time interval (T_0, T_1) with T_1 depending on $\|v(\cdot, T_0)\|_{\infty,\Omega}$. Continuing this process we obtain the sequence $\{T_k\}$ and a solution of problem (7.6) in the cylinders Q_{T_k}. If $T_k \to \infty$, the solution exists globally in time; otherwise, $\lim T_k = T^* < \infty$ and problem (7.6) admits a local-in-time solution.

Let us specially notice that in the cases mentioned in Corollary 7.1 the solutions of problem (7.6) exist globally in time and are bounded uniformly with respect to ϵ in every cylinder Q_T.

8. Proof of Theorem 3.1

8.1. **The limit as $\epsilon \to 0$.** Let us denote by $\{v_\epsilon\}$ the sequence of strong bounded solutions of problem (7.6).

Lemma 8.1. *Let the conditions of Theorem 7.1 be fulfilled. If $v_0 \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ and conditions (3.6) hold, then there exists T_0 such that in the cylinder Q_{T_0} problem (7.6) has a solution v_ϵ which satisfies the inequality*

$$\sup_{(0,T_0)} \int_\Omega \sum_{i=1}^n |D_i v_\epsilon|^{p_i(z)}\,dx + \int_0^{T_0} \int_\Omega A_\varepsilon(v_\epsilon, m)\,(\partial_t v_\epsilon)^2\,dx\,d\tau \le C \tag{8.1}$$

with a constant C independent of ϵ.

Proof. It is sufficient to fix K and T_K according to (7.4) and (7.5) and then let $k \to \infty$ in estimate (5.9). □

Lemma 8.2. *Under the foregoing conditions the sequence $\{v_\epsilon\}$ contains a subsequence that converges pointwise as $\epsilon \to 0$.*

Proof. We imitate the proof of Lemma 5.3 taking into account the uniform boundedness of v_ϵ. For the functions $w_\epsilon = |v_\epsilon|^{\alpha-1} v_\epsilon$ with $\alpha = (m^+ + 1)/2$,

$$\begin{aligned} |\partial_t w_\epsilon| &= \alpha |v_\epsilon|^{\alpha-1} |\partial_t v_\epsilon| \le C\,|v_\epsilon|^{\alpha-1-(m-1)/2} \left|\sqrt{A_\epsilon(v_\epsilon, m)}\,\partial_t v_\epsilon\right| \\ &\le C' \left|\sqrt{A_\epsilon(v_\epsilon, m)}\,\partial_t v_\epsilon\right|, \\ |\nabla w_\epsilon|^{p^-} &\le C\,|v_\epsilon|^{(\alpha-1)p^-} |\nabla v_\epsilon|^{p^-} \le C'\,|\nabla v_\epsilon|^{p^-}. \end{aligned}$$

By [28, Section 5, Theorem 8] there is a subsequence of $\{v_\epsilon\}$ that converges in $L^\nu(Q_{T_0})$ with some $\nu > 1$ and, respectively, pointwise in Q_{T_0}. □

The main a priori estimates derived for the solutions of problems (4.1) are independent of ϵ, do not depend on K, and remain true for the solutions v_ϵ of problem (7.6). Gathering the uniform estimates on v_ϵ we extract a subsequence (for which we keep the same notation) such that

$$\begin{cases} v_\epsilon \to v & \text{weakly in } \mathbf{W}(Q_{T_0}) \text{ and a.e. in } Q_{T_0}, \\ \Phi_\epsilon(v_\epsilon, m) \to |v|^{m(z)-1} v & \text{a.e. in } Q_{T_0}, \\ \dfrac{d}{dt}\Phi_\epsilon(v_\epsilon, m) \to \dfrac{d}{dt}\left(|v|^{m(z)-1} v\right) & \text{weakly in } \mathbf{W}'(Q_{T_0}), \\ \mathcal{F}_i(\nabla v_\epsilon, z) \to V_i & \text{weakly in } L^{p_i'(\cdot)}(Q_{T_0}), \\ f(v_\epsilon, z) \to f(v, z) & \star\text{-weakly in } L^\infty(Q_{T_0}). \end{cases}$$

Letting $\epsilon \to 0$ in (4.3) we conclude that for every $\phi \in L^{l+1}(Q_T) \cap \mathbf{W}(Q_{T_0})$

$$\int_{Q_{T_0}} \left[\phi \frac{d}{dt}\Phi(v,m) + \sum_{i=1}^{n} V_i \cdot D_i\phi - f(v,z)\,\phi\right] dx = 0. \tag{8.2}$$

