Advances in Differential Equations Volume 17, Numbers 9-10 (2012), 879–902

ASYMPTOTIC STABILITY OF SECOND-ORDER EVOLUTION EQUATIONS WITH INTERMITTENT DELAY

Serge Nicaise
Institut des Sciences et Techniques de Valenciennes
Laboratoire de Mathématiques et leurs Applications
Université de Valenciennes et du Hainaut Cambrésis
59313 Valenciennes Cedex 9, France

Cristina Pignotti
Dipartimento di Matematica Pura e Applicata
Università di L'Aquila
Via Vetoio, Loc. Coppito, 67010 L'Aquila Italy

(Submitted by: Viorel Barbu)

Abstract. We consider second-order evolution equations in an abstract setting with intermittently delayed/not-delayed damping and give sufficient conditions ensuring asymptotic and exponential stability results. Our abstract framework is then applied to the wave equation, the elasticity system, and the Petrovsky system. For the Petrovsky system with clamped boundary conditions, we further prove an internal observability estimate that was not available in the literature.

1. Introduction

Let H be a real Hilbert space and let $A : \mathcal{D}(A) \to H$ be a positive self-adjoint operator with a compact inverse in H. Denote by $V := \mathcal{D}(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$. Moreover, let U be a real Hilbert space and let $B_i(t) \in \mathcal{L}(U, H)$, $i = 1, 2$, be time-dependent operators satisfying

$$B_1(t)B_2(t) = 0, \quad \forall t > 0.$$

Let us consider the problem

$$u_{tt}(t) + Au(t) + B_1(t)B_1^*(t)u_t(t) + B_2(t)B_2^*(t)u_t(t-\tau) = 0, \quad t > 0 \tag{1.1}$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1, \tag{1.2}$$

where the constant $\tau > 0$ is the time delay.

Accepted for publication: May 2012.
AMS Subject Classifications: 35L05, 93D15.

We are interested in giving stability results for such a problem under suitable assumptions on the feedback operators B_1 and B_2.

Time delays are often present in applications and practical problems and it is by now well known that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly exponentially stable in absence of delay. For some examples in this sense we refer to [4, 5, 17, 21].

In [17] we considered the wave equation with both dampings acting simultaneously, that is, $B_1(t) = \mu_1$ and $B_2(t) = \mu_2$, with $\mu_1, \mu_2 \in \mathbb{R}^+$, and we proved that if $\mu_1 > \mu_2$ then the system is uniformly exponentially stable. Otherwise, if $\mu_2 \geq \mu_1$, that is, the delay term prevails on the not-delayed one, then there are instability phenomena, namely there are unstable solutions for arbitrarily small delays.

Here, we consider second-order evolution equations with intermittent delay; this means that the standard damping and the delayed one act in different time intervals. This is clearly related to the stabilization problem of second-order evolution equations damped by on/off feedbacks. We refer for this subject to [8]. See also [18] for the link between the wave equation with time delay in the damping and the wave equation with indefinite damping.

Assuming that an observability inequality holds for the conservative model associated with (1.1)–(1.2) and, through the definition of a suitable energy (see (3.2)), we obtain sufficient conditions ensuring asymptotic stability. Under more restrictive assumptions an exponential stability estimate is also obtained. Our abstract framework is then applied to some concrete examples, namely the wave equation, the elasticity system, and the Petrovsky system.

Our argument relies on the fact that on the time intervals without delay, where a standard dissipative damping acts, the energy is decreasing and it satisfies a suitable estimate. On the remaining intervals, where time delay is present, the energy is not decreasing in general; however, its derivative satisfies a certain bound. Combining these estimates we can obtain stability results under appropriate conditions. Roughly speaking, in order to have stability we assume that the total length of the time intervals with delay or/and the total *size* of the delayed dampings is small, in a suitable sense, with respect to the length of the intervals without time delay or/and the size of the corresponding dampings.

A similar problem has been considered in [1] for 1-d models for the wave equation but with a different approach. Indeed, here we give stability results under conditions that allow us to compensate for the destabilizing delay effect with the *good* behaviour of the system in the time intervals without delay. On the contrary, in [1] we obtain stability results for particular values of the time delays, related to the length of the domain, by using the D'Alembert formula.

Our stability results are crucially based on the existence of an observability inequality for the conservative problem associated with the damped model. For the wave equation with Dirichlet boundary conditions, the elasticity system with Dirichlet boundary conditions, and the Petrovsky system with hinged boundary conditions, such locally internal observability estimates are available in the literature. In the case of the Petrovsky system with clamped boundary conditions, we will prove an observability inequality for the associated conservative system since, to our best knowledge, it is not present in the literature. For the wave equation with distributed damping, we can obtain an additional asymptotic stability result (Theorem 4.6) by using an energy decay estimate on a short time interval (see Theorem 3.1 of [8]).

The paper is organized as follows. In Section 2 a well-posedness result of the abstract system is proved. In Section 3 we obtain asymptotic and exponential stability results for the abstract model under suitable conditions. Finally, in Sections 4, 5, and 6 we apply our abstract results to the wave equation, the elasticity system, and the Petrovsky system respectively.

2. Well-posedness

In this section we will give well-posedness results for problem (1.1)–(1.2) using semigroup theory.

Now we assume that for all $n \in \mathbb{N}$, there exists $t_n > 0$ with $t_n < t_{n+1}$ and such that

$$B_2(t) = 0 \ \forall \ t \in I_{2n} = [t_{2n}, t_{2n+1}),$$
$$B_1(t) = 0 \ \forall \ t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),$$

with $B_1 \in C^1([t_{2n}, t_{2n+1}]; \mathcal{L}(U,H))$ and $B_2 \in C([t_{2n+1}, t_{2n+2}]; \mathcal{L}(U,H))$. We further assume that $\tau \leq T_{2n}$ for all $n \in \mathbb{N}$, where T_n denotes the length of the interval I_n; that is,

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}. \tag{2.1}$$

Under these assumptions, we obtain the following result:

Theorem 2.1. *Under the above assumptions, for any $u_0 \in V$ and $u_1 \in H$, the system* (1.1)–(1.2) *has a unique solution* $u \in C([0,\infty); V) \cap C^1([0,\infty); H)$.

Proof. We prove the existence result on the interval $(0, t_2)$; the global existence result follows by translation. First, in the interval $(0, t_1)$, since B_2

is zero, the system (1.1)–(1.2) is a second-order problem with a bounded damping term that depends "smoothly" on time. Hence for initial data $u_0 \in V$ and $u_1 \in H$, the existence of a solution $u \in C([0,t_1];V) \cap C^1([0,t_1];H)$ is guaranteed by checking that the triplet $\{\mathcal{A}, V \times H, Y\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,t_1]\}$ and $Y = D(\mathcal{A}(0)) = D(A) \times V$ forms a CD-system (or constant domain system) in the sense of Kato (see [9, 10]), where

$$\mathcal{A}(t)(u,v)^\top = (v, -Au - B_1(t)B_1^*(t)v)^\top \quad \forall (u,v)^\top \in Y.$$

The situation is more delicate on (t_1,t_2). In that case, we decompose (t_1,t_2) into the successive intervals $(t_1+\ell\tau, t_1+(\ell+1)\tau)$, for ℓ from 0 to L where L is such that $t_1+(L+1)\tau \geq t_2$. The last interval is then reduced to $(t_1+L\tau, t_2)$.

As a consequence on the interval $(t_1, t_1+\tau)$, problem (1.1)–(1.2) can be written as

$$\begin{aligned} &u_{tt}(t) + Au(t) = -B_2(t)B_2^*(t)u_t(t-\tau) \quad t \in (t_1, t_1+\tau),\\ &u(t_1+) = u(t_1-) \quad \text{and} \quad u_t(t_1+) = u_t(t_1-). \end{aligned}$$

Since $(u(t_1-), u_t(t_1-))$ belongs to $V \times H$ and since $B_2(t)B_2^*(t)u_t(\cdot - \tau) \in C((t_1, t_1+\tau); H)$, we get a unique solution $u \in C^1([t_1,t_1+\tau];H) \cap C([t_1,t_1+\tau];V)$ of this problem.

