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A PRIORI ESTIMATES AND REDUCTION PRINCIPLES FOR QUASILINEAR ELLIPTIC PROBLEMS AND APPLICATIONS

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Abstract. Variants of Kato's inequality are proved for general quasilinear elliptic operators L. As an outcome we show that, dealing with Liouville theorems for coercive equations of the type

$$Lu = f(x, u, \nabla_L u) \quad \text{on } \Omega \subset \mathbb{R}^N,$$

where f is such that $f(x, t, \xi)\, t \geq 0$, the assumption that the possible solutions are nonnegative involves no loss of generality. Related consequences such as comparison principles and *a priori* bounds on solutions are also presented. An underlying structure throughout this work is the framework of Carnot groups.

Introduction

A priori estimates of solutions of quasilinear elliptic equations has been a subject of fundamental and remarkable interest in recent years. For quasilinear elliptic problems, significant and interesting results are dealing with nonnegative solutions associated to nonlinearities that grow faster than the differential part.

Recently, Serrin [44] considered quasilinear coercive equations and inequalities with source term changing sign and proved some interesting Liouville theorems. These results (see also [14, 15] for related contributions) are a consequence of appropriate *a priori* estimates on the possible solutions or on suitable functionals of them.

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It is well known that when looking for Liouville theorems of *noncoercive* nonlinear equations or inequalities, the fact that the nonlinearity has definite sign is of fundamental importance. This is because, in general, canonical examples of this type show that when the nonlinearity changes sign, the problem may possess infinitely many solutions with no *a priori* bound. A canonical example in this direction is the following:

$$-\Delta u = |u|^{q-1} u \quad \textit{on } \mathbb{R}^N. \tag{0.1}$$

Indeed, it is well known that if $1 < q < \frac{N+2}{N-2}$, $N > 2$, then (0.1) admits infinitely many radial solutions with increasing number of zeroes.

On the other hand, when the problem is *coercive*, then the situation may be completely different as the following striking result due to Brezis [7] shows.

Theorem (Brezis). *Let $q>1$. If $u \in L^q_{loc}(\mathbb{R}^N)$ is a distributional solution of*

$$\Delta u \geq |u|^{q-1} u \quad \textit{on } \mathbb{R}^N, \tag{0.2}$$

then $u \leq 0$ almost everywhere on $\mathbb{R}^N$. In particular if equality holds in (0.2), *then $u \equiv 0$ almost everywhere on $\mathbb{R}^N$.*

It is worth pointing out that, besides the quite general functional framework, there are no assumptions on the behavior of the possible solutions of (0.2) at infinity.

Brezis's technique is based on a form of Kato's inequality [24, 7, 2] and on a construction of a suitable Loewner–Nirenberg barrier function. See [27] and [25, 37].

Some generalizations of Brezis's result for quasilinear elliptic inequalities of second order have been obtained in [14, 15, 16] and more recently in a series of papers by Farina and Serrin [18, 19] and Pucci and Serrin [40].

One common aspect in these recent contributions is that from the technical point of view, none of them use a form of Kato's inequality.

Thus one natural question is the extent to which Kato's inequality might be satisfied in the quasilinear case. A positive answer to this problem will allow the development of a general strategy for proving positivity-type results as well as Liouville theorems for wide classes of quasilinear inequalities. This will bring together some aspects of qualitatively different problems, namely, coercive and noncoercive quasilinear elliptic inequalities of second order.

In this paper we shall prove Kato's inequality for a wide class of quasilinear weakly elliptic operators including those related to the sub–Riemannian setting (also known as Carnot geometry or nonholonomic Riemannian geometry). Plainly, this framework includes the canonical Euclidean case. To get an idea of some preliminary results contained in this paper we mention the following special cases of Theorem 2.1 proved in the next section.

Example 1. The p–Laplacian-type operator. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a solution of the inequality

$$\operatorname{div}\big(|\nabla u|^{p-2}\, \nabla u\big) \geq f \qquad \textit{on } \Omega.$$

Then,

$$\operatorname{div}\big(\left|\nabla u^+\right|^{p-2} \nabla u^+\big) \geq \operatorname{sign}^+(u) f \qquad \textit{on } \Omega.$$

Example 2. The 1–Laplacian-type operator. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,1}_{loc}(\Omega)$ be a solution of the inequality

$$\operatorname{div}\big(|\nabla u|^{-1}\, \nabla u\big) \geq f \qquad \textit{on } \Omega.$$

Then,

$$\operatorname{div}\big(\operatorname{sign}^+(u) \left|\nabla u^+\right|^{-1} \nabla u^+\big) \geq \operatorname{sign}^+(u)\, f \qquad \textit{on } \Omega.$$

Example 3. The mean curvature operator in nonparametric form. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,2}_{loc}(\Omega)$ be a solution of the inequality

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \Big) \geq f \qquad \textit{on } \Omega.$$

Then,

$$\operatorname{div}\Big(\frac{\nabla u^+}{\sqrt{1 + |\nabla u^+|^2}} \Big) \geq \operatorname{sign}^+(u) f \qquad \textit{on } \Omega.$$

We note that the conclusion of Example 1 enables us to prove the analogue of Brezis's theorem for the corresponding inequality (0.2) associated to the p-Laplacian operator under minimal assumption on the solutions, by using Brezis's original idea [7]. See Theorem 3.1 below and the recent contribution by Farina and Serrin [18, 19] and the Authors [14, 15, 16] for a different proof of this special case. More generally, consider an inequality of the type

$$\operatorname{div}_L \left(\mathcal{A}(x, u, \nabla_L u)\right) \geq f(x, u, \nabla_L u) \qquad \textit{on} \;\; \Omega \subset \mathbb{R}^N. \tag{0.3}$$

Here $\Omega \subset \mathbb{R}^N$ is an open set, $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Caratheodory vector field, $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ is a Caratheodory function, and ∇_L is a quite general vector field. As concrete examples we may consider as special cases the Euclidean gradient as well as the horizontal vector field on a Carnot group.

The main goal of this paper, besides the study of quasilinear versions of Kato's inequality, is to prove positivity-type results as well as Liouville theorems for (0.3). Our study of (0.3) can be briefly described as follows.

i) *Reduction of the problem* (0.3) *to an inequality that may possess only nonnegative solutions.*
ii) *Good* a priori *bounds of the possible nonnegative solutions of the reduced problem.*
iii) *Nonexistence of nonnegative solutions of the reduced problem.*
iv) *Nonexistence of nonnegative and sign-changing solutions of* (0.3).

In the above scheme, we shall see that point i) depends on the weak ellipticity of the differential operators. On the other hand, roughly speaking, ii) depends on the behavior of the nonlinearity at infinity. Notice that when dealing with noncoercive problems, step ii) depends only on the behavior of the nonlinearity near zero. See [16].

Altogether the above considerations suggest the following natural problem for elliptic equations and inequalities.

Problem A: *What kind of second-order elliptic inequalities of type* (0.3) *on* $\mathbb{R}^N$ *admits only solutions of definite sign*?

The possibility to exclude sign-changing solutions is of fundamental importance when looking for unconditionally Liouville theorems. We point out that an interesting consequence of the validity of Kato's inequality is that for a large class of differential inequalities associated to coercive operators, the nonexistence of positive solutions implies that all possible solutions of the given problem must be of definite (negative) sign. In other words, the problem cannot have oscillatory solutions. This fact is obviously false if the problem is *noncoercive;* see (0.1).

In this paper we will give an answer to Problem A for inequalities of type (0.3) and illustrate some general implications. We shall call these consequences *reduction principles*. As we will see during the course, these consequences imply some *maximum and comparison principles*, which are new in our general framework, and some of them are new even in the Euclidean setting (see Theorems 4.9, 4.13, 4.14, and 4.15).

Another point of interest is that our contribution shows that, when looking for Liouville theorems for coercive inequalities of type (0.3) with $f(x,t,\xi)\,t \geq 0$, the assumption that the possible solutions are nonnegative involves no loss of generality.

Consequently, to our knowledge, most of the Liouville theorems concerning positive solutions proved in the literature for *coercive problems* are indeed results on the *nonoscillatory character* of the possible solutions of (0.3); see for instance [25, 37, 4, 34, 32, 20, 13, 28] and the references therein.

This paper is organized as follows. In Part 1 we prove some variants of Kato's inequality for general classes of quasilinear operators, which include as a special case operators acting (roughly speaking) on Carnot groups. We will illustrate these inequalities with some canonical examples.

The goal of Part 2 is to point out one of the main consequences of the inequalities proved in the preceding section, namely *the reduction principles*. These results are described in detail in the related sections as well as some applications that will have important consequences in the following sections.

In Part 3, which is the core of the present work, we focus our attention on general *a priori* estimates of nonnegative solutions of coercive inequalities. Since we are mainly interested in Liouville theorems, we have chosen to consider classes of differential inequalities which are sufficiently general to illustrate the typical difficulties encountered during the course, avoiding nonessential generalizations.

Obviously more general problems can be considered. In these respects, the reader may refer to the recent interesting work by Serrin [44] and Farina and Serrin [18, 19] where, by using a different set of ideas, a deep and detailed study of various completely coercive problems is presented.

We point out that when dealing with Liouville theorems for coercive inequalities associated to weakly elliptic operators, one common key assumption on the possible solutions is that they belong to $L^\infty_{loc}(\mathbb{R}^N)$. See for instance [18, 13]. Indeed, in general, for nonnegative solutions associated to weakly elliptic problems of type (0.3), to the Authors' knowledge, no weak Harnack's inequality holds. Hence, in general, weak solutions do not belong to $L^\infty_{loc}(\mathbb{R}^N)$.

On the other hand, the assumption $u \in L^\infty_{loc}(\mathbb{R}^N)$ allows one to employ the nonlinear capacity method developed in [32]. The latter consists in using some special multiplier depending on the positive power of the solutions and deriving improved integral estimates. Thus, we begin this section by proving *a priori* estimates on the solutions that allow us to use those techniques without requiring that the solutions belong to $L^\infty_{loc}(\mathbb{R}^N)$. See Section 6.

As the main application of the results proved in this section we consider a quasilinear version of Gross–Pitaevskii- and Ginzburg–Landau-type equations; see Farina [17] and a recent contribution of Brezis [8]. In addition we give an answer to a problem posed by Peletier and Serrin (see [38, p. 80]).

We end this paper with some notes (see appendices A and B) that contain some basic facts on sub–Riemannian structures.

Note. The results of this paper were announced by the second author to the meeting Giornata di lavoro in ricordo di Bruno Pini, Bologna, November 26, 2010, Dipartimento di Matematica dell' Università di Bologna.

Part 1. Some variants of the Kato inequality

1. Notation and definitions

In this paper ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm. $\Omega \subset \mathbb{R}^N$ is open. Let $\mu \in \mathcal{C}(\mathbb{R}^N;\mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij})$, $i = 1,\dots,l$, $j = 1,\dots,N$ and assume that for any $i = 1,\dots,l$, $j = 1,\dots,N$ the derivative $\frac{\partial}{\partial x_j}\mu_{ij} \in \mathcal{C}(\Omega)$. For $i = 1,\dots,l$, let X_i and its formal adjoint X_i^* be defined as

$$X_i := \sum_{j=1}^N \mu_{ij}(\xi)\frac{\partial}{\partial \xi_j}, \qquad X_i^* := -\sum_{j=1}^N \frac{\partial}{\partial \xi_j}\left(\mu_{ij}(\xi)\cdot\right), \tag{1.1}$$

and let ∇_L be the vector field defined by

$$\nabla_L := (X_1,\dots,X_l)^T = \mu\nabla, \quad \text{and} \quad \nabla_L^* := (X_1^*,\dots,X_l^*)^T.$$

For any vector field $h = (h_1,\dots,h_l)^T \in \mathcal{C}^1(\Omega,\mathbb{R}^l)$, we shall use the notation $\operatorname{div}_L(h) := \operatorname{div}\left(\mu^T h\right)$; that is,

$$\operatorname{div}_L(h) = -\sum_{i=1}^l X_i^* h_i = -\nabla_L^*\cdot h.$$

Examples of vector fields, which we are interested in, are the usual gradient acting on $l(\leq N)$ variables (see Example B.1), vector fields related to the Bouendi–Grushin operator (see Example B.2), the Heisenberg–Kohn sub–Laplacian (see Example B.3), the Heisenberg–Greiner operator (see Example B.4), and the sub–Laplacian on Carnot Groups (see Appendix A). Another motivation for considering these kinds of operators is the following. Let $A = (a_{ij}(x))_{1\leq i,j\leq N}$ be a matrix with continuous entries. Consider the linear operator $Lu := \operatorname{div}(A(x)\nabla u)$. Assume that A is symmetric and positive semidefinite (that is, $a_{ij} = a_{ji}$ and $A(x)\xi\cdot\xi \geq 0$ for any $\xi \in \mathbb{R}^N$). With this assumption the operator L is weakly elliptic; see Definition 1.1 below. Since A is symmetric and positive semidefinite, there exists a matrix μ such that $A = \mu^T\mu$. Let l be the rank of μ. Since A may be singular, in general we shall have $l \leq N$. Therefore, setting $\nabla_L := \mu\nabla$ and $\operatorname{div}_L(\cdot) := \operatorname{div}\left(\mu^T\cdot\right)$, the operator L can be rewritten as $Lu = \operatorname{div}_L(\nabla_L u)$ (formally as the Laplace operator). Finally, even if the entries of the matrix A are smooth, in general then nothing can be said on the regularity of the entries of μ.

Since we are interested in weak solutions of the problems under consideration, we shall allow that the entries of the matrix μ are singular. However, for simplicity we shall assume that μ_{ij} are continuous.

Let $\delta := (\delta_1, \dots, \delta_N)$ be an N-tuple of positive real numbers. Let $R > 0$; we shall denote by δ_R the anisotropic dilation $\delta_R : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\delta_R(x) = \delta_R(x_1, \dots, x_N) := (R^{\delta_1} x_1, \dots, R^{\delta_N} x_N). \tag{1.2}$$

The Jacobian of the transformation δ_R is given by $J(\delta_R) = R^Q$, where $Q := \delta_1 + \delta_2 + \cdots + \delta_N$.

In Part 3 we shall require that ∇_L is *pseudo homogeneous of degree 1 with respect to dilation* δ_R, that there exist $\delta_i > 0$ $(i = 1, \dots, N)$ such that

$$\text{for each } \phi \in \mathcal{C}^1(\mathbb{R}^N) \text{ and } R > 0: \ \nabla_L(\phi(\delta_R(\cdot))) = R(\nabla_L \phi)(\delta_R)(\cdot). \tag{1.3}$$

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm*, if

i) $S(x) = 0$ if and only if $x = 0$, and

ii) it is homogeneous of degree 1 with respect to δ_R (i.e., $S(\delta_R(x)) = RS(x)$).

An example of a homogeneous norm which is differentiable for $x \neq 0$ is given by

$$S_\delta(x) := \Big(\sum_{i=1}^{N} (x_i^r)^{\frac{d}{\delta_i}} \Big)^{\frac{1}{rd}}, \tag{1.4}$$

where $d := \delta_1 \delta_2 \cdots \delta_N$ and r is the smallest even integer such that

$$r \geq \max\{\delta_1/d, \dots, \delta_N/d\}.$$

Notice that if S is a homogeneous norm differentiable almost everywhere and ∇_L is pseudo homogeneous of degree 1 with respect to δ_R, then $|\nabla_L S|$ is homogeneous of degree 0 with respect to δ_R. Hence the function $|\nabla_L S|$ is bounded.

In Part 3 we shall fix a homogeneous norm S differentiable away from 0 and we shall set

$$\psi := |\nabla_L S(\cdot)|. \tag{1.5}$$

We define B_R to be the ball of radius $R > 0$ generated by the norm S; i.e., $B_R := \{x : S(x) < R\}$ and A_R stands for the annulus $B_{2R} \setminus \overline{B_R}$. Therefore we have

$$|B_R| = \int_{B_R} dx = R^Q \int_{S(x)<1} dx = c_S R^Q \quad \text{and} \quad |A_R| = c_S(2^Q - 1)R^Q.$$

A canonical framework for which our results apply (see the next sections) is the Euclidean space $(\mathbb{R}^N, |\cdot|)$ with $|\cdot|$ the Euclidean norm. In this case $\mu = I_N$, the identity matrix in N dimensions, $\nabla_L = \nabla$ is the isotropic gradient, and div_L is the divergence operator. The dilation δ_R, defined by

$$\delta_R(x) = \delta_R(x_1, \dots, x_N) := (Rx_1, \dots, Rx_N),$$

is isotropic. Here, $Q = N$ is the dimension of the space. In this case, $\psi \equiv 1$ and B_R is the Euclidean open ball of radius R centered at the origin.

Another setting in which our results apply is the framework of Carnot groups. For more details see Appendix A. Further examples will be discussed in Appendix B below.

In what follows we shall assume that $\mathcal{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Caratheodory function; that is, for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^l$ the function $\mathcal{A}(\cdot, t, \xi)$ is measurable, and for almost every $x \in \mathbb{R}^N$, $\mathcal{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathcal{A}$; that is,

$$L(u)(x) = \mathrm{div}_L \left(\mathcal{A}(x, u(x), \nabla_L u(x))\right).$$

Our model cases are the p-Laplacian operator, the mean curvature operator, and some related generalizations. See Examples 1.3 below.

Definition 1.1. *Let $\mathcal{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be a Caratheodory function. The function $\mathcal{A}$ is called* weakly elliptic *if it generates a weakly elliptic operator L; i.e.,*

$$\begin{array}{c} \mathcal{A}(x,t,\xi)\cdot\xi \geq 0 \;\; \textit{for each } x \in \mathbb{R}^N,\, t \in \mathbb{R},\, \xi \in \mathbb{R}^l, \\ \mathcal{A}(x,0,\xi) = 0 \;\; \textit{or} \;\; \mathcal{A}(x,t,0) = 0. \end{array} \tag{WE}$$

Let $p \geq 1$; the function $\mathcal{A}$ is called **W**-p-**C** (*weakly-p-coercive*) (*see* [3]), *if $\mathcal{A}$ is* (WE) *and it generates a weakly-p-coercive operator L, i.e., if there exists a constant $k_2 > 0$ such that*

$$(\mathcal{A}(x,t,\xi)\cdot\xi)^{p-1} \geq k_2 \left|\mathcal{A}(x,t,\xi)\right|^p \text{ for each } x \in \mathbb{R}^N,\, t \in \mathbb{R},\, \xi \in \mathbb{R}^l. \tag{$\mathbf{W}$-p-$\mathbf{C}$}$$

Let $p > 1$; the function $\mathcal{A}$ is called **S-**p**-C** (*strongly-p-coercive*) (*see* [42, 3, 33]), *if there exist $k_1, k_2 > 0$ constants such that*

$$(\mathcal{A}(x,t,\xi)\cdot\xi) \geq k_1 \left|\xi\right|^p \geq k_2 \left|\mathcal{A}(x,t,\xi)\right|^{p'} \textit{ for each } x \in \mathbb{R}^N,\, t \in \mathbb{R},\, \xi \in \mathbb{R}^l. \tag{$\mathbf{S}$-p-$\mathbf{C}$}$$

Definition 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function. Let $p \geq 1$. We say that $u \in W^{1,p}_{loc}(\Omega)$ is a* weak solution *of*

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) \geq f(x,u,\nabla_L u) \qquad on \quad \Omega,$$

if $\mathcal{A}(\cdot,u,\nabla u) \in L^{p'}_{loc}(\Omega)$, $f(\cdot,u,\nabla_L u) \in L^1_{loc}(\Omega)$, and for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$ we have

$$-\int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L\phi \geq \int_\Omega f(x,u,\nabla_L u)\phi.$$

Examples 1.3. (1) Let $p > 1$. The p-Laplacian operator defined on suitable functions u by

$$\Delta_{p,L} u = \mathrm{div}_L\big(\,|\nabla_L u|^{p-2}\,\nabla_L u\big)$$

is an operator generated by $\mathcal{A}(x,t,\xi) := |\xi|^{p-2}\,\xi$ which is **S**-p-**C**.

(2) If $\mathcal{A}$ is of mean curvature type, that is, $\mathcal{A}$ can be written as $\mathcal{A}(x,t,\xi) := A(|\xi|)\xi$ with $A : \mathbb{R} \to \mathbb{R}$ a positive bounded continuous function (see [32, 3]), then $\mathcal{A}$ is **W**-2-**C**.

(3) The mean curvature operator in nonparametric form,

$$Tu := \mathrm{div}_L\Big(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\Big),$$

is generated by $\mathcal{A}(x,t,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}$. In this case $\mathcal{A}$ is **W**-p-**C** with $1 \leq p \leq 2$ and of mean curvature type but it is not **S**-2-**C**.
Let $m > 1$. The operator

$$T_m u := \mathrm{div}\Big(\frac{|\nabla u|^{m-2}\,\nabla u}{\sqrt{1+|\nabla u|^m}}\Big)$$

is **W**-p-**C** for $m \geq p \geq m/2$.

(4) Let T_M be the operator defined as

$$T_M u := \mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big).$$

The operator T_M is the mean curvature operator in the Lorentz–Minkowski space $L^{N+1} := \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$ endowed with the metric $-dt^2 + \sum_{j=1}^N\, dx_j^2$.

(5) Let $p > 1$ and define

$$Lu := \sum_{i=1}^{N} \partial_i\big(|\partial_i u|^{p-2} \partial_i u\big).$$

The operator L is **S**-p-**C**.

(6) The operator defined by $\operatorname{div}\big(\frac{|u|\nabla u}{|u|+|\nabla u|}\big)$ is **W**-2-**C**.

(7) Let $\nu > 0$ and define

$$B_\nu u := \nu \operatorname{div}\Big(\frac{|u|\,\nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}\,|\nabla u|^2}}\Big).$$

The operator B_ν is related to the so-called *"tempered diffusion equation"* or *"relativistic heat equation"* (here ν is a constant representing a kinematic viscosity and c the speed of light). See [6] and [41]. This operator is **W**-2-**C**.