Repeating the proof of Lemma 5.4 we show that the family $\{\Psi_\epsilon(v_\epsilon, m)\}$ is compact in $C^0([0,T]; L^1(\Omega))$ and

$$\Psi_\epsilon(v_\epsilon, m) \to \frac{m}{m+1}|v|^{m+1} \in C^0([0,T]; L^1(\Omega)).$$

Moreover, v satisfies estimate (7.1) and the formula of integration by parts holds:

$$\int_{t_1}^{t_2}\int_\Omega v\frac{d}{dt}\left(|v|^{m-1}v\right)\,dz = \int_\Omega \frac{m}{m+1}|v|^{m+1}\,dx\Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_\Omega \frac{\partial_t m}{(m+1)^2}\,[1-(m+1)\ln|v|]\,|v|^{m+1}\,dz.$$

The monotonicity arguments used to prove Theorem 7.2 show that $V_i = a_i|D_iv|^{p_i(z)-2}D_iv$. Condition (3.3) is proved in exactly the same way as in the proof of Theorem 4.1.

Given $\phi \in \mathbf{W}(Q_{T_0})$, there is a sequence $\phi_\delta \in L^{l+1}(Q_T) \cap \mathbf{W}(Q_{T_0})$ that converges to ϕ in $\mathbf{W}(Q_{T_0})$. Testing (8.2) with ϕ_δ we then pass to the limit as $\delta \to 0$, which gives the integral identity (3.2).

Let us assume now that $m^- \geq 1$. By virtue of (8.1) and (7.7),

$$\left|\frac{d}{dt}\Phi_\epsilon(v_\epsilon, m)\right|^2 \leq 2\,A_\epsilon(v_\epsilon,m)\left(A_\epsilon(v_\epsilon,m)\,|\partial_t v_\epsilon|^2\right) + 2\,|v_\epsilon||\ln|v_\epsilon||\,m_t \leq C$$

with a constant C independent of ϵ. It follows that the family $\left\{\frac{d}{dt}\Phi_\epsilon(v_\epsilon, m)\right\}$ contains a subsequence which converges weakly in $L^2(Q_{T_0})$. Finally, the possibility of continuation of the constructed solution to the arbitrary time interval follows from the maximum principle for the solutions of the regularized problems given in Theorem 7.1.

The proof of Theorem 3.1 is completed.

9. The energy relations for strong solutions

In this section we prove several energy estimates for the constructed strong solution of problem (1.1).

Theorem 9.1. *Under the conditions of Theorem* 3.1 *the solution of problem* (3.1) *satisfies the following energy relations:*

(1) *for every* $t \in (0, T_0)$,

$$\int_\Omega \frac{m}{m+1}|v(x,t)|^{m+1}\,dx + \sum_{i=1}^n \int_{Q_t} a_i |D_i v|^{p_i}\,dz = \int_{Q_t} v f(v,z)\,dz \tag{9.1}$$
$$+ \int_\Omega \frac{m_0}{m_0+1}|v_0|^{m_0+1}\,dx + \int_{Q_t} \frac{\partial_t m}{(m+1)^2}\,\left[1-(m+1)\ln|v|\right]\,|v|^{m+1}\,dz,$$

(2) *for almost every* $t \in (0, T_0)$,

$$\frac{d}{dt}\Big(\int_\Omega \frac{m}{m+1}|v|^{m+1}\,dx\Big) + \sum_{i=1}^n \int_\Omega a_i |D_i v|^{p_i}\,dz = \int_\Omega v f(v,z)\,dz \tag{9.2}$$
$$+ \int_\Omega \frac{\partial_t m}{(m+1)^2}\,\left[1-(m+1)\ln|v|\right]\,|v|^{m+1}\,dz.$$

Proof. To obtain (9.1) it suffices to take the solution v for the test function in identity (3.2); equality (9.2) follows from (9.1) by the Lebesgue differentiation theorem. □

Let us introduce the function

$$E(t,u) = \int_\Omega \sum_{i=1}^n a_i \frac{|D_i u|^{p_i}}{p_i}\,dx - \int_\Omega b\frac{|u|^\sigma}{\sigma}\,dx.$$

Theorem 9.2. *Let, in the conditions of Theorem* 3.1, $b(z) \geq 0$ *in* Q_T. *For almost every* $t \in (0,t)$,

$$E(t,v) + \frac{1}{2}\int_{Q_t} m(z)|v|^{m(z)-1}v_t^2\,dx\,d\tau \leq E(0,v_0) + C^*\,(1+t), \tag{9.3}$$

with the constant C^* *from* (5.6). $C^* = 0$ *in the special case when* p_i, σ, *and* m *are independent of* t *and the coefficients* a_i *and* b *satisfy the inequalities* $a_{it}(z) \leq 0$ *and* $b_t(z) \geq 0$ *in* Q_T.