By iteration, we get a solution on the interval $(t_1+\tau, t_1+2\tau)$ and then on the whole interval (t_1,t_2). □

3. Stability result

To get stability we assume that there exists a Hilbert space W such that H is continuously embedded into W; i.e.,

$$\|u\|_W^2 \leq C\|u\|_H^2, \quad \forall u \in H \text{ with } C>0 \text{ independent of } u. \tag{3.1}$$

Moreover, we assume that for all $n \in \mathbb{N}$, there exist three positive constants m_{2n}, M_{2n}, and M_{2n+1} with $m_{2n} \leq M_{2n}$ and such that for all $u \in H$, we have

i) $m_{2n}\|u\|_W^2 \leq \|B_1^*(t)u\|_U^2 \leq M_{2n}\|u\|_W^2$ for $t \in I_{2n} = [t_{2n}, t_{2n+1})$, $\forall\, n \in \mathbb{N}$;

ii) $\|B_2^*(t)u\|_U^2 \leq M_{2n+1}\|u\|_W^2$ for $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, $\forall\, n \in \mathbb{N}$.

Let us introduce the energy of the system

$$E(t) = E(u;t) := \frac{1}{2}\Big(\|u(t)\|_V^2 + \|u_t(t)\|_H^2 + \int_{t-\tau}^t \|B_2^*(s+\tau)u_t(s)\|_U^2\, ds\Big), \tag{3.2}$$

which is the standard energy for wave-type equations plus an integral term due to the presence of a time delay.

Let us assume

$$m_{2n} > \frac{M_{2n+1}}{2}, \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Then, we have the following estimates.

Proposition 3.1. *Assume* i), ii), *and* (3.3). *For any regular solution of problem* (1.1)–(1.2) *the energy is decreasing on the intervals* I_{2n}, $n \in \mathbb{N}$, *and*

$$E'(t) \le -\Big(m_{2n} - \frac{M_{2n+1}}{2}\Big)\|u_t\|_W^2. \tag{3.4}$$

Moreover, on the intervals I_{2n+1}, $n \in \mathbb{N}$,

$$E'(t) \le M_{2n+1}\|u_t\|_W^2. \tag{3.5}$$

Proof. Differentiating $E(t)$, we get

$$E'(t) = (u_t, u)_V + (u_{tt}, u_t)_H + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_U^2.$$

Hence using the definition of A and (1.1) we get successively

$$\begin{aligned} E'(t) &= \langle u_t, u_{tt} + Au\rangle_{V-V'} + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_U^2 \\ &= -\langle u_t, B_1(t)B_1^*(t)u_t(t) + B_2(t)B_2^*(t)u_t(t-\tau)\rangle_{V-V'} \\ &\qquad + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_U^2. \end{aligned}$$

By the definition of the dual operators, we arrive at

$$\begin{aligned} E'(t) = &-\|B_1^*(t)u_t(t)\|_U^2 - (B_2^*(t)u_t, B_2^*(t)u_t(t-\tau))_U \\ &+\frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_U^2. \end{aligned}$$

If $t \in I_{2n}$, then $B_2(t) = 0$ and the previous identity becomes

$$E'(t) = -\|B_1^*(t)u_t(t)\|_U^2 + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2.$$

Since $T_{2n} = |I_{2n}| \ge \tau$, it results that $t + \tau \in I_{2n} \cup I_{2n+1} \cup I_{2n+2}$. Now, if $t + \tau \in I_{2n} \cup I_{2n+2}$, then $B_2(t + \tau) = 0$. Therefore, $B_2(t + \tau) \ne 0$ only if $t + \tau \in I_{2n+1}$. In both cases, by our assumptions i) and ii), we get (3.4).

For $t \in I_{2n+1}$, as $B_1(t) = 0$, the previous identity becomes

$$E'(t)=-(B_2^*(t)u_t, B_2^*(t)u_t(t-\tau))_U+\frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2-\frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_U^2.$$

By Young's inequality we get

$$E'(t) \le \frac{1}{2}\|B_2^*(t)u_t(t)\|_U^2 + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_U^2.$$

This proves (3.5) using assumption ii) because $t+\tau$ is either in I_{2n+1} or in I_{2n+2}, and in that last case $B_2^*(t+\tau)=0$. □

Consider now the conservative system associated with (1.1)–(1.2)

$$w_{tt}(t)+Aw(t)=0 \quad t>0 \tag{3.6}$$

$$w(0)=w_0 \quad \text{and} \quad w_t(0)=w_1 \tag{3.7}$$

with $(w_0,w_1)\in V\times H$. For our stability result we need that an appropriate observability inequality holds. Namely we assume that there exists a time $\overline{T}>0$ such that for every time $T>\overline{T}$ there is a constant c, depending on T but independent of the initial data, such that

$$E_S(0)\le c\int_0^T \|w_t(s)\|_W^2\,ds, \tag{3.8}$$

for every weak solution of problem (3.6)–(3.7) with initial data $(w_0,w_1)\in V\times H$. Here $E_S(\cdot)$ denotes the standard energy for wave-type equations; that is,

$$E_S(t)=E_S(w,t):=\frac{1}{2}(\|w\|_V^2+\|w_t\|_H^2).$$

Proposition 3.2. *Assume* i), ii), *and* (3.3). *Moreover, we assume that the observability inequality* (3.8) *holds for every time* $T>\overline{T}$ *and that*

$$T^*:=\inf_n\{T_{2n}\}>\max\,\{\overline{T},\tau\}. \tag{3.9}$$

Then, for any solution of system (1.1)–(1.2) *we have*

$$E(t_{2n+1})\le c_nE(t_{2n}),\quad \forall\ n\in\mathbb{N}, \tag{3.10}$$

where

$$c_n=\frac{4c(1+4C^2T_{2n}^2M_{2n}^2)}{2m_{2n}-M_{2n+1}+4c(1+4C^2T_{2n}^2M_{2n}^2)}, \tag{3.11}$$

c being the observability constant in (3.8) *corresponding to the time T^* and C the constant in the norm embedding* (3.1) *between W and H.*

Proof. It is sufficient to prove the estimate (3.10) in the interval $I_0=[0,t_1)$. We can proceed analogously in the other intervals I_{2n}, $n\in\mathbb{N}$.

We can decompose $u=w+\tilde{w}$, where w is a solution of system (3.6)–(3.7) with $w_0=u_0$, $w_1=u_1$, while $\tilde{w}$ solves

$$\tilde{w}_{tt}(t)+A\tilde{w}(t)=-B_1(t)B_1^*(t)u_t(t) \quad t>0 \tag{3.12}$$

$$\tilde{w}(0)=0 \quad \text{and} \quad \tilde{w}_t(0)=0 \tag{3.13}$$

First we have

$$E(0) = E_S(w,0) + \frac{1}{2}\int_{-\tau}^{0} \|B_2^*(s+\tau)u_t(s)\|_U^2\, ds = E_S(w,0),$$

because for $s \in (-\tau, 0)$, $s + \tau < \tau < t_1$. Now using the observability inequality (3.8) we can estimate

$$E(0) = E_S(w,0) \le c\int_0^{T^*} \|w_t(s)\|_W^2\, ds. \tag{3.14}$$

Using the splitting $w = u - \tilde{w}$ and the fact that $T_0 = t_1 \ge T^*$, we deduce that

$$E(0) \le 2c\int_0^{T_0} (\|\tilde{w}_t(s)\|_W^2 + \|u_t(s)\|_W^2)\, ds. \tag{3.15}$$

Now, observe that from equation (3.12) we deduce

$$\frac{d}{dt}\frac{1}{2}(\|\tilde{w}_t(t)\|_H^2 + \|\tilde{w}(t)\|_V^2) = (\tilde{w}_t, \tilde{w}_{tt} + A\tilde{w})_H = -(\tilde{w}_t, B_1(t)B_1^*(t)u_t(t))_H.$$

