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EXISTENCE OF A FINITE-DIMENSIONAL GLOBAL ATTRACTOR FOR A DAMPED PARAMETRIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We study the existence of a global attractor for a damped parametric nonlinear Schrödinger equation. We provide sufficient conditions for this attractor to have finite dimension.

1. Introduction

In this article we are interested in some dynamical properties of the solutions to a Schrödinger-like equation which models the propagation of a light beam (see [19] and the references therein; see also [13]). Here, we are in the framework of infinite-dimensional dynamical systems corresponding to a mildly dissipative partial differential equation on the whole line (see [21], [16], and [20]). We are interested in proving the existence of a global attractor and to discuss if this global attractor has finite dimension.

To begin with we give an overview of some results concerned with dissipative NLS equations in this framework. Damped forced nonlinear Schrödinger equations are an infinite-dimensional dynamical system that has retained attention during the last three decades. The first result for the one-dimensional cubic NLS equation while the space variable belongs to the one-dimensional torus appeared in [7]. The author proved the existence of a global attractor for the weak topology in the energy space H^1. A well-known argument due to J. Ball implies that in fact this weak attractor is a global attractor in the usual sense, i.e., a global attractor for the strong topology ([22], [1]). To the best of our knowledge, the issue of the regularity of the attractor, also called the asymptotical smoothing effect, was first addressed in [9] for the cubic equation on the torus, then in [2] for the equation on the line. For the

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sub-critical NLS in dimension two we refer to [10] and [11]. It is worthwhile to point out that these results were proved considering smooth solutions, i.e., solutions that belong to the energy space H^1. For low regular L^2 solutions the existence and regularity of a global attractor was proved in [12] on the line and in [17] for the torus.

We now describe the equation under consideration in this article. Consider here a parametric nonlinear Schrödinger equation (PNLS) that reads

$$u_t + au - iu_{xx} + i\lambda u + i|u|^2 u + i\gamma\overline{u} = 0, \tag{1.1}$$

where the unknown u maps $\mathbb{R}_t \times \mathbb{R}_x$ into $\mathbb{C}$. Here $a > 0$ (the damping parameter) and $\lambda > 0$ competes with the term $\gamma\overline{u}$ that introduces energy in the low-frequency modes of the solution. The Schrödinger equation is a defocusing NLS equation: if $\gamma = 0$ any solution converges towards the rest state 0. The energy comes into the system through low-frequency modes that are unstable for the linearized equation, and propagates to all modes due to the nonlinear effects.

Throughout this article we assume that $x \mapsto \gamma(x)$ is a real-valued function in $L^\infty(\mathbb{R})$ whose derivative $\dot{\gamma}$ belongs to $L^2(\mathbb{R})$. At this stage γ could be a constant; actually, to ensure some dissipativity for the equation, we assume below that γ is a function that converges to 0 when $|x|$ goes to infinity.

Our main results are stated as follows:

Theorem 1.1. *Assume that*

$$\lim_{|x| \to +\infty} \gamma(x) = 0. \tag{1.2}$$

Then, the dynamical system associated with (1.1) *has a compact global attractor* $\mathcal{A}$ *in* $H^1(\mathbb{R})$*. Moreover,* $\mathcal{A}$ *is a compact subset of* $H^3(\mathbb{R})$.

Theorem 1.2. *Assume that*

$$\int_{\mathbb{R}} (1 + x^2)\gamma(x)^2 \, dx < +\infty. \tag{1.3}$$

Then, the dynamical system associated with (1.1) *has a compact global attractor* $\mathcal{A}$ *in* $H^1(\mathbb{R})$*, whose Hausdorff dimension is finite.*

The article is organized as follows. In the first section we give an overview of the initial-value problem in the energy space $H^1(\mathbb{R})$. In the second section we prove that if the function $\gamma(x)$ decays fast enough at infinity then there exists a compact global attractor in the energy space for the dynamical system associated with (1.1); moreover, we prove the regularity of this global attractor. In the last section we prove that under some further restrictions on $\gamma(x)$ this global attractor has finite dimension in the energy space. We then summarize our results in a short conclusion.

We end this introduction with some notation. We define a *weight function* as a nonnegative function $p(x)$ defined on the line. The Hilbert space $L^2_p(\mathbb{R})$ is the completion of the Schwartz class for the norm

$$\|u\|_{L^2_p} = \Big(\int_{\mathbb{R}} |u(x)|^2 p(x)\, dx \Big)^{\frac{1}{2}}.$$

Hence for $p = 1$ we have the usual $L^2(\mathbb{R})$ space equipped with the scalar product $\mathrm{Re} \int_{\mathbb{R}} u\bar{v}\, dx$. Classical Sobolev spaces such as $H^1(\mathbb{R})$ will be used. $H^1(\mathbb{R})$ is endowed with the usual scalar product

$$\mathrm{Re} \int_{\mathbb{R}} (u_x \bar{v}_x + u\bar{v})\, dx.$$

The letter c denotes a numerical constant that may vary from one line to another without notice. The letter K denotes a generic constant that depends on the data a, λ, and γ, which may also vary from one line to another.

2. The initial-value problem

2.1. **Solving locally in time.** The arguments below are standard (see [4], [8], and [18] for instance). We just give an overview of the method, pointing out the differences with the usual cubic nonlinear Schrödinger equation.