(8) Letting $\nu \to +\infty$ in B_ν above, we obtain the operator that appears in the so called *"diffusion equation in transparent media,"*

$$B_\infty u := c \operatorname{div}\Big(\frac{|u|\,\nabla u}{|\nabla u|}\Big).$$

See [6]. This operator is obviously (WE).

(9) A very interesting class of weakly elliptic operators was recently introduced by Farina and Serrin in [18], namely,

$$(\mathcal{A}(x,t,\xi)\cdot\xi)^{p-1}\,|x|^s\,|t|^r \geq k_2\,|\mathcal{A}(x,t,\xi)|^p\,, \qquad (FS)$$

where $s, r \in \mathbb{R}$, $p \geq 1$, and $k_2 > 0$. This class of operators allows the Authors to obtain several Liouville theorems for inequalities and equations associated to this kind of operator (see also [19]).

2. Quasilinear weakly elliptic operators

In this section we shall prove that suitable versions of the Kato's inequalities [24] hold for some *quasilinear weakly elliptic operators.* A somewhat complete study of Kato's inequalities for *linear differential operators* will be the subject of a forthcoming work. In these respects, we point out that one can handle second-order operators generated by a system of smooth vector fields in $\mathbb{R}^N$ satisfying the Hörmander's condition and left invariant differential operators on homogeneous groups. See [22]. A related version of Kato's inequality can be also be proved for operators in a sub–Riemannian setting. However we shall not discuss these kinds of generalizations here.

We point out that the following results hold for a wide class of differential operators for which no group invariance is required. Of course the price to pay for this generality is that we need to consider solutions that belong to the space $W^{1,p}_{loc}(\Omega)$. Under additional assumptions on the underlying group structure and suitable invariance, it is possible to handle solutions that belong to the more natural space $W^{1,p}_{L,loc}(\Omega)$. See Remark 2.5 for the exact meaning.

Let Ω be an open set contained in $\mathbb{R}^N$, $p \geq 1$, and $u \in W^{1,p}_{loc}(\Omega)$. As usual, we denote by sign, sign$^+$, and u^+ the functions defined as follows:

$$\operatorname{sign}(t) := \begin{cases} 1 & \text{if } t > 0; \\ 0 & \text{if } t = 0; \\ -1 & \text{if } t < 0; \end{cases} \qquad \operatorname{sign}^+(t) := \begin{cases} 1 & \text{if } t > 0; \\ \\ 0 & \text{if } t \leq 0; \end{cases}$$

$$u^+ := \operatorname{sign}^+(u)\, u.$$

Theorem 2.1 (Kato's inequality: The quasilinear case)**.** *Let $\mathcal{A}$ be such that*

$$\mathcal{A}(x,t,\xi) \cdot \xi \geq 0 \qquad \textit{for any } x \in \Omega,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l. \tag{2.1}$$

Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of

$$\operatorname{div}_L \left(\mathcal{A}(x,u,\nabla_L u)\right) \geq f \qquad \textit{on} \quad \Omega. \tag{2.2}$$

Then

$$\operatorname{div}_L \left(\operatorname{sign}^+ u \ \mathcal{A}(x,u,\nabla_L u)\right) \geq \operatorname{sign}^+ u \ f \qquad \textit{on} \quad \Omega. \tag{2.3}$$

Moreover, if

$$\operatorname{div}_L \left(\mathcal{A}(x,u,\nabla_L u)\right) = f \qquad \textit{on} \quad \Omega, \tag{2.4}$$

then

$$\operatorname{div}_L \left(\operatorname{sign} u \ \mathcal{A}(x,u,\nabla_L u)\right) \geq \operatorname{sign} u \ f \qquad \textit{on} \quad \Omega. \tag{2.5}$$

In particular, if $\mathcal{A}$ is (WE) *and u is a weak solution of* (2.2)*, then u^+ is a weak solution of*

$$\operatorname{div}_L \left(\mathcal{A}(x,u^+,\nabla_L u^+)\right) \geq \operatorname{sign}^+ u \ f \qquad \textit{on} \quad \Omega. \tag{2.6}$$

If in addition $\mathcal{A}$ is odd, i.e.,

$$\mathcal{A}(x,-t,-\xi) = -\mathcal{A}(x,t,\xi), \tag{2.7}$$

and u is a solution of (2.4)*, then $|u|$ satisfies*

$$\operatorname{div}_L \left(\mathcal{A}(x,|u|,\nabla_L |u|)\right) \geq \operatorname{sign} u \ f \qquad \textit{on} \quad \Omega. \tag{2.8}$$

The proof relies on the following lemma.

Lemma 2.2. *Let $\mathcal{A}$ satisfy* (2.1). *Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) \geq f \qquad on \quad \Omega.$$

Let $\gamma \in \mathcal{C}^1(\mathbb{R})$ be nonnegative and such that γ and γ' are bounded. Then,

$$\int_\Omega f\gamma(u)\phi + \int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L u\ \gamma'(u)\phi \leq -\int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L\phi\ \gamma(u). \tag{2.9}$$

In particular, if $\gamma' \geq 0$, we have

$$\mathrm{div}_L\,(\gamma(u)\mathcal{A}(x,u,\nabla_L u)) \geq \gamma(u)f \qquad on \quad \Omega. \tag{2.10}$$

Moreover, if

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) = f \qquad on \quad \Omega, \tag{2.11}$$

then (2.10) *holds provided $\gamma' \geq 0$ regardless of the nonnegativity assumption on γ.*

Proof. For simplicity we shall omit the arguments of $\mathcal{A}$. So we shall write $\mathcal{A}$ instead of $\mathcal{A}(x,u,\nabla_L u)$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a nonnegative $\mathcal{C}^1(\mathbb{R})$ function such that γ and γ' are bounded by a constant M. Let $u_\eta := u \star m_\eta$ be a mollified family of the solution u. Choosing $\gamma(u_\eta)\phi$, as a test function, from the definition of weak solution we get

$$\int_\Omega f\gamma(u_\eta)\phi \leq -\int_\Omega \mathcal{A}\cdot\nabla_L\phi\ \gamma(u_\eta) - \int_\Omega \mathcal{A}\cdot\nabla_L u_\eta\ \gamma'(u_\eta)\phi.$$

Now, it is easy to check that the following claims hold:

i) $|f\gamma(u_\eta)\phi| \leq |f|\,\phi M \in L^1(\Omega)$

ii) $|\mathcal{A}\ \nabla_L\phi\gamma(u_\eta)| \leq |\mathcal{A}|\,|\nabla_L\phi|\,M \in L^1_{loc}(\Omega)$

iii) $|\mathcal{A}\ \nabla_L u_\eta\gamma'(u_\eta)\phi| \leq |\mathcal{A}|\,M\,|\nabla_L u_\eta|$ with $|\mathcal{A}| \in L^{p'}_{loc}(\Omega)$, and $\nabla_L u_\eta \to \nabla_L u$ in $L^p_{loc}(\Omega)$,[1]

$$\mathcal{A}\ \gamma'(u_\eta) \to \mathcal{A}\ \gamma'(u) \quad in \quad L^{p'}_{loc}(\Omega). \tag{2.12}$$

Since $\mathcal{A} \in L^{p'}_{loc}(\Omega)$ we have the following pointwise convergence: $\gamma'(u_\eta) \to \gamma'(u)$, and

$$\left|\mathcal{A}\ \gamma'(u_\eta)\right| \leq |\mathcal{A}|\,M \in L^{p'}_{loc}(\Omega).$$

Thus,

$$\int_\Omega f\gamma(u)\phi + \int_\Omega \mathcal{A}\cdot\nabla_L u\ \gamma'(u)\phi \leq -\int_\Omega \mathcal{A}\cdot\nabla_L\phi\ \gamma(u).$$

---

[1] This follows from the fact that since $\nabla_L = \mu\nabla$ and μ has continuous entries and $\nabla u_\eta \to \nabla u$ in $L^p_{loc}(\Omega)$.

If $\gamma' \geq 0$, from (2.1) we get

$$\int_\Omega f\gamma(u)\phi \leq -\int_\Omega \mathcal{A}\cdot\nabla_L\phi\ \gamma(u). \tag{2.13}$$

This proves (2.10).

If u is a solution of (2.11) then the same proof above applies regardless of the nonnegativity of γ. □

Proof of Theorem 2.1. In order to prove (2.3) it suffices to approximate sign^+ with a family of nonnegative smooth bounded functions which are nondecreasing and with bounded derivative.

To this end we introduce

$$\gamma_\epsilon(t) := \begin{cases} \left(\frac{2}{\pi}\arctan(t/\epsilon)\right)^2, & \text{if } t \geq 0;\\ 0 & \text{if } t < 0.\end{cases}$$

Then $0 \leq \gamma_\epsilon < 1$ and $\gamma_\epsilon(t) \to \text{sign}^+(t)$. Applying Lemma 2.2, from (2.10) with γ replaced by γ_ϵ we obtain

$$\int_\Omega f\gamma_\epsilon(u)\phi \leq -\int_\Omega \gamma_\epsilon(u)\mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L\phi. \tag{2.14}$$

Passing to the limit $\epsilon \to 0$ in (2.14), by Lebesgue's dominated convergence theorem we finally obtain (2.3), i.e.,

$$\int_\Omega \text{sign}^+(u)f\phi \leq -\int_\Omega \text{sign}^+(u)\mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L\phi. \tag{2.15}$$

In addition, if $\mathcal{A}$ is (WE) from the identity

$$\text{sign}^+(u)\ \mathcal{A}(x,u,\nabla_L u) = \mathcal{A}(x,u^+,\nabla_L u^+) \text{ on } \Omega,$$

inequality (2.6) follows.

The proof of (2.5) follows once again by applying the above argument to the family of functions $\gamma_\epsilon(t) := \frac{2}{\pi}\arctan(t/\epsilon)$. □

Remark 2.3. Under the hypotheses of Theorem 2.1, inequality (2.3) holds by replacing sign^+ with $\text{sign}^+_h := \text{sign}^+(\cdot - h)$. More precisely, if $\mathcal{A} = \mathcal{A}(x,\xi)$ and $\mathcal{A}(x,0) = 0$, and $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (2.2), then for any $h \in \mathbb{R}$, $u^+_h := \text{sign}^+_h(u)$ is a weak solution of

$$\text{div}_L\left(\mathcal{A}(x,\nabla_L u^+_h)\right) \geq \text{sign}^+_h uf \qquad \text{on } \Omega. \tag{2.16}$$

This fact follows from the proof of Theorem 2.1 by replacing γ_ϵ with $\gamma_\epsilon(t-h)$.

The following result is fairly easy to prove.

Theorem 2.4. *Let $\mathcal{A}=\mathcal{A}(x,\xi)$ be* (WE). *Let $u\in W^{1,p}_{loc}(\Omega)$ be a weak solution of*

$$\mathrm{div}_L(\mathcal{A}(x,\nabla_L u))\geq f \qquad \text{on} \quad \Omega. \tag{2.17}$$

Then for any $h\in\mathbb{R}$, u_h^+ is a weak solution of

$$\mathrm{div}_L\left(\mathcal{A}(x,\nabla_L u_h^+)\right)\geq \mathrm{sign}_h^+ u\ f \qquad \text{on} \quad \Omega. \tag{2.18}$$

Remark 2.5. If ∇_L is the horizontal vector field in a Carnot group, then Theorem 2.1 and Lemma 2.2 hold for solutions that belong to the wider space $W^{1,p}_{L,loc}(\Omega):=\{u\in L^p_{loc}(\Omega):\ |\nabla_L u|\in L^p_{loc}(\Omega)\}$.

Definition 2.6. *Let $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^l\to\mathbb{R}^l$, be a Caratheodory function. We say that $\mathcal{A}$ is* monotone, *if for any $u,v\in W^1_{loc}(\Omega)$ we have*

$$(\mathcal{A}(x,u(x),\nabla_L u(x))-\mathcal{A}(x,v(x),\nabla_L v(x)))\cdot(\nabla_L u(x)-\nabla_L v(x))\geq 0 \tag{M}$$

for almost every $x\in\Omega$. We say that $\mathcal{A}$ is strictly monotone *if $\mathcal{A}$ is* monotone *and if the condition*

$$(\mathcal{A}(x,u(x),\nabla_L u(x))-\mathcal{A}(x,v(x),\nabla_L v(x)))\cdot(\nabla_L u(x)-\nabla_L v(x))=0$$

for almost every $x\in\Omega$, implies $u(x)=v(x)$ or

$$\nabla_L u(x)=\nabla_L v(x) \quad a.e.\ x\in\Omega.$$

Examples of strictly monotone operators are the mean curvature operator, the p-Laplacian operator (see Examples 1.3), or more generally, an operator generated by $\mathcal{A}(x,t,\xi)=A(|\xi|)\xi$, where the function $h(s):=sA(s)$ is positive and increasing (if h is nondecreasing, then $\mathcal{A}$ is monotone); see [39].

Dealing with the comparison of the solutions u and v of the inequalities

$$\begin{cases}\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right)\geq f & \text{on } \Omega,\\ \mathrm{div}_L\left(\mathcal{A}(x,v,\nabla_L v)\right)\leq g & \text{on } \Omega,\end{cases} \tag{2.19}$$

the following may be useful. See the proof of Theorem 4.8 in the next section.

Theorem 2.7. *Let $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^l\to\mathbb{R}^l$ be a monotone Caratheodory function. Let $f,g\in L^1_{loc}(\Omega)$ and let u,v be weak solutions of*

$$-\mathrm{div}_L\left(\mathcal{A}(x,v,\nabla_L v)\right)+g\geq-\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right)+f \quad \text{on } \Omega.$$

Let $\gamma\in\mathcal{C}^1(\mathbb{R})$ be such that $0\leq\gamma(t),\gamma'(t)\leq M$; then

$$\mathrm{div}_L\left(\gamma(u-v)(\mathcal{A}(x,u,\nabla_L u)-\mathcal{A}(x,v,\nabla_L v))\right)\geq\gamma(u-v)(f-g) \quad \text{on } \Omega.$$

Moreover,

$$\mathrm{div}_L\left(\mathrm{sign}^+(u-v)(\mathcal{A}(x,u,\nabla_L u)-\mathcal{A}(x,v,\nabla_L v))\right)\geq\mathrm{sign}^+(u-v)(f-g) \quad \text{on } \Omega. \tag{2.20}$$

The proof is a slight modification of the proof Theorem 2.1. We omit the details.

3. Examples

Kato's inequality (2.6) holds for all (WE) operators. In particular it holds for all operators listed in Examples 1.3. In this section we illustrate with some detail some classes of operators for which (2.6) holds.

3.1. **p-Laplacian-type operators.** Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a solution of the inequality

$$L_p u := \operatorname{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) \geq f \quad \text{ on } \Omega.$$

Then,

$$L_p u^+ \geq \operatorname{sign}^+(u) f \quad \text{ on } \Omega.$$

In particular, if ∇_L is the Euclidean gradient ∇ and $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of

$$\Delta_p u \geq f \quad \text{ on } \Omega,$$

then $u^+ \in W^{1,p}_{loc}(\Omega)$ is a weak solution of

$$\Delta_p u^+ \geq \operatorname{sign}^+(u) f \quad \text{ on } \Omega. \tag{3.1}$$

As a consequence of (3.1), we have the following simple quasilinear version of a result due to Brezis (see the Introduction of this paper and [7]).

For related results under stronger assumptions on the solutions see [44, 18].

Theorem 3.1. *Let $q > p - 1 > 0$. If $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ is a weak solution of*

$$\Delta_p u \geq |u|^{q-1} u \quad \text{on } \mathbb{R}^N, \tag{3.2}$$

then $u \leq 0$ almost everywhere on $\mathbb{R}^N$. In particular, if in (3.2) the equality sign holds, then $u \equiv 0$ almost everywhere on $\mathbb{R}^N$.

Proof. Let $q > p - 1$ and set

$$u_R(x) := \frac{cR^\beta}{(R^{p/(p-1)} - |x|^{p/(p-1)})^\alpha} \qquad x \in B_R,$$

with

$$\alpha := \frac{p}{q-p+1}, \qquad \beta := \begin{cases} 0 & \text{if } q \leq 1, \\ \frac{\alpha p}{p-1} - \frac{p}{q-1} & \text{if } q > 1, \end{cases}$$

and the positive constant c satisfying $c^{q-p+1} = (\frac{\alpha p}{p-1})^{p-1} \max\{N, p(\alpha+1)\}$.

The function u_R is a slight modification of the Loewner–Nirenberg [27] function used by Brezis in his original argument [7] for $p = 2$. It is easy to check that for $R > 0$, u_R is a solution of the inequality

$$-\Delta_p u_R + u_R^q \geq 0 \quad \text{on } B_R.$$

Indeed,

$$\frac{\Delta_p u_1}{u_1^q} = \Big(\frac{\alpha p}{p-1}\Big)^{p-1} c^{-q+p-1}[N + (p\alpha + p - N)r^{p/(p-1)}] \leq 1.$$

Now, since

$$u_R = \frac{cR^{\beta-\alpha p/(p-1)}}{(1-(\frac{|x|}{R})^{p/(p-1)})^\alpha} = R^{\beta-\alpha p/(p-1)} u_1(\frac{|x|}{R}),$$

for $R \geq 1$ we have

$$\frac{\Delta_p u_R}{u_R^q} = \frac{R^{\beta-\alpha p/(p-1)} R^{-p} (\Delta_p u_1)(|x|/R)}{R^{q(\beta-\alpha p/(p-1))} u_1^q(|x|/R)} \leq R^{(1-q)(\beta-\alpha p/(p-1))-p} \leq 1.$$

Let $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ be a weak solution of (3.2). Applying inequality (3.1) it follows that in the weak sense we have

$$\Delta_p u^+ \geq (u^+)^q \quad \text{on } \mathbb{R}^N.$$

Since u^+ is p-subharmonic, from [29] we deduce $u^+ \in L^\infty_{loc}(\mathbb{R}^N)$. By the weak comparison principle we deduce that for any $R > 1$ we have $u^+ \leq u_R$ almost everywhere on B_R. Since $u_R \to 0$ for $R \to +\infty$, it follows that $u^+ \leq 0$ almost everywhere on $\mathbb{R}^N$. This completes the proof. □

We point out that the above proof is based on three fundamental ingredients, namely the explicit knowledge of a Loewner–Nirenberg function u_R, weak Harnack inequality for subsolutions (see [29]), and a comparison principle. The explicit construction of a barrier function can be done when the differential operator satisfies some kind of homogeneity. See [43] for a thorough discussion on existence of barrier functions. In this respect it would be interesting to know whether the above proof can be adapted to the subelliptic framework.

We point out that an analogue of Theorem 3.1 still holds by using a different set of ideas; see Theorem 8.1 below.

Other related results for inequalities of the type (3.2) associated to more general nonlinearities appear in [16] and in the forthcoming sections; see Part 3.

3.2. 1**-Laplacian-type operators.** Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,1}_{loc}(\Omega)$ be a solution of the inequality

$$L_1 u := \operatorname{div}_L\big(|\nabla_L u|^{-1} \nabla_L u\big) \geq f \quad \text{on } \Omega.$$

In this case, we notice that $\mathcal{A}(x,u,\xi) = \frac{\xi}{|\xi|}$. Since $\mathcal{A}$ is not continuous we cannot apply directly Theorem 2.1. However, a slight modification of the proof of the latter allows us to conclude that

$$\operatorname{div}_L\Big(\operatorname{sign}^+ u \,\frac{\nabla_L u^+}{|\nabla_L u^+|}\Big) \geq \operatorname{sign}^+(u) f \quad \text{on } \Omega.$$

In particular if ∇_L is the Euclidean gradient ∇ and $u \in W^{1,1}_{loc}(\Omega)$ is a weak solution of

$$\Delta_1 u \geq f \quad \text{on } \Omega,$$

then $u^+ \in W^{1,1}_{loc}(\Omega)$ is a weak solution of

$$\operatorname{div}\Big(\operatorname{sign}^+ u \,\frac{\nabla u^+}{|\nabla u^+|}\Big) \geq \operatorname{sign}^+(u) f \quad \text{on } \Omega.$$

3.3. **Mean-curvature-type operators.** Now we shall consider the operator T modelled on the mean-curvature operator, namely

$$Tu := \operatorname{div}_L\Big(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\Big).$$

Here the function $A(x,u,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}$ is bounded. Therefore, a notion of solution of the inequality $Tu \geq f$ can be defined for functions belonging to $W^{1,p}_{loc}(\Omega)$ for any $p \geq 1$.

Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a solution of the inequality

$$Tu := \operatorname{div}_L\Big(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\Big) \geq f \quad \text{on } \Omega.$$

Then,

$$T(u^+) \geq \operatorname{sign}^+(u) f \quad \text{on } \Omega.$$

In particular, if ∇_L is the Euclidean gradient ∇ and $u \in W^{1,p}_{loc}(\Omega)$ satisfies

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \geq f \quad \text{on } \Omega,$$

then $u^+ \in W^{1,p}_{loc}(\Omega)$ is a weak solution of

$$\operatorname{div}\Big(\frac{\nabla u^+}{\sqrt{1+|\nabla u^+|^2}}\Big) \geq \operatorname{sign}^+(u) f \quad \text{on } \Omega.$$

Part 2. The reduction principles

In the following sections, unless otherwise stated, Ω stands for an open subset contained in $\mathbb{R}^N$ and $\mathcal{A}$ is (WE).

4. The role of positive solutions

Here we are going to develop the main ideas that we shall use throughout this paper when studying quasilinear elliptic inequalities of coercive type. It is known [16] that dealing with noncoercive problems of the form

$$-\operatorname{div}_L(\mathcal{A}(x,v,\nabla_L v)) \geq f(x,v), \qquad v \geq 0, \quad \text{on } \mathbb{R}^N, \tag{4.1}$$

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative function, the existence or nonexistence of positive solutions in a suitable functional space is determined only by the behavior of the nonlinearity f near zero. On the other hand, in the coercive case, that is,

$$\operatorname{div}_L(\mathcal{A}(x,v,\nabla_L v)) \geq g(x,v) \quad \text{on } \mathbb{R}^N, \tag{4.2}$$

where $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a given function, a first step for understanding the solutions set is to reduce our problem to an inequality with solutions having a definite sign. A remarkable fact is that this reduction is always possible for weakly elliptic quasilinear inequalities, even though, as we shall see during the course, this reduction leads to nontrivial problems in finding *good a priori* estimates on the possible nonnegative solutions of (4.2).