Integrating this identity in $[0,t]$ with $0 < t < T_0$, recalling (3.13), and using the assumption i), we get

$$\begin{aligned}\frac{1}{2}(\|\tilde{w}_t(t)\|_H^2 + \|\tilde{w}(t)\|_V^2) &= -\int_0^{t} (B_1^*(s)\tilde{w}_t(s), B_1^*(s)u_t(s))_H\, ds\\ &\le M_0\int_0^{T_0} \|\tilde{w}_t(s)\|_W\|u_t(s)\|_W\, ds.\end{aligned} \tag{3.16}$$

Integrating (3.16) on $[0,t_1]$, we deduce

$$\begin{aligned}\int_0^{T_0} \|\tilde{w}_t(t)\|_W^2\, dt &\le C\int_0^{T_0} \|\tilde{w}_t(t)\|_H^2\, dt \le 2CT_0M_0\int_0^{T_0} \|\tilde{w}_t(s)\|_W\|u_t(s)\|_W\, ds\\ &\le CT_0M_0\int_0^{T_0} \Big(\epsilon\|\tilde{w}_t(t)\|_W^2 + \frac{1}{\epsilon}\|u_t(t)\|_W^2\Big)\, dt,\end{aligned}$$

for all $\epsilon > 0$, and therefore choosing ϵ such that $CT_0M_0\epsilon = \frac{1}{2}$, we arrive at

$$\int_0^{T_0} \|\tilde{w}_t(t)\|_W^2\, dt \le 4C^2T_0^2M_0^2\int_0^{T_0} \|u_t(t)\|_W^2\, dt. \tag{3.17}$$

From (3.15) and (3.17) we obtain

$$\begin{aligned}E(0) &\le 2c(1 + 4C^2T_0^2M_0^2)\int_0^{T_0} \|u_t(t)\|_W^2\, dt\\ &\le \frac{4c(1 + 4C^2T_0^2M_0^2)}{2m_0 - M_1}\Big(m_0 - \frac{M_1}{2}\Big)\int_0^{T_0} \|u_t(t)\|_W^2\, dt.\end{aligned} \tag{3.18}$$

From (3.4) and (3.18) we deduce

$$E(t_1) \leq E(0) \leq \frac{4c(1+4C^2T_0^2M_0^2)}{2m_0 - M_1}(E(0) - E(t_1)),$$

where we used also the fact that $E(\cdot)$ is decreasing on the time interval $[0, t_1]$. This clearly implies $E(t_1) \leq c_0 E(0)$, with

$$c_0 = \frac{4c(1+4C^2T_0^2M_0^2)}{2m_0 - M_1 + 4c(1+4C^2T_0^2M_0^2)}. \qquad \square$$

Theorem 3.3. *Under the assumptions of Proposition* 3.2, *if*

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{2m_{2n} - M_{2n+1}}{1+4C^2T_{2n}^2M_{2n}^2} = +\infty, \tag{3.19}$$

then system (4.2)–(4.4) *is asymptotically stable; that is, any solution* u *of* (4.2)–(4.4) *satisfies* $E(u,t) \to 0$ *for* $t \to +\infty$.

Proof. Note that (3.5) implies

$$E'(t) \leq 2M_{2n+1}CE(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}), \ n \in \mathbb{N}.$$

Then we have

$$E(t_{2n+2}) \leq e^{2CM_{2n+1}T_{2n+1}} E(t_{2n+1}), \quad \forall\, n \in \mathbb{N}. \tag{3.20}$$

Combining Proposition 3.2 and (3.20) we obtain

$$E(t_{2n+2}) \leq e^{2CM_{2n+1}T_{2n+1}} c_n E(t_{2n}), \quad n \in \mathbb{N},$$

and therefore

$$E(t_{2n+2}) \leq \left(\Pi_{p=0}^n e^{2CM_{2p+1}T_{2p+1}} c_p\right) E(0). \tag{3.21}$$

Then, by (3.21), asymptotic stability occurs if

$$\sum_{p=0}^{\infty} [2CM_{2p+1}T_{2p+1} + \ln c_p] = -\infty. \tag{3.22}$$

In particular, (3.22) holds true if (3.19) is valid. Indeed, from (3.11),

$$c_p = \frac{1}{\frac{2m_{2p} - M_{2p+1}}{4c(1+4C^2T_{2p}^2M_{2p}^2)} + 1},$$

and then

$$\ln c_p = -\ln\left(1 + \frac{2m_{2p} - M_{2p+1}}{4c(1+4C^2T_{2p}^2M_{2p}^2)}\right). \tag{3.23}$$

So, if $\frac{2m_{2p}-M_{2p+1}}{1+4C^2T_{2p}^2M_{2p}^2}$ tends to 0 as $p\to\infty$, then

$$-\ln c_p \sim \frac{2m_{2p}-M_{2p+1}}{4c(1+4C^2T_{2p}^2M_{2p}^2)}.$$

Consequently, if (3.19) holds then

$$\sum_{p=0}^{\infty}\ln c_p = -\infty.$$

Otherwise, if $\frac{2m_{2p}-M_{2p+1}}{1+4C^2T_{2p}^2M_{2p}^2}$ does not tend to 0, then, by (3.23),

$$\sum_{p=0}^{\infty}\ln c_p = -\infty.$$

Therefore, conditions (3.19) imply (3.22). □

We now show that under additional assumptions on the coefficients T_n, m_n, and M_n an exponential stability result holds.

Theorem 3.4. *Assume* i), ii), *and* (3.3). *Assume also that the observability inequality* (3.8) *holds for every time* $T>\overline{T}$ *and that*

$$T_n = T^* > \max\,\{\overline{T},\tau\},\quad \forall\ n\in\mathbb{N}. \tag{3.24}$$

Moreover, assume that (3.19) *and*

$$\sup_{n\in\mathbb{N}}\ e^{2CM_{2n+1}T^*}c_n = d < 1, \tag{3.25}$$

hold, where c_n *is as in* (3.11). *Then, there exist two positive constants* γ *and* μ *such that*

$$E(t)\le \gamma e^{-\mu t}E(0),\quad t>0, \tag{3.26}$$

for any solution of problem (1.1)–(1.2).

Proof. From (3.25) and (3.21) we obtain $E(2T^*)\le dE(0)$, and also

$$E(2nT^*)\le d^nE(0),\quad \forall n\in\mathbb{N}.$$

Then, the energy satisfies an exponential estimate like (3.26) (see Lemma 1 of [7]). □

Remark 3.5. In the assumptions of Theorem 3.4, from (3.21) we can see that exponential stability also holds if instead of (3.25) we assume

$$\exists n\in\mathbb{N}\quad \text{such that}\quad \Pi_{p=k(n+1)}^{k(n+1)+n}e^{2CM_{2p+1}T^*}c_p\le d<1,\ \ \forall\ k=0,1,2,\ldots.$$

4. The wave equation

Our first application concerns the wave equation with locally internal damping. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a boundary $\partial\Omega$ of class C^2. Denoting by m the standard multiplier $m(x) = x - x_0,\ x_0 \in \mathbb{R}^n$, let ω be the intersection of Ω with an open neighborhood of the subset of $\partial\Omega$

$$\Gamma_0 = \{\, x \in \partial\Omega \ :\ m(x) \cdot \nu(x) > 0 \,\}, \tag{4.1}$$

where $\nu(x)$ is the outer unit normal vector at $x \in \partial\Omega$.

Below, if ω_1 is a subset of Ω, we denote by χ_{ω_1} the characteristic function of the set ω_1; that is, $\chi_{\omega_1}(x) = 1$ for $x \in \omega_1$, $\chi_{\omega_1}(x) = 0$ for $x \in \omega_1^c$.

Let us consider the initial–boundary-value problem

$$u_{tt}(x,t) - \Delta u(x,t) + b_1(t)\chi_\omega u_t(x,t) + b_2(t)\chi_\omega u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) \tag{4.2}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty) \tag{4.3}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{4.4}$$

with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and b_1, b_2 in $L^\infty(0,+\infty)$ such that $b_1(t)b_2(t) = 0$, $\forall\ t > 0$. Moreover, we assume

- i_w) $0 < m_{2n} \le b_1(t) \le M_{2n}$, $b_2(t) = 0$, for all $t \in I_{2n} = [t_{2n}, t_{2n+1})$, and $b_1 \in C^1(\bar{I}_{2n})$, for all $n \in \mathbb{N}$;
- ii_w) $|b_2(t)| \le M_{2n+1}$, $b_1(t) = 0$, for all $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, and $b_2 \in C(\bar{I}_{2n+1})$, for all $n \in \mathbb{N}$.