To solve equation (1.1), we perform a fixed-point argument on Duhamel's form of the equation. Let us introduce $U_a(t)$, the linear group of applications on $H^1(\mathbb{R})$ defined by the following: $u(t) = U(t)u_0$ if and only if u solves the linear equation

$$u_t + au - iu_{xx} + i\lambda u = 0, \tag{2.1}$$

on $[0, t]$, supplemented with initial data $u(0) = u_0$. It is standard to check that the norm of the linear operator $U_a(t)$ on any H^s space is $\|U_a(t)\| = \exp(-at)$. Fix $T > 0$. Consider a closed ball of $C([0, T]; H^1(\mathbb{R}))$ whose size R depends on the initial data u_0 through its H^1 norm. We seek a "mild" solution for the equation, that is, an element of the closed ball satisfying

$$u(t) = U_a(t)u_0 - i \int_0^t U_a(t-s) \big((\gamma + u^2)\bar{u} \big)\, ds, \tag{2.2}$$

for all $t \in [0, T]$. Since $H^1(\mathbb{R})$ is a Banach algebra, the mapping $u \mapsto u^2\bar{u}$ is locally Lipschitz. It is an exercise to check that if γ is a bounded function whose first derivative is in $L^2(\mathbb{R})$, then the map $u \mapsto \gamma\bar{u}$ is also locally Lipschitz in $H^1(\mathbb{R})$. Hence, if T is small enough depending on $\|u_0\|_{H^1(\mathbb{R})}$, we can perform the fixed-point argument and construct a local-in-time solution on $[0,T]$. This "mild" solution is a solution of the PDE (1.1) in the distributional sense (see [4]). Moreover, the map $u_0 \mapsto u(t)$ is continuous in H^1, for every $t \in [0,T]$.

2.2. **From local to global solutions.** The solutions constructed by the method quoted above satisfy the following alternative: for a given u_0 in $H^1(\mathbb{R})$, either the trajectory $u(t)$ blows up in $H^1(\mathbb{R})$ in finite positive time, or its lifespan is $[0,+\infty)$. We prove that the first assertion of this alternative cannot occur. The method proceeds as follows: we regularize the initial data (and indeed the solution), and we prove some a priori estimates or energy equalities on the regularized solutions; these estimates remain valid on the original solution using a limiting argument. Since this method is once again standard (see [4]), we just indicate below how to derive the energy equalities, which will be useful in the sequel.

For this purpose consider first the scalar product in $L^2(\mathbb{R})$ of (1.1) with $2u$. We then obtain

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R})} + 2a\|u\|^2_{L^2(\mathbb{R})} = 2\,\mathrm{Im}\int_{\mathbb{R}} \gamma\bar{u}^2\,dx \le 2\|\gamma\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})}. \tag{2.3}$$

The Gronwall lemma leads to

$$\|u\|^2_{L^2(\mathbb{R})} \le \|u_0\|^2_{L^2(\mathbb{R})} e^{2(\|\gamma\|_{L^\infty(\mathbb{R})}-a)t}. \tag{2.4}$$

This proves that the solution cannot blow up in $L^2(\mathbb{R})$ in finite time. Consider now the scalar product in $L^2(\mathbb{R})$ of (1.1) with $2i(u_t + au)$. This yields

$$\frac{d}{dt}J(u) + 2aJ(u) = -a\|u\|^4_{L^4(\mathbb{R})} \le 0, \tag{2.5}$$

where

$$J(u) = \|u_x\|^2_{L^2(\mathbb{R})} + \lambda\|u\|^2_{L^2(\mathbb{R})} + \mathrm{Re}\int \gamma\overline{u}^2 dx + \frac{1}{2}\|u\|^4_{L^4(\mathbb{R})}. \tag{2.6}$$

We then have an upper bound on $J(u(t))$ on any finite time interval. We then obtain

$$J(u) \ge \|u_x\|^2_{L^2(\mathbb{R})} + (\lambda - \|\gamma\|_{L^\infty(\mathbb{R})})\|u\|^2_{L^2(\mathbb{R})}. \tag{2.7}$$

The upper bounds on J and on the L^2 norm of u ensure that the L^2 norm of u_x cannot blow up in finite time.

We sum up and rephrase the results of the first section in the following proposition:

Proposition 2.1. *There exists a nonlinear semigroup* $S(t) : u_0 \mapsto u(t)$ *defined on* $H^1(\mathbb{R})$ *associated with the solution of* (1.1); $S(t)$ *is a continuous mapping which is defined for all* $t \geq 0$.

Remark 2.2. Actually we can also solve the equation (1.1) for $t < 0$ and prove that the mapping $S(t)$ is defined for any $t \in \mathbb{R}$, which makes it a group on $\mathbb{R}$.

3. Existence and regularity of the global attractor

3.1. **Dissipativity in $H^1(\mathbb{R})$.** To enforce some dissipativity on the equation, we now assume that the assertion (1.2) is valid, that is, that γ converges towards 0 when $|x|$ goes to the infinity. We first prove the existence of some *absorbing set* in $H^1(\mathbb{R})$

Proposition 3.1. *There exists a* (*closed*) *ball* B *in* $H^1(\mathbb{R})$ *that captures all trajectories in finite time; i.e., for any* u_0 *in* $H^1(\mathbb{R})$, *there exists a time* T *that depends on* $\|u_0\|_{H^1(\mathbb{R})}$ *such that* $S(t)u_0 \in B$ *for* $t \geq T$. *In the sequel,* M_1 *denotes the radius of this absorbing ball.*

Proof. Using (1.2) we know that there exists $R = R(\gamma) > 0$ such that for $|x| \geq R$, $|\gamma(x)| \leq \frac{\lambda}{2}$. We now use

$$\begin{aligned}\Big|\int \gamma\bar{u}^2\,dx\Big| &\leq \|\gamma\|_{L^\infty(\mathbb{R})}\int_{|x|\leq R}|u|^2\,dx + \frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R})} \\ &\leq \sqrt{R}\|\gamma\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^4(\mathbb{R})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{3.1}$$

Gathering (3.1) and (2.6) leads to

$$J(u) \geq \|u_x\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R})} - \frac{R\|\gamma\|^2_{L^\infty(\mathbb{R})}}{2}. \tag{3.2}$$

The energy dissipation (2.5), together with (3.2), yields the existence of a bounded set D in $H^1(\mathbb{R})$ that captures all the bounded sets of $H^1(\mathbb{R})$ in finite time. Enlarging this bounded set to a ball concludes the proof of the proposition. □

Remark 3.2. To ensure the existence of an absorbing ball, we can relax assumption (1.2) as

$$\limsup_{|x|\to+\infty}|\gamma(x)| < \lambda.$$

We prefer to deal with (1.2); this amounts to assuming that the operator $u \mapsto \gamma\bar{u}$ is a compact perturbation of $\mathrm{Id} - \Delta$. This will transpire in the next section.

