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# ISOLATED INITIAL SINGULARITIES FOR THE VISCOUS HAMILTON–JACOBI EQUATION

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**Abstract.** Here we study the nonnegative solutions of the viscous Hamilton–Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, where $q > 1$, $T \in (0,\infty]$, and $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$ containing 0, or $\Omega = \mathbb{R}^N$. We consider weak solutions with a possible singularity at the point $(x,t) = (0,0)$. We show that if $q \geq q_* = (N+2)/(N+1)$ the singularity is removable. For $1 < q < q_*$, we prove the uniqueness of a very singular solution without condition as $|x| \to \infty$; we also show the existence and uniqueness of a very singular solution of the Dirichlet problem in $Q_{\Omega,\infty}$, when $\Omega$ is bounded. We give a complete description of the weak solutions in each case.

## 1. Introduction

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ containing 0, or $\Omega = \mathbb{R}^N$, and $\Omega_0 = \Omega \backslash \{0\}$. Here we consider the nonnegative solutions of the viscous parabolic Hamilton–Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0 \tag{1.1}$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, where $q > 1$, with a possible singularity at the point $(x,t) = (0,0)$, in the following sense:

$$\lim_{t\to 0} \int_\Omega u(\cdot,t)\varphi \, dx = 0, \qquad \forall \varphi \in C_c(\Omega_0), \tag{1.2}$$

which means *formally* that $u(x,0) = 0$ for $x \neq 0$.

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Such a problem was first considered for the semilinear equation with a lower term or order 0 :

$$u_t - \Delta u + |u|^{q-1}u = 0 \quad \text{ in } Q_{\Omega,T}, \tag{1.3}$$

with $q > 1$. In the well-known article of Brezis and Friedman [14], it was shown that the problem admits a critical value $q_c = (N+2)/N$. For any $q < q_c$, and any bounded Radon measure $u_0 \in \mathcal{M}_b(\Omega)$, there exists a unique solution of (1.3) with Dirichlet conditions on $\partial\Omega$ with initial data $u_0$, in the weak * sense:

$$\lim_{t\to 0} \int_\Omega u(\cdot,t)\varphi\, dx = \int_\Omega \varphi\, du_0, \qquad \forall \varphi \in C_c(\Omega). \tag{1.4}$$

Moreover, from [15] and [19], there exists a *very singular solution* in $\mathbb{R}^N$, satisfying

$$\lim_{t\to 0} \int_{B_r} u(\cdot,t)\, dx = \infty, \qquad \forall\ B_r \subset \Omega, \tag{1.5}$$

and it is the limit as $k \to \infty$ of the solutions with initial data $k\delta_0$, where $\delta_0$ is the Dirac mass at 0; its uniqueness, obtained in [24], is also a consequence of the general results of [22]. For any $q \geqq q_c$, such solutions do not exist, and the singularity is *removable*; in other words, any solution of (1.3), (1.2) satisfies $u \in C^{2,1}(\Omega \times [0,T))$ and $u(x,0) = 0$ in $\Omega$ (see again [14]).

Concerning equation (1.1), up to now, the description was not yet complete. Here another critical value is involved:

$$q_* = \frac{N+2}{N+1}.$$

In the case $\Omega = \mathbb{R}^N$, we define a *very singular solution* (called VSS) in $Q_{\mathbb{R}^N,\infty}$ as any function $u \in L^1_{loc}(Q_{\mathbb{R}^N,\infty})$, such that $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,\infty})$, satisfying equation (1.1) in $\mathcal{D}'(Q_{\mathbb{R}^N,\infty})$, and conditions

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(\cdot,t)\varphi\, dx = 0, \qquad \forall \varphi \in C_c(\mathbb{R}^N \setminus \{0\}). \tag{1.6}$$

$$\lim_{t\to 0} \int_{B_r} u(\cdot,t)\, dx = \infty, \qquad \forall r > 0. \tag{1.7}$$

For $q \in (1, q_*)$, the existence of a solution with initial data $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$ was shown in [9], and uniqueness in a specific class, enlarged in [6]. The existence of a radial self-similar VSS $U$ in $Q_{\mathbb{R}^N,\infty}$, unique in its class, was obtained in [27]; independently in [10], the existence of a VSS as a limit as

$k \to \infty$ of the solutions with initial data $k\delta_0$ was proved. Uniqueness was proved in [11], in the class of functions $U$ such that

$$\begin{aligned}
&\lim_{t\to 0}\int_{\mathbb{R}^N\setminus B_r} U(\cdot,t)\,dx = 0, \qquad \forall r>0,\\
&U \in C^{2,1}\left(Q_{\mathbb{R}^N,\infty}\right)\cap C((0,\infty);L^1(\mathbb{R}^N))\cap L^q_{loc}((0,\infty);W^{1,q}(\mathbb{R}^N)), \qquad (1.8)\\
&\sup_{t>0}(t^{N/2}\left\|U(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} + t^{(q(N+1)-N)/2q}\left\|\nabla(U^{(q-1)/q}(\cdot,t))\right\|_{L^\infty(\mathbb{R}^N)}) < \infty.
\end{aligned}$$

If $q \geqq q_*$, it was proved in [10] that there is no solution $u$ in $Q_{\mathbb{R}^N,T}$ with initial data $\delta_0$, under the constraints

$$u \in C((0,T);L^1(\mathbb{R}^N))\cap L^q((0,T);W^{1,q}(\mathbb{R}^N)); \tag{1.9}$$

and the nonexistence of VSS was stated as an open problem.

In the case of the Dirichlet problem in $Q_{\Omega,T}$, with $\Omega$ bounded, similar results were obtained in [7]: for $q\in(1,q_*)$ and any $u_0\in\mathcal{M}_b(\Omega)$, there exists a solution $u$ such that

$$u \in C((0,T);L^1(\Omega))\cap L^1((0,T);W_0^{1,1}(\Omega)), \quad |\nabla u|^q \in L^1\left(Q_{\Omega,T}\right), \tag{1.10}$$

satisfying (1.4) for any $\varphi\in C_b(\Omega)$, and unique in that class; for $q\geqq q_*$ there exists no solution in this class when $u_0$ is a Dirac mass; the existence or nonexistence of a VSS was not studied.

In this article we answer these questions and complete the description of *all the weak solutions*.

In Section 2, we introduce the notion of weak solutions. When $q \leqq 2$, we show a $C^{2,1}$ property for any weak solution, improving some results of [11]; see Theorems 2.9 and 2.10. We point out some particular singular solutions or supersolutions, fundamental in the sequel. We also give some trace results, in the footsteps of [22], and apply them to the solutions of (1.1)–(1.2).

Our main result is the *removability* in the supercritical case $q \geqq q_*$, proved in Section 3, extending the results of [14] to equation (1.1).

**Theorem 1.1.** *Assume $q \geqq q_*$. Let $\Omega$ be any domain in $\mathbb{R}^N$. Let $u \in L^1_{loc}(Q_{\Omega,T})$, such that $|\nabla u| \in L^q_{loc}(Q_{\Omega,T})$, be any solution of the problem*

$$(P_\Omega) \quad \begin{cases} u_t - \Delta u + |\nabla u|^q = 0 & \text{in } \mathcal{D}'(Q_{\Omega,T}),\\ \lim_{t\to 0}\int_\Omega u(\cdot,t)\varphi\,dx = 0 & \forall\varphi\in C_c(\Omega_0). \end{cases}$$

*Then the singularity is removable, in the following sense:*

*If $q \leqq 2$, then $u \in C(\Omega\times[0,T))$ and $u(x,0)=0$, $\forall x\in\Omega$.*

*If* $q > 2$, *then* $u \in C([0,T); L^r_{loc}(\Omega))$, *for any* $r > 1$, $u$ *is locally bounded near* $0$, *and for any domain* $\omega \subset\subset \Omega$,

$$\lim_{t\to 0}(\sup_{Q_{\omega,t}} u) = 0. \tag{1.11}$$

Observe that our conclusions hold *without any condition as* $|x| \to \infty$ if $\Omega = \mathbb{R}^N$, or near $\partial\Omega$ when $\Omega \neq \mathbb{R}^N$. As a consequence, for $q \geqq q_*$,

(i) there exists *no VSS in* $Q_{\mathbb{R}^N,\infty}$ in the sense above,

(ii) there exists *no solution of* $(P_\Omega)$ *with a Dirac mass* at $(0,0)$, without assuming (1.9) or (1.10).

We give different proofs of Theorem 1.1 according to the values of $q$. For $q \leqq 2$, we take advantage of the regularity of the solutions shown in Section 2. When $q < 2$, we make use of supersolutions, and the difficult case is the critical one, $q = q_*$. When $q \geqq 2$, our proof is based on a change of unknown, and on our trace results; the case $q > 2$ is the most delicate, because of the lack of regularity.

Besides, if $\Omega = \mathbb{R}^N$, we can show a *global removability, without condition at* $\infty$:

**Theorem 1.2.** *Under the assumptions of Theorem* 1.1 *with* $\Omega = \mathbb{R}^N$,

$$u(x,t) \equiv 0, \qquad a.e. \text{ in } \mathbb{R}^N, \quad \text{for any } t > 0.$$

In Section 4, we complete the study of the subcritical case $q < q_*$. Our main result in this range is the uniqueness of the VSS in $Q_{\mathbb{R}^N,\infty}$ *without any condition*:

**Theorem 1.3.** *Let* $q \in (1, q_*)$. *Then there exists a unique VSS in* $Q_{\mathbb{R}^N,\infty}$.

Moreover, we give a complete description of the solutions:

**Theorem 1.4.** *Let* $q \in (1, q_*)$. *Let* $u \in L^1_{loc}(Q_{\mathbb{R}^N,\infty})$ *be any function such that* $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,\infty})$, *solution of equation* (1.1) *in* $\mathcal{D}'(Q_{\mathbb{R}^N,\infty})$, *and satisfying* (1.6). *Then*

- *either* (1.7) *holds and* $u = U$, *or*
- *there exists* $k > 0$ *such that* $u(\cdot,0) = k\delta_0$ *in the weak sense of* $\mathcal{M}_b(\mathbb{R}^N)$:

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(\cdot,t)\varphi\,dx = k\varphi(0), \qquad \forall\varphi \in C_b(\mathbb{R}^N), \tag{1.12}$$

*and* $u$ *is the unique solution satisfying* (1.12), *or*

- $u \equiv 0$.

We also consider the Dirichlet problem in $Q_{\Omega,T}$ when $\Omega$ is bounded:

$$(D_{\Omega,T})\begin{cases} u_t - \Delta u + |\nabla u|^q = 0 & \text{in } Q_{\Omega,T} \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty)\,. \end{cases} \tag{1.13}$$

We give a notion of VSS for this problem, generally nonradial, and show the parallel of Theorem 1.3:

**Theorem 1.5.** *Assume that $q \in (1, q_*)$ and $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$. Then there exists a unique VSS of problem $(D_{\Omega,\infty})$.*

Finally, we describe all the weak solutions as above. In conclusion, $q_*$ clearly appears as the upper bound for existence of solutions with an isolated singularity at time 0. We refer to [12] for the study of equation (1.1) or more general quasilinear parabolic equations with rough initial data, where we give new decay and uniqueness properties. The study of nonpunctual singularities will be the object of a further article.

## 2. Weak solutions and regularity

2.1. **First properties of the weak solutions.** We set $Q_{\Omega,s,\tau} = \Omega \times (s, \tau)$, for any domain $\Omega \subset \mathbb{R}^N$ and any $-\infty \leqq s < \tau \leqq \infty$; thus $Q_{\Omega,T} = Q_{\Omega,0,T}$.

**Definition 2.1.** *For any function $\Phi \in L^1_{loc}(Q_{\Omega,T})$, we say that a function $U$ is a weak solution (respectively subsolution, respectively supersolution) of the equation*

$$U_t - \Delta U = \Phi \qquad in\ Q_{\Omega,T}, \tag{2.1}$$

*if $U \in L^1_{loc}(Q_{\Omega,T})$ and for any $\varphi \in \mathcal{D}^+(Q_{\Omega,T})$,*

$$\int_0^T \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)\,dx\,dt = 0 \qquad (\textit{respectively}\ \geqq, \textit{respectively}\ \leqq).$$

**Remark 2.2.** Regularizing $U$ by $U_\varepsilon = U * \varrho_\varepsilon$, where $(\varrho_\varepsilon)$ is a sequence of mollifiers in $(x,t) \in \mathbb{R}^{N+1}$, we see that any solution (respectively subsolution) $U$ of (2.1) such that $U \in C((0,T); L^1_{loc}(\Omega))$ satisfies also for any $s, \tau \in (0,T)$

$$\int_\Omega U(\cdot,\tau)\varphi(\cdot,\tau)\,dx - \int_\Omega U(\cdot,s)\varphi(\cdot,s)\,dx - \int_s^\tau \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)\,dx\,dt = 0 \tag{2.2}$$

(respectively $\leqq 0$) for any $\varphi \in C_c^{\infty+}(\Omega \times [0,T])$, and for any $\psi \in C_c^{2+}(\Omega)$,

$$\int_\Omega U(\cdot,\tau)\psi\,dx - \int_\Omega U(\cdot,s)\psi\,dx - \int_s^\tau \int_\Omega (U\Delta\psi + \Phi\psi)\,dx\,dt = 0 \tag{2.3}$$

(respectively $\leqq 0$).

Next we make precise our notion of solution of equation (1.1).