In keeping with the notation and terminology introduced above, we have

Theorem 4.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function satisfying*

$$f(x,0,\xi) = 0 \quad \text{or} \quad f(x,t,0) = 0. \tag{4.3}$$

Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$, then $u^+ \in X$. Assume that the problem

$$\operatorname{div}_L\left(\mathcal{A}(x,v,\nabla_L v)\right) \geq f(x,v,\nabla_L v) \qquad v \geq 0 \quad \text{on } \Omega, \tag{4.4}$$

has no nontrivial weak solutions in X. Then any weak solution of the problem

$$\operatorname{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) \geq f(x,u,\nabla_L u) \qquad u \in X, \tag{4.5}$$

is nonpositive; i.e., $u(x) \leq 0$ for almost every $x \in \Omega$.

Proof. Let $u \in X$ be a solution of (4.5). By inequality (2.6) and by hypothesis (4.3) it follows that

$$\operatorname{div}_L\left(\mathcal{A}(x,u^+,\nabla_L u^+)\right) \geq \operatorname{sign}^+ u\, f(x,u,\nabla_L u) = f(x,u^+,\nabla_L u^+) \quad \text{on } \Omega.$$

Hence $u^+ \in X$ is a nonnegative solution of (4.4). Thus $u^+ \equiv 0$ almost everywhere on Ω. This completes the proof. □

Remark 4.2. Notice that in the above result we do not suppose that f is nonnegative.

In what follows, for a given function $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$, we shall denote with $\overline{\mathcal{A}}$ the function $\overline{\mathcal{A}} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ defined by

$$\overline{\mathcal{A}}(x,t,\xi) := -\mathcal{A}(x,-t,-\xi). \tag{4.6}$$

Notice that if $\mathcal{A}$ is weakly elliptic or **W**-p-**C** or **S**-p-**C** then $\overline{\mathcal{A}}$ has the same properties. Moreover, if $\mathcal{A}$ is odd (see (2.7)), then $\overline{\mathcal{A}} = \mathcal{A}$.

An immediate implication of the above theorems is the following obvious consequence for noncoercive problems.

Theorem 4.3. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function satisfying (4.3). Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$, then $-u, u^+ \in X$. Assume that the problem*

$$\operatorname{div}_L\left(\overline{\mathcal{A}}(x,v,\nabla_L v)\right) \geq f(x,-v,-\nabla_L v) \qquad v \geq 0 \quad \text{on } \Omega, \tag{4.7}$$

has no nontrivial weak solutions in X. Then any weak solution of the problem

$$-\operatorname{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) \geq f(x,u,\nabla_L u) \quad \text{on } \Omega, \quad u \in X, \tag{4.8}$$

is nonnegative i.e., $u(x) \geq 0$ for almost every $x \in \Omega$.

Proof. Let $u \in X$ be a solution of (4.8). The function $w := -u \in X$ is a solution of

$$\operatorname{div}_L\left(\overline{\mathcal{A}}(x,v,\nabla_L v)\right) \geq f(x,-v,-\nabla_L v) \quad \text{on } \Omega, \quad v \in X.$$

Since $\overline{\mathcal{A}}$ is weakly elliptic and f satisfies (4.3) we are in the position to apply Theorem 4.1, which yields $w \leq 0$ on Ω. This completes the proof. □

Theorem 4.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying (4.3) and set $\overline{f}(x,t,\xi) = -f(x,-t,-\xi)$. Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$, then $-u, u^+ \in X$. Assume that the problems*

$$\operatorname{div}_L\left(\mathcal{A}(x,v,\nabla_L v)\right) \geq f(x,v,\nabla_L v), \qquad v \geq 0, \quad \text{on } \Omega, \tag{4.9}$$

$$\mathrm{div}_L\left(\overline{\mathcal{A}}\,(x,v,\nabla_L v)\right) \geq \overline{f}(x,v,\nabla_L v), \qquad v \geq 0, \quad on\ \Omega, \tag{4.10}$$

have no nontrivial weak solutions in X. Then the problem

$$\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) = f(x,u,\nabla_L u) \quad on\ \Omega, \quad u \in X, \tag{4.11}$$

has no nontrivial weak solutions.

Proof. Let $u \in X$ be a weak solution of (4.11). An application of Theorem 4.1 implies that $u \leq 0$ on Ω. Therefore $w := -u$ is a solution of

$$-\mathrm{div}_L\left(\mathcal{A}(x,-w,-\nabla_L w)\right) = -f(x,-w,-\nabla_L w) \quad \text{on } \Omega.$$

In other words w solves (4.10). By applying again Theorem 4.1 we complete the proof. □

Corollary 4.5. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be an odd Caratheodory function. Suppose that $\mathcal{A}$ is an odd and weakly elliptic function. Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$, then $-u, u^+ \in X$. Assume that the inequality* (4.9) *has no nontrivial weak solutions in X. Then the problem* (4.11) *has no nontrivial weak solutions in X.*

Remark 4.6. Dealing with the horizontal gradient on a Carnot group, the above Theorems 4.1, 4.3, and 4.4 can be formulated for solutions belonging to $X \subset W^{1,p}_{L,loc}(\Omega)$.

4.1. **Applications: maximum and comparison principles.** Although it is not exactly the direction in which we have been going, it seems appropriate to include here some interesting examples and applications of the reduction ideas.

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u, v \in L^1_{loc}(\Omega)$. In what follows the inequality $u \leq v$ on $\partial\Omega$ should be understood in the sense that for every $\epsilon > 0$ there exists a neighborhood V of $\partial\Omega$ such that for almost every $x \in V$ we have $u(x) \leq v(x) + \epsilon$.

Moreover, we shall need of the following hypothesis on ∇_L.

$$\text{If } O \subset \mathbb{R}^N \text{ is an open connected set and } \nabla_L u \equiv 0 \;\Rightarrow u \equiv \text{const on } O. \tag{4.12}$$

This assumption obviously holds if $\nabla_L = \nabla$, the standard Euclidean gradient. It also holds in the Carnot group setting as well as in all the examples of Appendix B except for the gradient of l variables; see Example B.1. A general condition assuring the validity of (4.12) is related to the Hörmander condition and to Caratheodory–Chow–Rashevsky theorem; see [5].

Theorem 4.7 (The weak maximum principle)**.** *Let $\mathcal{A}$ be weakly elliptic such that for almost every $x \in \Omega$,*

$$\text{if } \mathcal{A}(x,t,\xi) = 0, \text{ then } t = 0 \text{ or } \xi = 0. \tag{4.13}$$

Assume that (4.12) *holds. Let $p \geq 1$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of*

$$\operatorname{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) \geq 0 \quad \text{on } \Omega.$$

Suppose that $\Omega' \subset\subset \Omega$ and $u \leq 0$ on $\partial\Omega'$. Then $u \leq 0$ almost everywhere on Ω'.

Proof. We apply the reduction principle 4.1 with a natural choice of X. Indeed, consider the subset of $W^{1,p}(\Omega')$ defined by

$$X(\Omega') := \{v \in W^{1,p}(\Omega') : v \leq 0, \text{ on } \partial\Omega'\}.$$

If we prove that the only nonnegative solution v of the inequality

$$\operatorname{div}_L\left(\mathcal{A}(x,v,\nabla_L v)\right) \geq 0 \qquad v \in X(\Omega'), \tag{4.14}$$

is $v \equiv 0$ on Ω', then an application of the reduction principle will imply $u \leq 0$ almost everywhere on Ω'.

Indeed, let $v \in X(\Omega')$ be a nonnegative solution of (4.14). Then for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega')$ we have

$$0 \leq -\int_{\Omega'} \mathcal{A}(x,v,\nabla_L v) \cdot \nabla_L \phi.$$

Since $v \leq 0$ on $\partial\Omega'$ and it is nonnegative in Ω', it follows that for $\epsilon > 0$ the function $(v-\epsilon)^+$ has compact support on Ω'. Now we can choose as test function ϕ_η a mollification of $(v-\epsilon)^+$. Letting $\eta \to 0$ we have

$$0 \leq -\int_{\Omega'} \mathcal{A}(x,v,\nabla_L v) \cdot \nabla_L (v-\epsilon)^+ \leq 0.$$

Hence,

$$\mathcal{A}(x,v(x),\nabla_L v(x)) \cdot \nabla_L (v(x)-\epsilon)^+ = 0 \qquad \text{a.e. } x \in \Omega'.$$

Therefore, if $v(x) \leq \epsilon$ then $\nabla_L(v(x)-\epsilon)^+ = 0$. If $v(x) > \epsilon$, then $\nabla_L(v(x)-\epsilon)^+ = 0$, or from (4.13), $\nabla_L v(x) = 0$; that is, $\nabla_L(v(x)-\epsilon)^+ = 0$. In any case we have that $\nabla_L(v(x)-\epsilon)^+ = 0$, which, by (4.12), implies that in any connected region of Ω' the function $(v-\epsilon)^+$ is equal to some constant C. Since $(v-\epsilon)^+$ vanishes on the boundary, we deduce that $C = 0$. Thus, for every $\epsilon > 0$ we have $v \leq \epsilon$. Hence $v = 0$ on Ω'. □

For a thorough study of maximum and comparison principles for quasilinear elliptic operators in the Euclidean framework we refer the reader to the recent interesting monograph [39].

Theorem 4.8 (The weak comparison principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $\mathcal{A}$ be a monotone function. Let $f, g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be Caratheodory functions such that*

$$f(x,t) \geq g(x,t), \qquad a.e.\ x \in \Omega, \quad t \in \mathbb{R}, \tag{4.15}$$

and at least one of them is nondecreasing with respect to the t variable. Assume that (4.12) *holds and one of the following conditions*

(1) *$\mathcal{A}$ is strictly monotone;*
(2) *$f(x,t)$ or $g(x,t)$ is increasing with respect to t variable;*

is satisfied.

Let $u, v \in W^{1,p}_{loc}(\Omega)$ be such that

$$-\operatorname{div}_L(\mathcal{A}(x,v,\nabla_L v)) + g(x,v) \geq -\operatorname{div}_L(\mathcal{A}(x,u,\nabla_L u)) + f(x,u). \tag{4.16}$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ almost everywhere in Ω.

Proof. The idea is to apply Theorem 2.7. Indeed, since the right-hand side of (2.20) is given by

$$\operatorname{sign}^+(u-v)(f(x,u(x)) - g(x,v(x))) \geq 0,$$

then for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$, from (2.20) we have

$$\int_\Omega \operatorname{sign}^+(u-v)(\mathcal{A}(x,u,\nabla_L u) - \mathcal{A}(x,v,\nabla_L v)) \cdot \nabla_L \phi \leq 0.$$

Let $\epsilon > 0$. Since $u \leq v$ on $\partial\Omega$ the function $\psi := (u-v-\epsilon)^+ \in W^{1,p}(\Omega)$ and it has compact support. Choosing ϕ_η a mollification of ψ as test function, we obtain

$$\int_\Omega \operatorname{sign}^+(u-v)(f(x,u(x)) - g(x,v(x)))\phi_\eta + \int_\Omega \operatorname{sign}^+(u-v)(\mathcal{A}(x,u,\nabla_L u) - \mathcal{A}(x,v,\nabla_L v)) \cdot \nabla_L \phi_\eta \leq 0.$$

Hence, by letting $\eta \to 0$,

$$\int_\Omega \operatorname{sign}^+(u-v)(f(x,u(x)) - g(x,v(x)))(u-v-\epsilon)^+ + \int_\Omega \operatorname{sign}^+(u-v)(\mathcal{A}(x,u,\nabla_L u) - \mathcal{A}(x,v,\nabla_L v)) \cdot \nabla_L (u-v-\epsilon)^+ \leq 0.$$

By the monotonicity of $\mathcal{A}$ and (4.15) we deduce that each integral appearing in the above inequality is nonnegative. Therefore, setting $P = \{x \in \Omega :$

$u(x) > v(x)\}$ we have

$$\int_P (\mathcal{A}(x,u,\nabla_L u) - \mathcal{A}(x,v,\nabla_L v)) \cdot \nabla_L (u-v) = 0, \tag{4.17}$$

$$\int_P (f(x,u(x)) - g(x,v(x)))(u-v-\epsilon)^+ = 0. \tag{4.18}$$

From (4.17) it follows that

$$(\mathcal{A}(x,u,\nabla_L u) - \mathcal{A}(x,v,\nabla_L v)) \cdot (\nabla_L u - \nabla_L v) = 0 \quad \text{on } P.$$

If (1) holds, then by the strict monotonicity of $\mathcal{A}$ we obtain $\nabla_L u = \nabla_L v$ on P. Therefore, $\nabla_L (u-v)^+ \equiv 0$ in Ω. Hence $\nabla_L (u-v-\epsilon)^+ \equiv 0$. That is, $(u-v-\epsilon)^+$ is constant on the connected components of Ω. Since $(u-v-\epsilon)^+$ has compact support contained in Ω, we deduce that $(u-v-\epsilon)^+ \equiv 0$ in Ω. This shows that $u \le v+\epsilon$ almost everywhere in Ω. This completes the proof of the claim if (1) holds.

Next, suppose that (2) holds. Assume that $f(x,t)$ is increasing with respect to t, the case when $g(x,t)$ is increasing being similar. Then from (4.18) we get that necessarily $u - v - \epsilon \le 0$ on P. Since this is true for any ϵ we get $u \le v$ almost everywhere on Ω. This completes the proof. □

Theorem 4.9 (A generalized weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and suppose* (4.12) *holds. Suppose that there exists a Caratheodory function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ and a constant $\overline{\lambda} > 0$ such that for any nonnegative $v \in W_0^{1,p}(\Omega)$ we have $G(\cdot, v(\cdot)) \in L^1(\Omega)$ and*

$$\int_\Omega \mathcal{A}(x,v,\nabla_L v) \cdot \nabla_L v \ge \overline{\lambda} \int_\Omega G(x,v) \quad \textit{for any } v \ge 0,\ v \in W_0^{1,p}(\Omega). \tag{4.19}$$

Assume that either (4.13) *holds or*

$$\textit{if } v \in W_0^{1,p}(\Omega), v \ge 0, v \not\equiv 0 \quad \Rightarrow \quad \int_\Omega G(x,v) > 0. \tag{4.20}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $g(x,0) = 0$, and $0 \le g(x,t)t \le G(x,t)$ for $t > 0$. Let $u \in W^{1,p}(\Omega)$ be a weak solution of

$$\operatorname{div}_L (\mathcal{A}(x,u,\nabla_L u)) + \lambda g(x,u) \ge 0, \quad \textit{on } \Omega, \quad u \le 0 \textit{ on } \partial\Omega.$$

i) *If $\lambda < \overline{\lambda}$, then $u \le 0$ almost everywhere on Ω.*
ii) *If $\lambda = \overline{\lambda}$ and the constant $\overline{\lambda}$ in* (4.19) *is not achieved in $W_0^{1,p}(\Omega)$, then $u \le 0$ almost everywhere on Ω.*

Proof. The idea is to apply again Theorem 4.1 with the natural choice $X = W_0^{1,p}(\Omega)$. To this end it is enough to show that the problem

$$\mathrm{div}_L\,(\mathcal{A}(x,v,\nabla_L v)) + \lambda g(x,v) \geq 0, \quad \text{on } \Omega, \quad v \geq 0,\ v \in X$$

has only the trivial solution. Let v be a nontrivial solution of the above inequality. For simplicity, in what follows we shall omit the arguments of $\mathcal{A}$. So we shall write $\mathcal{A}$ instead of $\mathcal{A}(x,v,\nabla_L v)$. For any nonnegative $\phi \in \mathcal{C}_0^1(\Omega)$ we have

$$\int_\Omega \mathcal{A}\cdot\nabla_L\phi \geq \lambda\int_\Omega g(v)\phi \geq 0.$$

Since $v \in W_0^{1,p}(\Omega)$, we can approximate v by a sequence of nonnegative test functions $(\phi_n)_n$. In this case we have $\nabla_L\phi_n \to \nabla_L v$ in $L^p(\Omega)$ and $\phi_n \to v$. Therefore, by Fatou's lemma we obtain

$$\int_\Omega \mathcal{A}\cdot\nabla_L v \geq \lambda\int_\Omega g(v)v,$$

and

$$0 \leq \lambda\int_\Omega g(v)v - \int_\Omega \mathcal{A}\cdot\nabla_L v \leq \lambda\int_\Omega G(v) - \int_\Omega \mathcal{A}\cdot\nabla_L v \leq (\lambda-\overline{\lambda})\int_\Omega G(v). \quad (4.21)$$

We claim that $\int G(v) = 0$. Indeed, suppose $\int G(v) > 0$. If $\lambda < c_g/\overline{\lambda}$, then from (4.21) we reach a contradiction. On the other end if $\lambda = \overline{\lambda}$, then from (4.21) it follows that

$$\int_\Omega \mathcal{A}\cdot\nabla_L v \ = \bar{\lambda}\int_\Omega G(v).$$

In other words, $\bar{\lambda}$ is achieved in $W_0^{1,p}(\Omega)$. This is again a contradiction with our assumption.

Now, if (4.20) holds, from $\int G(v) = 0$, we have $v \equiv 0$.

Next, if (4.13) holds, then from (4.21) we obtain that $\int \mathcal{A}\cdot\nabla_L v = 0$, which implies that $v \equiv 0$ (see the proof of Theorem 4.7). □

Remark 4.10. When dealing with the horizontal gradient on a Carnot group, the above results can be formulated for solutions belonging to $W_{L,0}^{1,p}(\Omega)$.

Remark 4.11. It is clear that in the above theorem if we replace the hypothesis $0 \leq g(x,t)t \leq G(x,t)$ with $0 \leq g(x,t)t \leq c_g G(x,t)$ for some $c_g > 0$ then in the conclusions i) and ii) we need to replace λ with λc_g.

As an example of (4.19) consider the following p-Laplacian-type operator,

$$L_p u := \mathrm{div}_L\left(|\nabla_L u|^{p-2}\,\nabla_L u\right).$$

The following Hardy inequality will play an important role in what follows (see [12] for the proof and several other results).

Theorem 4.12. *Let $p > 1$. Let $d : \Omega \to \mathbb{R}$ be a nonnegative nonconstant measurable function and $\alpha \in \mathbb{R}$, $\alpha \neq 0$ be such that*

$$d^{-p}\,|\nabla_L d|^p\,,\; d^{(\alpha-1)(p-1)}\,|\nabla_L d|^{p-1} \in L^1_{loc}(\Omega).$$

If $-L_p(d^\alpha) \geq 0$ in the weak sense, then for every $u \in \mathcal{C}^1_0(\Omega)$ we have

$$\Big(\frac{|\alpha|\,(p-1)}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p}\,|\nabla_L d|^p\,dx \leq \int_\Omega |\nabla_L u|^p\,dx. \tag{4.22}$$

In particular,

(1) *If ∇_L is the horizontal gradient on a Carnot group $\mathbb{G}$ and S is a homogeneous norm such that $L_p S^{\frac{p-Q}{p-1}} = c\,\delta_0{}^2$ on $\mathbb{G}$ with $Q > p > 1$, then*

$$\Big(\frac{Q-p}{p}\Big)^p \int_{\mathbb{G}} \frac{|u|^p}{S^p}\,|\nabla_L S|^p\,dx \leq \int_{\mathbb{G}} |\nabla_L u|^p\,dx, \quad u \in \mathcal{C}^1_0(\mathbb{G}), \tag{4.23}$$

where the constant $(\frac{Q-p}{p})^p$ is sharp and it is not achieved.

(2) *If the first column of the matrix μ is such that $\mu_{11} = 1$ and $\mu_{k1} = 0$ for $k = 2, \ldots, l^3$ and Ω is bounded in the x_1 direction, then there exists $c > 0$ such that*

$$c^p \int_\Omega |u|^p \leq \int_\Omega |\nabla_L u|^p\,, \quad u \in \mathcal{C}^1_0(\Omega). \tag{4.24}$$

Some direct consequences of Theorem (4.9) are the following:

Theorem 4.13. *Let ∇_L be the horizontal gradient on a Carnot group $\mathbb{G}$. Let $Q > p > 1$ and let S be a homogeneous norm such that $L_p S^{\frac{p-Q}{p-1}} = c\,\delta_0$ on $\mathbb{G}$.*

Let $\Omega \subset \mathbb{G}$ be a bounded open set. Let $u \in W^{1,p}_L(\Omega)$ be a weak solution of

$$L_p u + \lambda \frac{|\nabla_L S|^p}{S^p}\,|u|^{p-2}\,u \geq 0 \quad \text{on } \Omega, \quad u \leq 0 \text{ on } \partial\Omega,$$

with $\lambda \leq (\frac{Q-p}{p})^p$. Then $u \leq 0$ almost everywhere on Ω.

---

[2] In the Euclidean setting S is the Euclidean norm.

[3] This condition is satisfied if ∇_L is the horizontal gradient on a Carnot group; see [5].

Theorem 4.14. *Let $p > 1$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that the first column of the matrix μ is such that $\mu_{11} = 1$ and $\mu_{k1} = 0$ for $k = 2, \dots, l$. Then, there exists a constant $\lambda(\Omega, p) > 0$ such that if $\lambda < \lambda(\Omega, p)$ and $u \in W^{1,p}(\Omega)$ is a weak solution of*

$$L_p u + \lambda \, |u|^{p-2} \, u \geq 0 \quad \text{on } \Omega, \quad u \leq 0 \text{ on } \partial\Omega,$$

then $u \leq 0$ almost everywhere on Ω.

For related interesting results in the Euclidean framework, see the earlier contribution by Damascelli [11].

Another simple application of Theorem (4.9) is the following.

Theorem 4.15. *Let ∇ be the Euclidean gradient on $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and $p > 1$. Set*

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) \quad x \in \Omega.$$

Then there exists $\lambda(\Omega, p) > 0$ such that if $\lambda < \lambda(\Omega, p)$ and $u \in W^{1,p}(\Omega)$ is a weak solution of

$$\Delta_p u + \lambda \frac{|u|^{p-2} \, u}{\delta^p} \geq 0 \quad \text{on } \Omega, \quad u \leq 0 \text{ on } \partial\Omega,$$

then $u \leq 0$ almost everywhere on Ω. Moreover, $0 < \lambda(\Omega, p) \leq (\frac{p-1}{p})^p$.