3.2. **Asymptotic compactness for the trajectories.** The route to proving the existence of a global attractor in $H^1(\mathbb{R})$ consists of three steps. To begin with, we prove the compactness in the $L^2(\mathbb{R})$ strong topology using an argument due to P. Laurençot [14]. Then, we prove the existence of a global attractor in the weak topology of $H^1(\mathbb{R})$ following the method in [7]. Then we conclude using the well-known J. Ball's argument [3].

Lemma 3.3. *Consider B the $H^1(\mathbb{R})$ absorbing set. There exists a compact set $\mathcal{K}$ in $L^2(\mathbb{R})$ that attracts $S(t)B$ when t goes to $+\infty$.*

Proof. We follow here the method in [14]. Consider $u(t)$, a trajectory that belongs to the absorbing ball B for $t > t_0$. Set M_1 for the radius of this ball. Up to the translation $t \mapsto t - t_0$, we may assume for the sake of simplicity that $t_0 = 0$. For $\varepsilon > 0$ introduce γ_ε and u_0^ε in $C_0^\infty(\mathbb{R})$ such that

$$\|\gamma - \gamma_\varepsilon\|_{L^\infty(\mathbb{R})} + \|\dot\gamma - \dot\gamma_\varepsilon\|_{L^2(\mathbb{R})} + \|u_0 - u_0^\varepsilon\|_{H^1(\mathbb{R})} \le \varepsilon. \tag{3.3}$$

Then, introduce the nonautonomous splitting of $u(t)$, given by

$$v_t + av - iv_{xx} - \varepsilon v_{xx} + i\lambda v + i|u|^2 v + i\gamma_\varepsilon \overline{u} = 0, \tag{3.4}$$

$$w_t + aw - iw_{xx} - \varepsilon w_{xx} + i\lambda w + i|u|^2 w + i(\gamma - \gamma_\varepsilon)\overline{u} = -\varepsilon u_{xx}, \tag{3.5}$$

supplemented respectively with initial data $v(0) = u_0^\varepsilon$ and $w(0) = u_0 - u_0^\varepsilon$.

To begin with, we prove that w remains small for positive times. Consider the scalar product of (3.5) with $2w$. Thanks to (3.3), this leads to

$$\begin{aligned}
&\frac{d}{dt}\|w\|^2_{L^2(\mathbb{R})} + 2a\|w\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|w_x\|^2_{L^2(\mathbb{R})} \\
&= 2\operatorname{Im}\int_{\mathbb{R}} (\gamma - \gamma_\varepsilon)\overline{u}\bar{w}\,dx + 2\varepsilon \operatorname{Re}\int_{\mathbb{R}} u_x \bar{w}_x\,dx \\
&\le 2\varepsilon(\|u\|_{L^2(\mathbb{R})}\|w\|_{L^2(\mathbb{R})} + \|u_x\|_{L^2(\mathbb{R})}\|w_x\|_{L^2(\mathbb{R})}) \\
&\le \varepsilon(\|w\|^2_{H^1(\mathbb{R})} + \|u\|^2_{H^1(\mathbb{R})}).
\end{aligned} \tag{3.6}$$

Assuming without loss of generality that $\varepsilon \le a$, we infer from (3.6), from the fact that u remains in the absorbing ball B and from Gronwall's lemma that

$$\|w(t)\|^2_{L^2(\mathbb{R})} \le \max(\varepsilon^2, \frac{\varepsilon M_1^2}{a}). \tag{3.7}$$

Now we prove that v remains in a bounded set of $H^1(\mathbb{R})$. Using the identity $v = u - w$, we already know that v remains for $t \ge 0$ in a bounded set of $L^2(\mathbb{R})$. Then consider the scalar product of (3.4) with $-2v_{xx}$. This leads to

$$\frac{d}{dt}\|v_x\|^2_{L^2(\mathbb{R})} + 2a\|v_x\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|v_{xx}\|^2_{L^2(\mathbb{R})}$$

$$= -2\operatorname{Im}\int_{\mathbb{R}} |u|^2 v\overline{v}_{xx}\,dx - 2\operatorname{Im}\int_{\mathbb{R}} \overline{v}_{xx}\overline{u}\gamma_\varepsilon\,dx \tag{3.8}$$
$$\le (2\|u\|^2_{L^\infty(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} + 2\|\gamma_\varepsilon\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})})\|v_{xx}\|_{L^2(\mathbb{R})}.$$

From this inequality and the bounds for u in $H^1(\mathbb{R})$ and for v in $L^2(\mathbb{R})$, we easily infer that v, and then $w = u - v$, remain in a bounded set of $H^1(\mathbb{R})$.

We now prove an estimate for v in the weighted space L^2_p for $p(x) = 1+x^2$. Consider the scalar product of (3.4) with $-2x^2 v$. This leads to

$$\frac{d}{dt}\|xv\|^2_{L^2(\mathbb{R})} + 2a\|xv\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|xv_x\|^2_{L^2(\mathbb{R})}$$
$$= 2\varepsilon\|v\|^2_{L^2(\mathbb{R})} + 4\operatorname{Im}\int_{\mathbb{R}} xv\overline{v}_x\,dx + 2\operatorname{Im}\int_{\mathbb{R}} \overline{v}\overline{u}x^2\gamma_\varepsilon\,dx \tag{3.9}$$
$$\le 2\varepsilon\|v\|^2_{L^2(\mathbb{R})} + (4\|v\|_{L^2(\mathbb{R})} + 2\|x\gamma_\varepsilon\|_{L^2(\mathbb{R})}\|u\|_{L^\infty(\mathbb{R})})\|xv\|_{L^2(\mathbb{R})}.$$

We straightforwardly infer from this estimate and the previous bounds that xv remains in a bounded set of $L^2(\mathbb{R})$.