**Definition 2.3.** (*i*) *We say that a nonnegative function $u$ is a weak solution of equation* (1.1) *in $Q_{\Omega,T}$, if $u \in L^1_{loc}(Q_{\Omega,T})$, $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, and $u$ is a weak solution of the equation in the sense above,*

(*ii*) *We say that $u$ is a weak solution of the Dirichlet problem $(D_{\Omega,T})$ if it is a weak solution of* (1.1) *in $Q_{\Omega,T}$, such that*

$$u \in L^1_{loc}((0,T);W^{1,1}_0(\Omega))\cap C((0,T);L^1(\Omega)), \quad \text{and } |\nabla u| \in L^q_{loc}((0,T);L^q(\Omega)).$$

Next we recall some well-known properties:

**Lemma 2.4.** *Any weak nonnegative solution of equation* (1.1) *satisfies $u \in L^\infty_{loc}(Q_{\Omega,T})$, $\nabla u \in L^2_{loc}(Q_{\Omega,T})$, and $u \in C((0,T);L^r_{loc}(\Omega))$, for any $r \geqq 1$. Then*

(*i*) *for any $\varphi \in C^1_c(Q_{\Omega,T})$,*

$$\int_0^T \int_\Omega (-u\varphi_t + \nabla u \cdot \nabla\varphi + |\nabla u|^q \varphi)\,dx\,dt = 0; \tag{2.4}$$

(*ii*) *for any $s,\tau \in (0,T)$, and any $\varphi \in C^1((0,T);C^1_c(\Omega))$,*

$$\int_\Omega u(\cdot,\tau)\varphi(\cdot,\tau)dx - \int_\Omega u(\cdot,s)\varphi(\cdot,s)dx + \int_s^\tau\int_\Omega(-u\varphi_t+\nabla u\cdot\nabla\varphi+|\nabla u|^q\varphi)dx\,dt = 0; \tag{2.5}$$

(*iii*) *for any $s,\tau \in (0,T)$, and any $\psi \in C^1_c(\Omega)$,*

$$\int_\Omega u(\cdot,\tau)\psi\,dx - \int_\Omega u(\cdot,s)\psi\,dx + \int_s^\tau \int_\Omega (\nabla u\cdot\nabla\psi + |\nabla u|^q\psi)\,dx\,dt = 0. \tag{2.6}$$

**Proof.** The function $u$ is subcaloric; then $u \in L^\infty_{loc}(Q_{\Omega,T})$ (see for example [14]). Consider any domains $\omega \subset\subset \omega' \subset\subset \Omega$, and any $\psi \in C^1_c(\Omega)$ with support in $\omega'$ such that $\psi \equiv 1$ on $\omega$, $\psi(\Omega) \subset [0,1]$, and $0 < s < \tau < T$. The regularized function $u_\varepsilon$ is a subsolution of equation (1.1) in $Q_{\omega,s,\tau}$, for $\varepsilon$ small enough, from the convexity of $|\nabla u|^q$. Then

$$\int_\Omega u_\varepsilon^2(\cdot,\tau)\psi^2\,dx - \int_\Omega u_\varepsilon^2(\cdot,s)\psi^2\,dx + \int_s^\tau \int_\Omega |\nabla u_\varepsilon|^2\psi^2\,dx$$
$$\leqq 2\int_s^\tau \int_\Omega u_\varepsilon|\nabla u_\varepsilon|\,|\nabla\psi|\,dx \leqq \frac{1}{2}\int_s^\tau\int_\Omega |\nabla u_\varepsilon|^2\psi^2\,dx + 4\int_s^\tau\int_\Omega u_\varepsilon^2|\nabla\psi|^2\,dx;$$

hence $\nabla u \in L^2_{loc}(Q_{\Omega,T})$ and

$$\|\nabla u\|_{L^2(Q_{\omega,s,\tau})} \leqq C(\|u(\cdot,s)\|_{L^2(Q_{\omega',s,\tau})} + \|u\|_{L^2(Q_{\omega',s,\tau})}) \leqq C\,\|u\|_{L^\infty(Q_{\omega',s,\tau})},$$

with $C = C(N,\omega,\omega')$. Then (2.4) holds for any $\varphi \in \mathcal{D}(Q_{\Omega,T})$. Moreover, since $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, the function $u \in L^2_{loc}((0,T);W^{1,2}_{loc}(\Omega))$ and

$u_t \in L^2_{loc}((0,T);W^{-1,2}(\Omega)) + L^1_{loc}(Q_{\Omega,T})$. From a local version of [26, Theorem 1.1], we find $u \in C((0,T);L^1_{loc}(\Omega))$. Then (2.5) and (2.6) follow. Moreover, $u \in L^\infty_{loc}(Q_{\Omega,T})$; thus $u \in C((0,T);L^r_{loc}(\Omega))$ for any $r > 1$. □

In the case of the Dirichlet problem $(D_{\Omega,T})$, the regularization does not provide estimates up to the boundary; thus we use another argument: the notion of *entropy solution*. For any $k > 0$ and $r \in \mathbb{R}$, we define as usual $T_k(r) = \max(-k, \min(k,r))$ the truncation function, and $\Theta_k(r) = \int_0^r T_k(s)\,ds$, if $u \in C([s,\tau];L^1(\Omega))$. The solutions can be defined in three equivalent ways:

**Lemma 2.5.** *Let $0 \leqq s < \tau \leqq T$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u \in C([s,\tau);L^1(\Omega))$, $u_s = u(s)$. Denoting by $e^{t\Delta}$ the semigroup of the heat equation with Dirichlet conditions acting on $L^1(\Omega)$, these three properties are equivalent:*

*(i) $u \in L^1_{loc}((s,\tau);W^{1,1}_0(\Omega))$ and*

$$u_t - \Delta u = f, \qquad \text{in } \mathcal{D}'(Q_{\Omega,s,\tau});$$

*(ii) $u$ is an entropy solution of this problem in $Q_{\Omega,s,\tau}$: $T_k(u) \in L^2((s,\tau);W^{1,2}_0(\Omega))$ for any $k > 0$, and*

$$\begin{aligned}&\int_\Omega \Theta_k v(\cdot,\tau)\,dx + \int_s^\tau \langle \varphi_t, T_k(v)\rangle\,dt + \int_s^\tau \int_\Omega \nabla u \cdot \nabla T_k(v)\,dx\,dt \\ &= \int_\Omega \Theta_k(u_s - \varphi(\cdot,s))\,dx + \int_s^\tau \int_\Omega f T_k(v)\,dx\,dt\end{aligned}$$

*for any $v$ such that $\varphi = u - v \in L^2((s,\tau);W^{1,2}(\Omega)) \cap L^\infty(Q_{\Omega,\tau})$ and $\varphi_t \in L^2((s,\tau);W^{-1,2}(\Omega))$;*

$$(iii) \qquad u(\cdot,t) = e^{(t-s)\Delta}u_s + \int_s^t e^{(t-\sigma)\Delta} f(\sigma)\,d\sigma \ \text{ in } L^1(\Omega),\ \forall t \in [s,\tau].$$

**Proof.** It follows from the existence and uniqueness of the solutions of (i) from [5, Lemma 3.4], as noticed in [7], and of the entropy solutions; see [3] and [25]. □

We deduce properties of all the *bounded* solutions $u$ of $(D_{\Omega,T})$:

**Lemma 2.6.** *Any nonnegative weak solution of problem $(D_{\Omega,T})$, such that $u \in L^\infty_{loc}((0,T);L^\infty(\Omega))$ satisfies*

$$\nabla u \in L^2_{loc}((0,T);L^2(\Omega)) \text{ and } u \in C((0,T);L^r(\Omega)) \text{ for any } r \geqq 1.$$

**Proof.** Since $u \in C((0,T);L^1(\Omega))$, for any $0 < s < \tau < T$, $u$ is an entropy solution on $[s,\tau]$ from Lemma 2.5. Since $u$ is bounded, $u = T_k(u) \in$

$L^2((s,\tau);W_0^{1,2}(\Omega))$, and

$$\int_\Omega u^2(\cdot,\tau)\,dx - \int_\Omega u^2(\cdot,s)\,dx + \int_s^\tau \int_\Omega |\nabla u|^2\,dx + \int_s^\tau \int_\Omega u|\nabla u|^q\,dx\,dt = 0;$$

and $u \in C((0,T);L^r(\Omega))$ as in Lemma 2.4. □

2.2. **Estimates of the classical solutions of** $(D_{\Omega,T})$**.** Consider the Dirichlet problem $(D_{\Omega,T})$ in a smooth bounded domain $\Omega$ with regular initial data $u(x,0) = u_0 \in C^1\left(\overline{\Omega}\right) \cap C_0(\Omega)$: it admits a unique classical solution $u \in C^{2,1}(Q_{\Omega,\infty}) \cap C\left(\overline{\Omega}\times[0,\infty)\right)$ such that $|\nabla u| \in C\left(\overline{\Omega}\times[0,\infty)\right)$. Let us recall some fundamental universal estimates proved in [17]:

**Theorem 2.7** ([17])**.** *Let $\Omega$ be any smooth bounded domain. Let $q>1$, and $u$ be the classical solution of $(D_{\Omega,T})$ with initial data $u_0 \in C^1\left(\overline{\Omega}\right) \cap C_0(\Omega)$. Then there exist functions $B, D \in C((0,\infty))$ depending only on $N$, $q$, and $\Omega$, such that, for any $t \in (0,T)$,*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqq B(t)d(x,\partial\Omega), \tag{2.7}$$

$$\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \leqq D(t). \tag{2.8}$$

**Remark 2.8.** In fact, the term $B(t)$ can be made precise: under the assumptions above, there holds

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqq C(1+t^{-\frac{1}{q-1}})d(x,\partial\Omega) \tag{2.9}$$

with $C = C(N,q,\Omega)$. Indeed, (2.7) is obtained by using an explicit supersolution for any $z \in \partial\Omega$, of the form $w_z(x,t) = J(t)b_z(x)$, where $b_z(x)$ is constructed such that $\inf_{z\in\partial\Omega} b_z(x) \approx d(x,\partial\Omega)$, and $J$ can be chosen by $J(t) = M(\arctan t)^{-1/(q-1)}$ with $M = M(N,q,\Omega)$.

2.3. **Regularity for** $q \leqq 2$**.** First of all, we give a result of regularity $\mathcal{C}^{2,1}$ *for any weak solution* of equation (1.1) and for any $q \leqq 2$. Such a regularity was obtained in [11, Proposition 3.2] for the VSS when $q < q_*$, and the proof was valid up to $q = (N+4)/(N+2)$. We did not find a good reference in the literature under our weak assumptions, even if a priori estimates can be found in [21], and Hölderian properties in [4] and [29]. Our proof is based on a bootstrap technique, starting from the fact that $u$ is subcaloric.

We set $\mathcal{W}^{2,1,\rho}(Q_{\omega,s,\tau}) = \{u \in L^\rho(Q_{\omega,s,\tau}) : u_t, \nabla u, D^2u \in L^\rho(Q_{\omega,s,\tau})\}$, for any $0 \leqq s < \tau < T$ and $1 \leqq \rho \leqq \infty$. This space is endowed with its usual norm.

**Theorem 2.9.** *Let $1 < q \leqq 2$. Let $\Omega$ be any domain in $\mathbb{R}^N$.*

*(i) Let $u$ be any weak nonnegative solution of* (1.1) *in* $Q_{\Omega,T}$. *Then* $u \in \mathcal{C}^{2,1}(Q_{\Omega,T})$, *and there exists* $\gamma \in (0,1)$ *such that for any smooth domains* $\omega \subset\subset \omega' \subset\subset \Omega$, *and* $0 < s < \tau < T$,

$$\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{\omega,s,\tau})} \leqq C\Phi(\|u\|_{L^\infty(Q_{\omega',s/2,\tau})}), \tag{2.10}$$

*where* $\Phi$ *is a continuous increasing function and* $C = C(N,q,\omega,\omega',s,\tau)$.

*(ii) For any sequence* $(u_n)$ *of weak solutions of equation* (1.1) *in* $Q_{\Omega,T}$, *uniformly locally bounded, one can extract a subsequence converging in* $C^{2,1}_{loc}(Q_{\Omega,T})$ *to a weak solution* $u$ *of* (1.1) *in* $Q_{\Omega,T}$.

**Proof.** (i) • Case $q < 2$. We can write (1.1) in the form $u_t - \Delta u = f$, where $f = -|\nabla u|^q$. From Lemma 2.4, there holds $u, \nabla u, f \in L^{q_1}_{loc}(Q_{\Omega,T})$, with $q_1 = 2/q \in (1,2)$. Then $u \in \mathcal{W}^{2,1,q_1}_{loc}(Q_{\Omega,T})$; see [21, Theorem IV.9.1]. Choosing $\omega''$ such that $\omega \subset\subset \omega'' \subset\subset \omega'$ and denoting $Q = Q_{\omega,s,\tau}$, $Q' = Q_{\omega',s/2,\tau}$, and $Q'' = Q_{\omega'',3s/4,\tau}$, we deduce that

$$\begin{aligned}\|u\|_{\mathcal{W}^{2,1,q_1}(Q)} &\leqq C(\|f\|_{L^{q_1}(Q'')} + \|u\|_{L^{q_1}(Q'')}) \\ &\leqq C(\|\nabla u\|^q_{L^2(Q'')} + \|u\|_{L^\infty(Q')}) \leqq C(\|u\|^q_{L^\infty(Q')} + \|u\|_{L^\infty(Q')}),\end{aligned}$$

with $C = C(N,q,\omega,\omega',s,\tau)$. From the Gagliardo–Nirenberg inequality, for almost any $t \in (0,T)$,

$$\|\nabla u(\cdot,t)\|_{L^{2q_1}(\omega)} \leqq c\|u(t)\|^{1/2}_{W^{2,q_1}(\omega)}\|u(t)\|^{1/2}_{L^\infty(\omega)},$$

where $c = c(N,q,\omega)$. Then $|\nabla u| \in L^{2q_1}_{loc}(Q)$, and

$$\|\nabla u\|_{L^{2q_1}(Q)} \leqq c\|u(t)\|^{1/2}_{W^{2,q_1}(Q)}\|u\|^{1/2}_{L^\infty(Q)} \leqq C_1\Phi_1(\|u\|_{L^\infty(Q')})),$$

with a new constant $C_1$ as above, where $\Phi_1$ is a continuous increasing function. Thus $f \in L^{q_2}_{loc}(Q_{\Omega,T})$, with $q_2 = (2/q)^2 \in (q_1, 2q_1)$ and $u, \nabla u, f \in L^{q_2}_{loc}(Q_{\Omega,T})$; in turn, $u \in \mathcal{W}^{2,1,q_2}_{loc}(Q_{\Omega,T})$. By induction we find that $u \in \mathcal{W}^{2,1,q_1^k}_{loc}(\Omega \times (0,T))$, for any $k \geqq 1$. Choosing any $k$ so that $q_k > N+2$, we deduce that $|\nabla u| \in C^{\gamma,\gamma/2}(\omega \times (s,\tau))$ for any $\gamma \in (0,1)$; see [21, Lemma II.3.3]. Then $f$ is locally Hölderian; thus $u \in C^{2+\gamma,1+\gamma/2}(Q_{\omega,s,\tau})$, and (2.10) holds.