This problem enters into our previous framework, if we take $H = L^2(\Omega)$ and the operator A defined by $A : \mathcal{D}(A) \to H \ :\ u \to -\Delta u$, where $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)$. We then define $U = L^2(\omega)$ and the operators B_i, $i = 1, 2$, as

$$B_i : U \to H : \quad v \to \sqrt{b_i}\,\tilde{v}\chi_\omega, \tag{4.5}$$

where $\tilde{v} \in L^2(\Omega)$ is the extension of v by zero outside ω. It is easy to verify that $B_i^*(\varphi) = \sqrt{b_i}\varphi_{|_\omega}$ for $\varphi \in H$, and thus (3.1) holds with $W = L^2(\omega)$, while $B_iB_i^*(\varphi) = b_i\varphi\chi_\omega$, for $\varphi \in H$ and $i = 1, 2$. This shows that problem (4.2)–(4.4) enters in the abstract framework (1.1)–(1.2). Moreover, i_w) and ii_w) easily imply i) and ii) of Section 3. Therefore we can restate Proposition 3.1.

Now, the energy functional is

$$E(t) = \frac{1}{2}\int_\Omega \{u_t^2(x,t) + |\nabla u(x,t)|^2\}\,dx + \frac{1}{2}\int_{t-\tau}^t |b_2(s+\tau)| \int_\omega u_t^2(x,s)\,dx\,ds, \tag{4.6}$$

which is the standard energy for the wave equation

$$E_S(t) = E_S(w,t) := \frac{1}{2}\int_\Omega (w_t^2 + |\nabla w|^2)\,dx,$$

plus an integral term due to the presence of a time delay.

Proposition 4.1. *Assume* i_w), ii_w), *and* (3.3). *Then, for every regular solution of problem* (4.2)–(4.4) *the energy is decreasing on the intervals* I_{2n}, $n \in \mathbb{N}$, *and*

$$E'(t) \le -\Big(m_{2n} - \frac{M_{2n+1}}{2}\Big)\int_\omega u_t^2(x,t)\,dx. \tag{4.7}$$

Moreover, on the intervals I_{2n+1}, $n \in \mathbb{N}$,

$$E'(t) \le M_{2n+1}\int_\omega u_t^2(x,t)\,dx. \tag{4.8}$$

4.1. **Wave equation with local damping.** Consider now the conservative system

$$w_{tt}(x,t) - \Delta w(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{4.9}$$

$$w(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty) \tag{4.10}$$

$$w(x,0) = w_0(x) \quad \text{and} \quad w_t(x,0) = w_1(x) \quad \text{in} \quad \Omega \tag{4.11}$$

with $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$. It is well known that an observability inequality holds (see, e.g., [2, 11, 13, 14, 16, 23]): There exists a time $\overline{T} > 0$ such that for every time $T > \overline{T}$ there is a constant c, depending on T but independent of the initial data, such that

$$E_S(0) \le c\int_0^T \int_\omega w_t^2(x,s)\,dx\,ds, \tag{4.12}$$

for every weak solution of problem (4.9)–(4.11).

We can then rewrite Proposition 3.2.

Proposition 4.2. *Assume* i_w), ii_w), *and* (3.3). *Moreover, assume*

$$T^* := \inf_n\{T_{2n}\} > \max\,\{\overline{T}, \tau\}, \quad \forall\, n \in \mathbb{N}, \tag{4.13}$$

where $\overline{T}$ *is the observability time. Then, for every solution of system* (4.2)–(4.4)

$$E(t_{2n+1}) \le c_n E(t_{2n}), \quad \forall\, n \in \mathbb{N}, \tag{4.14}$$

where

$$c_n = \frac{4c(1+4T_{2n}^2 M_{2n}^2)}{2m_{2n} - M_{2n+1} + 4c(1+4T_{2n}^2 M_{2n}^2)}, \tag{4.15}$$

c being the observability constant in (4.12) *corresponding to the time T^*.*

Therefore, we restate the stability results of Theorems 3.3 and 3.4 as follows.

Theorem 4.3. *Under the assumptions of Proposition* 4.2, *if*

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{2m_{2n} - M_{2n+1}}{1+4T_{2n}^2 M_{2n}^2} = +\infty, \tag{4.16}$$

then system (4.2)–(4.4) *is asymptotically stable; that is, any solution u of* (4.2)–(4.4) *satisfies $E(u,t) \to 0$ for $t \to +\infty$.*

Theorem 4.4. *Assume i_w), ii_w), and* (3.3). *Assume also*

$$T_n = T^* > \max\,\{\overline{T}, \tau\}, \quad \forall\ n \in \mathbb{N}, \tag{4.17}$$

where $\overline{T}$ is the observability time. Moreover, assume (4.16) *and*

$$\sup_{n\in\mathbb{N}}\ e^{2M_{2n+1}T^*} c_n = d < 1, \tag{4.18}$$

where c_n is as in (4.15). *Then, there exist two constants γ and μ such that*

$$E(t) \le \gamma e^{-\mu t} E(0), \quad t > 0, \tag{4.19}$$

for any solution of problem (4.2)–(4.4).

4.2. **Wave equation with distributed damping.** Here we consider a uniformly distributed damping; namely, we take $\omega = \Omega$. Let the energy $E(\cdot)$ be defined by (4.6) with $\omega = \Omega$. Assume here

$$|I_{2n}| \ge \tau, \quad n \in \mathbb{N}. \tag{4.20}$$

The following result immediately follows from Theorem 3.1 of [8].

Proposition 4.5. *Assume i_w) and ii_w). There exists $c > 0$ (independent of T_{2n}) such that for any solution of* (4.2)–(4.4), *it results that*

$$E_S(t_{2n+1}) \le \frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3} + T_{2n}^{-1} + M_{2n} m_{2n} T_{2n}^{-1}}} E_S(t_{2n}) \quad n \in \mathbb{N}. \tag{4.21}$$

Theorem 4.6. *Assume i_w), ii_w),* (3.3), *and* (4.20). *If*

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty, \tag{4.22}$$

$$\sum_{n=0}^{\infty} \ln\Big(\frac{1}{1+c\frac{m_{2p}}{T_{2p}^{-3}+T_{2p}^{-1}+M_{2p}m_{2p}T_{2p}^{-1}}} + \tau M_{2p+1}\Big) = -\infty,$$

then system (4.2)–(4.4) *is asymptotically stable; that is, any solution* u *of* (4.2)–(4.4) *satisfies* $E_S(u,t) \to 0$ *for* $t \to +\infty$.

Proof. Note that (4.8) implies

$$E'(t) \le 2M_{2n+1}E(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),\ n \in \mathbb{N}.$$

Then we have

$$E(t_{2n+2}) \le e^{2M_{2n+1}T_{2n+1}} E(t_{2n+1}), \quad \forall\ n \in \mathbb{N}. \tag{4.23}$$

Now observe that

$$E(t_{2n+1}) = E_S(t_{2n+1}) + \frac{1}{2}\int_{t_{2n+1}-\tau}^{t_{2n+1}} |b_2(s+\tau)| \int_\Omega u_t^2(x,s)\,dx\,ds,$$

and then, as $|I_{2n}| \ge \tau$, $n \in \mathbb{N}$,

$$\begin{aligned} E(t_{2n+1}) &\le E_S(t_{2n+1}) + \tau M_{2n+1}E_S(t_{2n+1}-\tau) \\ &\le E_S(t_{2n+1}) + \tau M_{2n+1}E_S(t_{2n}). \end{aligned} \tag{4.24}$$