At this stage we have proved the following: for any $\varepsilon > 0$ there exists a bounded set $\mathcal{K}_\varepsilon$ in $L^2_p \cap H^1(\mathbb{R})$ such that for $t > T_\varepsilon$ we have $S(t)B \subset B_{L^2(\mathbb{R})}(0, M_1\sqrt{\varepsilon}) + \mathcal{K}_\varepsilon$. Since the embedding $L^2_p \cap H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ is compact, the proof of the lemma is complete. □

We now follow the guidelines in [7]. To begin with we state a lemma whose proof is a carbon copy of the proof in [7]; the proof, which uses essentially that $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is an algebra (and the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$), is omitted for the sake of conciseness.

Lemma 3.4. *The semigroup $S(t)$ is continuous on B (or in any bounded set of $H^1(\mathbb{R})$) for the $L^2(\mathbb{R})$ strong topology.*

With this lemma, we can endow the absorbing set B with the weak topology in $H^1(\mathbb{R})$, which is a complete metric space for this topology. A mere consequence of the previous lemma is that $S(t)$ is continuous for the weak topology in $H^1(\mathbb{R})$. Since for this topology B is a compact set, we apply directly Theorem I.1.1 in [21] and we have

Proposition 3.5. *The semigroup $S(t)$ restricted to B has a global attractor $\mathcal{A}$ that attracts all the trajectories for the weak $H^1(\mathbb{R})$ topology.*

We complete this subsection by proving that this weak global attractor is a compact global attractor in the usual $H^1(\mathbb{R})$ strong sense, using the so-called J. Ball's argument. We apply the following proposition (see [22]):

Proposition 3.6. *Consider $S(t)$ a semigroup asymptotically compact in $L^2(\mathbb{R})$ which satisfies an energy equality*

$$\frac{d}{dt}(\|u_x\|^2_{L^2(\mathbb{R})} + \tilde{J}(u)) + 2a\|u_x\|^2_{L^2(\mathbb{R})} + \tilde{J}(u)) + G(u) = 0, \tag{3.10}$$

where $\tilde{J}$ and G are continuous for the weak topology in $H^1(\mathbb{R})$. Then, the weak global attractor is actually an $H^1(\mathbb{R})$ attractor in the usual sense.

To check the assumptions is easy. Going back to (2.5), we have that $G(u) = \frac{a}{2}\|u\|^4_{L^4(\mathbb{R})}$ is continuous, since any weakly $H^1(\mathbb{R})$ convergent sequence which is compact in $L^2(\mathbb{R})$ converges in the $L^4(\mathbb{R})$ strong topology. The function

$$\tilde{J}(u) = \lambda\|u\|^2_{L^2(\mathbb{R})} + \mathrm{Re}\int \gamma\overline{u}^2\,dx + \frac{1}{2}\|u\|^4_{L^4(\mathbb{R})}$$

satisfies the same properties. Hence, we have completed the proof of the following statement:

Proposition 3.7. *The dynamical system associated to* (1.1) *has a compact global attractor $\mathcal{A}$ in $H^1(\mathbb{R})$.*

3.3. **Regularity of the attractor.** In this subsection, we complete the proof of Theorem 1.1; i.e., we prove that $\mathcal{A}$ is a compact subset of $H^3(\mathbb{R})$. We follow closely the method in [2].

To begin with, we prove that $\mathcal{A}$ is a bounded subset of $H^3(\mathbb{R})$. We mimic the strategy in [9]. Consider a complete trajectory that belongs to the global attractor. We have a bound for $\|u(t)\|_{H^1} + \|u_t(t)\|_{H^{-1}}$. Set K for this upper bound (throughout this section the letter K denotes a constant that depends on a, γ, and λ and that may vary from one line to another without notice). We split u using Fourier transform as follows:

$$\begin{aligned} u(t) &= y(t) + z(t) = P_N u(t) + Q_N u(t) \\ &= 2\pi\int_{|\xi|\le N} \hat{u}(\xi,t)e^{i\xi x}\,d\xi + 2\pi\int_{|\xi|\ge N} \hat{u}(\xi,t)e^{i\xi x}\,d\xi. \end{aligned} \tag{3.11}$$

The low-frequency part $y(t)$ belongs to all Sobolev spaces due to classical inverse (or Bernstein's) inequalities:

Lemma 3.8. *For any $s \ge 0$ and for all y that belong to $P_N L^2(\mathbb{R})$,*

$$\|y\|_{H^s(\mathbb{R})} \le cN^s\|y\|_{L^2(\mathbb{R})}.$$

Considering the projection of the equation (1.1) onto the high-frequencies space $Q_N H^1(\mathbb{R})$, we observe that z is solution to

$$z_t + az - iz_{xx} + i\lambda z + iQ_N(u^2\bar{u}) + iQ_N\gamma\bar{u} = 0, \tag{3.12}$$

supplemented with initial data $z(0) = Q_N u_0$. We track this high mode component of u with Z that solves

$$Z_t + aZ - iZ_{xx} + i\lambda Z + iQ_N((y+Z)^2(\overline{y+Z})) + iQ_N\gamma(\overline{y+Z}) = 0, \tag{3.13}$$

supplemented with $Z(0) = 0$. We state and prove

Proposition 3.9. *For $N \geq N_0(\gamma, a; \lambda)$ large enough depending on the data,* (3.13) *has a unique solution $Z \in C(\mathbb{R}_+, H^3(\mathbb{R}))$ that satisfies moreover*

$$\|Z(t)\|_{H^3(\mathbb{R})} \leq KN^2, \tag{3.14}$$

$$\|Z(t) - z(t)\|_{H^1(\mathbb{R})} \leq Ke^{-at}, \tag{3.15}$$

where K is a constant that depends on the data a, γ, and λ.