• Case $q = 2$. We define $Q$ and $Q'$ as above. Since $u$ is locally bounded, the regularized function $u_\varepsilon = u * \varrho_\varepsilon$ converges to $u$ in $L^s(Q')$ for any $s \geqq 1$, and by extraction almost everywhere in $Q$. Also,

$$(u_\varepsilon)_t - \Delta u_\varepsilon + |\nabla u|^2 * \varrho_\varepsilon = 0 \qquad \text{in } Q'.$$

Defining the functions $z = 1 - e^{-u}$ in $Q_{\Omega,T}$, and $z^\varepsilon = 1 - e^{-u_\varepsilon}$ in $Q'$, we obtain that

$$(z^\varepsilon)_t - \Delta(z^\varepsilon) + h_\varepsilon = 0,$$

where $h_\varepsilon = e^{-u_\varepsilon}(|\nabla u|^2 * \varrho_\varepsilon - |\nabla u_\varepsilon|^2) \geqq 0$ from convexity. Then $|\nabla u|^2 * \varrho_\varepsilon$ converges to $|\nabla u|^2$ and $|\nabla u_\varepsilon|^2$ converges to $|\nabla u|^2$ in $L^1_{loc}(Q_{\Omega,T})$; thus $h_\varepsilon$ tends to 0 in $L^1_{loc}(Q_{\Omega,T})$. As $\varepsilon \to 0$, $z^\varepsilon$ converges to $z$ in $L^s(Q)$ for any $s \geqq 1$, and $z$ is a solution of the heat equation in $\mathcal{D}'(Q')$, and hence also in $\mathcal{D}'(Q_{\Omega,T})$). Then $z \in C^\infty(Q_{\Omega,T})$; hence $\max_Q z < 1$, and thus $u \in C^\infty(Q_{\Omega,T})$. Also, $\|z\|_{L^\infty(Q')} < 1 - e^{-\|u\|_{L^\infty(Q')}}$; then (2.10) follows from analogous estimates on $z$.

(ii) From the estimate (2.10), one can extract a diagonal subsequence, converging almost everywhere to a function $u$ in $Q_{\Omega,T}$, and the convergence holds in $C^{2,1}_{loc}(Q_{\Omega,T})$. Then $u$ is a weak solution of (1.1) in $Q_{\Omega,T}$. □

In the case of the Dirichlet problem we obtain a corresponding regularity result for the *bounded* solutions. Our proof can be compared to the proof of [7, Proposition 4.1] relative to the case $q < 1$.

**Theorem 2.10.** *Let $1 < q \leqq 2$. Let $\Omega$ be a smooth bounded domain. Let $u$ be any weak nonnegative solution of problem $(D_{\Omega,T})$, such that $u \in L^\infty_{loc}((0,T)\,; L^\infty(\Omega))$.*

*(i) Then $u$ satisfies the local estimates of Theorem 2.9. Moreover, $u \in C^{1,0}(\overline{\Omega} \times (0,T))$ and there exists $\gamma \in (0,1)$ such that, for any $0 < s < \tau < T$,*

$$\|u\|_{C(\overline{\Omega}\times[s,\tau])} + \|\nabla u\|_{C^{\gamma,\gamma/2}(\overline{\Omega}\times[s,\tau])} \leqq C\Phi(\|u\|_{L^\infty(Q_{\Omega,s/2,\tau})}), \tag{2.11}$$

*where $C = C(N, q, \Omega, s, \tau, \gamma)$, and $\Phi$ is an increasing function.*

*(ii) For any sequence $(u_n)$ of weak solutions of $(D_{\Omega,T})$ uniformly bounded in $L^\infty_{loc}((0,T); L^\infty(\Omega))$, one can extract a subsequence converging in $C^{1,0}_{loc}(\overline{\Omega}\times(0,T))$ to a weak solution $u$ of $(D_{\Omega,T})$.*

**Proof.** (i) • Case $q < 2$. From Lemma 2.6, we have $\nabla u \in L^2_{loc}((0,T); L^2(\Omega))$ and $u \in C((0,T); L^1(\Omega))$. Then $f = -|\nabla u|^q \in L^{q_1}_{loc}((0,T); L^{q_1}(\Omega))$. For any $0 < s < \tau < T$, and $t \in [s/2, \tau]$, we can write $u(\cdot,t) = u_1(\cdot,t) + u_2(\cdot,t)$, from Lemma 2.5, where

$$u_1(\cdot,t) = e^{(t-s/2)\Delta}u(\frac{s}{2}), \qquad u_2(\cdot,t) = \int_{s/2}^t e^{(t-\sigma)\Delta} f(\sigma)\, d\sigma.$$

We get $u_1 \in C^\infty(\overline{Q_{\Omega,s,\tau}})$ from the regularizing effect of the heat equation, and $u_2 \in \mathcal{W}^{2,1,q_1}(Q_{\Omega,T})$, from [21, Theorem IV.9.1]. As above, from the Gagliardo estimate, we get $f \in L^{q_2}_{loc}((0,t); L^{q_2}(\Omega))$, and by induction $|\nabla u| \in C^{\gamma,\gamma/2}(\overline{Q_{\Omega,s,\tau}})$ for some $\gamma \in (0,1)$; see [21, Lemma II.3.3]. The estimates follow.

• Case $q = 2$. From Theorem 2.9, $u$ is smooth in $Q_{\Omega,T}$, and $z = 1 - e^{-u}$ is a solution of the heat equation, and $z \in C((0,T); L^1(\Omega))$. Then $z(\cdot,t) =$

$e^{(t-s/2)\Delta}z(s/2)$; thus $z \in C^\infty(\overline{Q_{\Omega,s,\tau}})$. This implies that $\max_{\overline{Q_{\Omega,s,\tau}}} z < 1$; thus $u \in C^\infty(\overline{Q_{\Omega,s,\tau}})$ and the estimates follow again.

(ii) It follows directly from (2.11). □

**Remark 2.11.** As a consequence, in the case $q \leqq 2$, we find again the estimate (2.8) for the problem $(D_{\Omega,T})$ without using the Bernstein argument, and it is valid for any weak solution $u \in L^\infty_{loc}((0,T)\,; L^\infty(\Omega))$.

2.4. **Singular solutions or supersolutions.** In our study some functions play a fundamental role. The first one was introduced in [9].

2.4.1. *A stationary supersolution.* Assume that $1 < q < 2$. Equation (1.1) admits a stationary solution whenever $N = 1$ or $N \geqq 2$, $1 < q < N/(N-1)$, defined by

$$\Gamma_N(x) = \gamma_{N,q}\,|x|^{-a}\,, \quad a = \frac{2-q}{q-1}, \quad \gamma_{N,q} = a^{-1}(a+2-N)^{1-q}. \qquad (*)$$

Moreover, setting

$$\Gamma(x) = \Gamma_1(|x|) = \gamma_{1,q}\,|x|^{-a}\,, \tag{2.12}$$

the function $\Gamma$ is a radial *supersolution* of equation (1.1) for any $N$.

2.4.2. *Large solutions.* Here we recall a main result of [17] obtained as a consequence of the universal estimates.

**Theorem 2.12** ([17])**.** *Let $G$ be any smooth bounded domain, and $\eta > 0$ such that $B_\eta \subset\subset G$. Then for any $q > 1$, there exists a (unique) solution $Y^G_\eta$ of the problem*

$$\begin{cases} (Y^G_\eta)_t - \Delta Y^G_\eta + |\nabla Y^G_\eta|^q = 0, & \text{in } Q_{G,\infty}, \\ Y^G_\eta = 0, & \text{on } \partial G \times (0,\infty), \\ Y^G_\eta(x,0) = \begin{cases} \infty & \text{if } x \in B_\eta, \\ 0 & \text{if not,} \end{cases} \end{cases} \tag{2.13}$$

*which is uniformly Lipschitz continuous in $\overline{G}$ for $t$ in compacts sets of $(0,\infty)$ and is a classical solution of the problem for $t > 0$, and satisfies the initial condition in the following sense:*

$$\begin{aligned} &\lim_{t\to 0}\inf_{x\in K} Y^G_\eta(x,t) = \infty, && \forall K \text{ compact} \subset B_\eta; \\ &\lim_{t\to 0}\sup_{x\in K} Y^G_\eta(x,t) = 0, && \forall K \text{ compact} \subset \overline{G}\backslash\overline{B_\eta}. \end{aligned} \tag{2.14}$$

*Also, $Y^G_\eta$ is the supremum of the solutions $y_{\varphi_{\eta,G}}$ with initial data $\varphi_{\eta,G} \in C^+(G)$ such that $\varphi_{\eta,G} = 0$ on $G\backslash\overline{B_\eta}$.*

A crucial point for existence was the construction of a supersolution for the problem in a ball:

**Lemma 2.13.** *For any ball $B_s \subset \mathbb{R}^N$ and any $\lambda > 0$, there exists a supersolution $w_{\lambda,s}$ of equation* (1.1) *in $B_s \times [0,\infty)$, such that*

$$w_{\lambda,s} = \infty \quad \text{on } \partial B_s \times [0,\infty), \qquad w_{\lambda,s} = \lambda e^{ct+1/\alpha_s(x)}, \quad c = c(\lambda) > 0,$$

*where $\alpha_s$ is the solution of $-\Delta\alpha_s = 1$ in $B_s$ and $\alpha_s = 0$ on $\partial B_s$.*

2.5. **Some trace results.** First we extend a trace result of [23].

**Lemma 2.14.** *Let $\Phi \in L^1_{loc}(Q_{\Omega,T})$ and $U \in C((0,T); L^1_{loc}(\Omega))$ be any nonnegative weak solution of equation*

$$U_t - \Delta U = \Phi, \ \text{ in } Q_{\Omega,T}.$$

*(i) Assume that $\Phi \geqq -F$, where $F \in L^1_{loc}(\Omega \times [0,T))$. Then $U(\cdot,t)$ converges weak* to some Radon measure $U_0$:*

$$\lim_{t\to 0} \int_\Omega U(\cdot,t)\varphi\,dx = \int_\Omega \varphi\,dU_0, \qquad \forall\varphi \in C_c(\Omega).$$

*Furthermore, $\Phi \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$, and for any $\varphi \in C^2_c(\Omega \times [0,T))$,*

$$-\int_0^T \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)\,dx\,dt = \int_\Omega \varphi(\cdot,0)\,dU_0. \tag{2.15}$$

*(ii) Assume that $\Phi$ has a constant sign. Then*

$$\Phi \in L^1_{loc}([0,T); L^1_{loc}(\Omega)) \Longleftrightarrow U \in L^\infty_{loc}(\,[0,T)\,; L^1_{loc}(\Omega)).$$

**Proof.** (i) Let $\omega \subset\subset \omega' \subset\subset \Omega$ and $0 < s < \tau < T$. We approximate $U$, $F$, and $\Phi$ by $U_\varepsilon$, $F_\varepsilon$, and $\Phi_\varepsilon$ so that for $\varepsilon$ small enough, $(U_\varepsilon)_t - \Delta U_\varepsilon = \Phi_\varepsilon$ in $Q_{\omega',s/2,\tau}$. Let $\phi_1$ be a positive eigenfunction associated to the first eigenvalue $\lambda_1$ of $-\Delta$ in $W^{1,2}_0(\omega)$. Taking $\phi_1$ as a test function, integrating over $\omega$, and setting

$$h^\varepsilon(t) = e^{\lambda_1 t}\int_\omega U_\varepsilon(\cdot,t)\phi_1\,dx - \int_t^\tau \int_\omega e^{\lambda_1\theta} F_\varepsilon\phi_1\,dx\,d\theta,$$

then

$$h_\varepsilon(\tau) \geqq h_\varepsilon(s) + \int_s^\tau \int_\omega e^{\lambda_1\theta}(\Phi_\varepsilon + F_\varepsilon)\phi_1\,dx\,d\theta.$$

As $\varepsilon \to 0$ we deduce that the function

$$t \longmapsto h(t) = e^{\lambda_1 t}\int_\omega U(\cdot,t)\phi_1\,dx - \int_t^\tau \int_\omega e^{\lambda_1 s} F(\cdot,s)\phi_1\,dx\,d\theta$$

is nondecreasing on $(0,T)$. From the assumption on $F$, $\int_\omega U(\cdot,t)\phi_1\,dx$ has a limit as $t\to 0$, and $\Phi\in L^1_{loc}([0,T);L^1_{loc}(\Omega))$. Otherwise, for any $\psi\in\mathcal{C}^{2+}_c(\Omega)$, and any $t<\tau$, there holds

$$\int_\Omega U(\cdot,\tau)\psi\,dx-\int_t^\tau\int_\Omega(U\Delta\psi+\Phi\psi)\,dx\,dt=\int_\Omega U(\cdot,t)\psi\,dx$$

from (2.3). Thus $\int_\Omega U(\cdot,t)\psi\,dx$ has a limit $\mu(\psi)$ as $t\to 0$. Then $\mu$ extends in a unique way as a Radon measure $U_0$ on $\Omega$. Finally, for any $\varphi\in C^\infty_c(\Omega\times[0,T))$ and any $t\in(0,T)$, we have

$$-\int_t^T\int_\Omega(U\varphi_t+U\Delta\varphi+\Phi\varphi)\,dx\,dt=\int_\Omega U(\cdot,t)\varphi(\cdot,t)\,dx.$$

Going to the limit as $t\to 0$, we deduce (2.15), since

$$\Big|\int_\Omega U(\cdot,t)(\varphi(\cdot,t)-\varphi(\cdot,0))\,dx\Big|\leqq Ct\int_{\operatorname{supp}\varphi}U(\cdot,t)\,dx.$$

(ii) If $U\in L^\infty_{loc}([0,T);L^1_{loc}(\Omega))$, then $\int_t^\tau\int_\Omega\Phi\psi\,dx\,dt$ is bounded as $t\to 0$, and $\Phi\in L^1_{loc}([0,T);L^1_{loc}(\Omega))$ from the Fatou lemma. The converse is a direct consequence of (i). □

We deduce a trace property for equation (1.1), inspired by the results of [22] for equation (1.3):

**Proposition 2.15.** *For any nonnegative weak solution $u$ of* (1.1) *in $Q_{\Omega,T}$, the following conditions are equivalent:*

*(i)* $u\in L^\infty_{loc}([0,T);L^1_{loc}(\Omega))$,

*(ii)* $\nabla u\in L^q_{loc}(\Omega\times[0,T))$,

*(iii) $u(\cdot,t)$ converges weak* to some nonnegative Radon measure $u_0$ in $\Omega$. Then for any $\tau\in(0,T)$, and any $\varphi\in C^1_c(\Omega\times[0,T))$,*

$$\int_\Omega u(\cdot,\tau)\varphi\,dx+\int_0^\tau\int_\Omega(-u\varphi_t+\nabla u\cdot\nabla\varphi-|\nabla u|^q\,\varphi)\,dx\,dt=\int_\Omega\varphi(\cdot,0)\,du_0.$$

**Remark 2.16.** If $q\geqq 2$, and $u$ admits a Radon measure $u_0$ as a trace, in the sense of condition (iii), then necessarily $u_0\in L^1_{loc}(\Omega)$, and $u\in C([0,T);L^1_{loc}(\Omega))$. Indeed, condition (ii) implies $u\in L^2_{loc}([0,T);W^{1,2}_{loc}(\Omega))$, and $u_t\in L^2_{loc}((0,T);W^{-1,2}_{loc}(\Omega))+L^1(Q_{\omega,T})$; then the conclusion holds from [26]. As a first consequence, there exists no weak solution of equation (1.1) with a Dirac mass as initial data. This had been shown in [1, Theorem 2.2 and Remark 2.1] for the Dirichlet problem $(D_{\Omega,T})$.