Then, from Proposition 4.5 and (4.24) we deduce

$$E(t_{2n+1}) \le \Big(\frac{1}{1+c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau M_{2n+1}\Big)E_S(t_{2n}), \tag{4.25}$$

and therefore

$$\begin{aligned} E_S(t_{2n+2}) &\le E(t_{2n+2}) \\ &\le e^{2M_{2n+1}T_{2n+1}}\Big(\frac{1}{1+c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau M_{2n+1}\Big)E_S(t_{2n}). \end{aligned} \tag{4.26}$$

Since (4.26) holds for any $n \in \mathbb{N}$, we conclude

$$E_S(t_{2n+2}) \le \Pi_{p=0}^n e^{2M_{2p+1}T_{2p+1}}\Big(\frac{1}{1+c\frac{m_{2p}}{T_{2p}^{-3}+T_{2p}^{-1}+M_{2p}m_{2p}T_{2p}^{-1}}} + \tau M_{2p+1}\Big)E_S(0). \tag{4.27}$$

Now observe that the standard energy $E_S(\cdot)$ is not decreasing. However, for $t \in [t_{2n}, t_{2n+1})$, only the standard dissipative damping acts, and so

$$E_S(t) \le E_S(t_{2n}), \quad \forall t \in [t_{2n}, t_{2n+1}). \tag{4.28}$$

Moreover, for $t \in [t_{2n+1}, t_{2n+2})$, it results that

$$E_S(t) \le E(t) \le e^{2M_{2n+1}T_{2n+1}}E(t_{2n+1}), \tag{4.29}$$

where in the second inequality we have used (4.8).

Then, by (4.27), (4.28), (4.29), and (4.25), asymptotic stability occurs if

$$\sum_{n=0}^{\infty} [2M_{2p+1}T_{2p+1} + \ln \tilde{c}_p] = -\infty, \tag{4.30}$$

where

$$\tilde{c}_p = \frac{1}{1 + c\frac{m_{2p}}{T_{2p}^{-3}+T_{2p}^{-1}+M_{2p}m_{2p}T_{2p}^{-1}}} + \tau M_{2p+1}.$$

Thus, if

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty, \quad \sum_{n=0}^{\infty} \ln \tilde{c}_n = -\infty,$$

asymptotic stability is ensured. □

Remark 4.7. Theorems 4.3 and 4.6 are far from being equivalent, since each allows us to give stability results in some cases not covered by the other one. Indeed, we can observe that assumption (4.22) is satisfied in some cases for which (4.16) does not hold. For instance, consider $m_{2n} = M_{2n} = T_{2n} = n$ and $M_{2n+1} \to \alpha$ where $\alpha > 0$ satisfies $\tau\alpha < \frac{c}{1+c}$. Then, as $n \to \infty$,

$$\frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau M_{2n+1} \to \frac{1}{1+c} + \tau\alpha < 1,$$

and so (4.22) is satisfied if $\sum_n T_{2n+1} < +\infty$. With the same choice of the coefficients m_i, M_i, and T_i, $i \in \mathbb{N}$, the second relation of (4.16) is not satisfied. Indeed,

$$\frac{2m_{2n} - M_{2n+1}}{1 + 4T_{2n}^2 M_{2n}^2} \sim \frac{1}{2n^3}.$$

But the converse also holds; that is, (4.16) is satisfied for some choices of the coefficients for which (4.22) does not hold. As an example fix $m_{2n} = T_{2n} = 1/n$, $M_{2n+1} = \frac{c}{2\tau}\frac{1}{n^4}$, and $T_{2n+1} = T$. In this case, (4.16) clearly holds while the second relation of (4.22) is not satisfied, since

$$\ln\left(\frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau M_{2n+1}\right) \sim -\frac{c}{2}\frac{1}{n^4}.$$

5. The elasticity system

In the same setting as in the previous section, we consider the following elastodynamic system:

$$u_{tt}(x,t) - \mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u$$

$$+ b_1(t)\chi_\omega u_t(x,t) + b_2(t)\chi_\omega u_t(x,t-\tau) = 0 \text{ in } \Omega \times (0,+\infty) \tag{5.1}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty) \tag{5.2}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{5.3}$$

with initial data $(u_0, u_1) \in H_0^1(\Omega)^n \times L^2(\Omega)^n$ and b_1 and b_2 satisfying the same assumptions as in Section 4. Note that in this case the state variable u is vector-valued and λ and μ are the Lamé coefficients that are positive real numbers.

As before, this problem enters into our abstract setting, once we take $H = L^2(\Omega)^n$, and A defined by

$$A : \mathcal{D}(A) \to H \,:\, u \to -\mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u,$$

where $\mathcal{D}(A) = H_0^1(\Omega)^n \cap H^2(\Omega)^n$.

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)^n$ equipped with the inner product

$$(u,v)_V = \int_\Omega \Big(\mu \sum_{i,j=1}^n \partial_i u_j \partial_i v_j + (\lambda+\mu) \operatorname{div} u \operatorname{div} v\Big)\, dx, \quad \forall u, v \in H_0^1(\Omega)^n.$$

We then define $U = L^2(\omega)^n$ and the operators B_i, $i = 1, 2$, as

$$B_i : U \to H : \quad v \to \sqrt{b_i}\tilde{v}\chi_\omega,$$

where $\tilde{v}$ is the extension of v by zero outside ω. As before,

$$B_i^*(\varphi) = \sqrt{b_i}\varphi_{|\omega} \quad \text{for } \varphi \in H,$$

and thus $B_i B_i^*(\varphi) = b_i \varphi \chi_\omega$, for $\varphi \in H$ and $i = 1, 2$. So, problem (5.1)–(5.3) enters in the abstract framework (1.1)–(1.2). Moreover, $i_w)$ and $ii_w)$ easily imply $i)$ and $ii)$ of Section 3.

Therefore, in order to apply the abstract results of Section 3, we only need to check the observability estimate of the associated conservative system: There exists a time $T > 0$ and a constant $C > 0$ such that

$$\frac{1}{2}((w_0, w_0)_V + \int_\Omega |w_1|^2\, dx) \le C \int_0^T \int_\omega |w_t|^2(x,s)\, dx\, ds,$$

for every weak solution of

$$w_{tt}(x,t) - \mu\Delta w(x,t) - (\lambda+\mu)\nabla \operatorname{div} w = 0 \quad \text{in} \quad \Omega \times (0,+\infty)$$

$$w(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty)$$

$$w(x,0) = w_0(x) \quad \text{and} \quad w_t(x,0) = w_1(x) \quad \text{in} \quad \Omega$$

with initial data $(w_0, w_1) \in H_0^1(\Omega)^n \times L^2(\Omega)^n$. This estimate being obtained in the proof of Theorem 3.1 of [3] (estimate (3.2) of [3]), the stability results from Section 3 can be applied to the above system.

6. The Petrovsky system

6.1. **Hinged boundary conditions.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a boundary $\partial\Omega$ of class C^4. As before, let m be the standard multiplier $m(x) = x - x_0,\ x_0 \in \mathbb{R}^n$, and let ω be the intersection of Ω with an open neighborhood of the subset Γ_0 defined by (4.1).

Let us consider the initial–boundary-value problem

$$u_{tt}(x,t) + \Delta^2 u(x,t) + b_1(t)\chi_\omega u_t(x,t) + b_2(t)\chi_\omega u_t(x,t-\tau) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{6.1}$$

$$u(x,t) = \Delta u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty) \tag{6.2}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{6.3}$$

with initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$ and b_1 and b_2 satisfying the same assumptions as in Section 4.

Now, we take $H = L^2(\Omega)$ and let A be the operator

$$A : \mathcal{D}(A) \to H : \qquad u \to \Delta^2 u, \tag{6.4}$$

where $\mathcal{D}(A) = \{v \in H_0^1(\Omega) \cap H^4(\Omega) : \Delta u = 0 \text{ on } \partial\Omega\}$. The operator A is self-adjoint and positive, has a compact inverse in H, and satisfies $\mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$. We then define $U = L^2(\omega)$ and the operators B_i, $i = 1, 2$, by (4.5). So, problem (6.1)–(6.3) enters in the abstract framework (1.1)–(1.2). Moreover, i_w) and ii_w) easily imply i) and ii) of Section 3. It is well known that an observability estimate of the associated conservative system holds (see Proposition 7.5.7 (see also Example 11.2.4) of [20]). Therefore, the results of Section 3 apply also to the plate model.