The proof is based on the Hardy inequality

$$\int_\Omega |\nabla u|^p \geq \lambda(\Omega, p) \int_\Omega \frac{|u|^p}{\delta^p}. \tag{4.25}$$

It is known that the best constant $\lambda(\Omega, p)$ in (4.25) is such that $\lambda(\Omega, p) \leq (\frac{p-1}{p})^p$ and if Ω is convex then $\lambda(\Omega, p) = (\frac{p-1}{p})^p$. See [30]. Notice that if $\lambda = \lambda(\Omega, p)$, then the above theorem holds provided $\lambda(\Omega, p)$ is not achieved. For important contributions on Hardy inequalities we refer the interested reader to [1, 30, 12, 26, 31] and the references therein.

Remark 4.16. Notice that in Theorem 4.8 we require that (4.19) holds only for nonnegative functions and we do not require any behavior on $g(x, t)$ for $t < 0$. Therefore, Theorems 4.13, 4.14, and 4.15 still hold if we replace the function $|u|^{p-2} \, u$ with

$$g(u) = \begin{cases} u^{p-1} & \text{if } u \geq 0, \\ -f(u) & \text{if } u < 0, \end{cases}$$

where $f : (-\infty, 0] \to \mathbb{R}$ is any nonnegative continuous function with $f(0) = 0$.

5. The role of the behavior at infinity of the nonlinearity

The goal of this section is to prove that a large class of elliptic inequalities of the type

$$\operatorname{div}_L(\mathcal{A}(x,\nabla_L u)) + \mathcal{B}(x,u,\nabla_L u) \geq f(x,u,\nabla_L u),$$

where $\mathcal{A}$ is weakly elliptic, can be reduced to the study of

$$\operatorname{div}_L(\mathcal{A}(x,\nabla_L u)) \geq u^q \quad \text{or} \quad \operatorname{div}_L(\mathcal{A}(x,\nabla_L u)) \geq |\nabla_L u|^r.$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Consider

$$\operatorname{div}_L(\mathcal{A}(x,\nabla_L v)) + \mathcal{B}(x,u,\nabla_L u) \geq f(v)g(x,\nabla_L v), \quad \text{on } \Omega. \tag{5.1}$$

We have the following:

Theorem 5.1. *Let $g : \Omega \times \mathbb{R}^l \to \mathbb{R}$ be a nonnegative Caratheodory function and let $b : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative continuous nondecreasing function such that $b(t) > 0$ for $t > 0$. Assume that either $g(x,0) = 0$ for almost every $x \in \Omega$, or $b(0) = 0$. Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be such that if $v \in X$, then for any constant $h \geq 0$, $(v-h)^+ \in X$.*

Assume that for any $c > 0$ the problem

$$\operatorname{div}_L(\mathcal{A}(x,\nabla_L v)) \geq c\, b(v)\, g(x,\nabla_L v), \quad v \geq 0, \quad \text{on } \Omega, \tag{5.2}$$

has no nontrivial solution belonging to X. Let $f \in \mathcal{C}(\mathbb{R})$ be such that

$$\liminf_{t\to+\infty} \frac{f(t)}{b(t)} > 0, \tag{5.3}$$

and set $Z(f) := \{z \in \mathbb{R} : f(z) = 0\}$. Let $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function such that $\mathcal{B}(x,t,\xi) \leq 0$ when $t \geq 0$.

If $u \in X$ is a solution of (5.1), then $u \leq \max(Z(f) \cup \{0\})$ almost everywhere on Ω.

Proof. Let $u \in X$ be a solution of (5.1). Let $\alpha := \max(Z(f) \cup \{0\})$ and for $h > \alpha$, define $u_h := u - h$. The function u_h solves

$$\operatorname{div}_L(\mathcal{A}(x,\nabla_L u_h)) + \mathcal{B}(x,u_h+h,\nabla_L u_h) \geq f(u_h+h)g(x,\nabla_L u_h), \quad \text{on } \Omega.$$

By the variant of Kato's inequality (2.16), it follows that

$$\operatorname{div}_L\left(\mathcal{A}(x,\nabla_L u_h^+)\right) \geq \operatorname{sign}_h^+ u\, f(u_h+h)g(x,\nabla_L u_h) - \operatorname{sign}_h^+ u\, \mathcal{B}(x,u_h+h,\nabla_L u_h)$$

on Ω, which implies

$$\operatorname{div}_L\left(\mathcal{A}(x,\nabla_L u_h^+)\right) \geq \operatorname{sign}_h^+ u\, f(u_h^+ + h)g(x,\nabla_L u_h^+) \quad \text{on } \Omega. \tag{5.4}$$

Let $l = \frac{1}{2} \liminf_{t\to+\infty} \frac{f(t)}{b(t)} > 0$. Then there exists $R_0 > 0$ such that for $t > R_0$ we have $f(t) \geq l\, b(t)$. Setting $m := \min f([h, R_0])$ and taking into account that $h > \alpha$, we get $m > 0$. Let $M := l \max b([h, R_0])$, and $c := l \min\{1, \frac{m}{M}\}$. Clearly, for $t > R_0$ we have $f(t) \geq l\, b(t) \geq c\, b(t)$, and for $R_0 \geq t \geq h$, $f(t) \geq m = \frac{m}{M} M \geq \frac{m}{M} l\, b(t) \geq c\, b(t)$. Therefore, since b is nondecreasing, for any $t \geq 0$ it follows that $f(t+h) \geq c\, b(t+h) \geq c\, b(t)$, which, from (5.4) implies

$$\mathrm{div}_L \left(\mathcal{A}(x, \nabla_L u_h^+)\right) \geq c \ \mathrm{sign}_h^+ u \ b(u_h^+)\, g(x, \nabla_L u_h^+) \quad \text{on } \Omega.$$

Since by hypothesis $b(0)g(x,0) = 0$, we deduce that u_h^+ is a nonnegative solution of

$$\mathrm{div}_L \left(\mathcal{A}(x, \nabla_L v)\right) \geq c\, b(v) g(x, \nabla_L v) \quad \text{on } \Omega.$$

In other words u_h^+ is a solution of (5.2). Since by assumption $u_h^+ \in X$, we deduce that $u_h^+ \equiv 0$; i.e., $u \leq h$ on Ω. Since this holds for any $h > \alpha$, we conclude the proof. $\square$

If we "enlarge" the set X and $\mathcal{B} \equiv 0$, then we are led to the following consequence of Theorem 5.1.

Corollary 5.2. *Let $g : \Omega \times \mathbb{R}^l \to \mathbb{R}$ be a nonnegative Caratheodory function and let $b : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative continuous nondecreasing function such that $b(t) > 0$ for $t > 0$. Assume that for almost every $x \in \Omega$, $g(x,0) = 0$ or $b(0) = 0$. Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be such that if $v \in X$, then for any $h \in \mathbb{R}$ we have $(v-h)^+ \in X$.*

Assume that for any $c > 0$ the problem (5.2) *has no nontrivial solutions belonging to X. Let $f \in \mathcal{C}(\mathbb{R})$ be such that*

$$\liminf_{t\to+\infty} \frac{f(t)}{b(t)} > 0, \tag{5.5}$$

and set $Z(f) := \{z \in \mathbb{R} : f(z) = 0\}$.

If $u \in X$ is a solution of (5.1) *with $\mathcal{B} \equiv 0$, then $u \leq \max Z(f)$ almost everywhere on Ω.*

In particular, if f is positive then (5.1) *has no nontrivial solutions belonging to X.*

Proof. If $Z(f) \neq \emptyset$, then we set $z_1 := \max Z(f)$; otherwise, if f is positive, we set $z_1 := -\infty$.

Without loss of generality we may assume that $z_1 < 0$. Notice that if $z_1 \geq 0$, then it suffices to apply Theorem 5.1.

Let $u \in X$ be a solution of (5.1). By Theorem 5.1 it follows that $u \leq 0$. Fix $0 > h > z_1$ and set $u_h := u - h$. By the variant of Kato's inequality (2.16), with the same notation as in the proof of Theorem 5.1, we have

$$\mathrm{div}_L\left(\mathcal{A}(x, \nabla_L u_h^+)\right) \geq \mathrm{sign}_h^+ u\, f(u_h^+ + h) g(x, \nabla_L u_h^+), \quad \text{on } \Omega.$$

Set $m := \min f([h, 0]) > 0$, and define

$$\tilde{f}(t) := \begin{cases} \dfrac{m}{|h|} t & \text{if } t \leq |h|\,, \\ \dfrac{m}{b(|h|)}\, b(t) & \text{if } t > |h|\,. \end{cases}$$

Since $0 \leq u_h^+ \leq |h|$, we have $f(u_h^+ + h) \geq m \geq \tilde{f}(u_h^+)$, and since $\tilde{f}(0) = 0$, it follows that u_h^+ solves

$$\mathrm{div}_L\left(\mathcal{A}(x, \nabla_L u_h^+)\right) \geq \tilde{f}(u_h^+) g(x, \nabla_L u_h^+) \quad \text{on } \Omega.$$

Now we apply Theorem 5.1 with $\tilde{f}$ instead of f. Indeed, $\tilde{f}$ is continuous and

$$\lim_{t \to +\infty} \frac{\tilde{f}(t)}{b(t)} = \frac{m}{b(|h|)} > 0.$$

Therefore, by Theorem 5.1 we have $u_h^+ \equiv 0$ almost everywhere on Ω; i.e., $u \leq h$ almost everywhere on Ω. Since this holds for any $h > z_1$, we get the claim. □

A simple application of the above result gives the following.

Corollary 5.3. *Let $\Omega \subset \mathbb{R}^N$ be an open connected set and suppose that* (4.12) *holds. Let $\mathcal{A}$ and g be as in Theorem* 5.1 *and in addition, for almost every $x \in \Omega$, $g(x, 0) = 0$. Assume that for any $c > 0$ the weak solutions of the problem*

$$\mathrm{div}_L\left(\mathcal{A}(x, \nabla_L v)\right) \geq c g(x, \nabla_L v), \quad v \geq 0, \quad \text{on } \Omega, \tag{5.6}$$

are such that $\nabla_L v \equiv 0$ on Ω.

Let $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function such that $\mathcal{B}(x, t, \xi) \leq 0$ when $t \geq 0$. Let $f \in \mathcal{C}(\mathbb{R})$ be such that

$$\liminf_{t \to +\infty} f(t) > 0. \tag{5.7}$$

(1) *If u is a weak solution of* (5.1)*, then either $u \leq \max(Z(f) \cup \{0\})$ almost everywhere on Ω or $\nabla_L u \equiv 0$ almost everywhere on Ω.*
(2) *If u is a weak solution of* (5.1) *with $\mathcal{B} \equiv 0$, then either $u \leq \max Z(f)$ almost everywhere on Ω or $\nabla_L u \equiv 0$ almost everywhere on Ω. In particular, if f is positive, then $\nabla_L u \equiv 0$ almost everywhere on Ω.*

Proof. The proof follows by applying Theorem 5.1 and Corollary 5.2 with the choice $X := (W^{1,p}_{loc}(\Omega) \setminus \{\nabla_L u \equiv 0\}) \cup \{0\}$. To this end we need to show that such a set X is admissible; that is, if $u \in X$ then $(u-h)^+ \in X$ for any $h \in \mathbb{R}$. We shall argue by contradiction. Let $u \in X$, $u \not\equiv 0$, and $h \in \mathbb{R}$ be such that $(u-h)^+ \notin X$; that is, $\nabla_L (u-h)^+ \equiv 0$. By assumption (4.12) this implies $(u-h)^+ \equiv c$. Notice that case $c = 0$ is not admissible since this would imply that $(u-h)^+ \equiv 0 \in X$, contradicting the fact that $(u-h)^+ \notin X$. Hence, $c > 0$. This, in turn implies that $u \equiv h + c$, contradicting the hypothesis $u \in X$. □

Remark 5.4. A result with the same conclusion as Corollary 5.2 with a potential $\mathcal{B} \not\equiv 0$ cannot hold. Indeed, choose $N = 1$, $Lu = u''$, $\mathcal{B}(x,u,\xi) := -u$, and $f(t) := |t+10|^{q-1}(t+10)$. If the conclusion of Corollary 5.2 were true then the possible solution of

$$u'' - u \geq \lambda f(u) \quad \text{on } \mathbb{R}, \quad q > 1, \tag{5.8}$$

would satisfy $u \leq -10$. This is due to the fact that the inequality $u'' \geq cu^q$ has no nontrivial nonnegative solutions for any $c > 0$. On the other hand, the function $u(x) := -3 + \sin(x)$ satisfies $-4 \leq u \leq -2$, and it is a solution of (5.8) for $\lambda > 0$ small enough. Indeed,

$$u'' - u = 3 - 2\sin(x) \geq 1 \geq \lambda f(-2) \geq \lambda f(u).$$

Part 3. A priori estimates, positivity results and Liouville theorems

In what follows we shall assume that $\mathcal{A}$ is **W**-p-**C** with $p > 1$. Throughout all sections, except Section 6, we shall assume that the vector field ∇_L satisfies (1.3) and that is it homogeneous of degree one with respect to a dilation δ_R as specified in Section 1. However, for the convenience of the reader we state our assumptions at the beginning of each section.

6. GENERAL A PRIORI ESTIMATES

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $V \in L^\infty_{loc}(\Omega)$ be nonnegative and let $\mathcal{A}$ be **W**-p-**C** with $p > 1$. The following preliminary lemmata will play an important role in the proof of our main result (see Theorem 6.5 below).

Lemma 6.1. *Let $g \in L^1_{loc}(\Omega)$ be nonnegative and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) + V u^{p-1} \geq g, \qquad u \geq 0, \quad \text{on } \Omega. \tag{6.1}$$

Let $s \geq 1$. *If* $u^{s+p-1} \in L^1_{loc}(\Omega)$, *then*

$$gu^s,\ \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L u\ u^{s-1} \in L^1_{loc}(\Omega), \tag{6.2}$$

and for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$ *we have*

$$\int_\Omega gu^s\phi + c_1 s\int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L u\ u^{s-1}\phi \leq c_2 s^{1-p}\int_\Omega u^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}} + \int_\Omega V u^{s+p-1}\phi, \tag{6.3}$$

where $c_1 = 1 - \frac{\epsilon^{p'}}{p'k_2} > 0$, $c_2 = \frac{p^p}{p\epsilon^p}$, *and* $\epsilon > 0$ *is sufficiently small.*

Remark 6.2. i) Notice that from the above result it follows that if $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (6.1), then $g\,u \in L^1_{loc}(\Omega)$.

ii) The above lemma still holds if we replace the function $g \in L^1_{loc}(\Omega)$ with a regular Borel measure on Ω.

Proof. Let $\gamma \in \mathcal{C}^1(\mathbb{R})$ be a bounded nonnegative function with bounded nonnegative first derivative and let $\phi \in \mathcal{C}^1_0(\Omega)$ be a nonnegative test function.

It is clear that u is a weak solution of

$$\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) \geq f, \qquad u \geq 0, \quad \text{on } \Omega,$$

where $f := g - Vu^{p-1}$. Applying Lemma 2.2, from (2.9), it follows that

$$\begin{aligned}
-\int_\Omega Vu^{p-1}\gamma(u)\phi &+ \int_\Omega g\gamma(u)\phi + \int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\ \nabla_L u\ \gamma'(u)\phi \\
&\leq \int_\Omega |\mathcal{A}(x,u,\nabla_L u)|\ \ |\nabla_L\phi|\,\gamma(u) \\
&\leq \Big(\int_\Omega |\mathcal{A}(x,u,\nabla_L u)|^{p'}\ \gamma'(u)\phi\Big)^{1/p'}\Big(\int_\Omega \frac{\gamma(u)^p}{\gamma'(u)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}\Big)^{1/p} \\
&\leq \frac{\epsilon^{p'}}{p'k_2}\int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L u\ \gamma'(u)\phi + \frac{1}{p\epsilon^p}\int_\Omega \frac{\gamma(u)^p}{\gamma'(u)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}},
\end{aligned}$$

where $\epsilon > 0$ and all integrals are well defined provided $\frac{\gamma(u)^p}{\gamma'(u)^{p-1}} \in L^1_{loc}(\Omega)$. With a suitable choice of $\epsilon > 0$, for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$ and $\gamma \in \mathcal{C}^1(\mathbb{R})$ as above such that $\frac{\gamma(u)^p}{\gamma'(u)^{p-1}} \in L^1_{loc}(\Omega)$, it follows that

$$\int_\Omega g\gamma(u)\phi + c_1\int_\Omega \mathcal{A}\ \nabla_L u\gamma'(u)\phi \leq \frac{1}{p\epsilon^p}\int_\Omega \frac{\gamma(u)^p}{\gamma'(u)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}} + \int_\Omega Vu^{p-1}\gamma(u)\phi. \tag{6.4}$$

Now for $s \geq 1$ and $n \geq 1$, define

$$\gamma_n(t) := \begin{cases} t^s & \text{if } 0 \leq t < n, \\ cn^s - \dfrac{s}{\beta - 1} n^{\beta+s-1} t^{1-\beta} & \text{if } t \geq n, \end{cases} \tag{6.5}$$

where $c := \frac{\beta-1+s}{\beta-1}$ and $\beta > 1$ will be chosen later. Clearly $\gamma_n \in \mathcal{C}^1$,

$$\gamma_n'(t) = \begin{cases} st^{s-1} & \text{if } 0 \leq t < n, \\ sn^{\beta+s-1} t^{-\beta} & \text{if } t \geq n, \end{cases}$$

and γ_n and γ_n' are nonnegative and bounded with $\|\gamma_n\|_\infty = cn^s$ and $\|\gamma_n'\|_\infty = sn^{s-1}$. Moreover,

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} = \begin{cases} s^{1-p} t^{s+p-1} & \text{for } t < n, \\ \theta(t,n) & \text{for } t \geq n, \end{cases}$$

where

$$\theta(t,n) := \frac{(cn^s - \frac{s}{\beta-1} n^{\beta+s-1} t^{1-\beta})^p}{(sn^{\beta+s-1} t^{-\beta})^{p-1}} \leq (cn^s)^p s^{1-p} n^{-(\beta+s-1)(p-1)}\, t^{\beta(p-1)}.$$

Choosing $\beta := \frac{s+p-1}{p-1}$ we have $c = p$, and

$$\theta(t,n) \leq p^p s^{1-p} n^{sp-(\beta+s-1)(p-1)} t^{s+p-1} = p^p s^{1-p} t^{s+p-1}.$$

Therefore, for $t \geq 0$ we have

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} \leq p^p s^{1-p} t^{s+p-1}.$$

Since by assumption $u^{s+p-1} \in L^1_{loc}(\Omega)$, from (6.4) with $\gamma = \gamma_n$ it follows that

$$\int_\Omega g\gamma_n(u)\phi + c_1 \int_\Omega \mathcal{A}(x,u,\nabla_L u) \cdot \nabla_L u \; \gamma_n'(u)\phi \leq \frac{p^p s^{1-p}}{p\epsilon^p} \int_\Omega u^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} + \int_\Omega V u^{p-1} \gamma_n(u)\phi.$$

Now, noticing that $\gamma_n(t) \to t^s$ and $\gamma_n'(t) \to st^{s-1}$ as $n \to +\infty$, $g \geq 0$, and $\mathcal{A} \cdot \nabla_L u \geq 0$, by the Beppo Levi theorem we obtain

$$\int_\Omega gu^s\phi + c_1 s \int_\Omega \mathcal{A}(x,u,\nabla_L u) \cdot \nabla_L u \; u^{s-1}\phi \leq c_2 s^{1-p} \int_\Omega u^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} + \int_\Omega V u^{s+p-1} \phi.$$

This completes the proof. □

Remark 6.3. i) The above lemma holds for $s > 0$. Indeed, if $0 < s < 1$ the proof follows the same arguments as above. To this end in (6.4) it is enough to choose $\gamma := \gamma_n(u+\delta)$ where γ_n is defined by (6.5).

ii) If $V \le 0$, then the assumption $u^{s+p-1} \in L^1_{loc}(\Omega)$ is not needed for the validity of (6.2). Indeed, what really matters is that $u^{s+p-1} \in L^1_{loc}(S)$, where S is the support of $\nabla_L \phi$. This remark will be useful when dealing with inequalities on unbounded sets.

Let $a : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Let u be a weak solution of

$$\operatorname{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) + V u^{p-1} \ge a(x)\, u^q, \quad u \ge 0, \quad \text{on } \Omega. \tag{6.6}$$

The main strategy to obtain *a priori* estimates is to use the family of test functions $u^\alpha \phi$, where $\alpha > 0$ is a suitable constant that will be chosen according to our needs. See [32]. However, *a priori* it is not clear why, after multiplying the inequality by $u^\alpha\phi$, this family is admissible, i.e., why $u^{q+\alpha} \in L^1_{loc}(\Omega)$. A sufficient condition for the admissibility of the family $u^\alpha\phi$ is contained in the following.

Lemma 6.4. *Let u be a weak solution of* (6.6) *with $q > p-1$. Assume that there exists $\overline{\alpha} > 1$ such that $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\Omega)$. If $1 \le \alpha < \overline{\alpha}$, then*

$$a\, u^{q+\alpha},\ u^{\alpha+p-1} \in L^1_{loc}(\Omega), \tag{6.7}$$

and for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$, the following inequalities hold:

$$\int_\Omega a\, u^{q+\alpha}\phi + c_1\alpha \int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L u\ u^{\alpha-1}\phi \\ \le c_2\alpha^{1-p}\int_\Omega u^{\alpha+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}} + \int_\Omega V u^{\alpha+p-1}\phi, \tag{6.8}$$

$$\int_\Omega a\, u^{q+\alpha}\phi + c_1\alpha \int_\Omega \mathcal{A}(x,u,\nabla_L u)\cdot\nabla_L u\ u^{\alpha-1}\phi \tag{6.9}$$

$$\le c_2\alpha^{1-p}\Big(\int_S a\, u^{q+\alpha}\phi\Big)^{1/\chi}\Big(\int_S \frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}}\Big)^{1/\chi'} + \int_\Omega V u^{\alpha+p-1}\phi,$$

where $\chi := \frac{q+\alpha}{\alpha+p-1}$, $\chi' := \frac{q+\alpha}{q-p+1}$, and S is the support of $\nabla_L\phi$.