2.6. **Behavior of solutions of (1.1) and (1.2) in $\Omega_0$.** Next we come to problem (1.1)–(1.2). In order to see what occurs at $t = 0$, we extend the solutions on $(-T, T)$ as in [14].

**Proposition 2.17.** *Let $u$ be any weak solution of* (1.1)–(1.2). *Then the function $\overline{u}$ defined almost everywhere in $Q_{\Omega,-T,T}$ by*

$$\overline{u}(x,t) = \begin{cases} u(x,t), & \text{if } (x,t) \in Q_{\Omega,T}, \\ 0 & \text{if } (x,t) \in Q_{\Omega,-T,0}, \end{cases} \tag{2.16}$$

*is a weak solution of the equation* (1.1) *in $Q_{\Omega_0,-T,T}$. If moreover*

$$\lim_{t\to 0} \int_\Omega u(\cdot,t)\varphi\,dx = 0, \qquad \forall \varphi \in C_c(\Omega), \tag{2.17}$$

*then $\overline{u}$ is a weak solution of* (1.1) *in $Q_{\Omega,-T,T}$.*

**Proof.** From assumption (1.2), $u \in L^\infty_{loc}([0,T) \times \Omega_0)$, and $|\nabla u| \in L^q_{loc}(\Omega_0 \times [0,T))$, from Proposition 2.15. For any $k \geqq 1$, we consider a function $t \longmapsto \zeta_k(t) = \zeta(kt)$, where $\zeta \in C^\infty([0,\infty)), \zeta([0,\infty)) \subset [0,1]$, $\zeta \equiv 0$ in $[0,1]$, and $\zeta \equiv 1$ in $[2,\infty)$. For any $\varphi \in \mathcal{D}(Q_{\Omega_0,-T,T})$, we have from the Lebesgue theorem

$$\begin{aligned} \langle \nabla\overline{u}, \varphi\rangle &= -\int_0^T \int_\Omega u\nabla\varphi\,dx\,dt = -\lim \int_0^T \int_\Omega u\nabla(\varphi\zeta_k)\,dx\,dt \\ &= \lim \int_0^T \int_\Omega \varphi\zeta_k\nabla u\,dx\,dt = \int_0^T \int_\Omega \varphi\nabla u\,dx\,dt; \end{aligned}$$

thus, $\nabla\overline{u} \in L^q_{loc}(Q_{\Omega_0,-T,T})$ and $\nabla\overline{u}(x,t) = \chi_{(0,T)}\nabla u(x,t)$; hence, $\nabla\overline{u} \in L^2_{loc}(Q_{\Omega_0,-T,T})$ from Lemma 2.4, and

$$\int_{-T}^T \int_\Omega (-\overline{u}\varphi_t + \nabla\overline{u}\cdot\nabla\varphi + |\nabla\overline{u}|^q\varphi)\,dx\,dt = \int_0^T\int_\Omega (-u\varphi_t + \nabla u\cdot\nabla\varphi + |\nabla u|^q\varphi)\,dx\,dt. \tag{2.18}$$

Moreover,

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (-u(\varphi\zeta_k)_t + \nabla u\cdot\nabla(\varphi\zeta_k) + |\nabla u|^q\varphi\zeta_k\,dx\,dt \\ &= -\int_0^T\int_\Omega u\varphi(\zeta_k)_t\,dx\,dt + \int_0^T\int_\Omega (-u\varphi_t\zeta_k + \nabla u\cdot\nabla(\varphi\zeta_k) + |\nabla u|^q\varphi\zeta_k\,dx\,dt. \end{aligned}$$

As $k \to \infty$, the first term in the right-hand side tends to 0 from (1.2), as in [14], and we can go to the limit in the second term, since $|\nabla u| \in L^q_{loc}(\Omega_0 \times [0,T))$. Thus from (2.18), $\overline{u}$ is a weak solution of equation (1.1) in $Q_{\Omega_0,-T,T}$. If (2.17) holds, the same result holds in $\Omega$ instead of $\Omega_0$. □

From Proposition 2.17 and Theorem 2.9 applied to $\overline{u}$, we deduce directly the following:

**Corollary 2.18.** *Assume* $1 < q \leqq 2$. *Then any weak solution* $u$ *of* (1.1)–(1.2) *satisfies* $u \in C^{2,1}(\Omega_0 \times [0,T))$ *and* $u(x,0) = 0$, $\forall x \in \Omega_0$.

*If* (2.17) *holds, then* $u \in C^{2,1}(\Omega \times [0,T))$ *and* $u(x,0) = 0$, $\forall x \in \Omega$.

## 3. The critical or supercritical case

3.1. **Removability in the range** $q_* < q < 2$**.** For any $1 < q < 2$ we can compare the solutions with the function $\Gamma$ defined at (2.12).

**Lemma 3.1.** *Let* $1 < q < 2$. *Let* $u$ *be any nonnegative weak solution of* (1.1) *in* $Q_{\Omega,T}$, *satisfying* (1.2).

(*i*) *Let* $r > 0$ *such that* $B_r \subset \Omega$. *Then there exists* $\tau_1 > 0$ *such that*

$$u(x,t) \leqq \Gamma(x) \qquad \forall (x,t) \in Q_{B_r \backslash \{0\},\tau_1}. \tag{3.1}$$

(*ii*) *If* $\Omega = \mathbb{R}^N$, *then*

$$u(x,t) \leqq \Gamma(x) \qquad \forall (x,t) \in Q_{\mathbb{R}^N \backslash \{0\},T}. \tag{3.2}$$

**Proof.** (i) For any $\eta \in (0,r)$, we put $\Omega_\eta = B_r \backslash \overline{B_\eta}$, and we set $F_\eta(x) = \Gamma(|x| - \eta)$, for any $x \in \Omega_\eta$. Then $F_\eta$ is a supersolution of (1.1) in $Q_{\Omega_\eta,\infty}$. From Corollary 2.18, there exists $\tau_1 < T$ such that

$$\max_{\substack{t \in [0,\tau_1] \\ |x|=r}} u(t,x) < 1,$$

and $u$ is bounded in $\overline{\Omega_\eta} \times [0,\tau_1]$. From the comparison principle there holds $u(x,t) \leqq F_\eta(x)$ in $\Omega_\eta \times [0,\tau_1]$. As $\eta \to 0$, we deduce (3.1).

(ii) From Lemma 2.13, for any $x_0 \in \mathbb{R}^N \backslash B_2$, the function $x \mapsto w_{1,1}(x - x_0)$ is a supersolution of equation (1.1) in $Q_{B(x_0,1),\infty}$; then in particular $u(t,x_0) \leqq e^{c(1)t + 1/\alpha_1(0)}$; thus $u$ bounded in $Q_{\mathbb{R}^N \backslash B_2,T}$. From the comparison principle in $\mathbb{R}^N \backslash \overline{B_\eta}$ for any $\eta \in (0,1)$ (see [18]), we find $u(x,t) \leqq F_\eta(x)$ in $Q_{\mathbb{R}^N \backslash \overline{B_\eta},T}$; hence (3.2) holds as $\eta \to 0$. □

As a direct consequence we get a simple proof of Theorem 1.1 in this range of $q$:

**Proof of Theorem 1.1 for** $q_* < q < 2$**.** The assumption $q_* < q$ is equivalent to $a < N$, where $a$ is defined in $(*)$. Let $B_r \subset \Omega$ and $\tau_1$ be defined as in Lemma 3.1; we find for any $t \in (0,\tau_1)$,

$$\int_{B_r} u(\cdot,t)\,dx \leqq \int_{B_r} \Gamma\,dx \leqq \frac{\gamma_q\,|\partial B_1|\,r^{N-a}}{N-a};$$

then $u \in L^{\infty}((0,\tau_1);L^1(B_r))$. From Proposition 2.15, $u(\cdot,t)$ converges weak* to a measure $\mu$ on $B_r$:

$$\lim_{t\to 0}\int_{B_r} u(\cdot,t)\psi\,dx = \int_{B_r}\psi\,d\mu, \qquad \forall\psi\in C_c\left(B_r\right).$$

From (1.2), $\mu$ is concentrated at 0, and then $\mu = k\delta_0$ for some $k \geqq 0$. Suppose that $k>0$, and choose $\psi_\eta$ such that $\psi_\eta(0)=1$, $\psi_\eta(B_r)\subset[0,1]$, and $\operatorname{supp}\psi_\eta\subset B_\eta$, with $\eta\in(0,r)$ small enough such that $\gamma_q\left|\partial B_1\right|\eta^{N-a}\leqq (N-a)k/2$. For any $t\in(0,\tau_1)$, Lemma 3.1 yields

$$\int_{B_r} u(\cdot,t)\psi_\eta\,dx \leqq \int_{B\eta}\Gamma\,dx \leqq \frac{k}{2}.$$

As $t$ tends to 0 the left-hand side tends to $k$, which is a contradiction. Then $k=0$; hence for any $\psi\in C_c^{\infty}(B_r)$, there holds $\lim_{t\to 0}\int_{B_r} u(\cdot,t)\psi\,dx = 0$. We conclude from Corollary 2.18. □

3.2. **Removability in the whole range** $q_* \leqq q < 2$**.** The above proof is not valid in the critical case $q=q_*$, since $\Gamma\notin L^1_{loc}\left(\mathbb{R}^N\right)$. Then we use the large solutions constructed at Theorem 2.12, and prove a comparison property, valid for any $1<q<2$.

**Proposition 3.2.** *Let* $1<q<2$. *Under the assumptions of Theorem* 2.12 *with* $G=B_n$ $(n\geqq 1)$ *the functions* $Y_\eta^{B_n}$ *converge as* $n\to\infty$ *to a radial solution* $Y_\eta$ *of the problem*

$$\left\{\begin{array}{l} (Y_\eta)_t - \Delta Y_\eta + |\nabla Y_\eta|^q = 0, \qquad \text{in } Q_\infty, \\ \qquad Y_\eta(x,0) = \left\{\begin{array}{ll} \infty & \text{if } x\in B_\eta, \\ 0 & \text{if not.}\end{array}\right.\end{array}\right. \tag{3.3}$$

*Then, as* $\eta\to 0$, $Y_\eta$ *converges to a radial self-similar solution* $Y$ *of equation* (1.1) *in* $Q_{\mathbb{R}^N,\infty}$, *such that*

$$Y(x,t)\leqq\Gamma\left(x\right), \qquad Y(x,t)\leqq C(1+t^{-\frac{1}{q-1}}), \qquad \text{in } Q_\infty, \tag{3.4}$$

*where* $C=C(N,q)$, *and*

$$\lim_{t\to 0}(\sup_{|x|\geqq r} Y(x,t)) = 0. \tag{3.5}$$

*If* $q_*\leqq q<2$, *then* $Y=0$.

**Proof.** Let $\eta\in(0,1/2)$. For any $n\geqq 1$, $Y_\eta^{B_n}$ is the supremum of the solutions $y_{\varphi_{\eta,B_n}}$ with initial data $\varphi_{\eta,B_n}\in C^+(B_n)$ such that $\varphi_{\eta,B_n}=0$ on $B_n\backslash\overline{B_\eta}$; from the comparison principle in $Q_{B_n\backslash\overline{B_\eta},\infty}$, see for example [28],

$$y_{\varphi_{\eta,B_n}}(x,t)\leqq\Gamma_1\left(|x|-\eta\right) \qquad \text{in } (B_n\backslash\overline{B_\eta})\times[0,\infty). \tag{3.6}$$

Next we compare $y_{\varphi_{\eta,B_n}}$ in $Q_{B_1,\infty}$ with the classical solution $w$ of the Dirichlet problem $(D_{B_{1,\infty}})$ with initial data $\varphi_{\eta,B_n}$. We deduce that, for any $(x,t) \in \overline{B_1} \times (0,\infty)$,

$$y_{\varphi_{\eta,B_n}}(x,t) \leqq C(1+t^{-\frac{1}{q-1}}) + \gamma_q\{1-\eta)\}^{-a} \leqq C(1+t^{-\frac{1}{q-1}}) + \gamma_q 2^a, \quad (3.7)$$

with $C = C(N,q)$, from Theorem 2.7 and Remark 2.8. Also, for any $(x,t) \in (B_n\backslash\overline{B_1}) \times (0,\infty)$, we have $y_{\varphi_{\eta,B_n}}(x,t) \leqq \Gamma(1-\eta)$, since $\Gamma$ is decreasing; hence (3.7) holds in $B_n \times [0,\infty)$. The same majoration holds for $Y_\eta^{B_n}$:

$$Y_\eta^{B_n}(\cdot,t) \leqq C(1+t^{-\frac{1}{q-1}}), \qquad \text{in } Q_{B_n,\infty},$$

with a new $C = C(N,q)$. Then we can go to the limit as $n \to \infty$, for fixed $\eta$. Since $Y_\eta^{B_n} \leqq Y_\eta^{B_{n+1}}$ in $Q_{B_n,\infty}$, $(Y_\eta^{B_n})$ converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a weak solution $Y_\eta$ of equation (1.1), from Theorem 2.9. Then $Y_\eta = \sup Y_\eta^{B_n}$ satisfies

$$Y_\eta \leqq C(1+t^{-\frac{1}{q-1}}), \qquad \text{in } Q_\infty, \quad (3.8)$$

and $Y_\eta$ solves the problem (3.3) in the following sense:

$$\lim_{t\to 0}\inf_{x\in K} Y_\eta(x,t) = \infty, \qquad \forall K \text{ compact} \subset B_\eta;$$
$$\lim_{t\to 0}\sup_{x\in K} Y_\eta(x,t) = 0, \qquad \forall K \text{ compact} \subset \mathbb{R}^N\backslash\overline{B_\eta}. \quad (3.9)$$

Indeed, from Lemma 2.13, for any ball $B(x_0,s) \subset \mathbb{R}^N\backslash\overline{B_\eta}$, and any $\lambda > 0$, there holds $Y_\eta^{B_n} \leqq w_{\lambda,s}(x-x_0)$ in $Q_{B(x_0,s),\infty}$ for any $n > |x_0| + |r|$; in turn, $Y_\eta \leqq w_{\lambda,s}(x-x_0)$; hence $\lim_{t\to 0}\sup_{B(x_0,s/2)} Y_\eta(\cdot,t) \leqq \lambda e^{1/\alpha(s/2)}$ for any $\lambda > 0$. Moreover, (3.6) implies that

$$Y_\eta(x,t) \leqq \Gamma_1(|x|-\eta) \qquad \text{in } Q_{\mathbb{R}^N\backslash\overline{B_\eta},\infty}.$$

Then for any $r > \eta$ and any $p > r$,

$$\sup_{|x|\geqq r} Y_\eta(x,t) \leqq \sup_{x\in B_p\backslash\overline{B_\eta}} Y_\eta(x,t) + \sup_{x\in\mathbb{R}^N\overline{\backslash B_p}} Y_\eta(x,t) \leqq \sup_{x\in B_p\backslash\overline{B_\eta}} Y_\eta(x,t) + \Gamma(|p|-\eta).$$

Since $\lim_{r\to\infty}\Gamma_1(r) = 0$, we deduce that

$$\lim_{t\to 0}(\sup_{|x|\geqq r} Y_\eta(x,t)) = 0. \quad (3.10)$$

Next we let $\eta \to 0$: observing that $Y_\eta \leqq Y_{\eta'}$ for $\eta \leqq \eta'$, in the same way, from Theorem 2.9, the function $Y = \inf_{\eta>0} Y_\eta$ is a weak solution of equation (1.1) in $Q_{\mathbb{R}^N,\infty}$, satisfying (3.4) and (3.5), which implies in particular

(1.6). By their construction, all the functions $Y_\eta^{B_n}$ are radial, and satisfy the relation of similarity,

$$\kappa^a Y_\eta^{B_n}(\kappa x, \kappa^2 t) = Y_{\eta/\kappa}^{B_{n/\kappa}}(x,t), \qquad \forall \kappa > 0, \quad \forall (x,t) \in B_{n/k};$$

then $Y$ is radial and self-similar.