Proof. To begin with, we prove

$$\|Z(t)\|_{H^1(\mathbb{R})} \leq K. \tag{3.16}$$

It is worthwhile to point out that K is independent of N and of $t \geq 0$. Consider the scalar product of (3.13) with $i(Z_t + aZ)$. This yields

$$\begin{aligned} &\frac{1}{2}\frac{dJ_1(Z)}{dt} + aJ_1(Z) + \frac{a}{2}\|y+Z\|_{L^4}^4 \\ &= \operatorname{Re}\int_{\mathbb{R}} (\overline{y_t + ay})(|y+Z|^2(y+Z) + \gamma\overline{y+Z})\,dx, \end{aligned} \tag{3.17}$$

where

$$J_1(Z) = \|Z_x\|_{L^2}^2 + \lambda\|Z\|_{L^2}^2 + \frac{1}{2}\|y+Z\|_{L^4}^4 + \operatorname{Re}\int_{\mathbb{R}} \gamma(\overline{y+Z})^2\,dx. \tag{3.18}$$

We now prove that this functional is coercive on $H^1(\mathbb{R})$, namely,

$$J_1(Z) \geq \frac{1}{2}\|Z_x\|_{L^2}^2 - K.$$

The key argument is the following enhanced Poincaré inequality: for any $Z \in Q_N H^1(\mathbb{R})$,

$$\|Z\|_{L^2(\mathbb{R})} \leq N^{-1}\|Z_x\|_{L^2(\mathbb{R})}. \tag{3.19}$$

Considering for instance the last term on the right-hand side of (3.18), we have the following inequality:

$$|\operatorname{Re}\int_{\mathbb{R}} \gamma(\overline{y+Z})^2\,dx| \leq \gamma_\infty\|y+Z\|_{L^2}^2 \leq K(1 + \frac{\|Z_x\|_{L^2}^2}{N^2}). \tag{3.20}$$

From this we infer the coercivity of the functional on $H^1(\mathbb{R})$ assuming that N is large enough.

Now we seek an upper bound for (3.17). The more difficult term to handle is

$$q = \mathrm{Re} \int_{\mathbb{R}} y_t(\overline{y+Z})(|y+Z|^2)\, dx.$$

We proceed as follows:

$$|q| \leq \|y_t\|_{H^{-1}} \|(y+Z)|y+Z|^2\|_{H^1} \leq c\|y_t\|_{H^{-1}} \|y+Z\|_{H^1} \|y+Z\|^2_{L^\infty}. \quad (3.21)$$

On the one hand,

$$\|y_t\|_{H^{-1}} \leq \|u_t\|_{H^{-1}} \leq K.$$

On the other hand, due to the Agmon inequality, we have

$$\|y+Z\|^2_{L^\infty} \leq c\|y+Z\|_{L^2}\|y_x+Z_x\|_{L^2} \leq K(1+\frac{\|Z_x\|^2_{L^2}}{N}). \quad (3.22)$$

The right-hand side of (3.17) can be bounded above by

$$K(1+\frac{\|Z_x\|^3_{L^2}}{N}).$$

Now we infer from (3.17)–(3.22) and the coercivity of J on H^1 that

$$\frac{1}{2}\frac{dJ_1(Z)}{dt} + aJ_1(Z) \leq K(1+\frac{J_1(Z)^{\frac{3}{2}}}{N}). \quad (3.23)$$

Integrating this between 0 and t,

$$J_1(Z(t)) \leq J(Z(0))e^{-2at} + \frac{K}{N}\int_0^t e^{2a(s-t)} J_1(Z(s))^{\frac{3}{2}}(s)\, ds. \quad (3.24)$$

On the one hand,

$$J_1(Z(0)) = \frac{1}{2}\|y\|^4_{L^4} + \mathrm{Re}\int_{\mathbb{R}} \gamma\overline{y}^2\, dx \leq K.$$

On the other hand, using once again the coercivity of J_1,

$$\|Z_x(t)\|^2_{L^2} \leq K + \frac{\tilde{K}}{N}\int_0^t e^{2a(s-t)}\|Z_x(s)\|^3_{L^2}\, ds. \quad (3.25)$$

Introduce $\mu(t) = \sup_{0\leq s\leq t}\|Z_x(s)\|_{L^2}$. Now we have

$$\mu(t)^2 \leq K(1+\frac{\mu(t)^3}{N}) \leq K_0(1+\frac{\mu(t)^4}{N}). \quad (3.26)$$

Since $\mu(0) = 0$, if $N > 4K_0$, then $\mu(t)^2$ is trapped in $[0, k]$ where k is the first root of

$$X^2 - \frac{N}{K_0} X + N = 0,$$

which is uniformly bounded with respect to N. Hence (3.16) is proved.

Now we establish (3.14). Proving an a priori estimate on Z in H^3 amounts to proving an H^1 estimate on Z_t, since $Z_t = i\Delta Z +$ lower-order terms. The function $\xi = Z_t$ is solution to

$$\begin{aligned} \xi_t + a\xi - i\xi_{xx} + i\lambda\xi + iQ_N((y+Z)^2(\overline{y_t + \xi})) \\ + 2iQ_N|y+Z|^2\overline{y_t + \xi} + iQ_N\gamma(\overline{y_t + \xi}) = 0, \end{aligned} \tag{3.27}$$

supplemented with initial condition

$$\xi(0) = -iQ_N(y^2\overline{y}) + iQ_N\gamma\overline{y}$$

in H^1. The proof for (3.14) is similar to that of (3.16). Indeed, considering the scalar product of (3.27) with $i(\xi_t + a\xi)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}J_2(\xi) + aJ_2(\xi) \\ &= \operatorname{Re}\int_{\mathbb{R}} ((y+Z)^2\overline{y}_t)_t\overline{\xi}\,dx + \operatorname{Re}\int_{\mathbb{R}} \overline{\xi}^2(y_t + Z_t)(y+Z)\,dx + \cdots, \end{aligned} \tag{3.28}$$

where

$$J_2(\xi) = \|\xi_x\|_{L^2}^2 + \lambda\|\xi\|_{L^2}^2 + \operatorname{Re}\int_{\mathbb{R}} (y+Z)^2(\overline{y}_t\overline{\xi} + \frac{1}{2}\overline{\xi}^2\,dx + \cdots, \tag{3.29}$$

where $\cdots$ denote similar or lower-order terms. We just have to prove the coercivity of J_2 on H^1 and to provide an upper bound for the right-hand side of (3.28). For instance, we have

$$\begin{aligned} &\left|\operatorname{Re}\int_{\mathbb{R}} ((y+Z)^2\overline{y}_t)_t\overline{\xi}\,dx\right| \\ &\le c(\|y_{tt}\|_{H^{-1}}\|(y+Z)^2\xi\|_{H^1} + \|y_t + Z_t\|_{H^{-1}}\|y_t(y+Z)\xi\|_{H^1}). \end{aligned} \tag{3.30}$$