Proof. To derive (9.3) we revert to the sequence of finite-dimensional approximations for the solution of the regularized problem (4.1).

Lemma 9.1. *Let, in the conditions of Theorem* 4.1, $b(z) \geq 0$. *For every* $t \in [0,T]$ *and* $k \in \mathbb{N}$,

$$\int_\Omega b\,S^{(k)}\,dx \leq \int_\Omega \frac{b}{\sigma}\left(\min\{|v^{(k)}|, K\}\right)^\sigma\,dx$$
$$+ C\left(\operatorname{meas}\{x \in \Omega : |v^{(k)}| > K\}\right)^{\frac{\alpha}{m^+(t)+\alpha+1}}.$$

Proof. Fix some $t > 0$ and denote by $\chi_{\{|v^{(k)}|>K\}}$ the characteristic function of the set $\{x \in \Omega : |v^{(k)}| > K\}$. Since $K > 1$, by the definition of $S^{(k)}$

$$S^{(k)}(z) \leq \frac{1}{\sigma(z)} \left(\min\{|v^{(k)}|, K\}\right)^{\sigma(z)} + \frac{K^{\sigma^+(t)-m^-(t)-1}}{m^-(t)+1}|v^{(k)}|^{m^+(t)+1}\chi_{\{|v^{(k)}|>K\}} + \frac{1}{\sigma^-(t)}|v^{(k)}|^{m^+(t)+1}\chi_{\{|v^{(k)}|>K\}}.$$

It follows that for $b(z) \geq 0$,

$$\int_\Omega bS^{(k)}dx \leq C\int_\Omega |v^{(k)}|^{m^+(t)+1}\chi_{\{|v^{(k)}|>K\}}\,dx + \int_\Omega \frac{b}{\sigma}\left(\min\{|v^{(k)}|, K\}\right)^{\sigma(z)}\,dx \equiv I_1^{(k)} + I_2^{(k)}.$$

Using Hölder's inequality and Lemma 2.1

$$I_1^{(k)} \leq C\,\|v^{(k)}\|_{m^+(t)+\alpha+1,\Omega}\left(\text{meas}\{x \in \Omega : |v^{(k)}| > K\}\right)^{\frac{\alpha}{m^+(t)+\alpha+1}} \leq C\Big(\int_\Omega \sum_{i=1}^n |D_i v^{(k)}|^{p_i^-(t)}\Big)^{\frac{1}{p^-(t)}}\left(\text{meas}\{x \in \Omega : |v^{(k)}| > K\}\right)^{\frac{\alpha}{m^+(t)+\alpha+1}},$$

and the assertion follows now from the uniform-in-k estimate (4.8). □

Lemma 9.2. *Let $b(z) \geq 0$, $v_\epsilon = \lim v^{(k)}$ be the solution of problem* (7.6) *constructed in Theorem* 7.2, *and $\mathcal{E}(t, v_\epsilon)$ be the corresponding energy function defined in* (5.2). *For every $t \in [0, T_0)$,*

$$\mathcal{E}(t, v_\epsilon) + \frac{1}{2}\int_{Q_t} A_\epsilon(v_\epsilon, m)v_{\epsilon t}^2\,dz \leq \mathcal{E}(0, v_0) + C^*(1+t) \tag{9.4}$$

with the constant C^* from (5.6).

Proof. The assertion follows from (5.6) as $k \to \infty$. By the choice of the basis $\mathcal{E}(0, v_0^{(k)}) \to \mathcal{E}(0, v_0)$. By virtue of the Fatou lemma we may let $k \to \infty$ in the nonnegative terms on the left-hand side of (5.6). By the dominated convergence theorem, for every $K > 1$

$$\lim_{k\to\infty}\int_\Omega b\,S^{(k)}\,dx \leq \int_\Omega \frac{b}{\sigma}\left(\min\{|v_\epsilon|, K\}\right)^\sigma\,dx + C\left(\text{meas}\{x \in \Omega : |v_\epsilon| > K\}\right)^{\frac{\alpha}{m^+(t)+\alpha+1}}.$$

For $K > 2(1 + \|v_0\|_{\infty,\Omega})$ we have $\|v_\epsilon(t)\|_{\infty,\Omega} < K$ on $[0, T_0)$ (see the proof of Theorem 7.1), whence the required estimate. □

To complete the proof of Theorem 9.1 we repeat these arguments for the sequence $\{v_\epsilon\}$ of solutions of problems (7.6), which are uniformly bounded and satisfy estimate (9.4). □

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