Suppose $q \geqq q_*$ and $Y \not\equiv 0$; writing $Y$ in the similar form $Y(x,t) = t^{-a/2} f(t^{-1/2}|x|)$, from [27, Theorem 2.1], we find $\underline{\lim}_{r\to\infty} r^a f(r) > 0$, which contradicts (3.5); thus $Y \equiv 0$. □

**Proposition 3.3.** *Let $1 < q < 2$. Let $\Omega$ be any domain in $\mathbb{R}^N$. Let $u$ be any weak solution of (1.1)–(1.2) in $Q_{\Omega,T}$. Then for any $\tau \in (0,T)$ and any ball $B_r \subset\subset \Omega$, there holds*

$$u \leqq Y + \max_{\partial B_r \times [0,\tau]} u, \qquad \text{in } Q_{B_r,\tau}. \tag{3.11}$$

*Moreover, if $\Omega = \mathbb{R}^N$, then*

$$u \leqq Y, \qquad \text{in } Q_{\mathbb{R}^N,T} \tag{3.12}$$

*and $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty}) \cap C((0,\infty); C_b^2(\mathbb{R}^N))$.*

**Proof.** Let $u$ be such a solution in $Q_{\Omega,T}$. Let $\tau \in (0,T)$, $B_r \subset\subset \Omega$, and $M_r = \max_{\partial B_r \times [0,\tau]} u$ and $\varepsilon > 0$ be fixed. From Corollary 2.18, for any $0 < \eta < r/2$, there is $\delta_\eta > 0$ such that $u(x,t) < \varepsilon$, for $\eta \leqq |x| \leqq r$, $t \in (0,\delta_\eta)$. Let $R > r$. Next, for any $\delta \in (0,\delta_\eta)$, we make a comparison in $Q_{B_r,\delta,\tau}$ between $u(x,t)$ and $y_{2\eta,R,\delta}(x,t) = Y_{2\eta}^{B_R}(x,t-\delta) + M_r + \varepsilon$ as follows. On the parabolic boundary of $Q_{B_r,\delta,\tau}$, it is clear that $u \leqq y_{2\eta,\delta,R}$, since $u \leqq M_r$ on $\partial B_r \times [\delta,\tau]$, $u(x,\delta) \leqq \varepsilon$ for $x \in \overline{B_r} \backslash \overline{B_\eta}$, and $u(x,\delta) \leqq \infty = y_{2\eta,R,\delta}$, for $x \in \overline{B_\eta}$. Also, $y_{2\eta,R,\delta}$ converges to $+\infty$ uniformly on $\overline{B_\eta}$ as $t \to \delta$, and $u(\cdot,\delta)$ is bounded on $\overline{B_\eta}$. Then, from the comparison principle,

$$u \leqq y_{2\eta,R,\delta}, \qquad \text{in } Q_{B_r,\delta,\tau}.$$

As $\delta$ tends to 0 we deduce that

$$u \leqq Y_{2\eta}^{B_R} + M_r + \varepsilon, \qquad \text{in } Q_{B_r,\tau}, \tag{3.13}$$

by the continuity of $Y_{2\eta}^{B_R}$ in $Q_{B_r,T}$. Since (3.13) holds for any $\eta < r/2$, and any $\varepsilon > 0$, we obtain (3.11). Moreover, if $\Omega = \mathbb{R}^N$, then $M_r \leqq \Gamma_1(r)$ from Lemma 3.1, and we get (3.12) by letting $r \to \infty$. Moreover, $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty})$ from Theorem 2.9; then from (3.4), $u \in C_b(Q_{\mathbb{R}^N,\epsilon,\infty})$ for any $\epsilon > 0$, and then from [18, Theorems 3 and 6], $u \in C((0,\infty); C_b^2(\mathbb{R}^N))$. □

As a direct consequence, we deduce a new proof of Theorem 1.1, valid in the range $q_* \leqq q < 2$:

**Proof of Theorem 1.1 for $q_* \leqq q < 2$.** Since $q \geqq q_*$, we have $Y = 0$, from Proposition 3.2; thus $u$ is bounded in $Q_{B_r,\tau}$ from Proposition 3.3. Then $\lim_{t\to 0} \int_{B_r} u(\cdot,t)\psi\, dx = 0$ still holds for any $\psi \in C_c^\infty(B_r)$, and we conclude again from Corollary 2.18. □

3.3. **Removability for $q \geqq 2$.** When $q > 2$, the regularity of the solutions of equation (1.1), in particular the continuity property, is not known up to now. We can only mention the recent result of [16]: *if* a solution in the viscosity sense is continuous, then it is Hölderian. Then it is difficult to apply comparison theorems. Here we use the transformation $u \longmapsto z = 1 - e^{-u}$, which reduces classically equation (1.1) to the heat equation when $q = 2$. We gain the fact that $z$ is bounded. For $p > 2$, our proof requires regularization arguments.

**Proof of Theorem 1.1 for $q \geqq 2$.** Let us set $v = e^{-u}$, and $z = 1 - v$. Notice that $z$ is an increasing function of $u$, taking its values in $[0, 1]$.

(i) Case $q = 2$. From Theorem 2.9, $u$ is a classical solution in $Q_{\Omega,T}$, and $u(x, 0) = 0$ in $\Omega_0$ from Corollary 2.18. Then $z$ is a classical solution of the heat equation in $Q_{\Omega,T}$, and $z \in C(\Omega_0 \times [0,T))$ and $z(x, 0) = 0$ for $x \neq 0$. From Lemma 2.14, $z$ converges weak$^*$ to a Radon measure $\mu$ as $t \to 0$, necessarily concentrated at 0, from (1.2), since $z \leqq u$. Then $\mu = 0$, because $z$ is bounded. As for $u$, defining the extension $\overline{z}$ of $z$ by 0 for $t \in (-T, 0)$ , we find that $\overline{z}$ is a solution of the heat equation in $Q_{\Omega,-T,T}$; then $\overline{z} \in C^\infty(Q_{\Omega,-T,T})$. Hence $\overline{z}$ is strictly locally bounded by 1; thus also $\overline{u} \in C^\infty(Q_{\Omega,-T,T})$, and thus $u(0, 0) = 0$, and the proof is done. Moreover, $u \in C^\infty(\Omega \times [0,T))$.

(ii) Case $q > 2$. We regularize $u$ by $u_\varepsilon$ and obtain

$$(u_\varepsilon)_t - \Delta u_\varepsilon + (|\nabla u|^q)_\varepsilon = 0,$$

and we set $v^\varepsilon = e^{u_\varepsilon}$. Then $v^\varepsilon$ satisfies the equation

$$v_t^\varepsilon - \Delta v^\varepsilon = v^\varepsilon \left( (|\nabla u|^q)_\varepsilon - |\nabla u_\varepsilon|^2 \right).$$

Observe that $v^\varepsilon$ is not the regularization of $v$, but it has the same convergence properties. Going to the limit as $\varepsilon \to 0$, $v$ satisfies

$$v_t - \Delta v = \Phi, \text{ where } \Phi = v(|\nabla u|^q - |\nabla u|^2) \in L^1_{loc}(Q_{\Omega,T}),$$

in $\mathcal{D}'(Q_{\Omega,T})$. Next from Hölder's inequality, we can apply Lemma 2.14 to $v$ with $F = 1$. Then $z(\cdot, t)$ converges weak$^*$ to a Radon measure $\mu$ as $t \to 0$, and $\Phi \in L^1_{loc}(\Omega \times [0,T))$; and for any $\varphi \in C^2_c(\Omega \times [0,T))$ there holds

$$\int_0^T \int_\Omega z(\varphi_t + \Delta\varphi)\, dx\, dt = \int_0^T \int_\Omega \Phi\varphi\, dx\, dt + \int_\Omega \varphi(x,0)\, d\mu, \tag{3.14}$$

from (2.15). We claim that $\mu = 0$ and the extension of $z$ by 0 for $t = 0$ satisfies $z \in C([0,T), L^1_{loc}(\Omega))$. Indeed, from assumption (1.2), $u(\cdot,t)$converges to 0 in $L^1_{loc}(\Omega_0)$ as $t \to 0$, thus also $z(\cdot,t)$. For any sequence $(t_n)$ tending to 0, we can extract a (diagonal) subsequence such that $u(\cdot,t_\nu)$ converges to 0, almost everywhere in $\Omega$. Since $z$ is bounded, it follows that $(z(\cdot,t_\nu))$ converges to 0 in $L^1_{loc}(\Omega)$ from the Lebesgue theorem. And then $z(\cdot,t)$ converges to 0 in $L^1_{loc}(\Omega)$ as $t \to 0$.

We still consider the extension $\overline{z}$ of $z$ by 0 on for $t \in (-T,0)$. For any $\phi \in \mathcal{D}^+(Q_{\Omega,-T,T})$, we have from (3.14),

$$\begin{aligned}-\int_{-T}^T \int_\Omega \overline{z}(\phi_t + \Delta\phi)\, dx\, dt = -\int_0^T \int_\Omega z(\phi_t + \Delta\phi)\, dx\, dt = -\int_0^T \int_\Omega \Phi\varphi\, dx\, dt \\ \leqq \int_0^T \int_\Omega (1-z)\varphi\, dx\, dt \leqq \int_{-T}^T \int_\Omega (1-\overline{z})\varphi\, dx\, dt.\end{aligned}$$

Then $\overline{z}$ is a subsolution of equation

$$w_t - \Delta w + w = 1 \quad \text{in } \mathcal{D}'(Q_{\Omega,-T,T}). \tag{3.15}$$

Otherwise $\overline{u}$ is the weak solution of equation (1.1) in $Q_{\Omega_0,-T,T}$; then $\overline{u}$ is subcaloric. As a consequence, for any $\tau \in (0,T)$, and any ball $B_{2r} \subset\subset \Omega$, the function $\overline{u}$ is essentially bounded on $Q_{B_{2r}\setminus\overline{B_{r/2}},-\tau,\tau}$ by a constant $M_{r,\tau}$. Then $\overline{z} \leqq 1 - e^{-M_{r,\tau}} = m_{r,\tau} < 1$ on this set. For any $K > 0$ the function $y_K(t) = 1 - Ke^{-t}$ is a solution of equation (3.15). Taking $K = e^{-(M_{r,\tau}+\tau+1)}$, we can apply the comparison principle in $Q_{B_r,-\tau,\tau}$ to the regularization $\overline{z}_\varepsilon$ of $\overline{z}$ for $\varepsilon$ small enough, and deduce that $\overline{z} \leqq y_K$ almost everywhere in $Q_{B_r,-\tau,\tau}$, and then $\overline{z} \leqq 1 - e^{-(M_{r,\tau}+2\tau+1)} < 1$ in $Q_{B_r,-\tau,\tau}$. Hence $\overline{u} = -\ln(1-\overline{z})$ is essentially bounded in $Q_{B_r,-\tau,\tau}$. Finally, $\overline{u} \in L^\infty_{loc}(Q_{\Omega,-T,T})$, from the subcaloricity; hence $u \in L^\infty_{loc}(Q_{\Omega,T})$. Besides, for any $0 < s < t < \tau$, and any domain $\omega \subset\subset \Omega$,

$$|u(\cdot,t) - u(\cdot,s)| \leqq e^{\|\overline{u}\|_{L^\infty(Q_{\omega,-\tau,\tau})}} |z(\cdot,t) - z(\cdot,s)|.$$

Then $u \in \mathcal{C}([0,T); L^1_{loc}(\Omega))$, and $u \in C([0,T); L^r_{loc}(\Omega))$, for any $r > 1$, since $u$ is locally bounded. Furthermore, for any ball $B(x_0, 2\rho) \subset \Omega$, and any

$t \in \left(\rho^2 - T, T\right)$,

$$\sup_{B(x_0,\rho)\times(t-\rho^2,t))} \overline{u} \leqq C\rho^{-(N+2)} \int_{t-\rho^2}^{t} \int_{B(x_0,2\rho)} \overline{u}\, dx\, ds,$$

where $C = C(N)$; see for example [20, Theorem 6.17]. Hence for any $t \in (0,\tau)$ and $\rho < T^{1/2}$, we find

$$\sup_{B(x_0,\rho)\times(0,t))} u \leqq C\rho^{-(N+2)} \int_0^t\int_{B(x_0,2\rho)} u\, dx\, ds \leqq C\rho^{-(N+2)} t \|u\|_{L^\infty(Q_{B(x_0,2\rho),\tau})};$$

hence (1.11) holds, which achieves the proof. □

3.4. **Global removability in $\mathbb{R}^N$.** Next we show Theorem 1.2 relative to $\Omega = \mathbb{R}^N$. It is a consequence of Proposition 3.3 in the case $1 < q < 2$. In fact, the result is general, as shown below:

**Proposition 3.4.** *Let $q > 1$. Let $u$ be any nonnegative weak subsolution of equation* (1.1) *in $Q_{\mathbb{R}^N,T}$ such that $u \in C((0,T, L^1_{loc}(\mathbb{R}^N))$, and*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(\cdot,t)\psi\, dx = 0, \tag{3.16}$$