Then we make use of the inverse inequality

$$\|y_{tt}\|_{H^{-1}} + \|y_t\|_{H^1} \le c\|y\|_{H^3} \le KN^2.$$

We skip the rest of the proof for the sake of conciseness.

Now we proceed to the proof of (3.15). Set $v = y + Z$ and seek an upper bound for $w = v - u = Z - z$ for large times. The function w is solution to

$$w_t + aw - iw_{xx} + i\lambda w + iQ_N(v^2\overline{v} - u^2\overline{u}) + iQ_N\gamma\overline{w} = 0. \tag{3.31}$$

Considering the scalar product of (3.31) with $\overline{w_t} + a\overline{w}$, we obtain

$$\frac{1}{2}\frac{d}{dt}G(w) + aG(w) = \frac{1}{2}\operatorname{Re}\int_{\mathbb{R}} (uv)_t \overline{w}^2\,dx + \operatorname{Re}\int_{\mathbb{R}} (\overline{u}u_t + \overline{v}v_t)|w|^2\,dx, \quad (3.32)$$

where

$$G(w) = \|w_x\|_{L^2}^2 + \lambda\|w\|_{L^2}^2 + \operatorname{Re}\int_{\mathbb{R}} (\gamma + uv)\overline{w}^2\,dx + \operatorname{Re}\int_{\mathbb{R}} (|u|^2 + |v|^2)|w|^2\,dx.$$

Proceeding as above and using that w satisfies the enhanced Poincaré inequality (3.19), we prove that if N is large enough, then $G(w)$ is coercive on H^1. Indeed, if M_∞ denotes the L^∞ upper bound on the attractor, we have

$$\left|\operatorname{Re}\int_{\mathbb{R}} (\gamma + uv)\overline{w}^2\,dx\right| \leq (\|\gamma\|_{L^\infty} + M_\infty^2)\|w\|_{L^2}^2 \leq K\frac{\|w_x\|_{L^2}^2}{N^2}. \quad (3.33)$$

It remains to control the right-hand side in (3.32). We proceed as above. For instance

$$\begin{aligned} \operatorname{Re}\int_{\mathbb{R}} \overline{u}u_t|w|^2\,dx &\leq \|u_t\|_{H^{-1}}\|u|w|^2\|_{H^1} \quad (3.34)\\ &\leq \|u_t\|_{H^{-1}}\|u\|_{H^1}\|w\|_{L^\infty}\|w\|_{H^1}. \end{aligned}$$

Due to the Agmon inequality and to the estimates on the attractor, this term is bounded above by $\|w_x\|_{L^2}^2 KN^{-\frac{1}{2}}$. Therefore, if N is large enough we infer from (3.32) that

$$\frac{1}{2}\frac{dG(w)}{dt} + aG(w) \leq KN^{-\frac{1}{2}}G(w) \leq \frac{a}{2}G(w).$$

This yields $G(t) \leq e^{-at}G(0)$, and then (3.15) is proved.

It remains to prove that the global attractor is bounded in $H^3(\mathbb{R})$. We proceed as follows. Consider a complete trajectory $u(t)$ in $\mathcal{A}$. For $m > 0$, we approximate $u(t)$ for $t \geq -m$ as follows. Solve for $t \geq -m$ the equation (3.13) with initial data $Z(-m) = 0$. Let Z^m denote the solution for this equation. Set

$$v^m(t) = y(t) + Z^m(t).$$

Due to Proposition 3.9, the sequence $v^m(0)$ is bounded in $H^3(\mathbb{R})$ and converges towards $u(0)$ when m converges to infinity. This implies that $u(0)$ belongs to a bounded set in $H^3(\mathbb{R})$, and then the whole attractor satisfies the same property.

To complete the proof of Theorem 1.2, we have to prove the compactness of this set in $H^3(\mathbb{R})$. This is standard due to J. Ball's well-known argument (we already know that $\mathcal{A}$ is a bounded set in $H^3(\mathbb{R})$). Therefore, we omit the proof for the sake of conciseness.

4. The global attractor has finite dimension

This section is devoted to the proof of Theorem 1.2. We assume here that γ belongs to a weighted space. Hence it appears that the global attractor is also a subset of this weighted space. Then we use this information to prove, following the method described in [21], that the semigroup contracts m-dimensional small volumes. This requires some information on a min-max problem. We handle this min-max problem introducing a suitable Hilbert–Schmidt operator.