In particular, if for any $C \subset\subset \Omega$: $\operatorname{ess\,inf}_C a > 0$, *then for any $\alpha > 0$, we have $a\, u^{q+\alpha} \in L^1_{loc}(\Omega)$.*

Proof. It is enough to prove that $u^{\alpha+p-1} \in L^1_{loc}(\Omega)$. Knowing this, an application of Lemma 6.1 with $g := a\,u^q$ yields the claim. To this end we shall use a recursion/bootstrap procedure.

Assume that $u^{s+p-1} \in L^1_{loc}(\Omega)$ for some $s \geq 1$. From Lemma 6.1 we know that $a\,u^{q+s} \in L^1_{loc}(\Omega)$ and

$$\int_\Omega a\,u^{q+s}\phi \leq c_2 s^{1-p} \int_S u^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} + \int_\Omega V u^{s+p-1}\phi. \tag{6.10}$$

Set $\beta := (q+s)\frac{p-1+\overline{\alpha}}{q+\overline{\alpha}}$. By Hölder's inequality with exponent $y := \frac{q+s}{\beta} = \frac{q+\overline{\alpha}}{p-1+\overline{\alpha}} > 1$, we have

$$\int_\Omega u^\beta \phi \leq \Big(\int_\Omega a\,u^{q+s}\phi\Big)^{1/y} \Big(\int_\Omega a^{-\frac{\beta}{q+s-\beta}}\phi\Big)^{1/y'} \tag{6.11}$$

$$\leq \Big(c_3 \int_S u^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} + \int_\Omega V u^{s+p-1}\phi\Big)^{1/y} \Big(\int_\Omega a^{-\frac{\beta}{q+s-\beta}}\phi\Big)^{1/y'}. \tag{6.12}$$

Since by assumption $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\Omega)$, and $u^{s+p-1} \in L^1_{loc}(\Omega)$ with $\frac{\beta}{q+s-\beta} = \frac{\overline{\alpha}+p-1}{q-p+1}$, it follows that the integrals on the right-hand side of (6.12) are finite.

It is easy to check that if $\overline{\alpha} > s$, then $\beta > s+p-1$.

Thus the recursion argument assures that if $u^{s+p-1} \in L^1_{loc}(\Omega)$ for some $\overline{\alpha} > s \geq 1$, then we gain summability with exponent $\beta = (q+s)\sigma$, where $\sigma := \frac{p-1+\overline{\alpha}}{q+\overline{\alpha}}$.

Now we begin bootstrapping. Since $u^p \in L^1_{loc}(\Omega)$, we choose $s = \beta_0 = 1$ in the recursion argument, obtaining that $u^{\beta_1} \in L^1_{loc}(\Omega)$ with $\beta_1 := (q+1)\sigma$. Iterating this procedure we see that, if $u^{\beta_n} \in L^1_{loc}(\Omega)$, then by recursion with $s = \beta_n - p + 1$, it follows that $u^{\beta_{n+1}} \in L^1_{loc}(\Omega)$ with $\beta_{n+1} = \sigma\beta_n + \sigma(q-p+1)$. Solving this difference equation we have

$$\beta_n = \sigma^n(q+1) + \sigma(q-p+1)(1+\sigma+\sigma^2+\cdots+\sigma^{n-2}).$$

Since $\sigma < 1$, by letting $n \to \infty$, we obtain $\beta_n \nearrow \overline{\alpha} + p - 1$.

Now, if $\alpha < \overline{\alpha}$, for n large enough we deduce that $\alpha + p - 1 < \beta_n$. Hence $u^{\alpha+p-1} \in L^1_{loc}(\Omega)$. This completes the proof of (6.8).

Inequality (6.9) follows from (6.8) by applying Hölder's inequality with exponent $\chi := \frac{q+\alpha}{\alpha+p-1}$. □

Theorem 6.5. *Let u be a weak solution of* (6.6) *with $q > p-1$ and $V \leq 0$. Assume that there exists $\overline{\alpha} > 1$ such that $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\Omega)$. Then*

$$a\,u^{q+\overline{\alpha}},\ u^{\overline{\alpha}+p-1} \in L^1_{loc}(\Omega),$$

and for any $1 \le \alpha \le \overline{\alpha}$ *and for any nonnegative* $\phi \in \mathcal{C}_0^1(\Omega)$ *the inequalities* (6.8) *and* (6.9) *hold, and*

$$\int_\Omega a\, u^{q+\alpha}\phi \le (c_2\alpha^{1-p})^{\chi'} \int_\Omega \frac{|\nabla_L \phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}}, \tag{6.13}$$

where $\chi' := \frac{q+\alpha}{q-p+1}$.

Proof. From Lemma 6.4, for any $1 < \alpha < \overline{\alpha}$ we have

$$\int_\Omega a\, u^{q+\alpha}\phi \le c_2\alpha^{1-p}\Big(\int_S a\, u^{q+\alpha}\phi\Big)^{1/\chi}\Big(\int_S \frac{|\nabla_L \phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}}\Big)^{1/\chi'},$$

which in turns implies

$$\int_\Omega a\, u^{q+\alpha}\phi \le (c_2\alpha^{1-p})^{\chi'} \int_S \frac{|\nabla_L \phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}}. \tag{6.14}$$

By Lebesgue's dominated convergence theorem we obtain

$$\lim_{\alpha\to\overline{\alpha}} \int_S \frac{|\nabla_L \phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}} = \int_S \frac{|\nabla_L \phi|^{p\overline{\chi}'}}{\phi^{p\overline{\chi}'-1}} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}},$$

where $\overline{\chi}' := \frac{q+\overline{\alpha}}{q-p+1}$. Now, by applying Fatou's lemma to (6.14) (if necessary, by passing to a subsequence), we get

$$\int_\Omega a\, u^{q+\alpha}\phi \to \int_\Omega a\, u^{q+\overline{\alpha}}\phi.$$

Using this information in (6.11) and letting $s \to \overline{\alpha}$, it follows that $u^{\overline{\alpha}-p+1} \in L^1_{loc}(\Omega)$. Finally, an application of Lemma 6.1 completes the proof. $\square$

Remark 6.6. (1) If $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a weak solution of (6.6), then it is easy to check that all the results of this section hold assuming only that $a, V \in L^1_{loc}(\Omega)$, without any further assumption on a and V.

(2) If ∇_L is the horizontal vector field on a Carnot group then the the results of this section hold for a weak solution belonging to the wider space $W^{1,p}_{L,loc}(\Omega)$.

(3) We emphasize that if $V \le 0$ and the differential operator satisfies the **S**-p-**C** condition, then from Harnack's inequality (see [10]) it follows that the solutions of (6.6) belong to $L^\infty_{loc}(\Omega)$.

7. Universal a priori estimates

In this section $\Omega \subset \mathbb{R}^N$ is an open set, ∇_L is the horizontal gradient on a Carnot group $\mathbb{G}$ and $\mathcal{A}$ is **S**-p-**C** (see Definition 1.1). We shall denote with $|\cdot|_L$ a homogeneous norm on $\mathbb{G}$.

Theorem 7.1. *Let $q > p-1$ and $c > 0$. Assume that $f \in \mathcal{C}(\mathbb{R})$ satisfies $f(t) \geq ct^q$ for $t > 0$. Then there exists a constant $C = C(f, \mathbb{G}, \mathcal{A}) > 0$ such that if u is a weak solution of*

$$\operatorname{div}_L (\mathcal{A}(x,u,\nabla_L u)) + \mathcal{B}(x,u,\nabla_L u) \geq f(u) \quad \text{on } \Omega, \tag{7.1}$$

with $\mathcal{B}(x,t,\xi) \leq 0$ for $t \geq 0$, then

$$u(x) \leq C \operatorname{dist}(x, \partial\Omega)^{-\frac{p}{q-p+1}} \quad \text{a.e. } x \in \Omega. \tag{7.2}$$

In particular, if u is a weak solution of

$$\operatorname{div}_L (\mathcal{A}(x,u,\nabla_L u)) + \mathcal{B}(x,u,\nabla_L u) = f(u), \quad \text{on } \Omega. \tag{7.3}$$

with $\mathcal{B}(x,t,\xi)t \leq 0$ and $f(t)t \geq ct^{q+1}$ for $t \in \mathbb{R}$, then

$$|u(x)| \leq C \operatorname{dist}(x, \partial\Omega)^{-\frac{p}{q-p+1}} \quad \text{a.e. } x \in \Omega. \tag{7.4}$$

Proof. Let u be a weak solution of (7.1). If $u(x) \leq 0$, then (7.2) is obviously satisfied. Therefore it is enough to show that (7.2) holds for u^+. To this end by applying Kato's inequality (2.6) to (7.1) we obtain

$$\operatorname{div}_L \big(\mathcal{A}(x,u^+,\nabla_L u^+)\big) \geq (u^+)^q \quad \text{on } \Omega. \tag{7.5}$$

For simplicity assume that $x = 0$. Set $R_0 := \operatorname{dist}(x, \partial\Omega)/2$. Let $\phi_0 \in \mathcal{C}^1_0(\mathbb{R})$ be a standard cut-off function, and define $\phi(x) := \phi_0(|\delta_{1/R} x|_L)$. Clearly if $R < R_0$, the function $\phi \in \mathcal{C}^1_0(\Omega)$. From (6.13) of Theorem 6.5 applied to (7.5), it follows that for any $\alpha > 1$ we have

$$\int_{B_R} u^{q+\alpha} \leq C_1 R^{Q - p\frac{q+\alpha}{q-p+1}},$$

where $C_1 = C_1(p,q,\alpha,\mathcal{A})$. In other words

$$\Big(\fint_{B_R} u^{q+\alpha}\Big)^{\frac{1}{q+\alpha}} \leq C_2 R^{-\frac{p}{q-p+1}}.$$

Next by Harnack's inequality (see [10]) we get

$$\sup_{B_{R/2}} u \leq C_H \Big(\fint_{B_R} u^{q+\alpha}\Big)^{\frac{1}{q+\alpha}} \leq C_H C_2 R^{-\frac{p}{q-p+1}}. \tag{7.6}$$

Letting $R \to R_0$ in (7.6) we obtain (7.2). Now, if u is a solution of (7.3), then arguing with $v := -u$ and $\overline{\mathcal{A}}\,(x, v, \nabla_L v) := -\mathcal{A}(x, u, \nabla_L u)$ (see (4.6)), we obtain that $-u$ satisfies (7.2). This concludes the proof. □

Remark 7.2. In general, inequality (7.2) is sharp, as the following examples show. For $q > \frac{N}{N-2}$ the function $u(x) := c\,|x|^{-\frac{2}{q-1}}$, for a suitable $c > 0$, is a solution of

$$\Delta u = u^q \quad \text{on } \mathbb{R}^N \setminus \{0\}.$$

For $q > 1$ the function $u(x) := c{x_1}^{-\frac{2}{q-1}}$, for a suitable $c > 0$, is a solution of

$$\Delta u = u^q \quad \text{on } (0, +\infty) \times \mathbb{R}^{N-1}.$$

8. Some Liouville theorems for coercive inequalities

In this section we study Liouville theorems for a class of quasilinear elliptic inequalities on $\mathbb{R}^N$.

Recently, a wide class of weakly elliptic quasilinear problems were also considered by Farina and Serrin [19] and Pucci and Serrin [40], where sharp interesting cases were handled. The main technique we use throughout this section is a combination of three ingredients: the Kato inequalities (2.3) and (2.5), and a slight modification of the test-functions method together with an idea introduced in [32].

More precisely, we shall consider problems of the type

$$\operatorname{div}_L(\mathcal{A}(x, u, \nabla_L u)) + V(x)\,|u|^{p-2}\,u = a(x) f(u) \quad \text{on } \mathbb{R}^N, \tag{8.1}$$

where $V \le 0$ and $a : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function. The proof or our main results will be organized into two steps. The first is to apply Kato's inequality (2.3) and (2.5) to (8.1), reducing the problem to the study of the nonnegative solutions of

$$\operatorname{div}_L(\mathcal{A}(x, u, \nabla_L u)) \ge a(x)\,u^q, \quad u \ge 0, \quad \text{on } \mathbb{R}^N. \tag{8.2}$$

A second one will be the application of *a priori* estimates proved in Section 6 to (8.2). These estimates depend on two parameters α and R. By using an idea first introduced in [32, proof of Theorem 4.1], we can choose α large enough and then by letting $R \to +\infty$ we conclude.

We point out that when dealing with equations or inequalities other fine techniques based on Keller's and Osserman's ideas ([25] and [37] respectively) are available. However, the application of these later ideas needs special stronger assumptions on the differential operator and on the nonlinearity. For recent contributions see [34, 20, 28].

Throughout this section we shall assume that $\mathcal{A}$ is **W**-p-**C** with $p > 1$, the vector field ∇_L satisfies (1.3) (that is, ∇_L is homogeneous of degree one with respect to a dilation δ_R as specified in Section 1) and $|\cdot|_L$ stands for a homogeneous norm.

Theorem 8.1. *Let $V \in L^\infty_{loc}(\mathbb{R}^N)$ be such that $V \le 0$. Suppose that $f \in \mathcal{C}(\mathbb{R})$ satisfies*

$$f(t) \ge c\, t^q, \quad \text{for } t > 0,$$

where $q > p-1$ and $c > 0$. Assume that there exists $\overline{\alpha} > 1$ such that $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\mathbb{R}^N)$ and

$$\liminf_{R\to+\infty} R^{-p\frac{q+\overline{\alpha}}{q-p+1}} \int_{A_R} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} < +\infty. \tag{8.3}$$

Let u be a weak solution of

$$\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) + V(x)\,|u|^{p-2}\,u \ge a(x)\,f(u) \quad \text{on } \mathbb{R}^N. \tag{8.4}$$

Then $u \le 0$ almost everywhere on $\mathbb{R}^N$. Moreover, if $f(t)\,t \ge c\,|t|^{q+1}$, $t \in \mathbb{R}$, and u is a weak solution of the equation

$$\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) + V(x)\,|u|^{p-2}\,u = a(x)\,f(u) \quad \text{on } \mathbb{R}^N, \tag{8.5}$$

then $u \equiv 0$ almost everywhere on $\mathbb{R}^N$.

Proof. The main idea is based on the application of the reduction principles. Thus, first we investigate the nonexistence of nonnegative solutions, then we apply Theorems 4.1 and 4.4.

Step 1. Let u be a nonnegative weak solution of (8.4). Since $V \le 0$, then u solves the inequality (8.2).

Now from Lemma 6.5, inequality (6.9) holds. In particular for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$ we have

$$\int_{\mathbb{R}^N} a\,u^{q+\overline{\alpha}}\phi \le c_2\alpha^{1-p}\Big(\int_S a\,u^{q+\overline{\alpha}}\phi\Big)^{1/\chi}\Big(\int_S \frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}}\Big)^{1/\chi'}, \tag{8.6}$$

where $\chi := \frac{q+\overline{\alpha}}{\overline{\alpha}+p-1}$ and S is the support of $\nabla_L\phi$.

Next, let $\phi_0 \in \mathcal{C}^1_0(\mathbb{R})$ be an even nonnegative standard cut-off function, that is,

$$\phi_0(t) = 0 \text{ for } |t| \ge 2, \quad \phi_0(t) = 1 \text{ for } |t| \le 1 \text{ and } 0 \le \phi_0 \le 1.$$

For $R > 0$ choose $\phi(x) := \phi_0(\left|\delta_{1/R}x\right|_L)$. Therefore, we have

$$\int_S \frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \le c(\phi_0) R^{-p\frac{q+\overline{\alpha}}{q-p+1}} \int_{A_R} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}}.$$

Let $(R_j)_j$ be a sequence satisfying (8.3). For any j we have

$$R_j^{-p\frac{q+\overline{\alpha}}{q-p+1}} \int_{A_{R_j}} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} < M < +\infty,$$

which from (8.6) yields

$$\int_{B_{R_j}} a\, u^{q+\overline{\alpha}} \le c_2 \overline{\alpha}^{1-p} c(\phi_0)\, M.$$

Letting $j \to +\infty$, it follows that $a\, u^{q+\overline{\alpha}} \in L^1(\mathbb{R}^N)$. Using again (8.6) we deduce that

$$\int_{B_{R_j}} a\, u^{q+\overline{\alpha}}\phi \le c_2 \alpha^{1-p} \Big(\int_{A_{R_j}} a\, u^{q+\overline{\alpha}}\phi \Big)^{1/\chi} (c(\phi_0)M)^{1/\chi'},$$

which in turn implies $a\, u^{q+\overline{\alpha}} \equiv 0$ almost everywhere on $\mathbb{R}^N$. Now from the assumption $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\mathbb{R}^N)$, we know that $a \neq 0$ almost everywhere and then $u \equiv 0$ almost everywhere on $\mathbb{R}^N$.

Step 2. Applying Theorem 4.1, it follows that $u \le 0$ almost everywhere.

Finally, let u be a solution of (8.5). The idea is to apply Theorem 4.4. To this end it is enough to notice that since $\mathcal{A}$ is **W**-p-**C**, then also $\overline{\mathcal{A}}$ is **W**-p-**C**. Moreover the function $\overline{f}$ of Theorem 4.4 is such that

$$\overline{f}(x,t,\xi) = V(x)\, |t|^{p-2}\, t - a(x) f(-t) \ge V(x)\, |t|^{p-2}\, t + c\, a(x)\ |t|^q .$$

Therefore from Step 1 and Step 2, problems (4.9) and (4.10) have no nontrivial solutions in $W^{1,p}_{loc}(\mathbb{R}^N)$. Hence $u \equiv 0$ almost everywhere. □

Remark 8.2. Nonexistence results related to **W**-p-**C** operators with $p = 1$ have been studied in [16]. Notice that the mean curvature operator can be viewed as a **W**-p-**C** operator with $p = 1$.

Remark 8.3. The assumption $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\mathbb{R}^N)$ implies that $a(x) > 0$ almost everywhere on $\mathbb{R}^N$. We note that if (4.12) is satisfied then Theorem 8.1 still holds without the positivity property on a and the assumption that the solutions belong to $L^\infty_{loc}(\mathbb{R}^N)$. For this purpose, it is enough that (8.3) is satisfied. Indeed, arguing as above we get that $a\, u^{q+\overline{\alpha}} \equiv 0$ almost everywhere. Hence there exists $R_0 > 0$ such that $u(x) \equiv 0$ almost everywhere on $\mathbb{R}^N \setminus B_{R_0}$. Therefore inequality (8.2) becomes

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) \ge 0 \quad \text{on } B_{R_0}, \quad u \ge 0, \quad u = 0 \text{ on } \partial B_{R_0}.$$

On the other hand, by the maximum principle 4.7 it follows that $u \equiv 0$ almost everywhere on B_{R_0}. We leave the remaining details to the interested reader.

Theorem 8.4. *Let $\mathcal{A} = \mathcal{A}(x, \nabla_L u)$ and let $V \in L^\infty_{loc}(\mathbb{R}^N)$ be such that $V \leq 0$. Assume that there exist $q > p-1$ and $\overline{\alpha} > 1$ such that $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\mathbb{R}^N)$ and* (8.3) *holds. Suppose that $f \in \mathcal{C}(\mathbb{R})$ satisfies*

$$\liminf_{t\to+\infty} \frac{f(t)}{t^q} > 0. \tag{8.7}$$

Then,

(1) *If u is a weak solution of inequality* (8.4)*, then $u \leq \max(Z(f) \cup \{0\})$*[4] *almost everywhere on $\mathbb{R}^N$.*
(2) *If $V \equiv 0$ and u is a weak solution of inequality* (8.4)*, then $u \leq \max Z(f)$ almost everywhere on $\mathbb{R}^N$.*
(3) *If $V \equiv 0$ and f is positive, then inequality* (8.4) *has no weak solutions.*
(4) *If $V \equiv 0$ and u is a weak solution of equation* (8.5) *with f satisfying*

$$\liminf_{t\to+\infty} \frac{f(t)}{|t|^q} > 0, \qquad \limsup_{t\to-\infty} \frac{f(t)}{|t|^q} < 0, \tag{8.8}$$

then

$$\min Z(f) \leq u \leq \max Z(f) \quad \textit{a.e. on } \mathbb{R}^N.$$

Proof. First we prove cases (2) and (3). If u is a weak solution of (8.4), then u satisfies

$$\operatorname{div}_L (\mathcal{A}(x, \nabla_L u)) \geq a(x) f(u), \quad \text{on } \mathbb{R}^N. \tag{8.9}$$

Now it is enough to apply Corollary 5.2 with $b(t) := t^q$ and $g(x, \xi) := a(x)$. Indeed, from Theorem 8.1 it follows that the inequality (8.2) has only the trivial solution. This completes the proof of cases (2) and (3).

Proof of case (4). In order to prove that $\min Z(f) \leq u$ almost everywhere we argue as in case (2). Indeed, the function $w := -u$ satisfies the equation

$$\operatorname{div}_L \left(\overline{\mathcal{A}}\,(x, \nabla_L w)\right) = a(x) \bar{f}(w),$$

where $\bar{f}(t) := -f(-t)$. Since $\bar{f}$ satisfies (8.7) and $\overline{\mathcal{A}}$ is **W**-p-**C**, from (2) we get $w \leq \max Z(\bar{f})$ almost everywhere, completing the proof.

The proof of case (1) can be obtained as the proof of cases (2), (3), and (4) together with an application of Theorem 5.1. □

[4] We recall that we have denoted by $Z(f)$ the set of the zeros of f.