Note that for such boundary conditions, the observability estimate of the associated conservative system is also valid if Ω is a hypercube and ω is an arbitrary non-empty open subset of Ω; see Example 11.2.4 of [20] or Proposition 8.8 of [12].

6.2. **Clamped boundary conditions.** Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a boundary $\partial\Omega$ of class C^4, and let ω be the intersection of Ω with an open neighborhood of the subset Γ_0 defined by (4.1).

Here we consider the initial–boundary-value problem

$$u_{tt}(x,t) + \Delta^2 u(x,t) + b_1(t)\chi_\omega u_t(x,t) + b_2(t)\chi_\omega u_t(x,t-\tau) = 0$$

$$\text{in} \quad \Omega \times (0, +\infty) \tag{6.5}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty) \tag{6.6}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{6.7}$$

with initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, where

$$H_0^2(\Omega) := \{\varphi \in H^2(\Omega) \ : \ u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\},$$

and b_1 and b_2 satisfy the same assumptions as in the previous subsection.

Again we take $H = L^2(\Omega)$ and consider the operator A defined by (6.4) but with $\mathcal{D}(A) = H_0^2(\Omega) \cap H^4(\Omega)$.

The operator A is self-adjoint and positive with a compact inverse in H and such that $V = \mathcal{D}(A^{1/2}) = H_0^2(\Omega)$. We then define $U = L^2(\omega)$ and the operators B_i, $i = 1, 2$, by (4.5). Again problem (6.5)–(6.7) enters in the abstract framework (1.1)–(1.2). Moreover, i_w) and ii_w) easily imply i) and ii) of Section 3. The observability estimate of the associated conservative system seems to be not proved in the literature; hence, the next subsection is devoted to its proof. Therefore, the results of Section 3 apply also to this model.

6.3. **An observability estimate for the Petrovsky system with clamped boundary conditions.** For any $T > 0$, we consider the conservative system

$$\phi_{tt}(x,t) + \Delta^2\phi(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{6.8}$$

$$\phi(x,t) = \frac{\partial \phi}{\partial \nu}(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{6.9}$$

$$\phi(x,0) = \phi_0(x) \quad \text{and} \quad \phi_t(x,0) = \phi_1(x) \quad \text{in} \quad \Omega \tag{6.10}$$

with initial data $(\phi_0, \phi_1) \in H_0^2(\Omega) \times L^2(\Omega)$. A weak solution of that problem exists in the space $X := L^\infty(0,T; H_0^2(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$; see for instance Theorem IV.3.1 of [15].

The natural energy of this solution is

$$E(t) = \frac{1}{2}\int_\Omega (|\phi_t|^2 + |\Delta\phi|^2)\,dx.$$

Recall that the energy is constant in time; namely, $E(t) = E(0)$, $\forall t > 0$. Our goal is to prove the next observability estimate.

Theorem 6.1. *For all $T > 0$, there exists a positive constant $C(T)$ (depending on T) such that*

$$E(0) \leq C(T) \int_0^T \int_\omega |\phi_t|^2 \, dx, \tag{6.11}$$

for any $\phi \in X$ a solution of (6.8)–(6.10).

Proof. First fix $T > 0$ and $\epsilon > 0$ small enough such that $\omega_\epsilon \subset \omega$, where ω_ϵ is defined by (see the identity (2.34) in Chapter VII of [15])

$$O_\epsilon = \bigcup_{x \in \Gamma_0} B(x, \epsilon), \quad \omega_\epsilon = O_\epsilon \cap \Omega,$$

where as usual $B(x, \epsilon)$ means the (open) ball of center x and radius ϵ.

Step 1: For $\epsilon > 0$ fixed there exists $C_1 = C_1(T, \epsilon) > 0$ (depending on T and ϵ) such that

$$E(0) \leq C_1 \int_0^T \int_{\omega_\epsilon} (|\phi_t|^2 + |\Delta\phi|^2) \, dx \, dt. \tag{6.12}$$

First we recall the estimate (2.23) of [22], which stated that for all $T_0 \in (0, T)$, there exists a positive constant C that may depend on T_0 and Γ_0 such that

$$E(0) \leq C \int_0^{T_0} \int_{\Gamma_0} |\Delta\phi|^2 \, dx. \tag{6.13}$$

Now we fix $\alpha > 0$ small enough such that $T - 2\alpha > T_0$, and by applying the estimate (6.13) in the interval $(\alpha, T - \alpha)$ (allowed since our system is invariant by translation), we get

$$E(0) = E(\alpha) \leq C \int_\alpha^{T-\alpha} \int_{\Gamma_0} |\Delta\phi|^2 \, dx, \tag{6.14}$$

where here and below C means a positive constant that may depend on T, Γ_0, α, and ϵ but that does not depend on ϕ.

To estimate this right-hand side, we use a multiplier technique, by choosing as multiplier $q(x, t) = t(T - t)h(x)$, where $h \in C^1(\bar{\Omega})$ is a vector field (introduced in Remark 3.2 of Chapter I in [15]) that satisfies

$$h(x) \cdot \nu(x) = 1 \text{ on } \Gamma_0, \quad h(x) \cdot \nu(x) \geq 0 \text{ on } \partial\Omega, \quad \operatorname{supp} h \subset \omega_\epsilon.$$

Multiplying the identity (6.8) by $q \cdot \nabla\phi$, integrating in $Q_T := \Omega \times (0, T)$, and integrating by parts in time and space, we get (see the identity (60) in Lemma 8 of [6])

$$\frac{1}{2} \int_0^T \int_{\partial\Omega} q \cdot \nu |\Delta\phi|^2 \, dx = \frac{1}{2} \int_{Q_T} \operatorname{div} q (|\phi_t|^2 - |\Delta\phi|^2) \, dx \, dt$$

$$+\int_{Q_T}\Delta q\cdot\nabla\phi\Delta\phi\,dx\,dt+2\int_{Q_T}\Delta\phi\sum_{j,k=1}^{n}\partial_j q_k\partial_{jk}^2\phi\,dx\,dt-\int_{Q_T}\phi_t q_t\cdot\nabla\phi\,dx\,dt.$$

Hence, by the properties of h, we see that

$$\frac{1}{\alpha(T-\alpha)}\int_\alpha^{T-\alpha}\int_{\Gamma_0}|\Delta\phi|^2\,dx\le\int_\alpha^{T-\alpha}\int_{\Gamma_0}q\cdot\nu|\Delta\phi|^2\,dx\le\int_0^T\int_{\partial\Omega}q\cdot\nu|\Delta\phi|^2\,dx.$$

Using this estimate and Young's inequality in the previous identity we find for all $\eta>0$ that

$$\int_\alpha^{T-\alpha}\int_{\Gamma_0}|\Delta\phi|^2\,dx\le C\Big(\int_{q_{\epsilon,T}}(|\phi_t|^2+|\Delta\phi|^2)\,dx\,dt$$
$$+\int_{q_{\epsilon,T}}(\frac{1}{\eta}|\Delta\phi|^2+\eta(\sum_{j,k=1}^{n}|\partial_{jk}^2\phi|^2+|\nabla\phi|^2))\,dx\,dt+\int_{q_{\epsilon,T}}(\frac{1}{\eta}|\phi_t|^2+\eta|\nabla\phi|^2)\,dx\,dt\Big),$$

where $q_{\epsilon,T}=\omega_\epsilon\times(0,T)$ (we here emphasize the fact that the constant C above does not depend on η). By the definition of the $H^2(\Omega)$-norm, this estimate implies that for all $\eta\in(0,1)$

$$\int_\alpha^{T-\alpha}\int_{\Gamma_0}|\Delta\phi|^2\,dx\le\frac{C}{\eta}\int_{q_{\epsilon,T}}(|\phi_t|^2+|\Delta\phi|^2)\,dx\,dt+C\eta\int_0^T\|\phi(\cdot,t)\|_{H^2(\Omega)}^2\,dt. \tag{6.15}$$