*for any $\psi \in C_c\left(\mathbb{R}^N\right)$. Then $u \equiv 0$.*

**Proof.** From Remark 2.2 and Lemma 2.4, since $u \in C((0,T, L^1_{loc}(\mathbb{R}^N))$, there holds

$$\int_{\mathbb{R}^N} u(\cdot,t)\psi\, dx - \int_{\mathbb{R}^N} u(\cdot,s)\psi\, dx + \int_s^\tau \int_{\mathbb{R}^N} (\nabla u \cdot \nabla\psi + |\nabla u|^q \psi\, dx\, dt \leqq 0,$$

for any $\psi \in C_c^{2,+}(\mathbb{R}^N)$, and any $(s,t) \subset (0,T)$. Taking $\psi = \xi^{q'}$ with $\xi \in \mathcal{D}^+(\mathbb{R}^N)$ and using Hölder's inequality, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(\cdot,t)\psi\, dx - \int_{\mathbb{R}^N} u(\cdot,s)\psi\, dx + \int_s^t \int_{\mathbb{R}^N} |\nabla u|^q \psi\, dx\, dt \\
&\leqq q'(\int_s^t \int_{\mathbb{R}^N} |\nabla u|^q \psi\, dx)^{\frac{1}{q}} (\int_s^t \int_{\mathbb{R}^N} |\nabla \xi|^{q'}\, dx)^{\frac{1}{q'}} \\
&\leqq \frac{1}{2} \int_s^t \int_{\mathbb{R}^N} |\nabla u|^q \psi\, dx + C_q \int_s^t \int_{\mathbb{R}^N} |\nabla \xi|^{q'}\, dx
\end{aligned}$$

with $C_q = (2(q-1))^{q'}$. For given $R > r > 0$, we choose $\xi(x) = \phi(|x|/R)$, where $\phi([0,\infty)) \subset [0,1]$, $\phi \equiv 1$ in $[0,1]$, and $\phi \equiv 0$ in $[2,\infty)$, and go to the limit as $s \to 0$ from (3.16). It follows that

$$\int_{B_r} u(\cdot,t)\, dx + \frac{1}{2}\int_0^t \int_{B_r} |\nabla u|^q\, dx\, dt \leqq C_q t R^{N-q'}.$$

(i) Case $q < N/(N-1)$. Here $N - q' < 0$. Letting $R \to \infty$, we deduce that $\int_{B_r} u(\cdot,t)\,dx = 0$, for any $r > 0$; thus $u \equiv 0$.

(ii) Case $q \geqq N/(N-1)$. Here we fix some $k \in (1, N/(N-1))$; for any $\eta \in (0,1)$, there holds $\eta|\nabla u|^k \leqq \eta + |\nabla u|^q$; hence the function $w_\eta = \eta^{1/(k-1)}(u - \eta t)$ satisfies

$$(w_\eta)_t - \Delta w_\eta + |\nabla w_\eta|^k \leqq 0$$

in the weak sense. Thanks to the Kato inequality (see for example [24] or [5]), we deduce that

$$(w_\eta^+)_t - \Delta w_\eta^+ + |\nabla w_\eta^+|^k \leqq 0,$$

in $\mathcal{D}'(Q_{\mathbb{R}^N,T})$. Moreover, $w_\eta \in C([0,T), L^1_{loc}(\mathbb{R}^N))$, and, for any $r > 0$,

$$\lim_{t\to 0^+} \int_{B_r} w_\eta^+(\cdot,t)\,dx = \eta^{-\frac{1}{k-1}} \lim_{t\to 0^+} \int_{B_r} (u(\cdot,t) - \eta t)^+\,dx = 0.$$

By the above proof, $w_\eta^+ \equiv 0$. Letting $\eta$ tend to 0 we get again $u \equiv 0$. □

3.5. **Behavior of the approximating sequences.** When $q \geqq q_*$, a simple question is to know what can happen to a sequence of solutions with smooth initial data converging to the Dirac mass, and one can expect that it converges to 0. We get more generally the following:

**Theorem 3.5.** *Assume that $q \geqq q_*$. Let $(\varphi_\varepsilon)$ be any sequence in $\mathcal{D}^+(\mathbb{R}^N)$, with* $\operatorname{supp}\varphi_\varepsilon \in B_\varepsilon$. *Then the sequence $(u_\varepsilon)$ of solutions of* (1.1) *in $Q_{\mathbb{R}^N,\infty}$, with initial data $\varphi_\varepsilon$, converges to* 0 *in $C_{loc}(Q_{\mathbb{R}^N,\infty})$. In the same way, if $\Omega$ is bounded, the sequence $(u_\varepsilon^\Omega)$ of solutions of $(D_{\Omega,\infty})$, with initial data $\varphi_\varepsilon$, converges to* 0 *in $C_{loc}(\overline{\Omega} \times (0,\infty))$.*

**Proof.** Let $\varepsilon \in (0,1)$. Since $u_\varepsilon^\Omega \leqq u_\varepsilon$, we only need to prove the result in the case $\Omega = \mathbb{R}^N$.

(i) Case $q < 2$. From the comparison principle there holds $u_\varepsilon \leqq Y_{2\varepsilon}$, where $Y_{2\varepsilon}$ defined at (3.3), and $Y_{2\varepsilon}$ converges to 0 in $C^1_{loc}(Q_{\mathbb{R}^N,\infty})$ from Proposition 3.2; then also $u_\varepsilon$.

(ii) Case $q \geqq 2$. Let us fix some $k$ such that $q_* < k < 2$. As in the proof of Proposition 3.4, for any $\eta \in (0,1)$, $w_{\varepsilon,\eta} = \eta^{1/(k-1)}(u_\varepsilon - \eta t)$ satisfies

$$(w_{\varepsilon,\eta})_t - \Delta w_{\varepsilon,\eta} + |\nabla w_{\varepsilon,\eta}|^k \leqq 0$$

in $\mathcal{D}'(Q_{\mathbb{R}^N,\infty})$, and $w_{\varepsilon,\eta} \in L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^N))$. From the comparison principle we find that $w_{\varepsilon,\eta} \leqq v_\varepsilon$, where $v_\varepsilon$ is the solution of equation (1.1) with $q$ replaced by $k$ and $v_\varepsilon(\cdot,0) = \rho_\varepsilon$; hence, $u_\varepsilon \leqq \eta t + \eta^{-1/(k-1)} v_\varepsilon$. Also, $(v_\varepsilon)$

converges to 0 in $C_{loc}(Q_{\mathbb{R}^N,\infty})$ from (i). Let $\mathcal{K} = [s,\tau] \times K$ be any compact in $Q_{\mathbb{R}^N,\infty}$. Then

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\mathcal{K})} \leqq \eta\tau + \eta^{1/(k-1)} \limsup_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty(\mathcal{K})} = \eta\tau$$

for any $\eta > 0$; then $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\mathcal{K})} = 0$. □

## 4. The subcritical case $1 < q < q_*$

We first recall the following results of [7, Theorem 3.2 and Proposition 5.1] for the Dirichlet problem.

**Theorem 4.1** ([7])**.** *Let $1 < q < q_*$ and $\Omega$ be a smooth bounded domain. Then for any $u_0 \in \mathcal{M}_b(\Omega)$ there exists a weak solution of problem $(D_{\Omega,\infty})$ such that $u(\cdot,0) = u_0$ in the weak sense of $\mathcal{M}_b(\Omega)$:*

$$\lim_{t\to 0} \int_\Omega u(\cdot,t)\varphi\, dx = \int_\Omega \varphi\, du_0, \qquad \forall \varphi \in C_b(\Omega), \tag{4.1}$$

*and $u$ is given equivalently by the semigroup formula*

$$u(\cdot,t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot,s)|^q (s)\, ds \qquad \text{in } L^1(\Omega). \tag{4.2}$$

*Moreover, $u \in C^{2,1}(Q_{\Omega,\infty})$, and $u \in C\left(\overline{Q_{\Omega,\epsilon,\infty}}\right)$ for any $\epsilon > 0$. Also, $u$ is the unique weak solution of problem $(D_{\Omega,T})$ for any $T \in (0,\infty)$.*

This solution was obtained from the Banach fixed-point theorem. The existence was also obtained by approximation in [1], from the pioneer results of [13]. Here we give a shorter proof of Theorem 4.1 when $u_0$ is nonnegative, and state precisely the convergence:

**Proposition 4.2.** *Suppose $1 < q < q_*$. Let $u_0 \in \mathcal{M}_b^+(\Omega)$, and let $(u_{0,n})$ be any sequence of functions of $C_b^1(\overline{\Omega}) \cap C_0(\Omega)$ converging weak$^*$ to $u_0$, such that $\|u_{0,n}\|_{L^1(\Omega)} \leqq \|u_0\|_{\mathcal{M}_b(\Omega)}$. Let $u_n$ be the classical solution of $(D_{\Omega,\infty})$ with initial data $u_{0,n}$.*

*Then $(u_n)$ converges in $C^{2,1}_{loc}(Q_{\Omega,\infty}) \cap C^{1,0}_{loc}(\overline{\Omega} \times (0,\infty))$ to a function $u \in L^q_{loc}([0,\infty); W_0^{1,q}(\Omega))$, and $u$ is the unique solution of $(D_{\Omega,T})$, (4.1) for any $T > 0$. Also, $u$ satisfies the estimates (2.9) and (2.8) in $Q_{\Omega,\infty}$.*

**Proof.** There holds

$$u_n(\cdot,t) = e^{t\Delta}u_{0,n} - \int_0^t e^{(t-s)\Delta} |\nabla u_n(\cdot,s)|^q (s)\, ds \qquad \text{in } L^1(\Omega).$$

From estimate (2.9) and Theorem 2.10, since $q<2$, one can extract a subsequence, still denoted $(u_n)$, converging in $C^{2,1}_{loc}(Q_{\Omega,\infty})\cap C^1_{loc}(\overline{\Omega}\times(0,\infty))$ to a weak solution $u$ of $(D_{\Omega,\infty})$. Also,

$$\int_\Omega u_n(\cdot,t)\,dx+\int_0^t\int_\Omega |\nabla u_n(\cdot,s)|^q\,(s)\,dx\,ds-\int_0^t\int_{\partial\Omega}\frac{\partial u_n}{\partial\nu}(\cdot,s)\,dx\,ds=\int_\Omega u_{0,n}\,dx. \tag{4.3}$$

Hence, $|\nabla u_n|^q$ is bounded in $L^1(Q_{\Omega,\infty})$ by $\|u_0\|_{\mathcal{M}_b(\Omega)}$. Then from [5, Lemma 3.3], $(u_n)$ is bounded in $L^\gamma((0,\tau),W^{1,\gamma}_0(\Omega))$ for any $\gamma\in[1,q_*)$. Thus, $(|\nabla u_n|^q)$ converges to $|\nabla u|^q$ in $L^1_{loc}([0,\infty),L^1(\Omega))$, and $\left(e^{t\Delta}u_{0,n}\right)$ converges almost everywhere to $e^{t\Delta}u_0$, and $u$ satisfies (4.2). Moreover, $u$ is the unique solution of $(D_{\Omega,T})$, (4.1). Indeed, let $v$ be any other solution; taking $\gamma\in(q,q_*)$, there holds from [5, Lemma 3.3], with constants $C=C(\gamma,\Omega)$,

$$\begin{aligned}\|\nabla(u-v)\|_{L^\gamma(Q_{\Omega,\tau})} &\leqq C\,\||\nabla u|^q-|\nabla v|^q\|_{L^1(Q_{\Omega,\tau})}\\ &\leqq C(\|\nabla u\|^{q-1}_{L^q(Q_{\Omega,T})}+\|\nabla v\|^{q-1}_{L^q(Q_{\Omega,T})})\,\|\nabla(u-v)\|_{L^q(Q_{\Omega,\tau})}\\ &\leqq C\,\|u_0\|^{\frac{q-1}{q}}_{\mathcal{M}_b(\Omega)}\,\|\nabla(u-v)\|_{L^\gamma(Q_{\Omega,\tau})}\,\tau^{\frac{\gamma-q}{\gamma q}};\end{aligned}$$

hence $v=u$ on $(0,\tau)$ for $\tau\leqq C=C(\gamma,\Omega,u_0)$, and then on $(0,T)$. Then the whole sequence $(u_n)$ converges to $u$. □

**Remark 4.3.** Applying Proposition 4.2 on $(\epsilon,T)$ for $\epsilon>0$, we deduce regularity results: any weak solution $u$ of $(D_{\Omega,T})$ extends as a solution of the problem $(D_{\Omega,\infty})$, and $u\in C^{2,1}(Q_{\Omega,\infty})$, $u\in C\left(\overline{Q_{\Omega,\epsilon,\infty}}\right)$ for any $\epsilon>0$, and $u$ satisfies the universal estimates (2.9) and (2.8) in $Q_{\Omega,\infty}$. In turn $u\in C^{1,0}_{loc}(Q_{\Omega,\infty})$ from Theorem 2.10.

**Notation 4.4.** For any $k>0$, we denote by $u^{k,\Omega}$ the above solution of $(D_{\Omega,\infty})$ with initial data $k\delta_0$.