Remark 8.5. Condition (8.3) can be equivalently reformulated as

$$\liminf_{R\to+\infty} R^{\frac{Q-p}{\beta}}\left(\fint_{A_R}(|x|_L^p\, a(x))^{-\beta}\right)^{1/\beta} < +\infty \quad \text{for } \beta > \frac{p}{q-p+1}.$$

Corollary 8.6. *Let $\mathcal{A} = \mathcal{A}(x, \nabla_L u)$ and $V \in L^\infty_{loc}(\mathbb{R}^N)$ be such that $V \le 0$. Let a be a continuous positive function satisfying*

$$a(x) \ge c\,|x|_L^{-\theta} \quad \textit{for } |x|_L \textit{ large},$$

with $\theta < p$. Let $f \in \mathcal{C}(\mathbb{R})$ be such that (8.7) holds for some $q > p-1$. Then,

1. *If u is a weak solution of inequality (8.4), then $u \le \max(Z(f) \cup \{0\})$ almost everywhere on $\mathbb{R}^N$.*
2. *If $V \equiv 0$ and u is a weak solution of inequality (8.4), then $u \le \max Z(f)$ almost everywhere on $\mathbb{R}^N$.*
3. *If $V \equiv 0$ and f is positive, then inequality (8.4) has no weak solutions.*
4. *If $V \equiv 0$ and u is a weak solution of equation (8.5) with f satisfying (8.8), then $\min Z(f) \le u \le \max Z(f)$ almost everywhere on $\mathbb{R}^N$.*

Proof. For R large we have

$$R^{\frac{Q-p}{\beta}}\left(\fint_{A_R}(|x|_L^p\, a(x))^{-\beta}\right)^{1/\beta} \le cR^{\frac{Q-p}{\beta}+\theta-p}.$$

This implies that for β large enough the left-hand side in the above inequality vanishes when $R \to +\infty$. Therefore, from Remark 8.5 it follows that (8.3) holds and the hypotheses of Theorem 8.4 are fulfilled. $\square$

By using the same computations made for the proof of Theorem 3.23 of [16] we see that the following result holds. We leave to the interested reader the additional details for completing the proof.

Corollary 8.7. *Assume that ∇_L is the horizontal gradient on a Carnot group. Let $\mathcal{A} = \mathcal{A}(x, \nabla_L u)$ and let $V \in L^\infty_{loc}(\mathbb{R}^N)$ be such that $V \le 0$. Let $a \in \mathcal{C}(\mathbb{R}^N)$ be an almost-everywhere-positive function satisfying*

$$a(x) \ge c\frac{\pi(|x|_L)}{|x|_L^\theta} \quad \textit{for } |x|_L \textit{ large},$$

where $\theta < p$ and $\pi : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous periodic function with zeros of order at most $\mu > 0$.

Let $f \in \mathcal{C}(\mathbb{R})$ be such that (8.7) holds for some $q > \mu p + p - 1$ and $p > \theta + \mu(Q-p)$. Then

1. *If u is a weak solution of inequality (8.4), then $u \le \max(Z(f) \cup \{0\})$ almost everywhere on $\mathbb{R}^N$.*

(2) *If* $V \equiv 0$ *and* u *is a weak solution of inequality* (8.4), *then* $u \le \max Z(f)$ *almost everywhere on* $\mathbb{R}^N$.
(3) *If* $V \equiv 0$ *and* f *is positive, then inequality* (8.4) *has no weak solutions.*
(4) *If* $V \equiv 0$ *and* u *is a weak solution of equation* (8.5) *with* f *satisfying* (8.8), *then* $\min Z(f) \le u \le \max Z(f)$ *almost everywhere on* $\mathbb{R}^N$.

Remark 8.8. It would be be interesting to investigate the case $\mu p + p - 1 \ge q > p - 1$.

From Corollary 8.6 with $\theta = 0$ and Theorem 3.12 in [16], we have the following.

Theorem 8.9. *Let* $\mathcal{A} = \mathcal{A}(x, \nabla_L u)$ *and let* $f \in \mathcal{C}(\mathbb{R})$ *be such that*

$$\liminf_{t \to -\infty} \frac{f(t)}{|t|^q} > 0,$$

where $q > p - 1$. *Let* $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ *be a weak solution of*

$$-\operatorname{div}_L (\mathcal{A}(x, \nabla_L u)) \ge f(u) \quad \text{on } \mathbb{R}^N. \tag{8.10}$$

Then, f *has a zero and* $u \ge \min Z(f)$ *almost everywhere on* $\mathbb{R}^N$. *Moreover, if* $\mathcal{A}$ *is* **S**-p-**C**, ∇_L *is the horizontal gradient on a Carnot group and* $f \ge 0$*; setting* $m := \operatorname{ess\,inf}_{\mathbb{R}^N} u$, *it follows that* $f(m) = 0$ *and*

$$\liminf_{t \to m^+} \frac{f(t)}{(t - m)^{\frac{Q(p-1)}{Q-p}}} = 0.$$

Theorem 8.10. *Assume that* ∇_L *is the usual gradient* ∇ *on* $\mathbb{R}^N$ *or the horizontal gradient on the Heisenberg group* $\mathbb{H}^n (= \mathbb{R}^{2n+1} = \mathbb{R}^N)$. *Let* $f \in \mathcal{C}(\mathbb{R})$ *be such that*

$$\begin{gathered}\text{for some } c > 0 \ f \text{ is nondecreasing, positive on } [c, +\infty), \text{ and} \\ \int_c^{+\infty} \Big(\int_c^t f(s)\ ds \Big)^{-\frac{1}{p}}\ dt < +\infty.\end{gathered} \tag{8.11}$$

Assume that $V \in L^\infty_{loc}(\mathbb{R}^N)$ *satisfies* $V \le 0$. *Then,*

(1) *If* u *is a weak solution of*

$$\operatorname{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) + V(x) \, |u|^{p-2} \, u \ge f(u) \quad \text{on } \mathbb{R}^N, \tag{8.12}$$

then $u \le \max(Z(f) \cup \{0\})$ *almost everywhere on* $\mathbb{R}^N$.
(2) *If* $V \equiv 0$ *and* u *is a weak solution of* (8.12), *then* $u \le \max Z(f)$ *almost everywhere on* $\mathbb{R}^N$.
(3) *If* $V \equiv 0$ *and* f *is positive, then* (8.12) *has no weak solutions.*

(4) *If u is a weak solution of*

$$\mathrm{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) = f(u) \quad \textit{on } \mathbb{R}^N, \tag{8.13}$$

with f and $\bar{f}(t) := -f(-t)$ *satisfying* (8.11), *then*

$$\min Z(f) \le u \le \max Z(f) \quad \textit{a.e. on } \mathbb{R}^N.$$

Remark 8.11. i) The above theorem is a generalization of some results proved in [14], [15], and [16].

ii) Notice that the above result can be formulated for a general Carnot group for which there exists a smooth homogeneous norm $|\cdot|_L$ such that $\Delta_{p,G} |x|_L^{\frac{p-Q}{p-1}} = c\,\delta_0$. This can be done also for polarizable Carnot groups such as the H-type groups (see [5, 12] and references therein).

Proof of Theorem 8.10. The main idea is to apply Corollary 5.2.

Step 1. Let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive increasing function such that $b(0) = 0$ and $b(t) = f(t)$ for $t > c$. Let $c > 0$ and $w \in W^{1,p}_{loc}(\mathbb{R}^N)$ be a nonnegative weak solution of

$$\mathrm{div}_L \left(|\nabla_L w|^{p-2} \nabla_L w \right) \ge cb(w), \quad w \ge 0, \quad \text{on } \mathbb{R}^N. \tag{8.14}$$

We shall argue by contradiction assuming that $w \not\equiv 0$. Then there exists a constant $a > 0$ and a set U of positive measure such that $\inf_U w > 2a > 0$.

Let $|\cdot|_L$ be the Euclidean norm or the canonical homogeneous norm in the Heisenberg group (see Appendix A), and set $\psi := |\nabla_L |\cdot|_L|$. In these cases it is known that $\psi \le 1$. Let $D = Q > 1$ be the homogeneous dimension. For $a > 0$ as above, there exist $R > 0$ and φ a solution of

$$\left(r^{D-1} \left|\varphi'(r)\right|^{p-2} \varphi'(r) \right)' = r^{D-1} cb(\varphi(r)), \quad \varphi(0) = a/2, \quad \varphi'(0) = 0, \tag{8.15}$$

such that $\varphi(r) \to +\infty$ as $r \to R$; see [37] for a proof in the case $p = 2$ and [34] for the quasilinear case $p \ne 2$. Set $v(x) := \varphi(|x|_L)$. By computation we have

$$\begin{aligned} \mathrm{div}_L \left(|\nabla_L v|^{p-2} \nabla_L v \right) &= (p-1)\psi^p \left|\varphi'\right|^{p-2} \left(\varphi''(r) + \frac{Q-1}{p-1} \frac{\varphi'(r)}{r} \right)_{r=|\cdot|_L} \\ &= \psi^p \, |\cdot|_L^{1-Q} \left(r^{Q-1} \left|\varphi'(r)\right|^{p-2} \varphi'(r) \right)'_{r=|\cdot|_L}. \end{aligned}$$

Therefore, the function v satisfies the differential equation

$$\mathrm{div}_L \left(|\nabla_L v|^{p-2} \nabla_L v \right) = g_2(x, v) := \psi^p c\, b(v) \le c\, b(v) \quad \text{on } B_R.$$

Since w is a solution of (8.14), it follows that $w \in L^\infty_{loc}(\mathbb{R}^N)$. Indeed, in the Euclidean setting this follows from [29], while for the Heisenberg group see [10].

Clearly, the fact that $w \in L^\infty_{loc}(\mathbb{R}^N)$ implies that for $|x|_L$ close to R we have $w(x) \le v(x)$.

Now, an application of the weak comparison principle Theorem 4.8 gives $w \le v$ on B_R. Since the function v is continuous, there exists a neighborhood of the origin, say B_r, such that $w \le a$ on B_r. Taking into account that the problem (8.14) is translation invariant, then $w \le a$ on $\mathbb{R}^N$, contradicting $\inf_U w > 2a$.

Step 2. An application of Corollary (5.2) with $g \equiv 1$ completes the proof. □

In the Euclidean setting a more general operator then the p-Laplacian can be considered. Indeed, assume that $\mathcal{A}$ is **S**-p-**C** and has the form $\mathcal{A} = A(|\xi|)\xi$ with

$$\begin{cases} A \in \mathcal{C}(]0,+\infty[), \quad A(t) > 0 \quad \text{for } t > 0, \\ tA(|t|) \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1((0,+\infty)) \text{ and } (tA(t))' > 0 \quad \text{for } t > 0. \end{cases} \tag{8.16}$$

Let G be defined as

$$G(t) := t^2 A(t) - \int_0^t sA(s)\,ds, \quad t \ge 0.$$

It is easily seen that G is continuous and increasing, $G(0) = 0$, and, since $\mathcal{A}$ is **S**-p-**C**, $G(+\infty) = +\infty$. Let $H := G^{-1}$; then the function H is increasing and $H(+\infty) = +\infty$. See [35, 36, 34].

Theorem 8.12. *Let $f \in \mathcal{C}(\mathbb{R})$ be such that*

for some $c > 0$, f is nondecreasing, positive on $[c, +\infty)$, and

$$\int_c^\infty \left(H\left(\int_c^t f(s)\ ds \right) \right)^{-1} dt < \infty. \tag{8.17}$$

Assume that $V \in L^\infty_{loc}(\mathbb{R}^N)$ satisfies $V \le 0$. Then

(1) *If u is a weak solution of*
$$\operatorname{div}(A(|\nabla u|)\nabla u) + V(x)\,|u|^{p-2}\,u \ge f(u) \quad \text{on } \mathbb{R}^N, \tag{8.18}$$
then $u \le \max(Z(f) \cup \{0\})$ almost everywhere on $\mathbb{R}^N$.

(2) *If $V \equiv 0$ and u is a weak solution of* (8.18), *then $u \le \max Z(f)$ almost everywhere on $\mathbb{R}^N$.*

(3) *If $V \equiv 0$ and f is positive, then* (8.18) *has no weak solutions.*

(4) *If u is a weak solution of*

$$\operatorname{div}(A(|\nabla u|)\nabla u) = f(u) \quad \text{on } \mathbb{R}^N, \tag{8.19}$$

with f and $\bar{f}(t) := -f(-t)$ satisfying (8.17), *then*

$$\min Z(f) \le u(x) \le \max Z(f) \quad \text{a.e. on } \mathbb{R}^N.$$

The proof follows the same ideas as the proofs of Theorem 8.10 above, and of Theorems 4.1–4.3 in [16].

Remark 8.13. If f satisfies (8.11), then (8.17) is fulfilled.

Remark 8.14. All the results of this section can be formulated for the equation/inequality

$$\operatorname{div}_L(\mathcal{A}(x,u,\nabla_L u)) + V(x)\mathcal{B}(x,u,\nabla_L u) = (\ge) a(x) f(u), \quad \text{on } \mathbb{R}^N, \tag{8.20}$$

where the function $\mathcal{B}$ satisfies $\mathcal{B}(x,t,\xi) \le 0$ for $t \ge 0$.

9. Half space and exterior domain

Throughout this section we shall assume that $\mathcal{A}$ is **W**-p-**C** with $p > 1$, the vector field ∇_L satisfies (1.3), and $|\cdot|_L$ stands for a homogeneous norm.

If $\Omega = \mathbb{R}^N \setminus B_{R_0}$ and $q > 1$, it is well known that the inequality

$$\Delta u \ge u^q \quad \text{on } \Omega \tag{9.1}$$

has nontrivial positive solutions. Indeed, let $u(x) := ce^{-|x|}$ with $c^{q-1} = 1/N$. For $r = |x| > N$, we have

$$\Delta u = ce^{-r}(1 - \frac{N-1}{r}) \ge ce^{-r}(1 - \frac{N-1}{N}) = \frac{c}{N}e^{-r} \ge c^q e^{-rq} = u^q.$$

In a similar way the function $u(x) := e^{-x_1}$ is a solution of (9.1) for $q > 1$ on the half space $\Omega := (0+\infty) \times \mathbb{R}^{N-1}$.

More generally, if v is a nonnegative bounded solution of

$$\Delta v + \lambda^2 v \ge v^q, \quad \text{on } \mathbb{R}^{N-1},$$

then the problem

$$\Delta u \ge u^q, \quad (x_1, y) \in (0+\infty) \times \mathbb{R}^{N-1}, \quad u(0,y) = v(y), \quad \text{on } \mathbb{R}^{N-1}$$

admits a nonnegative bounded solution $u(x_1, y) := e^{-\lambda x_1} v(y)$. For instance, we can choose v satisfying one of the following:

(1) Any nonnegative bounded subharmonic function,
(2) For $N = 2$, $v(y) := c(1 + \sin(y))$, with $0 < c \le 2^{-\frac{q}{q-1}}$.

From above examples it is apparent that when $\Omega \neq \mathbb{R}^N$, in order to prove Liouville theorems for coercive inequalities, some extra conditions are needed. In this section for simplicity we shall consider the cases $\Omega = \mathbb{R}^N \setminus B_{R_0}$ or $\Omega := (0 + \infty) \times \mathbb{R}^{N-1}$. In particular, we shall assume that the possible solutions of our inequalities satisfy an homogeneous Dirichlet condition.

Definition 9.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function. Let $p \geq 1$. We say that u is a* weak solution *of*

$$\operatorname{div}_L (\mathcal{A}(x, u, \nabla_L u)) \geq f(x, u, \nabla_L u) \quad \text{on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{9.2}$$

if $u \in \mathcal{C}(\overline{\Omega})$, $u(x) = 0$ for $x \in \partial\Omega$, and u is a weak solution in the sense of Definition 1.2; that is, $u \in W^{1,p}_{loc}(\Omega)$, $\mathcal{A}(\cdot, u, \nabla u) \in L^{p'}_{loc}(\Omega)$, $f(\cdot, u, \nabla_L u) \in L^1_{loc}(\Omega)$, and for any nonnegative $\phi \in \mathcal{C}^1_0(\Omega)$, we have

$$-\int_\Omega \mathcal{A}(x, u, \nabla_L u) \cdot \nabla_L \phi \geq \int_\Omega f(x, u, \nabla_L u)\phi.$$

Let u be a nonnegative weak solution of (9.2). Let $\epsilon > 0$ and define $v_{\epsilon,\eta} := m_\eta \star (u - \epsilon)^+$ where m_η is a standard family of mollifiers. Clearly, for any nonnegative $\phi \in \mathcal{C}^1_0(\mathbb{R}^N)$ and $\alpha > 0$ the function $v^\alpha_{\epsilon,\eta}\phi$ has compact support in Ω and

$$\int_\Omega f v^\alpha_{\epsilon,\eta}\phi \leq -\alpha \int_\Omega \mathcal{A}(x, u, \nabla_L u) \cdot \nabla_L v_{\epsilon,\eta}\, v^{\alpha-1}_{\epsilon,\eta}\phi - \int_\Omega \mathcal{A}(x, u\nabla_L u) \cdot \nabla_L \phi\, v^\alpha_{\epsilon,\eta}.$$

A standard argument shows that by letting $\eta \to 0$ and $\epsilon \to 0$ in the above inequality, for any nonnegative $\phi \in \mathcal{C}^1_0(\mathbb{R}^N)$ we have

$$\int_\Omega f u^\alpha \phi + \alpha \int_\Omega \mathcal{A}(x, u, \nabla_L u) \cdot \nabla_L u\, u^{\alpha-1}\phi \leq -\int_\Omega \mathcal{A} \cdot \nabla_L \phi\, u^\alpha.$$

Therefore, we can deduce *a priori* estimates on nonnegative solutions of (9.2) which are the analogue of those contained in Lemmas 6.1 and 6.4. Notice that since $\mathcal{A}$ is **W**-p-**C**, estimate (6.3) holds. Moreover, if $f(t) \geq at^q$ with a a positive constant, then (6.9) and (6.13) are satisfied.

Since most of the results like Theorems 8.1 and 8.4 are based on those estimates, we can easily deduce that the same holds for (9.2).

In what follows we shall assume that $a \equiv 1$. The case when a is non-constant can be easily obtained by a slight modification of the argument below.

Theorem 9.2. *Let $\Omega = \mathbb{R}^N \setminus B_{R_0}$ or $\Omega := (0 + \infty) \times \mathbb{R}^{N-1}$. Suppose that $V \in L^\infty_{loc}(\Omega)$ satisfies $V \leq 0$ and $f \in \mathcal{C}(\mathbb{R})$ is such that $f(t) \geq c\,t^q$ for $t > 0$,* *with $q > p-1$ and $c > 0$. If u is a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) + V(x)\,|u|^{p-2}\,u \geq a(x)f(u) \quad \text{on } \Omega,\ u=0 \text{ on } \partial\Omega, \tag{9.3}$$

then $u \leq 0$ on Ω. Moreover, if $t\,f(t) \geq c\,|t|^{q+1}$, $t \in \mathbb{R}$, and u is a solution the equation

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) + V(x)\,|u|^{p-2}\,u = a(x)\,f(u) \quad \text{on } \Omega,\ u=0 \text{ on } \partial\Omega, \tag{9.4}$$

then $u \equiv 0$ on Ω.

Proof. The proof follows the same steps as the proof of Theorem 8.1. Indeed, we first prove that

$$\mathrm{div}_L\,(\mathcal{A}(x,u,\nabla_L u)) \geq c\,u^q, \quad u \geq 0 \text{ on } \Omega, \quad u=0 \text{ on } \partial\Omega, \tag{9.5}$$

has no nontrivial solutions for any $c > 0$ and then we apply the reduction principles Theorem 4.1.

Step 1. From (6.13) with $a \equiv 1$ we have

$$\int_\Omega u^{q+\alpha}\phi \leq (c_2\alpha^{1-p})^{\chi'} \int_\Omega \frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}},$$

for any $\alpha > 1$ and $\chi' := \frac{q+\alpha}{q-p+1}$. Choosing $\phi \in \mathcal{C}_0^1(\mathbb{R}^N)$ as in the proof of Theorem 8.1 we get

$$\int_{\Omega\cap B_R} u^{q+\alpha} \leq \int_\Omega u^{q+\alpha}\phi \leq c(\phi_0,p,\alpha,Q)R^{Q-p\frac{q+\alpha}{q-p+1}}.$$

Now for α large enough, the exponent $Q - p\frac{q+\alpha}{q-p+1}$ is negative. Thus, by letting $R \to +\infty$, it follows that

$$\int_\Omega u^{q+\alpha} = 0.$$

Hence, $u \equiv 0$.

Step 2. By applying Theorem 4.1 and 4.4, we complete the proof. □

By using Theorem 5.1 we also have the following.

Theorem 9.3. *Let $\Omega = \mathbb{R}^N \setminus B_{R_0}$ or $\Omega :=]0{+}\infty[\times\mathbb{R}^{N-1}$ and $\mathcal{A} = \mathcal{A}(x,\nabla_L u)$. Suppose that $V \in L^\infty_{loc}(\Omega)$ satisfies $V \leq 0$ and let $f \in \mathcal{C}(\mathbb{R})$ be such that*

$$\liminf_{t\to+\infty} \frac{f(t)}{|t|^q} > 0,$$

for $q > p-1$. Then

(1) *If u is a weak solution of inequality* (9.3), *then $u \leq \max(Z(f)\cup\{0\})$.*

(2) *If $V \equiv 0$ and u is a weak solution of equation* (9.4) *and f satisfies*

$$\liminf_{t\to+\infty} \frac{f(t)}{|t|^q} > 0, \qquad \limsup_{t\to-\infty} \frac{f(t)}{|t|^q} < 0,$$

then

$$\min(Z(f)\cup\{0\}) \le u \le \max(Z(f)\cup\{0\}).$$

10. Some conditionally Liouville theorems

Throughout this section we shall assume that $\mathcal{A}$ is **W**-p-**C** with $p > 1$, the vector field ∇_L satisfies (1.3), and $|\cdot|_L$ stands for a homogeneous norm.

In what follows we shall study some Liouville theorems under additional weak integral conditions on the solutions at infinity.

We point out that similar results in the Euclidean framework are considered in [18, 19, 40] assuming pointwise *a priori* conditions on the solutions.