4.1. **Functions in the global attractor belong to some weighted space.**

We state and prove

Proposition 4.1. *Set $p(x) = 1 + x^2$. If γ belongs to $L^2_p(\mathbb{R})$, then the global attractor $\mathcal{A}$ is included into a bounded set of $L^2_p(\mathbb{R})$.*

Proof. Consider a complete trajectory u in $\mathcal{A}$. We split $u = v + w$ such that

$$\begin{cases} v_t - iv_{xx} = i|u|^2 v - av - i\lambda v, \\ v(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{4.1}$$

and

$$\begin{cases} w_t - iw_{xx} = i|u|^2 w - aw - i\lambda w - i\gamma \overline{u}\ , \\ w(0,x) = 0. \end{cases} \tag{4.2}$$

Considering the scalar product of (4.1) with v, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2} + a\|v\|^2_{L^2} = 0. \tag{4.3}$$

Integrating in time, we then infer that for positive times, $v(t)$ remains trapped in a bounded set of $L^2(\mathbb{R})$. Moreover,

$$\|v(t)\|_{L^2(\mathbb{R})} \le \exp(-\frac{a}{2}t).$$

Considering the scalar product of (4.1) with v_{xx}, we also have

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2_{L^2} + a\|v_x\|^2_{L^2} \le c\|u_x\|_{L^2}\|u\|_{L^\infty}\|v_x\|_{L^2}\|v\|_{L^\infty}. \tag{4.4}$$

Since u remains in a bounded set of $H^1(\mathbb{R})$, we obtain by direct computations that for positive times, v remains in a bounded set of $H^1(\mathbb{R})$.

Now we prove a weighted estimate for w. For this purpose we consider the scalar product of (4.2) with $x^2 w$. This leads to

$$\frac{1}{2}\frac{d}{dt}\|xw\|^2_{L^2} + a\|xw\|^2_{L^2} = \int x^2\,\mathrm{Im}(\overline{w}w_{xx}) + \int x^2\,\mathrm{Im}(\gamma\overline{uw}). \tag{4.5}$$

Integrating by parts and using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|xw\|_{L^2}^2 + a\|xw\|_{L^2}^2 \\ &\leq c(\|x\gamma\|_{L^2}\|w\|_{L^\infty}\|u\|_{L^\infty} + \|w_x\|_{L^2})\|xw\|_{L^2(\mathbb{R})}.\end{aligned} \tag{4.6}$$

We infer from this and from the fact that u and w remain bounded in $H^1(\mathbb{R})$ that there exists K such that for any t positive, $\|xw(t)\|_{L^2} \leq K$. Then we have a splitting of the semigroup on the global attractor that reads

$$S(t)u_0 = \tilde{S}_1(t)u_0 + \tilde{S}_2(t)u_0 = v(t) + w(t)$$

where, when t goes to $+\infty$, the first part converges uniformly to 0 and the second one is trapped in a bounded set of $L^2_p(\mathbb{R})$. This concludes the proof of the proposition. □

4.2. **Contracting the m-dimensional infinitesimal volumes.** We follow here [21]. By the fundamental theorem of calculus, proving that $S(t)$ contracts m-dimensional small volumes amounts to proving that the derivative $DS(t)u_0$ contracts m-volumes. To begin with, we state

Proposition 4.2. *The map $u_0 \mapsto S(t)u_0$ has a derivative $v = DS(t)u_0 \cdot h$ in $H^1(\mathbb{R})$ that satisfies the nonautonomous PDE*

$$v_t + av - iv_{xx} + i\lambda v + i\gamma\overline{v} + i|u|^2 v + 2i\,\mathrm{Re}(\overline{u}v)u = 0, \tag{4.7}$$

supplemented with initial data $v(0) = h$.

Proof. The proof is lengthy but straightforward since $H^1(\mathbb{R})$ is a Banach algebra. We skip the proof for the sake of conciseness. We refer the reader to [15] for a complete proof. □

Consider now $h_1, \ldots, h_m$ that are orthogonal in $H^1(\mathbb{R})$. We flow the parallelogram $0, h_1, \ldots, h_m$ by $DS(t)u_0$. Set $v^j(t) = DS(t)u_0 \cdot h_j$. Introduce the volume element as the Gram determinant

$$G_m = \| v^1(t) \wedge \cdots \wedge v^m(t) \|_{H^1}^2 = \det_{1\leq i,j\leq m}(v^i(t), v^j(t))_{H^1}. \tag{4.8}$$

Consider, for any j, the scalar product of (4.7) with $i(v_t + av)$. This leads to

$$\frac{1}{2}\frac{d}{dt}J_0(v) + aJ_0(v) = 2\int \mathrm{Re}(\overline{u}v)\,\mathrm{Re}(\overline{u_t}v) + \int \mathrm{Re}(\overline{u}u_t)|v|^2, \tag{4.9}$$

where

$$J_0(v) = \|v_x\|_{L^2}^2 + \lambda\|v\|_{L^2}^2 + \mathrm{Re}\int \gamma\overline{v}^2 + \int |u|^2|v|^2 + 2\int(\mathrm{Re}\,\overline{u}v)^2.$$

Set

$$J_\beta(v) = J_0(v) + \beta\|v\|_{L^2}^2,$$

with $\beta \geq 3M_\infty + \|\gamma\|_{L^\infty}$, where M_∞ is the L^∞ upper bound for trajectories in $\mathcal{A}$. The function $\sqrt{J_\beta}$ defines a norm on $H^1(\mathbb{R})$ which is equivalent to the usual H^1 norm, but which depends on t. We shall use this norm, as in [7], to compute the Gram determinant defined above. The corresponding scalar product reads

$$\Lambda(v,w) = \frac{J_\beta(v+w) - J_\beta(v-w)}{4}. \tag{4.10}$$

We now use as Gram determinant the following:

$$G_m(t) = \det_{1\leq i,j\leq m}(\Lambda(v^i(t), v^j(t))). \tag{4.11}$$

We have

$$\begin{aligned} &\frac{d}{dt}J_\beta(v) + 2aJ_\beta(v) \qquad (4.12)\\ &= 2\beta\int \operatorname{Re}(\overline{u}v)\operatorname{Im}(u\overline{v}) + \beta\int \operatorname{Im}(\gamma\overline{v}^2) + 2\int \operatorname{Re}(\overline{u}v)\operatorname{Re}(\overline{u_t}v). \end{aligned}$$

We seek an upper bound for the right-hand side of this equality. We have

$$\left|\int \operatorname{Re}(\overline{u_t}v)\operatorname{Re}(u\overline{v})\right| \leq \|u_t\|_{L^\infty}\|u\|_{L^2_p}\|v\|_{L^\infty}\|v\|_{L^2_{\frac{1}{p}}}. \tag{4.13}$$