4.1. **The case $\Omega=\mathbb{R}^N$.** We first show that the function $Y$ constructed in Proposition 3.2 is a VSS:

**Lemma 4.5.** *The function $Y$ is a maximal VSS in $Q_{\mathbb{R}^N,\infty}$, and coincides with the radial self-similar solution constructed in* [27]*. It satisfies*

$$\lim_{t\to 0}\int_{\mathbb{R}^N\setminus B_r} Y(\cdot,t)\,dx=0,\qquad \forall r>0. \tag{4.4}$$

**Proof.** Consider any ball $B_p$ with $p\in\mathbb{N}^*$. We can approximate the function $u^{k,B_p}$ by $u^{k,B_p}_\varepsilon$, a solution with initial data $k\rho_\varepsilon$, where $(\rho_\varepsilon)$ is a sequence of mollifiers with support in $B_\varepsilon\subset B_1$. For any $\eta\in(0,1)$, there holds

$u_\varepsilon^{k,B_p} \leqq Y_\eta$ for $\varepsilon < \eta$. Then we find $u^{k,B_p} \leqq Y$. As a first consequence, $Y \neq 0$, and for any ball $B_r$ such that $r < 1$, taking $\varphi \in C_c(B_r)$ with values in $[0,1]$, such that $\varphi \equiv 1$ on $B_{r/2}$,

$$\underline{\lim}_{t\to 0} \int_{B_r} Y(\cdot,t)\,dx \geqq \lim_{t\to 0} \int_{B_r} u^{k,B_p}(\cdot,t)\varphi\,dx = k; \tag{4.5}$$

thus $Y$ satisfies (1.7). From (3.10), $Y$ is the unique radial self-similar VSS constructed in [27]. It satisfies (4.4), since $Y(x,t) = t^{-a/2} f(t^{-1/2}|x|)$, and $\lim_{r\to\infty} r^{a-N} e^{r^2/4} f(r) > 0$, from [27, Theorem 2.1], which implies (1.6). And $Y$ is a maximal VSS, since $Y$ is greater than any weak solution of (1.1)–(1.2), from Proposition 3.3. □

In [10], a VSS $U$ is obtained as the limit of a sequence of solutions $u^k$ of (1.1) in $Q_{\mathbb{R}^N,\infty}$ with initial data $k\delta_0$, constructed in [9]. The proof is based on difficult estimates of the gradient obtained from the Bernstein technique by derivation of equation, showing that $U$ satisfies (1.8) and is minimal in that class, from [11, Theorem 3.8]. Here we prove again the existence of the $u^k$ and $U$ in a very simple way:

**Lemma 4.6.** (*i*) *For any $k > 0$ there exists a weak solution $u^k$ of* (1.1) *in $Q_{\mathbb{R}^N,\infty}$, such that $u^k \in L^\infty((0,\infty); L^1(\mathbb{R}^N))$ and $|\nabla u^k| \in L^q(Q_{\mathbb{R}^N,\infty})$, with initial data $k\delta_0$, in the weak sense of $\mathcal{M}_b(\mathbb{R}^N)$,*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u^k(\cdot,t)\psi\,dx = k\psi(0), \qquad \forall \psi \in C_b(\mathbb{R}^N), \tag{4.6}$$

*and $u^k = \sup_{p\in\mathbb{N}^*} u^{k,B_p}$, where $u^{k,B_p}$ is the solution of the Dirichlet problem $(D_{B_p,\infty})$ with initial data $k\delta_0$.*

(*ii*) *As $k \to \infty$, $u^k$ converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a VSS $U$ in $Q_{\mathbb{R}^N,\infty}$.*

**Proof.** (i) Let $k > 0$ be fixed. Consider the sequence $\left(u^{k,B_p}\right)_{p\geqq 1}$ and notice that it is nondecreasing. We have, from Proposition 3.2,

$$u^{k,B_p}(\cdot,t) \leqq Y(\cdot,t) \leqq C(1 + t^{-\frac{1}{q-1}}). \tag{4.7}$$

From Theorem 2.9 the sequence converges in $C^{2,1}_{loc}(Q_{\Omega,\infty})$ to a solution $u^k$ of equation (1.1) in $Q_{\mathbb{R}^N,\infty}$, and $u^k \leqq Y$; thus $u^k$ satisfies (1.6) from (3.5). Moreover, for any $t > 0$, there holds $\int_{B_p} u^{k,B_p}(\cdot,t)\,dx \leqq k$ from (4.2); then $\int_{\mathbb{R}^N} u^k(\cdot,t)\,dx \leqq k$ from the Fatou lemma. In turn, from Proposition 2.15, $u^k(\cdot,t)$ converges weak* to a Radon measure $\mu$, concentrated at 0; then $\mu =$

$k'\delta_0$, $k' > 0$. Otherwise, $u^{k,B_p} \leqq u^k$; then $\int_{B_p} u^{k,B_p}(\cdot,t)\,dx \leqq \int_{\mathbb{R}^N} u^k(\cdot,t)\,dx$. Thus from (4.5)

$$k \leqq \liminf_{t\to 0} \int_{\mathbb{R}^N} u^k(\cdot,t)\,dx;$$

then

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u^k(\cdot,t)\,dx = k.$$

Taking $\varphi_p \in \mathcal{D}^+(\mathbb{R}^N)$, with values in $[0,1]$, such that $\varphi_p = 1$ on $B_p$, we get

$$\int_{B_p} u^{k,B_p}(\cdot,t)\,dx \leqq \int_{\mathbb{R}^N} u^k(\cdot,t)\varphi_p\,dx \leqq \int_{\mathbb{R}^N} u^k(\cdot,t)\,dx;$$

hence $k' = k$. Thus $u^k(\cdot,t)$ converges weak* to $k\delta_0$ as $t \to 0$. In fact, the convergence holds in the weak sense of $\mathcal{M}_b(\mathbb{R}^N)$. Indeed, for any $\psi \in C_b^+(\mathbb{R}^N)$, using a function $\varphi \in C_c(\mathbb{R}^N)$ with values in $[0,1]$ such that $\varphi \equiv 1$ on a ball $B_r$, we can write

$$\int_{\mathbb{R}^N} u^k(\cdot,t)\psi\,dx = \int_{\mathbb{R}^N} u^k(\cdot,t)\psi\varphi\,dx + \int_{\mathbb{R}^N} u^k(\cdot,t)\psi(1-\varphi)\,dx$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} u^k(\cdot,t)\psi(1-\varphi)\,dx &\leqq \|\psi\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N\setminus B_r} u^k(\cdot,t)\,dx \\ &\leqq \|\psi\|_{L^\infty(\Omega)} \int_{\mathbb{R}^N\setminus B_r} Y(\cdot,t)\,dx,\end{aligned}$$

and the right-hand side tends to 0 from (4.4). From (4.3), we find

$$\left\| \left|\nabla u_\varepsilon^{k,B_p}\right|^q \right\|_{L^1(Q_{B_p,\infty})} \leqq k\,\|\rho_\varepsilon\|_{L^1(B_p)} = k;$$

hence $\left\| \left|\nabla u^{k,B_p}\right|^q \right\|_{L^1(Q_{B_p,\infty})} \leqq k$, and finally $\left\| \left|\nabla u^k\right|^q \right\|_{L^1(Q_{\mathbb{R}^N,\infty})} \leqq k$.

(ii) From (4.7) or from Proposition (3.3), there holds

$$u^k(\cdot,t) \leqq Y(\cdot,t) \leqq C(1 + t^{-\frac{1}{q-1}}).$$

From Theorem 2.9, $u^k$ converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a weak solution $U$ of equation (1.1). Then $u^k \leqq U \leqq Y$; thus $U$ satisfies (1.7) and (4.4) as $Y$. Hence $U$ is a VSS in $Q_{\mathbb{R}^N,\infty}$. □

Next we prove the uniqueness of the VSS:

**Proof of Theorem 1.3.** Let us show that $U$ is minimal among all the VSS. Any VSS $u$ in $Q_{\mathbb{R}^N,\infty}$ satisfies $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty}) \cap C((0,\infty); C_b^2(\mathbb{R}^N))$ and $u \leqq Y$, from Proposition 3.3 and (3.4). For fixed $k > 0$ and $p > 1$, one constructs a sequence of functions $u_{0,n}^k \in \mathcal{D}^+(\mathbb{R}^N)$ with support in $B_1$ such that

$$u_{0,n}^k \leqq u(\cdot, \tfrac{1}{n}) \quad \text{in } \mathbb{R}^N, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} u_{0,n}^k \, dx = k.$$

Indeed, $\|u(\cdot,1/n)\|_{L^1(\mathbb{R}^N)}$ tends to $\infty$; then, for $n$ large enough, there exists $s_{n,k} > 0$ such that $\left\|T_{s_{n,k}}(u)(\cdot,1/n)\right\|_{L^1(\mathbb{R}^N)} = k$. And

$$\varepsilon_n = \|u(\cdot,1/n)\|_{L^1(\mathbb{R}^N\setminus B_1)} + \|u(\cdot,1/n)\|_{L^\infty(\mathbb{R}^N\setminus B_1)}$$

tends to 0, from (4.4) and (3.5). Then $v_n^k = (T_{s_{n,k}}(u)(\cdot,1/n) - 2\varepsilon_n)^+$ has a compact support in $B_1$, and we can take for $u_{0,n}^k$ a suitable regularization of $v_n^k$. Let us call $u_n^{k,B_p}$ the solution of $(D_{B_p,\infty})$ with initial data $u_{0,n}^k$. Then $u_n^{k,B_p}(\cdot,t) \leqq u(\cdot,t+1/n)$ from the comparison principle. As $n \to \infty$, $u_{0,n}^k$ converges to $k\delta_0$ weakly in $\mathcal{M}_b(B_p)$, since for any $\psi \in C_b^+(B_p)$, and any $r \in (0,1)$,

$$\Big|\int_{B_p} u_{0,n}^k \psi \, dx - k\psi(0)\Big| \leqq \psi(0)\Big|\int_{B_p} (u_{0,n}^k - k)\, dx\Big|$$
$$+ 2\|\psi\|_{L^\infty(\mathbb{B}_p)} \int_{\mathbb{R}^N\setminus B_r} u(\cdot,\frac{1}{n})\, dx + \sup_{B_r} |\psi - \psi(0)| \int_{\mathbb{R}^N} u_{0,n}^k \, dx.$$

Then $u_n^{k,B_p}$ converges to $u^{k,B_p}$ from Proposition 4.2, and $u^{k,B_p} \leqq u$. From Lemma 4.6, we get $u^k \leqq u \leqq Y$. As $k \to \infty$, we deduce that $U \leqq u \leqq Y$. Moreover, $U$ is radial and self-similar; then $U = Y = u$ from [27]. □

Finally, we describe all the solutions:

**Proof of Theorem 1.4.** Let $u$ be any weak solution of (1.1) and (1.6). Either (1.7) holds (then $u = Y$), or there exists a ball $B_r$ such that $\int_{B_r} u(\cdot,t)\, dx$ stays bounded as $t \to 0$. Then $u \in L^\infty_{loc}([0,T); L^1_{loc}(\mathbb{R}^N))$, from Corollary 2.18. From Proposition 2.15, $u(\cdot,t)$ converges weak* to a measure $\mu$ as $t \to 0$. Then $\mu$ is concentrated at 0 from (1.6); hence there exists $k \geqq 0$ such that $\mu = k\delta_0$, and (1.12) holds as in Lemma 4.6, since $u \leqq Y$. If $k = 0$, then $u \equiv 0$ from Theorem 1.2.

Next we show the uniqueness, namely that of $u = u^k$ constructed in Lemma 4.6. *Here only* we use the gradient estimates obtained by the Bernstein technique. We have $u \in C((0,\infty); C_b^2(\mathbb{R}^N))$ from Proposition (3.3), and

$u \in L^\infty((0,\infty);L^1(\mathbb{R}^N))$ from (3.2) or (4.4); thus $u \in C((0,\infty);L^1(\mathbb{R}^N))$. From [9] and [8], for any $\epsilon > 0$, and any $t \geqq \epsilon$, we have the semigroup formula

$$u(\cdot,t) = e^{(t-\epsilon)\Delta}u(\cdot,\epsilon) - \int_\epsilon^t e^{(t-s)\Delta} |\nabla u|^q (s)\, ds \qquad \text{in } L^1(\mathbb{R}^N), \tag{4.8}$$

and there exists $C(q)$ such that for any $t > 0$,

$$|\nabla u(\cdot,t)|^q \leqq C(q)(t-\epsilon)^{-1}u(\cdot,t)$$

and $u \leqq Y$; then as $\epsilon \to 0$ we obtain

$$\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(q)t^{-1/q} \|Y(\cdot,t)\|^{1/q}_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-(N+2)/2q},$$

where $C = C(N,q)$.

From (1.12) and (4.8) there holds $|\nabla u|^q \in L^1_{loc}([0,\infty)\,;L^1(\mathbb{R}^N))$. Otherwise $e^{(t-\epsilon)\Delta}u(x,\epsilon)$ converges to $kg$ in $\mathcal{C}'_b(\mathbb{R}^N)$, where $g$ is the heat kernel; then

$$u(\cdot,t) = kg - \int_0^t e^{(t-s)\Delta} |\nabla u|^q (s)\, ds \qquad \text{in } \mathcal{C}'_b(\mathbb{R}^N).$$

Then

$$(u-u^k)(\cdot,t) = -\int_0^t e^{(t-s)\Delta}(|\nabla u|^q - |\nabla u^k|^q)(s)\, ds \qquad \text{in } L^1(\mathbb{R}^N),$$

$$\begin{aligned} &\left\|\nabla(u-u^k)(\cdot,t)\right\|_{L^q(\mathbb{R}^N)} \\ &\leqq \int_0^t \left\|\nabla e^{(t-s)\Delta}\right\|_{L^1(\mathbb{R}^N)} \left\||\nabla u(\cdot,s)|^q - \left|\nabla u^k(\cdot,s)\right|^q\right\|_{L^q(\mathbb{R}^N)} ds \\ &\leqq C\int_0^t (t-s)^{-1/2}s^{-(q-1)(N+2)/2q} \left\|\nabla(u-u^k)(\cdot,s)\right\|_{L^q(\mathbb{R}^N)} ds. \end{aligned}$$

Thus $\nabla(u-u^k)(\cdot,t) = 0$ in $L^q(\mathbb{R}^N)$, from the singular Gronwall lemma, valid since $q < \frac{N+2}{N+1}$; hence $u = u^k$. □

**Remark 4.7.** This uniqueness result is a special case of a general one given for measure data in [12, Theorem 3.27].

4.2. **The Dirichlet problem** $(D_{\Omega,\infty})$**.** Here $\Omega$ is bounded, and we consider the weak solutions of the problem $(D_{\Omega,\infty})$ such that

$$\lim_{t\to 0}\int_\Omega u(\cdot,t)\varphi\, dx = 0, \qquad \forall\varphi \in C_c(\overline{\Omega}\setminus \{0\}). \tag{4.9}$$

First, we give regularity properties of these solutions.

**Lemma 4.8.** *Any weak solution $u$ of $(D_{\Omega,\infty})$, (4.9), in $Q_{\Omega,\infty}$ satisfies*

$$u \in C^{1,0}(\overline{\Omega}\backslash\{0\} \times [0,\infty)) \cap C^{1,0}(\overline{\Omega} \times (0,\infty)) \cap C^{2,1}(Q_{\Omega,\infty}).$$

**Proof.** We know that $u \in C^{1,0}(\overline{\Omega} \times (0,\infty)) \cap C^{2,1}(Q_{\Omega,\infty})$; see Remark 4.3. Moreover, $u \in C^{2,1}(\Omega_0 \times [0,\infty))$ and $u(x,0) = 0$, $\forall x \in \Omega_0$, from Corollary 2.18. Let $B_\eta \subset\subset \Omega$ be fixed, and $\Omega_\eta = \Omega\backslash\overline{B_\eta}$. Then $u \in C^1(\partial B_\eta \times [0,\infty))$; thus for any $T \in (0,\infty)$, there exists $C_\tau > 0$ such that $u(\cdot,t) \leqq C_\tau t$ on $\partial B_\eta \times [0,T)$. Then the function $w = u - C_\tau t$ solves

$$w_t - \Delta w = -|\nabla u|^q - C_\tau \qquad \text{in } \mathcal{D}'\left(Q_{\Omega_\eta,T}\right),$$

and then $w^+ \in C((0,T); L^1(\Omega_\eta) \cap L^1_{loc}((0,T); W^{1,1}_0(\Omega_\eta))$, and

$$w^+_t - \Delta w^+ \leqq 0 \qquad \text{in } \mathcal{D}'\left(Q_{\Omega_\eta,T}\right)$$

from the Kato inequality.