But it is well known that

$$\int_\Omega|\Delta w|^2\,dx=\int_\Omega(|\partial_1^2w|^2+|\partial_2^2w|^2+2|\partial_{12}^2w|^2)\,dx,\quad\forall w\in H_0^2(\Omega).$$

Indeed, this identity is easily checked (by integration by parts) for all $w\in\mathcal{D}(\Omega)$; by density it then holds in the whole of $H_0^2(\Omega)$. From this identity we deduce by Poincaré's inequality that

$$\|w\|_{H^2(\Omega)}^2\le C\int_\Omega|\Delta w|^2\,dx,\quad\forall w\in H_0^2(\Omega).$$

This property in (6.15) implies that for all $\eta\in(0,1)$

$$\int_\alpha^{T-\alpha}\int_{\Gamma_0}|\Delta\phi|^2\,dx\le\frac{C}{\eta}\int_{q_{\epsilon,T}}(|\phi_t|^2+|\Delta\phi|^2)\,dx\,dt+C\eta\int_0^T\int_\Omega|\Delta\phi|^2\,dx\,dt.$$

Using the definition of the energy we arrive at

$$\int_\alpha^{T-\alpha}\int_{\Gamma_0}|\Delta\phi|^2\,dx\le\frac{C}{\eta}\int_{q_{\epsilon,T}}(|\phi_t|^2+|\Delta\phi|^2)\,dx\,dt+C\eta TE(0),$$

for all $\eta\in(0,1)$.

Finally, using this estimate in (6.14) and choosing η small enough (namely $C\eta T < 1$), we arrive at (6.12).

Step 2: There exists $C_2 = C_2(T, \epsilon) > 0$ (depending on T and ϵ) such that

$$E(0) \leq C_2 \int_0^T \int_\omega (|\phi_t|^2 + |\nabla\phi|^2 + |\phi|^2)\, dx. \tag{6.16}$$

We start from (6.12) but on $(\alpha, T-\alpha)$ and in the domain $\omega_{\epsilon/2}$ that yields

$$E(0) = E(\alpha) \leq C_1 \int_\alpha^{T-\alpha} \int_{\omega_{\epsilon/2}} (|\phi_t|^2 + |\Delta\phi|^2)\, dx\, dt. \tag{6.17}$$

Hence, it remains to estimate the term

$$\int_\alpha^{T-\alpha} \int_{\omega_{\epsilon/2}} |\Delta\phi|^2\, dx\, dt.$$

For that purpose we use an argument similar to the one for the wave equation (see Lemma 2.4 of Chapter VII of [15]); namely, we use a multiplier method: we multiply the identity (6.8) by $\xi(x,t) = t(T-t)\varphi_\epsilon(x)\phi(x,t)$, where $\varphi_\epsilon \in W_0^{2,\infty}(O_\epsilon)$ is defined by (the definition of this function is different from the one in [15], since our spatial operator is a fourth-order one; see page 414 of [15])

$$\varphi_\epsilon(x) = \begin{cases} 1 & \text{on } O_{\epsilon/2}, \\ \frac{(\epsilon - 2d(x))^4}{\epsilon^4} & \text{on } O_\epsilon \setminus O_{\epsilon/2}, \\ 0 & \text{elsewhere,} \end{cases}$$

where $d(x)$ is the distance from x to $\partial O_{\epsilon/2}$. Simple calculations show that

$$0 \leq \varphi_\epsilon \leq 1 \text{ in } \Omega, \tag{6.18}$$

$$\frac{|\nabla\varphi_\epsilon|^2}{\varphi_\epsilon} \leq \frac{C}{\epsilon^4} \text{ in } \omega_\epsilon, \tag{6.19}$$

$$\frac{|\Delta\varphi_\epsilon|^2}{\varphi_\epsilon} \leq \frac{C}{\epsilon^4} \text{ in } \omega_\epsilon. \tag{6.20}$$

As said before, multiplying the identity (6.8) by ξ, integrating in $Q_T := \Omega \times (0,T)$ and performing some integrations by parts in time and space, we get

$$\begin{aligned} 0 &= -\int_{Q_T} \phi_t \varphi_\epsilon (t(T-t)\phi_t + (T-2t)\phi)\, dx\, dt \\ &+ \int_{Q_T} \Delta\phi\, t(T-t)(\Delta\varphi_\epsilon \phi + 2\nabla\varphi_\epsilon \cdot \nabla\phi + \varphi_\epsilon \Delta\phi)\, dx\, dt. \end{aligned}$$

Consequently, since φ_ϵ is zero outside ω_ϵ, we get

$$\begin{aligned}\int_{Q_T} |\Delta\phi|^2 t(T-t)\varphi_\epsilon\,dx\,dt &= +\int_{q_{\epsilon,T}} \phi_t\varphi_\epsilon(t(T-t)\phi_t + (T-2t)\phi)\,dx\,dt \\ &- \int_{q_{\epsilon,T}} \Delta\phi t(T-t)(\Delta\varphi_\epsilon\phi + 2\nabla\varphi_\epsilon\cdot\nabla\phi)\,dx\,dt.\end{aligned}$$

Hence by Young's inequality we obtain for all $\eta > 0$

$$\begin{aligned}\int_{Q_T} |\Delta\phi|^2 t(T-t)\varphi_\epsilon\,dx\,dt &\le C\Big(\int_{q_{\epsilon,T}} (|\phi_t|^2 + |\phi|^2)\,dx\,dt \\ &+ \int_{q_{\epsilon,T}} |\Delta\phi| t(T-t)\sqrt{\varphi_\epsilon}\big(\frac{|\Delta\varphi_\epsilon|}{\sqrt{\varphi_\epsilon}}|\phi| + \frac{|\nabla\varphi_\epsilon|}{\sqrt{\varphi_\epsilon}}|\nabla\phi|\big)\,dx\,dt\Big) \\ &\le C\Big(\int_{q_{\epsilon,T}} (|\phi_t|^2 + |\phi|^2)\,dx\,dt + \eta\int_{q_{\epsilon,T}} |\Delta\phi|^2\varphi_\epsilon t(T-t)\,dx\,dt \\ &+ \frac{1}{\eta}\int_{q_{\epsilon,T}} t(T-t)\big(\frac{|\Delta\varphi_\epsilon|^2}{\varphi_\epsilon}|\phi|^2 + \frac{|\nabla\varphi_\epsilon|^2}{\varphi_\epsilon}|\nabla\phi|^2\big)\,dx\,dt\Big).\end{aligned}$$

Using the properties (6.19) and (6.20) we arrive at

$$\begin{aligned}&\int_{Q_T} |\Delta\phi|^2 t(T-t)\varphi_\epsilon\,dx\,dt \le C\Big(\int_{q_{\epsilon,T}} (|\phi_t|^2 + |\phi|^2)\,dx\,dt \\ &+ \eta\int_{q_{\epsilon,T}} |\Delta\phi|^2\varphi_\epsilon t(T-t)\,dx\,dt + \frac{1}{\eta}\int_{q_{\epsilon,T}} t(T-t)(|\phi|^2 + |\nabla\phi|^2)\,dx\,dt\Big).\end{aligned}$$

Choosing η small enough and recalling that φ_ϵ is zero outside ω_ϵ, we arrive at

$$\int_{Q_T} |\Delta\phi|^2 t(T-t)\varphi_\epsilon\,dx\,dt \le C\int_{q_{\epsilon,T}} (|\phi_t|^2 + |\phi|^2 + |\nabla\phi|^2)\,dx\,dt.$$

Using (6.18) and the fact that $\varphi_\epsilon = 1$ in $\omega_{\epsilon/2}$, we arrive at

$$\int_\alpha^{T-\alpha}\int_{\omega_{\epsilon/2}} |\Delta\phi|^2\,dx\,dt \le C\int_{q_{\epsilon,T}} (|\phi_t|^2 + |\phi|^2 + |\nabla\phi|^2)\,dx\,dt.$$

Inserting this estimate in (6.17) we obtain (6.16) since $\omega_\epsilon \subset \omega$.