10.1. **Homogeneous inequalities.**

Theorem 10.1. *Let $p \ge Q$ and let $\mathcal{A}$ be* **W**-p-**C** *satisfying* (4.13). *Assume that* (4.12) *holds. Let $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ be a weak solution of*

$$\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) \ge 0 \qquad \text{on } \mathbb{R}^N. \tag{10.1}$$

Assume that there exists $\sigma > p-1$ such that

$$\liminf_{R\to+\infty} R^{\frac{Q-p}{\sigma}} \left(\fint_{A_R} |u|^\sigma \right)^{1/\sigma} < +\infty. \tag{10.2}$$

Then $u \equiv const$ almost everywhere on $\mathbb{R}^N$.

The proof of the above result is based on the following lemma. Notice that we do not assume that $p \ge Q$.

Lemma 10.2. *Let $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ be a weak nonnegative solution of* (10.1). *Assume that there exists $\sigma > p-1$ such that* (10.2) *holds.*

Then $\mathcal{A}(x,u(x),\nabla_L u(x)) \equiv 0$ almost everywhere on $\mathbb{R}^N$. In particular, if (4.12) *and* (4.13) *hold, then $u \equiv const$ almost everywhere on $\mathbb{R}^N$.*

Proof. Let u be a nonnegative solution of (10.1). Then u is a solution of (6.1) with $g \equiv V \equiv 0$.

Let $\phi \in \mathcal{C}^1_0(\mathbb{R}^N)$ be a nonnegative test function and let $\epsilon, \alpha > 0$. Arguing as in the proof of Lemma 6.1, by choosing as test function $u_\epsilon^\alpha \phi$, where $u_\epsilon := u + \epsilon$, we obtain

$$\alpha \int_{\mathbb{R}^N} \mathcal{A}\cdot\nabla_L u u_\epsilon^{\alpha-1}\phi \le -\int_{\mathbb{R}^N} \mathcal{A}\cdot\nabla_L\phi\, u_\epsilon^\alpha.$$

Since $\mathcal{A}$ is **W**-p-**C**, by Hölder's inequality it follows that

$$\alpha k_1 \int_{\mathbb{R}^N} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1}\phi \le \Big(\int_S |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1}\phi\Big)^{1/p'} \Big(\int_S \frac{|\nabla_L \phi|^p}{\phi^{p-1}} u_\epsilon^{\alpha+p-1}\Big)^{1/p}, \quad (10.3)$$

where S is the support of $\nabla_L \phi$. The integrals of the right-hand side of (10.3) are finite provided $\alpha + p - 1 \le \sigma$ and $\alpha - 1 \le p^5$.

Let $\phi(x) := \phi_0(|\delta_{1/R} x|)$, where ϕ_0 is a standard cut-off function, and choose $\alpha := \sigma - p + 1$. Inequality (10.3) implies

$$\int_{B_R} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1} \le cR^{-p} \int_{A_R} u_\epsilon^\sigma, \quad (10.4)$$

and

$$\int_{B_R} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1} \le c\Big(\int_{A_R} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1}\Big)^{1/p'} \Big(R^{-p} \int_{A_R} u_\epsilon^\sigma\Big)^{1/p}. \quad (10.5)$$

Next we note that (10.2) is equivalent to

$$\liminf_{R\to+\infty} R^{-p} \int_{A_R} u_\epsilon^\sigma < M < +\infty. \quad (10.6)$$

Let $(R_j)_j$ be a sequence such that

$$\lim_{j\to+\infty} R_j^{-p} \int_{A_{R_j}} u_\epsilon^\sigma < M < +\infty. \quad (10.7)$$

From (10.4) and Beppo Levi's theorem, it follows that

$$\int_{\mathbb{R}^N} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1} \le cM;$$

that is, $|\mathcal{A}|^{p'} u_\epsilon^{\alpha-1} \in L^1(\mathbb{R}^N)$. This implies

$$\lim_{j\to+\infty} \int_{A_{R_j}} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1} = 0.$$

This information together with (10.5) and (10.7) gives

$$\int_{\mathbb{R}^N} |\mathcal{A}|^{p'} u_\epsilon^{\alpha-1} = 0.$$

Since $u_\epsilon > 0$, from the above fact we get the claim.

Finally, if (4.13) holds we have that $u(x)\nabla_L u(x) = 0$ for almost every x; that is, $\nabla_L u^p \equiv 0$. The validity of (4.12) implies the claim. □

[5]Notice that if $\alpha \le 1$ then $u_\epsilon^{\alpha-1} \le \epsilon^{1-\alpha}$.

Proof of Theorem 10.1. From Lemma 10.2 we are in a position to apply Theorem 4.1 with the choice

$$X := W^{1,p}_{loc}(\mathbb{R}^N) \setminus \{\text{constant functions}\} \cup \{0\},$$

deducing that the nonconstant solutions of (10.1) are nonpositive[6].

Now, the function $w := -u$ is a nonnegative solution of the inequality

$$-\mathrm{div}_L\left(\overline{\mathcal{A}}\,(x, w, \nabla_L w)\right) \geq 0, \quad \text{on } \mathbb{R}^N.$$

Since $p \geq Q$ and $\overline{\mathcal{A}}$ is **W**-p-**C**, from [13, Theorem 4.2], it follows that $\overline{\mathcal{A}} \equiv 0$, which together with (4.13) and (4.12) gives the claim. □

Remark 10.3. Let $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ be such that

$$\left(\fint_{A_R} |u|^\sigma\right)^{1/\sigma} = O(R^\theta),$$

for $\sigma > p-1$ and $\theta \in \mathbb{R}$. Then we have the following:

(1) If $p > Q$, then u satisfies (10.2) provided $\theta < \frac{p-Q}{p-1}$. The exponent $\frac{p-Q}{p-1}$ is sharp. Indeed, in the Euclidean case, i.e., $N = Q$, considering $\Gamma(x) := |x|^{\frac{p-N}{p-1}}$ (the fundamental solution of the p-Laplacian), then

$$\left(\fint_{A_R} |\Gamma|^\sigma\right)^{1/\sigma} = O(R^{\frac{p-N}{p-1}}),$$

and for a positive constant $c = c(N,p)$, there holds

$$\Delta_p u = c\,\delta_0 \geq 0.$$

(2) If $Q = p$, then u satisfies (10.2) provided $\theta = 1$. We point out that it would be interesting to know whether condition (10.2) can be relaxed to

$$\left(\fint_{A_R} |u|^\sigma\right)^{1/\sigma} = o(\ln R).$$

By using Lemma 10.2 and Theorem 4.4 we can prove the following. We omit the details.

Theorem 10.4. *Let $p \geq Q$ and let $\mathcal{A}$ be* **W**-p-**C** *such that* (4.13) *and* (4.12) *hold. Let $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ be a weak solution of*

$$\mathrm{div}_L\left(\mathcal{A}(x, u, \nabla_L u)\right) = f(x, u, \nabla_L u), \quad \text{on } \mathbb{R}^N, \tag{10.8}$$

where f is a Caratheodory function satisfying $f(x,t,\xi)t \geq 0$.

[6] To see that X satisfies the hypotheses of Theorem 4.1, see the proof of Corollary 5.3.

If $\sigma > p-1$ *and* (10.2) *holds, then* $u \equiv c = const$ *and* $f(x,c,0) = 0$.

10.2. **Sublinear inequalities.**

Theorem 10.5. *Let* q *be such that* $-\infty < q < p-1$. *The problem*

$$\operatorname{div}_L(\mathcal{A}(x,u,\nabla_L u)) \geq u^q, \quad u \geq 0, \quad \text{on } \mathbb{R}^N, \tag{10.9}$$

has no nontrivial solution u *satisfying*

$$\Big(\fint_{A_R} |u|^\sigma\Big)^{1/\sigma} R^{-\frac{p}{p-q-1}} \leq MR^{-\theta} \quad \text{for } R \text{ large}, \tag{10.10}$$

where $\sigma, \theta > 0$ *are such that*

$$\sigma\theta > Q + \frac{qp}{p-q-1} \quad \text{and} \quad \sigma > p-1. \tag{10.11}$$

Proof. Let $\alpha > 0$ be small enough such that

$$\sigma > \alpha + p - 1, \qquad \sigma\theta > Q + \frac{qp}{p-q-1} + \alpha\frac{p}{p-q-1}.$$

Choosing $\phi(x) := \phi_0(\left|\delta_{1/R}x\right|_L)$ with ϕ_0 a standard cut-off function, from (6.3) and Remark 6.3, we deduce that for large R we have

$$\int_{\mathbb{R}^N} u^{q+\alpha}\phi \leq c\int_{\mathbb{R}^N} u^{\alpha+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}.$$

By Hölder's inequality with exponent $y := \frac{\sigma-q-\alpha}{p-q-1}$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} u^{q+\alpha}\phi &\leq c\int_{\mathbb{R}^N} u^{\frac{q+\alpha}{y'}} u^{\frac{q+\alpha}{y}+p-q-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}} \\ &\leq c\Big(\int_{\mathbb{R}^N} u^{q+\alpha}\phi\Big)^{1/y'}\Big(\int_{\mathbb{R}^N} u^\sigma\frac{|\nabla_L\phi|^{py}}{\phi^{py-1}}\Big)^{1/y}.\end{aligned}$$

Thus, by our choice of ϕ and (10.10) we obtain

$$\int_{B_R} u^{q+\alpha} \leq c(y,\phi_0)R^{-py}\int_{A_R} u^\sigma \leq c(y,\phi_0,M)R^k, \tag{10.12}$$

where $k := -py - \theta\sigma + \dfrac{\sigma p}{p-q-1} + Q$. On the other hand,

$$k = -p\frac{\sigma-q-\alpha}{p-q-1} + \frac{\sigma p}{p-q-1} + Q - \theta\sigma = \frac{pq+p\alpha}{p-q-1} + Q - \theta\sigma < 0.$$

By letting $R \to +\infty$ in (10.12), we complete the proof. □

Corollary 10.6. *Let $-\infty < q < p-1$. Problem* (10.9) *has no nontrivial solution u satisfying*

$$u(x) \leq M\,|x|_L^{\gamma} \quad \text{for } |x|_L \text{ large and } \gamma < \tfrac{p}{p-q-1}.$$

Proof. It is enough to show that the assumptions of Theorem 10.5 are fulfilled. Assumption (10.10) is satisfied for any $\sigma > 0$ and $\theta = \frac{p}{p-q-1} - \gamma$. This implies that for σ large, (10.11) holds, completing the proof. □

An application of Theorem 4.4 and Theorem 5.2 gives the following.

Corollary 10.7. *Let u be a weak solution of*

$$\mathrm{div}_L\left(\mathcal{A}(x,u,\nabla_L u)\right) = f(u) \quad \text{on } \mathbb{R}^N,$$

where $f \in \mathcal{C}(\mathbb{R})$ is such that $f(t)t \geq c\,|t|^{q+1}$ for $0 < q < p-1$ and $c > 0$. Assume there exist $\sigma, \theta > 0$ such that (10.11) *and* (10.10) *hold. Then $u \equiv 0$ almost everywhere on $\mathbb{R}^N$.*

Theorem 10.8. *Let $f \in \mathcal{C}(\mathbb{R})$ be such that*

$$\liminf_{t\to+\infty} \frac{f(t)}{t^q} > 0 \quad \text{and} \quad \limsup_{t\to-\infty} \frac{f(t)}{|t|^q} < 0$$

for $-\infty < q < p-1$. Let u be a weak solution of

$$\mathrm{div}_L\left(\mathcal{A}(x,\nabla_L u)\right) = f(u) \quad \text{on } \mathbb{R}^N. \tag{10.13}$$

Assume there exist $\sigma, \theta > 0$ such that (10.11) *and* (10.10) *hold. Then*

$$\min Z(f) \leq u \leq \max Z(f) \quad \text{a.e. on } \mathbb{R}^N.$$

11. LIOUVILLE THEOREMS RELATED TO INEQUALITIES INVOLVING A GRADIENT TERM

Throughout this section we shall assume that $\mathcal{A}$ is **W**-p-**C** with $p > 1$, the vector field ∇_L satisfies (1.3), and $|\cdot|_L$ stands for a homogeneous norm.

Definition 11.1. *Let $f : \Omega \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function. A vector field $H : \mathbb{R}^N \to \mathbb{R}^l$ is a weak solution of*

$$\pm\,\mathrm{div}_L\left(H(x)\right) \geq f(x, H(x)), \quad \text{on } \mathbb{R}^N, \tag{11.1}$$

if $|H|, f(\cdot, H(\cdot)) \in L^1_{loc}(\mathbb{R}^N)$ and for any nonnegative $\phi \in \mathcal{C}^1_0(\mathbb{R}^N)$ there holds

$$\mp \int_{\mathbb{R}^N} H(x)\cdot\nabla_L\phi \geq \int_{\mathbb{R}^N} f(x,H(x))\phi.$$

Lemma 11.2. *Let $r>1$. Let $H \in L^1_{loc}(\mathbb{R}^N;\mathbb{R}^l)$ be a solution of*

$$\pm \operatorname{div}_L (H(x)) \geq a(x)\ |H(x)|^r, \quad \text{on } \mathbb{R}^N, \tag{11.2}$$

where $a:\mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function such that for R large $\psi^{\frac{r}{r-1}} a^{-\frac{1}{r-1}} \in L^1_{loc}(A_R)$.

Then there exists $C>0$ such that for R large, we have

$$\int_{B_R} a(x)\,|H(x)| \leq C R^{-\frac{r}{r-1}} \int_{A_R} \psi^{\frac{r}{r-1}} a^{-\frac{1}{r-1}}. \tag{11.3}$$

Moreover, if

$$\liminf_{R\to+\infty}\ R^{-\frac{r}{r-1}} \int_{A_R} \psi^{\frac{r}{r-1}} a^{-\frac{1}{r-1}} \leq M < +\infty, \tag{11.4}$$

then $a(x)H(x)=0$ for almost every $x \in \mathbb{R}^N$.

In particular, if a is positive and there exist $\theta<1$ and $c>0$ such that

$$a(x) \geq \frac{c}{|x|_L^\theta} \quad \text{for } |x|_L \text{ large,}$$

with $1<r\leq \frac{Q-\theta}{Q-1}$, then (11.2) has no nontrivial solutions.

Proof. Let $\phi := \phi_R := \phi_0(\left|\delta_{1/R} x\right|_L)$ with ϕ_0 a standard cut-off function. Since $r>1$, we have

$$\begin{aligned}
&\int_{B_{2R}} a(x)\ |h(x)|^r\,\phi \leq \int_{A_R} |H(x)|\,|\nabla_L \phi| \\
&\leq \Big(\int_{A_R} a(x)\ |H(x)|^r\,\phi\Big)^{1/r} \Big(\int_{A_R} a(x)^{-r'/r} \frac{|\nabla_L \phi|^{r'}}{\phi^{r'-1}}\Big)^{1/r'} \\
&\leq \Big(\int_{A_R} a(x)\ |H(x)|^r\,\phi\Big)^{1/r} \Big(C(\phi_0) \int_{A_R} R^{-\frac{r}{r-1}} \psi^{\frac{r}{r-1}} a(x)^{-\frac{1}{r-1}}\Big)^{1/r'}.
\end{aligned} \tag{11.5}$$

Therefore, for R sufficiently large we have (11.3).

Assume that (11.4) holds, and let $(R_j)_j$ be a sequence such that

$$\lim_{j\to+\infty}\ R_j^{-\frac{r}{r-1}} \int_{A_{R_j}} \psi^{\frac{r}{r-1}} a^{-\frac{1}{r-1}} \leq M < +\infty.$$

From (11.5) we obtain

$$\int_{B_{R_j}} a(x)\ |H(x)|^r \leq (CM)^{1/r'} \Big(\int_{A_{R_j}} a(x)\ |H(x)|^r\Big)^{1/r}. \tag{11.6}$$

Now, from (11.3) and (11.4), we deduce that $a\,|H|^r \in L^1(\mathbb{R}^N)$. Using this fact in (11.6) it follows that $a\,|H| \equiv 0$ almost everywhere on $\mathbb{R}^N$. □

As a consequence we have the following.

Theorem 11.3. *Let $r > p-1 > 0$ and let $\mathcal{A} = \mathcal{A}(x,\xi)$ be* **W**-*p*-**C**. *Let $a : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative measurable function and let $h \in L^{p-1}_{loc}(\mathbb{R}^N;\mathbb{R}^l)$ be a solution of*

$$\pm\,\mathrm{div}_L\left(\mathcal{A}(x,h(x))\right) \geq a(x)\ |h(x)|^r\,, \quad \text{on } \mathbb{R}^N. \tag{11.7}$$

If

$$\liminf_{R\to+\infty} R^{-\frac{r}{r-p+1}} \int_{A_R} \psi^{\frac{r}{r-p+1}} a^{-\frac{p-1}{r-p+1}} < +\infty, \tag{11.8}$$

then $a(\cdot)\mathcal{A}(\cdot,h(\cdot)) \equiv 0$ almost everywhere on $\mathbb{R}^N$.

In particular, if a is positive and there exist $\theta < 1$ and $c > 0$ such that

$$a(x) \geq \frac{c}{|x|_L^{\theta}}, \quad \text{for } |x|_L \text{ large},$$

with $1 < r \leq \frac{Q-\theta}{Q-1}$, then (11.7) has no nontrivial solutions. Moreover, if $\mathcal{A} = \mathcal{A}(x,\xi)$ is **W**-*p*-**C** *and $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ is a weak solution of*

$$\pm\,\mathrm{div}_L\left(\mathcal{A}(x,\nabla_L u(x))\right) \geq |\nabla_L u(x)|^r\,, \quad \text{on } \mathbb{R}^N, \tag{11.9}$$

with $p-1 < r \leq \frac{Q}{Q-1}(p-1)$, then $\nabla_L u \equiv 0$ almost everywhere on $\mathbb{R}^N$.

Proof. Since $\mathcal{A}$ is **W**-*p*-**C** one easily gets that $|\mathcal{A}(x,\xi)| \leq k_2^{1-p}\,|\xi|^{p-1}$. Therefore the condition $|\mathcal{A}(\cdot,h)| \in L^1_{loc}(\mathbb{R}^N)$ is satisfied and h solves the inequality

$$\pm\mathrm{div}_L\left(\mathcal{A}(x,h(x))\right) \geq a(x)\ |\mathcal{A}(x,h(x))|^{\frac{r}{p-1}} \quad \text{on } \mathbb{R}^N.$$

An application of Lemma 11.2 completes the proof. □

Theorem 11.4. *Suppose that (4.12) holds. Let $\mathcal{A} = \mathcal{A}(x,\xi)$ be* **W**-*p*-**C** *and let $g : \mathbb{R}^l \to \mathbb{R}$ be such that $g(\xi) \geq c\,|\xi|^r$ with $r > p-1$. Let $a : \mathbb{R}^N \to \mathbb{R}$ be a positive measurable function satisfying (11.8). Let $f \in \mathcal{C}(\mathbb{R})$ be such that*

$$\liminf_{t\to+\infty} f(t) > 0, \tag{11.10}$$

and let $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function such that for any $t \geq 0$ we have $\mathcal{B}(x,t,\xi) \leq 0$.

(1) *Let u be a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,\nabla_L u)) + \mathcal{B}(x,u,\nabla_L u) \geq a(x)f(u)g(\nabla_L u). \tag{11.11}$$

Then either $u \leq \max(Z(f)\cup\{0\})$ *almost everywhere on* $\mathbb{R}^N$ *or* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$.

(2) *Let u be a weak solution of* (11.11) *with* $\mathcal{B} \equiv 0$. *Then either* $u \leq \max Z(f)$ *almost everywhere on* $\mathbb{R}^N$ *or* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$. *In particular, if f is positive then* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$.

(3) *Let u be a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,\nabla_L u)) + \mathcal{B}(x,u,\nabla_L u) = a(x)f(u)g(\nabla_L u), \tag{11.12}$$

with f satisfying

$$\limsup_{t\to-\infty} f(t) < 0, \qquad \liminf_{t\to+\infty} f(t) > 0, \tag{11.13}$$

and $\mathcal{B}(x,t,\xi)t \leq 0$. *Then either* $\min(Z(f)\cup\{0\}) \leq u \leq \max(Z(f)\cup\{0\})$ *almost everywhere on* $\mathbb{R}^N$ *or* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$.

(4) *Let u be a weak solution of* (11.12) *with* $\mathcal{B} \equiv 0$ *and f satisfying* (11.13). *Then either* $\min Z(f) \leq u \leq \max Z(f)$ *almost everywhere on* $\mathbb{R}^N$ *or* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$.

Proof. Claims (1) and (2) are a consequence of Corollary 5.3 and Theorem 11.3, while (3) and (4) follow from (1) and (2) applied to the function $-u$ and the function $\overline{\mathcal{A}}$. □

Corollary 11.5. *Suppose that* (4.12) *holds. Let* $\mathcal{A} = \mathcal{A}(x,\xi)$ *be* **W**-*p*-**C** *and let* $g : \mathbb{R}^l \to \mathbb{R}$ *be such that* $g(\xi) \geq c\,|\xi|^r$ *for* $r > p-1$. *Assume that* $a : \mathbb{R}^N \to \mathbb{R}$ *is a positive measurable function satisfying*

$$a(x) \geq \frac{c}{|x|_L^\theta}, \quad \text{for } |x|_L \text{ large},$$

where $\theta < 1$ *and* $c > 0$. *Let f and $\mathcal{B}$ be as in Theorem* 11.4. *If* $0 < p-1 < r \leq \frac{Q-\theta}{Q-1}(p-1)$, *then the conclusions* (1), (2), (3), *and* (4) *of Theorem* 11.4 *hold.*

Remark 11.6. We notice that in Theorem 11.4 and Corollary 11.5 the assumption (11.10) is necessary in order to have a Liouville theorem.

Indeed, the function $u(x) := \ln(1+|x|^2)$ is a $\mathcal{C}^\infty$ solution of

$$\Delta_p u = f(u)\,|\nabla u|^{(p-1)\frac{N}{N-1}} \quad \text{on } \mathbb{R}^N, \quad N > p > 1,$$

where

$$f(u) = 2^{\frac{1-p}{N-1}} \frac{(N-p)e^u + 2(p-1)}{e^{\frac{N-p}{N-1}u}(e^u-1)^{\frac{N+p-2}{2(N-1)}}}.$$

If $0 < p-1 < r \leq \frac{Q-\theta}{Q-1}(p-1)$ and f does not satisfy (11.10), then solutions of (11.11) may exist. In this case they must be unbounded, as the following result shows.