Moreover, from assumption (4.9), $w^+ \in L^\infty((0,T); L^1(\Omega_\eta))$ and $w^+(\cdot,t)$ converges to 0 in the weak sense of $\mathcal{M}_b(\Omega_\eta)$ as $t \to 0$. As a consequence, $w \leqq 0$, from [5, Lemma 3.4]; thus $u(\cdot,t) \leqq C_T t$ in $\Omega_{\eta,T}$. Then the function $\overline{u}$ defined by (2.16) is bounded in $Q_{\Omega_\eta,\tau}$. Hence $\overline{u} \in C^{1,0}(\overline{\Omega_\eta} \times (-T,T))$ from Theorem 2.10; thus $u \in C^{1,0}(\overline{\Omega}\backslash\{0\} \times [0,\infty))$. □

**Definition 4.9.** *Let $T \in (0,\infty]$. We call VSS in $Q_{\Omega,T}$ any weak solution $u$ of the Dirichlet problem $(D_{\Omega,T})$, (4.9), such that*

$$\lim_{t\to 0} \int_{B_r} u(\cdot,t)\,dx = \infty, \qquad \forall B_r \subset \Omega. \tag{4.10}$$

**Remark 4.10.** From Remark 4.3, any VSS in $Q_{\Omega,T}$ extends as a VSS in $Q_{\Omega,\infty}$, and satisfies (2.9) and (2.8).

Next we prove the existence and uniqueness of the VSS. Our proof is based on the uniqueness of the VSS in $\mathbb{R}^N$, and does not use the uniqueness of the function $u^k$.

**Proof of Theorem 1.5.** (i) *Existence of a minimal VSS.* For any $k > 0$ we consider the solution $u^{k,\Omega}$ of $(D_{\Omega,\infty})$ with initial data $k\delta_0$. By regularization as in Lemma 4.6, we obtain that $u^{k,\Omega} \leqq Y$. The sequence $\left(u^{k,\Omega}\right)$ is nondecreasing. From estimate (2.9) and Theorem 2.10, $\left(u^{k,\Omega}\right)$ converges in $C^{2,1}_{loc}(Q_{\Omega,\infty}) \cap C^{1,0}_{loc}(\overline{\Omega} \times (0,\infty))$ to a weak solution $U^\Omega$ of $(D_{\Omega,\infty})$, and then $U^\Omega \leqq Y$. Hence $U^\Omega$ satisfies (4.10), and (4.9) from (4.4); thus $U^\Omega$ is a VSS in $\Omega$. Next we show that $U^\Omega$ is minimal. Consider any VSS $u$ in $Q_{\Omega,\infty}$. Let

$k > 0$ be fixed. As in the proof of Theorem 1.3, one constructs a sequence $u_n^{k,\Omega}$ of solutions of $(D_{\Omega,\infty})$ with initial data functions $u_{0,n}^{k,\Omega} \in \mathcal{D}(\Omega)$ such that

$$0 \leqq u_{0,n}^{k,\Omega} \leqq u(\cdot, \frac{1}{n}) \quad \text{in } \Omega, \qquad \lim_{n\to\infty} \int_\Omega u_{0,n}^{k,\Omega}\, dx = k.$$

We still find $u_n^{k,,\Omega}(\cdot,t) \leqq u(\cdot, t+1/n)$ from the comparison principle, valid from Lemma 4.8. As $n \to \infty$, $u_{0,n}^{k,\Omega}$ converges to $k\delta_0$ weakly in $\mathcal{M}_b(\Omega)$; then $u_n^{k,\Omega}$ converges to $u^{k,\Omega}$ from Proposition 4.2. Thus $u^{k,\Omega} \leqq u$ for any $k > 0$; hence $U^\Omega \leqq u$.

(ii) *Existence of a maximal VSS.* For any ball $B_\eta \subset\subset \Omega$, we consider the function $Y_\eta^\Omega$ defined in Theorem 2.12. Consider again any VSS $u$ in $\Omega$, and follow the proof of Proposition 3.3, replacing $B_r$ by $\Omega$. Let $\varepsilon > 0$ be fixed. From Lemma 4.8, for any ball $B_\eta \subset\subset \Omega$, setting $\Omega_\eta = \Omega \backslash \overline{B_\eta}$ there exists $\delta_\eta > 0$ such that $u(x,t) < \varepsilon$ in $Q_{\Omega_\eta,\delta_\eta}$. For any $\delta \in (0,\delta_\eta)$, from the comparison principle in $Q_{\Omega,\delta,\tau}$ we obtain

$$u(x,t) \leqq Y_{2\eta}^\Omega(x, t-\delta) + \varepsilon \quad \text{in} \quad Q_{\Omega,\delta,\tau}.$$

As $\delta$ tends to 0, and then $\varepsilon \to 0$, we deduce that $u \leqq Y_{2\eta}^\Omega$ in $Q_{\Omega,\infty}$. Note that $Y_\eta^\Omega \leqq Y_{\eta'}^\Omega$ for any $\eta \leqq \eta'$. From the estimate (2.9) and Theorem 2.9, $Y_\eta^\Omega$ converges in $C_{loc}^{1,0}(\overline{\Omega} \times (0,\infty))$ to a classical solution $Y^\Omega$ of $(D_{\Omega,\infty})$, and $u \leqq Y^\Omega$. Moreover, $Y^\Omega$ satisfies (4.10), since $Y^\Omega \geqq U$, and (4.9) since $Y^\Omega \leqq Y$; then $Y^\Omega$ is a maximal VSS in $\Omega$.

(iii) *Uniqueness.* For fixed $k > 0$, we intend to compare $u^{k,\Omega}$ with $u^k$, by approximation. Let $0 < \eta < r$ be fixed such that $B_r \subset\subset \Omega$. Consider again the function $Y_\eta$ defined by (3.3). Let $\delta > 0$ be fixed. From (3.10), there exists $\tau_\delta > 0$ such that $\sup_{(\mathbb{R}^N \backslash B_r)\times[0,\tau_\delta]} Y_\eta \leqq \delta$. Let $(\rho_\varepsilon)$ be a sequence of mollifiers with support in $B_\varepsilon \subset B_\eta$. Let $u_\varepsilon^{k,\Omega}$ be the solution of $(D_{\Omega,\infty})$ in $Q_{\Omega,\infty}$ with initial data $k\rho_\varepsilon$. For any $p > 1$ such that $\Omega \subset B_p$, let $u_\varepsilon^{k,B_p}$ be the solution of $(D_{B_p,\infty})$ with the same initial data. By definition of $Y_\eta^{B_p}$ and $Y_\eta$, there holds $u_\varepsilon^{k,B_p} \leqq Y_\eta^{B_p} \leqq Y_\eta$; hence $\sup_{\partial\Omega\times[0,\tau_\delta]} u_\varepsilon^{k,B_p} \leqq \delta$. From the comparison principle we find

$$u_\varepsilon^{k,B_p} \leqq u_\varepsilon^{k,\Omega} + \delta \quad \text{in} \quad \overline{\Omega} \times [0,\tau_\delta]\,.$$

Going to the limit as $\varepsilon \to 0$ from Proposition 4.2, then as $p \to \infty$ from Lemma 4.6, then as $k \to \infty$, we find $U \leqq U^\Omega + \delta$ in $\overline{\Omega} \times (0,\tau_\delta]$. The function $W^\Omega = Y^\Omega - U^\Omega \in C^{1,0}(\overline{\Omega} \backslash \{0\} \times [0,\infty)) \cap C^{1,0}(\overline{\Omega} \times (0,\infty))$ from Lemma 4.8, and $W^\Omega = 0$ on $\partial\Omega \times [0,\infty)$. Since $Y^\Omega \leqq Y = U$, $W^\Omega \leqq \delta$ in $\overline{\Omega} \times (0,\tau_\delta]$.

Thus $W^{\Omega}(\cdot,t)$ converges uniformly to 0 as $t \to 0$. Then from the comparison principle, for any $\epsilon \in (0,\delta)$, $\sup_{\overline{\Omega}\times[\epsilon,T)} W^{\Omega} \leqq \max_{\overline{\Omega}} W^{\Omega}(\cdot,\epsilon)$; thus $W^{\Omega}=0$, and hence $Y^{\Omega}=U^{\Omega}$. □

Finally, we describe all the solutions as in the case of $\mathbb{R}^N$:

**Theorem 4.11.** *Let $u$ be any weak solution of* $(D_{\Omega,\infty})$, (4.9)*. Then either $u=U^{\Omega}$, or there exists $k>0$ such that $u=u^{k,\Omega}$, or $u\equiv 0$.*

**Proof.** Either $u=Y^{\Omega}$, or there exists a ball $B_r$ such that $\int_{B_r} u(\cdot,t)\,dx$ stays bounded as $t \to 0$. Then from (4.9), $u \in L^{\infty}_{loc}([0,\infty)\,;L^1(\Omega))$. From Proposition 2.15, $u(\cdot,t)$ converges weak* to a measure $\mu$ as $t \to 0$, concentrated at $\{0\}$ from (4.9). Hence the exists $k \geqq 0$ such that $\mu = k\delta_0$; thus

$$\lim_{t\to 0}\int_{\Omega} u(\cdot,t)\varphi\,dx = k\varphi(\cdot,0), \qquad \forall \varphi \in C_c(\Omega),$$

and it holds for any $\varphi \in C_b(\Omega)$, from (4.9). If $k>0$, then $u=u^{k,\Omega}$ from uniqueness; see Proposition 4.2. If $k=0$, then $u\equiv 0$ from Theorem 1.2. □

## References


[1] M. Alaa, *Solutions faibles d'équations paraboliques quasilinéaires avec données mesures*, Ann. Math. Blaise Pascal, 3 (1996), 1–15.

[2] L. Amour and M. Ben-Artzi, *Global existence and decay for viscous Hamilton–Jacobi equations,* Nonlinear Analysis, Methods and Applications, 31 (1998), 621–628.

[3] F. Andreu, J. Mazon, S. Segura de Leon, and J. Toledo, *Existence and uniqueness for a degenerate parabolic equation with $l^1$ data*, Trans. Amer. Math. Soc., 351 (1999), 285–306.

[4] D.G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rat. Mech. Anal., 25 (1967), 81–122.

[5] P. Baras and M. Pierre, *Problemes paraboliques semi-linéaires avec données mesures,* Applicable Anal., 18 (1984), 111–149.

[6] M. Ben Artzi, P. Souplet, and F. Weissler, *The local theory for viscous Hamilton–Jacobi equations in Lebesgue spaces*, J. Math. Pures Appl., 81 (2002), 343–378.

[7] S. Benachour and S. Dabuleanu, *The mixed Cauchy–Dirichlet problem for a viscous Hamilton–Jacobi equation*, Adv. Differential Equations, 8 (2003), 1409–1452.

[8] S. Benachour, M. Ben Artzi, and P. Laurençot, *Sharp decay estimates and vanishing viscosity for diffusive Hamilton–Jacobi equations,* Adv. Differential Equations, 14 (2009), 1–25.

[9] S. Benachour and P. Laurençot, *Global solutions to viscous Hamilton–Jacobi equations with irregular initial data,* Comm. Partial Differential Equations, 24 (1999), 1999–2021.

[10] S. Benachour and P. Laurençot, *Very singular solutions to a nonlinear parabolic equation with absorption, I—Existence,* Proc. Roy. Soc. Edinburgh, Sect. A, 131 (2001), 27–44.

[11] S. Benachour, H. Koch, and P. Laurençot, *Very singular solutions to a nonlinear parabolic equation with absorption,* II—Uniqueness, Proc. Roy. Soc. Edinburgh, Sect. A, 134 (2004), 39–54.

[12] M.F. Bidaut-Véron and A.N. Dao, *$L^\infty$ estimates and uniqueness results for nonlinear parabolic equations with gradient absorption terms,* Arxiv N° 1202 2674.

[13] L. Boccardo and T. Gallouet, *Nonlinear elliptic and parabolic equations involving measure data,* J. Funct. Anal., 87 (1989), 149–169.

[14] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions,* J. Math. Pures Appl., 62 (1983), 73–97.

[15] H. Brezis, L.A. Peletier, and D. Terman, *A very singular solution of the heat equation with absorption,* Arch. Ration. Mech. Analysis, 95 (1986), 185–209.

[16] P. Cannarsa and P. Cardaliaget, *Hölder estimates in space-time for viscosity solutions of Hamilton–Jacobi equations,* Comm. Pure Appl. Math., 63 (2010) 590–629.

[17] M. Crandall, P. Lions, and P. Souganidis, *Maximal solutions and universal bounds for some partial differential equations of evolution,* Arch. Rat. Mech. Anal., 105 (1989), 163–190.

[18] B. Gilding, M. Guedda, and R. Kersner, *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$,* J. Math. Anal. Appl., 284 (2003), 733–755.

[19] S. Kamin and L.A. Peletier, *Singular solutions of the heat equation with absorption,* Proc. Amer. Math. Soc., 95 (1985), 205–210.

[20] G. Lieberman, "Second Order Parabolic Differential Equations," World Scientific Publishing Co. Pte. Ltd., 1996.

[21] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'Ceva, "Linear and Quasilinear Equations of Parabolic Type," Transl. Math. Monogr., 23, Amer. Math. Soc., Providence, 1968.

[22] M. Marcus and L.Véron, *Initial trace of positive solutions of some nonlinear parabolic equations,* Comm. Part. Diff. Equ., 24 (1999), 1445–1499.

[23] I. Moutoussamy and L. Véron, *Source type positive solutions of nonlinear parabolic inequalities,* Ann. Normale Sup. Di Pisa, 4 (1989), 527–555.

[24] L. Oswald, *Isolated positive singularities for a nonlinear heat equation*, Houston J. Math., 14 (1988), 543–572.

[25] A. Prignet, *Existence and uniqueness of "entropy" solutions of parabolic prolems with $L^1$ data,* Nonlinear Anal., 28 (1997), 1943–1954.

[26] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of trucations,* Ann. Mat. Pura Appl., 177 (1999), 143–172.

[27] Y. Qi and M. Wang, *The self-similar profiles of generalized KPZ equation,* Pacific J. Math., 201 (2001), 223–240.

[28] P. Souplet and Q. Zhang, *Global solutions of inhomogeneous Hamilton–Jacobi equations,* J. Anal. Math., 99 (2006), 355–396.

[29] N. Trudinger, *Pointwise estimates and quasilinear parabolic equations,* Comm. Part. Diff. Equ., 21 (1968), 205–226.