Step 3: There exists $C_3 = C_3(T) > 0$ (depending on T) such that

$$\|\phi\|_{L^\infty(0,T;H_0^1(\Omega))} \le C_3\Big(\int_0^T\int_\omega |\phi_t|^2\,dx\,dt + \|\phi\|^2_{L^\infty(0,T;L^2(\Omega))}\Big), \qquad (6.21)$$

for all $\phi \in X$ a solution of (6.8)–(6.10).

We use a contradiction argument (compare with [22, pp. 471–472]): if (6.21) does not hold, then there exists a sequence $\phi_n \in X$ solution of (6.8)–(6.9) for all $n \in \mathbb{N}^*$ such that

$$\|\phi_n\|_{L^\infty(0,T;H_0^1(\Omega))} = 1, \tag{6.22}$$

$$\int_0^T \int_\omega |\phi_t|^2 \, dx \, dt + \|\phi_n\|^2_{L^\infty(0,T;L^2(\Omega))} \le \frac{1}{n}. \tag{6.23}$$

But using (6.16), the sequence $(\phi_n)_n$ will be bounded in X. According to the compact embedding of X into $L^\infty(0,T;H_0^1(\Omega))$ (cf. [19]), we deduce that up to a subsequence still denoted by $(\phi_n)_n$, there exists $\phi \in L^\infty(0,T;H_0^1(\Omega))$ such that $\phi_n \to \phi$ strongly in $L^\infty(0,T;H_0^1(\Omega))$, and satisfying, thanks to (6.22), $\|\phi\|_{L^\infty(0,T;H_0^1(\Omega))} = 1$. On the other hand owing to (6.23) we also have $\|\phi\|_{L^\infty(0,T;L^2(\Omega))} = 0$, and therefore $\phi = 0$, which is a contradiction.

Steps 2 and 3 imply that there exists $C > 0$ (depending on T) such that

$$E(0) \le C(\int_0^T \int_\omega |\phi_t|^2 \, dx \, dt + \|\phi\|_{L^\infty(0,T;L^2(\Omega))}), \tag{6.24}$$

for all $\phi \in X$ a solution of (6.8)–(6.10).

Step 4: We prove (6.11), namely that there exists $C_4 = C_4(T) > 0$ (depending on T) such that

$$E(0) \le C_4 \int_0^T \int_\omega |\phi_t|^2 \, dx \, dt, \tag{6.25}$$

for all $\phi \in X$ a solution of (6.8)–(6.10).

We again use a contradiction argument: if (6.25) does not hold, then there exists a sequence $\phi_n \in X$, solutions of (6.8)–(6.9) for all $n \in \mathbb{N}^*$, such that

$$E(\phi_n(0)) = 1, \tag{6.26}$$

$$\int_0^T \int_\omega |\phi_{n,t}|^2 \, dx \, dt \le \frac{1}{n}. \tag{6.27}$$

As system (6.8)–(6.9) is conservative, we deduce that $(\phi_n)_n$ is bounded in X. According to the compact embedding of X into $L^\infty(0,T;H_0^1(\Omega))$ (cf. [19]), we deduce that up to a subsequence still denoted by $(\phi_n)_n$, there exists $\phi \in L^\infty(0,T;H_0^1(\Omega))$, a weak solution of (6.8)–(6.10) such that $\phi_n \to \phi$ strongly in $L^\infty(0,T;H_0^1(\Omega))$. This convergence, (6.27), and (6.16) imply that $\phi_n \to \phi$ strongly in X. Furthermore, due to (6.26) and (6.27), ϕ satisfies

$$E(\phi(0)) = 1, \tag{6.28}$$

$$\int_0^T \int_\omega |\phi_t|^2 \, dx \, dt = 0. \tag{6.29}$$

Now we notice that ϕ_t belongs to $L^\infty(0,T;L^2(\Omega))$ and satisfies

$$(\phi_t)_{tt} + \Delta^2 \phi_t = 0 \text{ in } Q_T.$$

Thanks to (6.29) and to the estimate (6.24), we deduce that (see [22, p. 472] for a similar argument) $\phi_t \in X$. But (6.29) also yields that

$$\Delta \phi_t = 0 \text{ in } \omega \times (0,T),$$

and therefore by taking the trace on Γ_0, we get

$$\Delta \phi_t = 0 \text{ in } \Gamma_0 \times (0,T).$$

By Proposition 2.1 of [22], we deduce that

$$\phi_t = 0 \text{ in } \Omega \times (0,T).$$

Coming back to ϕ, we can say that $\phi(x,t) = \chi(x)$, with $\chi \in H_0^2(\Omega)$ a weak solution of (recalling that ϕ is a solution of (6.8)–(6.10))

$$\Delta^2 \chi = 0 \text{ in } \Omega.$$

Therefore $\chi = 0$, and then $\phi = 0$. This contradicts (6.28). □

References

[1] K. Ammari, S. Nicaise, and C. Pignotti, *Stabilization by switching control methods,* preprint (2011), submitted.

[2] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary,* SIAM J. Control Optim., 30 (1992), 1024–1065.

[3] M. Cavalcanti and J.S. Prates Filho, *Exact internal controllability of the elasticity system,* Southern J. Pure and Applied Math., 1 (1998), 29–43.

[4] R. Datko, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks,* SIAM J. Control Optim., 26 (1988), 697–713.

[5] R. Datko, J. Lagnese, and M.P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations,* SIAM J. Control Optim., 24 (1986), 152–156.

[6] L. De Teresa and E. Zuazua, *Controllability of the linear system of thermoelastic plates,* Adv. Differential Equations, 1 (1996), 369–402.

[7] M. Gugat, *Boundary feedback stabilization by time delay for one-dimensional wave equations,* IMA J. Math. Control Inform., 27 (2010), 189–203.

[8] A. Haraux, P. Martinez, and J. Vancostenoble, *Asymptotic stability for intermittently controlled second-order evolution equations,* SIAM J. Control Optim., 43 (2005), 2089–2108.

[9] T. Kato, *Linear and quasilinear equations of evolution of hyperbolic type,* C.I.M.E., II ciclo (1976), 125–191.

[10] T. Kato, "Abstract Differential Equations and Nonlinear Mixed Problems," Lezioni Fermiane [Fermi Lectures], Scuola Normale Superiore, Pisa, 1985.

[11] V. Komornik, "Exact Controllability and Stabilization, the Multiplier Method," volume 36 of *RMA*, Masson, Paris, 1994.

[12] V. Komornik and P. Loreti, "Fourier Series in Control Theory," Springer Monographs in Mathematics, Springer-Verlag, New York, 2005.

[13] J. Lagnese, *Control of wave processes with distributed control supported on a subregion,* SIAM J. Control Optim., 21 (1983), 68–85.

[14] I. Lasiecka and R. Triggiani, *Uniform exponential decay in a bounded region with* $L_2(0,T;L_2(\Sigma))$*-feedback control in the Dirichlet boundary conditions,* J. Differential Equations, 66 (1987), 340–390.

[15] J.L. Lions, "Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués, Tome 1," volume 8 of *RMA*, Masson, Paris, 1988.

[16] K. Liu, *Locally distributed control and damping for the conservative systems,* SIAM J. Control and Optim., 35 (1997), 1574–1590.

[17] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks,* SIAM J. Control Optim., 45 (2006), 1561–1585.

[18] C. Pignotti, *A note on stabilization of locally damped wave equations with time delay,* Systems Control Lett., 61 (2012), 92–97.

[19] J. Simon, *Compact sets in the space* $L^p(0,T;B)$*,* Ann. Mat. Pura Appl., 146 (1987), 65–96.

[20] M. Tucsnak and G. Weiss, "Observation and Control for Operator Semigroups," Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2009.

[21] G.Q. Xu, S.P. Yung, and L.K. Li, *Stabilization of wave systems with input delay in the boundary control,* ESAIM: Control Optim. Calc. Var., 12 (2006), 770–785.

[22] E. Zuazua, "Contrôlabilité exacte en un temps arbitrairement petit de quelques modèles de plaques," Appendix 1 in RMA 8, Masson, Paris, 1988.

[23] E. Zuazua, *Exponential decay for the semi-linear wave equation with locally distributed damping,* Comm. Partial Differential Equations, 15 (1990), 205–235.