Theorem 11.7. *Suppose that* (4.12) *holds. Let* $\mathcal{A} = \mathcal{A}(x,\xi)$ *be* **W**-p-**C** *and assume that* $g : \mathbb{R}^l \to \mathbb{R}$ *is such that* $g(\xi) \geq c\,|\xi|^r$. *Let* $a : \mathbb{R}^N \to \mathbb{R}$ *be a positive measurable function satisfying*

$$a(x) \geq \frac{c}{|x|_L^\theta}, \quad \text{for } |x|_L \text{ large},$$

where $\theta < 1$ *and* $c > 0$. *If* $0 < p-1 < r \leq \frac{Q-\theta}{Q-1}(p-1)$, *then we have the following:*

(1) *Let* u *be a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,\nabla_L u)) \geq a(x)f(u)g(\nabla_L u), \tag{11.14}$$

where $f \in \mathcal{C}(\mathbb{R})$ *is such that* $f(t) > 0$ *for* $t > z_1$. *If* u *is bounded from above, then either* $u \leq z_1$ *almost everywhere on* $\mathbb{R}^N$ *or* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$.

(2) *Let* u *be a weak solution of*

$$\mathrm{div}_L\,(\mathcal{A}(x,\nabla_L u)) = a(x)f(u)g(\nabla_L u), \tag{11.15}$$

where $f \in \mathcal{C}(\mathbb{R})$ *is such that* $f(t) > 0$ *for* $t > z_1$ *and* $f(t) < 0$ *for* $t < z_0$. *If* u *is bounded, then either* $z_0 \leq u \leq z_1$ *almost everywhere on* $\mathbb{R}^N$ *or* $\nabla_L u \equiv 0$ *almost everywhere on* $\mathbb{R}^N$.

Proof. We prove the case (1). Case (2) follows similarly.

Let u be a solution of (11.14) such that $u \leq M$. If $M \leq z_1$ we are done. Otherwise, if $M > z_1$, define $f_M \in \mathcal{C}(\mathbb{R})$ as follows:

$$f_M(t) := \begin{cases} f(t) & \text{if } t \leq M; \\ f(M) & \text{if } t > M. \end{cases}$$

Clearly, u is a solution of

$$\mathrm{div}_L\,(\mathcal{A}(x,\nabla_L u)) \geq a(x)f_M(u)g(\nabla_L u),$$

with f_M satisfying the hypotheses of Corollary 11.5. This completes the proof. □

Remark 11.8. In general, the exponent $\frac{Q}{Q-1}(p-1)$ in Theorem 11.4, Corollary 11.5, and Theorem 11.7 is sharp. To see this consider the special case where $\nabla_L = \nabla$, the usual Euclidean gradient, and $Q = N$. The function defined by $u(x) := \exp(-1/|x|^{\frac{p-r}{r-p+1}})$, is a $\mathcal{C}^\infty$ bounded solution of

$$\Delta_p u \geq c\,|\nabla u|^r \quad \text{on } \mathbb{R}^N, \tag{11.16}$$

with $p > r > \frac{N}{N-1}(p-1)$. Indeed,

$$\frac{\Delta_p u}{|\nabla u|^r} = \frac{\alpha^{p-1-r}}{r-p+1}\Big(r(N-1) - N(p-1) + \frac{(p-1)(p-r)}{|x|^\alpha}\Big)u^{p-1-r},$$

where $\alpha := \frac{p-r}{r-p+1}$. Since $p > r > \frac{N(p-1)}{N-1}$ it follows that $p-1-r < 0$. From the bound $u < 1$ we have that u solves (11.16) with

$$c = \frac{\alpha^{p-1-r}}{r-p+1}\left(r(N-1) - N(p-1)\right).$$

Corollary 11.9. *Let $p-1 < r \leq \frac{N}{N-1}(p-1)$ and let f be a positive real continuous function such that* $\liminf_{t\to+\infty} f(t) > 0$. *Then, the weak solutions of the inequality*

$$\Delta_p u \geq f(u)\,|\nabla u|^r, \quad \text{on } \mathbb{R}^N,$$

are almost everywhere constant on $\mathbb{R}^N$.

Corollary 11.10. *Let $p-1 < r \leq \frac{N}{N-1}(p-1)$ and let f be a negative real continuous function such that* $\liminf_{t\to-\infty} f(t) < 0$. *Then, the weak solutions of the equation*

$$\Delta_p u = f(u)\,|\nabla u|^r, \quad \text{on } \mathbb{R}^N, \tag{11.17}$$

are almost everywhere constant on $\mathbb{R}^N$.

Proof. Let $v := -u$ and $\bar{f}(t) := -f(-t)$. The function v solves the equation

$$\Delta_p v = \bar{f}(v)\,|\nabla v|^r, \quad \text{on } \mathbb{R}^N.$$

Since $\bar{f}$ satisfies the assumptions of Corollary 11.9, we conclude the proof. □

The above results give an answer to a problem raised by Peletier and Serrin in [38, p. 80] where, among other things, the authors prove that the solutions of (11.17) with $p = 2$ are constants provided $r > 1$, $f' \geq 0$, and $\inf(f' + f^2) > 0$. In [38] the authors ask if the assumption $\inf(f' + f^2) > 0$ can be weakened to $f' + f^2 > 0$. An affirmative answer in this direction is given by the following.

Corollary 11.11. *Let $p-1<r\le \frac{N}{N-1}(p-1)$ and let $f\in \mathcal{C}(\mathbb{R})$ such that*

$$f'(t)\ge 0,\quad f'(t)+f^2(t)>0,\quad t\in\mathbb{R}. \tag{11.18}$$

Then, the weak solutions of the equation (11.17) *are almost-everywhere constant on* $\mathbb{R}^N$.

Proof. First we consider the case when f has no zeros. If f is positive, since f is nondecreasing, it follows that $\lim_{t\to+\infty} f(t)>0$, and the claim follows from Corollary 11.9. Analogously, if f is negative the claim follows from Corollary 11.10.

Now assume that f possesses a zero. Since f is monotone $Z(f)$ is an interval. We claim that $Z(f)$ has only one point. Indeed, arguing by contradiction, if $t_1,t_2\in Z(f)$ with $t_1<t_2$ then for any $t\in(t_1,t_2)$ we have $f(t)=f'(t)=0$, contradicting (11.18). An application of Corollary 11.5 concludes the proof. □

12. APPLICATIONS TO GROSS–PITAEVSKII- AND GINZBURG–LANDAU-TYPE EQUATIONS

The aim of this last section is to illustrate some applications of the results obtained in the previous sections to study solutions of equation of the form

$$\Delta_p u = f(u)-\lambda u, \quad \text{on } \mathbb{R}^N. \tag{12.1}$$

For $p=2$, (12.1) appears when dealing with the so-called *defocusing Schrödinger equation* or the Gross–Pitaevskii equation, as well as the Ginzburg–Landau-type equation (see (12.5) below).

We point out that our results can be formulated also for equations involving more general operators as in(8.19). For additional interesting results on the semilinear case see [17] and the recent paper [8].

In what follows we shall assume that f has the form

$$f(t)=f_1(|t|)t, \text{ with } \lim_{t\to+\infty} f_1(t)=+\infty.$$

With f_∞ we denote the function defined as $f_\infty(t):=\inf\{f(s),\ s\ge t\}$. Clearly f_∞ is nondecreasing, and if f is nondecreasing on $[c,+\infty)$, then $f_\infty\equiv f$ on $[c,+\infty)$.

Theorem 12.1. *Let $p>1$ and $\lambda\in\mathbb{R}$, and assume that f_∞ satisfies* (8.11). *Let $u\in W^{1,p}_{loc}(\mathbb{R}^N)$ be a solution of*

$$\Delta_p u = f(u)-\lambda u, \quad \textit{on } \mathbb{R}^N. \tag{12.2}$$

Then we have the following:

(1) *If* $f_1(t) > \lambda$ *for any* $t > 0$, *then* $u \equiv 0$.
(2) *Let* T_λ *be the largest solution of the equation* $f_1(t) = \lambda$; *then* $|u| \leq T_\lambda$ *almost everywhere on* $\mathbb{R}^N$.

In particular, if $f := |u|^{q-1} u$ *with* $q > \max\{1, p-1\}$, *then we have the following:*

(1) *If* $\lambda \leq 0$, *then* $u \equiv 0$ *almost everywhere.*
(2) *If* $\lambda > 0$, *then* $|u| \leq \lambda^{\frac{1}{q-1}}$ *almost everywhere on* $\mathbb{R}^N$.

Proof. Since

$$\liminf_{t \to +\infty} \frac{f(t) - \lambda t}{f_\infty(t)} \geq 1,$$

we can apply Corollary 5.2 and Theorem 8.10. □

Here we describe some results (probably well known) for which we were unable to find a reference in the literature and whose proof is an easy consequence of the results proved in this paper.

The above result for $p = 2$ is related to the Gross–Pitaevskii equation and to the search of standing-wave solutions of the equation

$$u_{tt} - \Delta_p u + f(u) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{12.3}$$

As usual, a standing-wave solution (or solitary wave) of (12.3) is a solution of the form

$$u(t, x) := e^{i\omega t} v(x), \tag{12.4}$$

where ω is called the propagation parameter and v is a real-valued function.

Theorem 12.2. *Let* u *be a standing-wave solution with propagation parameter* ω *of* (12.3); *that is,* u *has the form* (12.4) *with* $v \in W^{1,p}_{loc}(\mathbb{R}^N)$. *Then* u *is bounded and the conclusions of Theorem* 12.1 *hold with* $\lambda = \omega^2$.

Proof. Let u have the form (12.4). Then by computation the function v is a solution of (12.2) with $\lambda = \omega^2$. Since $|u| = |v|$, the conclusion follows from Theorem 12.1. □

Next result deals with the standing-wave solution or stationary-wave solution of the equation

$$iu_t + \Delta_p u = f(u), \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{12.5}$$

This equation is the quasilinear version of the so-called *defocusing Schrödinger equation*. Equation (12.5) for $p = 2$ and $f(u) = |u|^2 u - \lambda u$ is the so-called Gross–Pitaevskii equation. This kind of equation describes several physical phenomena; see [9] and the relevant references therein.

By a standing-wave solution of (12.5) we mean a solution of the form

$$u(t,x) := e^{-iEt}v(x),$$

where the real parameter E is called the *energy* of the wave and v is a real-valued function.

Theorem 12.3. *Let u be a standing-wave solution of* (12.5). *Then the conclusions of Theorem* 12.1 *hold with $\lambda = E$. In particular, if $f(u) = |u|^{q-1}u - u$ with $q > \max\{1, p-1\}$, then $|u|^{q-1} \leq \max\{0, 1+E\}$.*

Other generalizations of the Gross–Pitaevskii equation involve a nonlinearity f of the form $f(t) = |t|^{q-1}t + \alpha|t|^{s-1}t + \beta t$. We leave the related results on the boundedness of the standing-wave solution to the interested reader. For similar results in the linear case for polynomial nonlinearity see [17].

Remark 12.4. All the results stated in Theorems 12.1, 12.2, and 12.3 hold in the framework of a Heisenberg group as well as dealing with power growth nonlinearity in the framework of Carnot groups. We leave the details to the interested reader.

APPENDIX A. CARNOT GROUPS

A.1. **Basic facts.** Here, we quote some facts on Carnot groups and refer the interested reader to [21, 22, 5] for more detailed information on these structures.

A Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ of dimension N with graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ such that $[V_1, V_i] = V_{i+1}$ for $i = 1, \ldots, r-1$ and $[V_1, V_r] = 0$. Such an integer r is called the *step* of the group. We set $l = n_1 = \dim V_1$ and $n_2 = \dim V_2, \ldots, n_r = \dim V_r$. A Carnot group $\mathbb{G}$ of dimension N can be identified, up to an isomorphism, with the structure of a *homogeneous Carnot group* $(\mathbb{R}^N, \circ, \delta_R)$ defined as follows: we identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with a Lie group law $\circ$. We consider $\mathbb{R}^N$ split into r subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$ and $\xi = (\xi^{(1)}, \ldots, \xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$. We shall assume that for any $R > 0$ the dilation $\delta_R(\xi) = (R\xi^{(1)}, R^2\xi^{(2)}, \ldots, R^r\xi^{(r)})$ is a Lie group automorphism. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N, \circ)$ is $\mathcal{G}$. For $i = 1, \ldots, n_1 = l$ let X_i be the unique vector field in $\mathcal{G}$ that coincides with $\partial/\partial\xi_i^{(1)}$ at the origin. We require that the Lie algebra generated by $X_1, \ldots, X_l$ is the whole of $\mathcal{G}$.

We denote with ∇_L the vector field $\nabla_L := (X_1, \dots, X_l)^T$ and we call it the *horizontal vector field* in $\mathbb{G}$. Moreover, the vector fields $X_1, \dots, X_l$ are homogeneous of degree 1 with respect to δ_R. In this case $Q = \sum_{i=1}^{r} i n_i = \sum_{i=1}^{r} i\,\mathrm{dim} V_i$ is called the *homogeneous dimension* of $\mathbb{G}$. The *canonical sub–Laplacian* on $\mathbb{G}$ is the second-order differential operator defined by $\Delta_G = \sum_{i=1}^{l} X_i^2$, and for $p > 1$ the p-sub–Laplacian operator is given by $\sum_{i=1}^{l} X_i(|\nabla_L u|^{p-2} X_i u)$. Since $X_1, \dots, X_l$ generate the whole graded Lie algebra $\mathcal{G}$, the sub–Laplacian Δ_G satisfies the Hörmander hypoellipticity condition.

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm* on $\mathbb{G}$, if $S(\xi^{-1}) = S(\xi)$, $S(\xi) = 0$ if and only if $\xi = 0$, and it is homogeneous of degree 1 with respect to δ_R (i.e., $S(\delta_R(\xi)) = RS(\xi)$). A homogeneous norm S defines on $\mathbb{G}$ a *pseudo-distance* defined as $d(\xi, \eta) := S(\xi^{-1}\eta)$, which in general is not a distance. If S and $\tilde{S}$ are two homogeneous norms, then they are equivalent; that is, there exists a constant $C > 0$ such that $C^{-1}S(\xi) \le \tilde{S}(\xi) \le CS(\xi)$. Let S be a homogeneous norm; then there exists a constant $C > 0$ such that $C^{-1}|\xi| \le S(\xi) \le C|\xi|^{1/r}$, for $S(\xi) \le 1$. Examples of homogeneous norms are $S_\delta(\cdot)$ defined in (1.4) or $S(\xi) := \left(\sum_{i=1}^{r} |\xi_i|^{2r!/i}\right)^{1/2r!}$.

Notice that if S is a homogeneous norm differentiable almost everywhere, then $|\nabla_L S|$ is homogeneous of degree 0 with respect to δ_R; hence $|\nabla_L S|$ is bounded.

Special examples of Carnot groups are the Euclidean spaces $\mathbb{R}^Q$. Moreover, if $Q \le 3$ then any Carnot group is the ordinary Euclidean space $\mathbb{R}^Q$.

The simplest nontrivial example of a Carnot group is the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$. For an integer $n \ge 1$, the Heisenberg group $\mathbb{H}^n$ is defined as follows: let $\xi = (\xi^{(1)}, \xi^{(2)})$ with $\xi^{(1)} := (x_1, \dots, x_n, y_1, \dots, y_n)$ and $\xi^{(2)} := t$. We endow $\mathbb{R}^{2n+1}$ with the group law $\hat{\xi} \circ \tilde{\xi} := (\hat{x} + \tilde{x}, \hat{y} + \tilde{y}, \hat{t} + \tilde{t} + 2\sum_{i=1}^{n}(\tilde{x}_i\hat{y}_i - \hat{x}_i\tilde{y}_i))$. We consider the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t},\ Y_i := \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}, \quad \text{for } i = 1, \dots, n,$$

and the associated Heisenberg gradient $\nabla_H := (X_1, \dots, X_n, Y_1, \dots, Y_n)^T$. The Kohn Laplacian Δ_H is then the operator defined by $\Delta_H := \sum_{i=1}^{n} X_i^2 + Y_i^2$. The family of dilations is given by $\delta_R(\xi) := (Rx, Ry, R^2 t)$ with homogeneous dimension $Q = 2n + 2$. In $\mathbb{H}^n$ a canonical homogeneous norm is defined as $|\xi|_H := \left(\left(\sum_{i=1}^{n} x_i^2 + y_i^2\right)^2 + t^2\right)^{1/4}$.

A.2. **Mollifiers.** On a Carnot group there is a "good" notion of *mollifier*. Let $\mathbb{G}$ be a homogeneous Carnot group on $\mathbb{R}^N$ and let S be a fixed homogeneous norm on $\mathbb{G}$. For every $x \in \mathbb{G}$ and every $r > 0$, the set

$$B_S(x,r) := \{y \in \mathbb{G} : S(x^{-1} \circ y) < r\}$$

is called the *S-ball with center at x and radius r*. For a fixed point $x \in \mathbb{G}$ and a set $A \subset \mathbb{G}$, the number

$$\text{dist}_S(x,A) := \inf\{S(x^{-1} \circ a) : a \in A\}$$

is called *S-distance of x from A*. Let $\Omega \subset \mathbb{G}$ and $\epsilon > 0$; we define

$$\Omega_{S,\epsilon} := \{x \in \Omega : \text{dist}_S(x, \partial\Omega) > \epsilon\}.$$

In order to avoid cumbersome notation we shall omit the norm S in the above symbols.

Let $m \in \mathcal{C}_0^\infty(\mathbb{G})$, $m \geq 0$ be given such that

$$\text{supp}(m) \subset B_S(0,1) \text{ and } \int m = 1.$$

For any $\eta > 0$ we set $m_\eta := \eta^{-Q} m(\delta_{1/\eta}(x))$. The family $(m_\eta)_\eta$ will be called a *family of mollifiers*.

Let $\Omega \subset \mathbb{G}$ be an open set and let $u \in L^1_{loc}(\Omega)$. For any $x \in \Omega_\eta$ we define

$$u_\eta := (u \star_{\mathbb{G}} m_\eta)(x) := \int_{B(x,\eta)} u(y) m_\eta(x \circ y^{-1})\, dy = \int_{B(0,\eta)} u(y^{-1} \circ x) m_\eta(y)\, dy,$$

calling u_η a *mollified of u related to the homogeneous norm S*. It is easy that check that if $u \in L^1_{loc}(\Omega)$, then $u_\eta \to u$ as $\eta \to 0$ in $L^1_{loc}(\Omega)$. See [5].

APPENDIX B. EXAMPLES

In this section, for easy reference, we shall present some other examples of operators for which our results for differential inequalities apply.

B.1. **The gradient of l variables on $\mathbb{R}^N$.** Let $l \leq N$ be a positive natural number and let $\mu^l \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R}^l)$ be the matrix defined by $\mu^l := (I_l \quad 0)$. The corresponding vector field ∇^l is the usual gradient acting only on the first l variables $\nabla^l = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_l})$. Clearly $\nabla^N = \nabla$ and ∇^l is homogeneous with respect to the dilation

$$\delta_R(x) = (Rx_1, Rx_2, \ldots, Rx_l, R^{\delta_{l+1}} x_{l+1}, \ldots, R^{\delta_N} x_N);$$

here, $\delta_{l+1}, \ldots, \delta_N$ are arbitrary real positive numbers.

B.2. **Baouendi–Grushin-type operator.** Let $\xi = (x, y) \in \mathbb{R}^n \times \mathbb{R}^k (= \mathbb{R}^N)$. Let $\gamma \geq 0$ and let μ be the following matrix:

$$\begin{pmatrix} I_n & 0 \\ 0 & |x|^\gamma I_k \end{pmatrix}.$$

The corresponding vector field is given by $\nabla_\gamma = (\nabla_x, |x|^\gamma \nabla_y)^T$ and the linear operator $L = \mathrm{div}_L (\nabla_L \cdot) = \Delta_x + |x|^{2\gamma} \Delta_y$ is the so-called Baouendi–Grushin operator. Notice that if $k = 0$ or $\gamma = 0$, then L coincides with the usual Laplacian operator. The vector field ∇_γ is homogeneous with respect to the dilation $\delta_R(x) = (Rx_1, \ldots, Rx_n, R^{1+\gamma} y_1, \ldots, R^{1+\gamma} y_k)$ and $Q = N + k\gamma$.

B.3. **Heisenberg–Kohn operator.** Let $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{H}^n (= \mathbb{R}^N)$ and let μ be defined as

$$\begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.$$

The corresponding vector field ∇_H is the Heisenberg gradient on the Heisenberg group $\mathbb{H}^n$. The vector field ∇_H is homogeneous with respect to $\delta_R(\xi) = (Rx, Ry, R^2 t)$ and $Q = 2n + 2$.

In $\mathbb{H}^1$ the corresponding vector fields are $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$. In this case $Q = 4$.

B.4. **Heisenberg–Greiner operator.** Let $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} (= \mathbb{R}^N)$, $r := |(x, y)|$, and $\gamma \geq 1$, and let μ be defined as

$$\begin{pmatrix} I_n & 0 & 2\gamma y r^{2\gamma-2} \\ 0 & I_n & -2\gamma x r^{2\gamma-2} \end{pmatrix}.$$

The corresponding vector fields are $X_i = \partial_{x_i} + 2\gamma y_i r^{2\gamma-2} \partial_t$ and $Y_i = \partial_{y_i} - 2\gamma x_i r^{2\gamma-2} \partial_t$ for $i = 1, \ldots, n$.

For $\gamma = 1$, $L = \mathrm{div}_L (\nabla_L \cdot)$ is the sub–Laplacian Δ_H on the Heisenberg group $\mathbb{H}^n$. If $\gamma = 2, 3, \ldots$, L is a Greiner operator (see [23]). The vector field associated to μ is homogeneous with respect to $\delta_R(\xi) = (Rx, Ry, R^{2\gamma} t)$ and $Q = 2n + 2\gamma$.

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