Since the global attractor is bounded in $H^3(\mathbb{R})$, u_t remains bounded in $H^1(\mathbb{R})$. The other terms can be handled similarly, and we have

$$\frac{d}{dt}J_\beta(v) + 2aJ_\beta(v) \leq K\|v\|_{L^\infty}\|v\|_{L^2_{\frac{1}{p}}}. \tag{4.14}$$

Set $\xi(v) = \|v\|^2_{L^2_{\frac{1}{p}}}$ and $\Gamma(v,w) = \frac{\xi(v+w)-\xi(v-w)}{4}$. We use the following statement:

Lemma 4.3. *There exists S, a positive symmetric matrix $m \times m$ that depends on t and m, such that $\Gamma(v,w) = \Lambda(Sv,w)$.*

Introduce $S^{\frac{1}{2}}$ the square root of S. Then G_m satisfies

$$\frac{dG_m}{dt} + (am - TrS^{\frac{1}{2}})G_m = 0. \tag{4.15}$$

It remains to provide an upper bound for $TrS^{\frac{1}{2}}$, using the following min-max principle:

$$TrS^{\frac{1}{2}} = \sum_{l=1}^{l=m} (\lambda_l(t))^{\frac{1}{2}} = \sum_{l=1}^{l=m} \min_{F\subset\mathbb{R}^m,\, \dim F=l} \max_{v\in F,\, v\neq 0} (\frac{\xi(v)}{J_\beta(v)})^{\frac{1}{2}},$$

where the λ are the eigenvalues of S.

For this purpose, we follow the method in [2]. In fact, we have to compute

$$\sum_{l=1}^{l=m} \min_{F\subset\mathbb{R}^m,\, \dim F=l} \max_{v\in F,\, v\neq 0} (\frac{\xi(v)}{\|v\|_{H^1}^2})^{\frac{1}{2}}.$$

Set $H = L^2_{\frac{1}{p}}$ for the Hilbert space corresponding to the weight function $\frac{1}{p}$. Set $(\cdot,\cdot)_H$ for the corresponding scalar product. Introduce $L = p(\mathrm{Id} - \Delta)$, the unbounded operator on H defined by

$$\mathrm{Re} \int (u_x \overline{v}_x + u\overline{v})\, dx = (Lu, v)_H. \tag{4.16}$$

Then L is a self-adjoint unbounded operator such that $D(L^{\frac{1}{2}}) = H^1(\mathbb{R})$. Since the embedding $H^1 \subset H$ is compact, L has compact inverse L^{-1}. Actually, we have

Lemma 4.4. *L^{-1} belongs to the Hilbert–Schmidt class.*

Before moving to the proof of this lemma, let us make the connection with the min-max problem above. Introduce $\lambda_j \leq \lambda_{j+1}$, the increasing sequence of eigenvalues of L. Then, on the one hand,

$$TrS^{\frac{1}{2}} \leq c \sum_{j=1}^{m} \frac{1}{\lambda_j^{\frac{1}{2}}}.$$

On the other hand, if L^{-1} is a Hilbert–Schmidt operator [5], then

$$\sum_{j=1}^{m} \frac{1}{\lambda_j^2} \leq \|L^{-1}\|_{HS} < +\infty.$$

Going back to the computations above, if L^{-1} is Hilbert–Schmidt, then we infer from (4.15) that

$$\frac{dG_m}{dt} + (am - Km^{\frac{3}{4}})G_m \leq 0, \tag{4.17}$$

and G_m decays to 0 if m is large enough; the proof of Theorem 2 is complete. It remains to prove Lemma 4.4.

Proof of Lemma 4.4. To begin with, let us recall that due to the kernel theorem of L. Schwartz there exists a function $k(x,y)$ such that

$$(L^{-1}u,v)_H = \iint k(x,y)u(x)\overline{v(y)}\frac{dx}{1+x^2}\frac{dy}{1+y^2}. \tag{4.18}$$

In fact, proving that L^{-1} belongs to the Hilbert–Schmidt class amounts to proving that (see [23])

$$\iint k^2(x,y)\frac{dx}{1+x^2}\frac{dy}{1+y^2} < \infty. \tag{4.19}$$

We now compute $k(x,y)$ as follows:

$$(L^{-1}u,v)_H = \int (\mathrm{Id}-\Delta)^{-1}(\frac{u(x)}{1+x^2})\frac{\overline{v(x)}}{1+x^2};$$

Using Plancherel's theorem, we have

$$\begin{aligned} 2\pi(L^{-1}u,v)_H &= \int \frac{1}{1+|\tau|^2}\mathcal{F}(\frac{u}{p})\overline{\mathcal{F}(\frac{v}{p})}\,d\tau \\ &= \int \frac{d\tau}{1+|\tau|^2}[\int e^{-ix\tau}\frac{u(x)}{p(x)}\,dx][\int e^{iy\tau}\frac{\overline{v(y)}}{p(y)}\,dy] \\ &= \iiint \frac{e^{-i(x-y)\tau}}{1+|\tau|^2}u(x)\overline{v(y)}\,d\tau\frac{dx}{p(x)}\frac{dy}{p(y)}. \end{aligned}$$

We then obtain

$$k(x,y) = \frac{1}{2\pi}\int \frac{e^{-i(x-y)\tau}}{1+|\tau|^2}\,d\tau = \frac{1}{2}e^{-|x-y|}.$$

This function satisfies (4.19) and the proof of the lemma is complete.

5. Conclusion

In this article, we first have established that the PNLS equation features a compact global attractor in the energy space $H^1(\mathbb{R})$, assuming that the term $\gamma\overline{u}$ providing energy into the system satisfies (1.2). This assumption is restrictive and does not include the simple case $\gamma =$ constant. In this case it is an interesting issue to address the infinite-dimensional dynamical system in the so-called Zhidkov space $X^1(\mathbb{R}) = \{u \text{ in } L^\infty(\mathbb{R}) : u_x \text{ in } L^2(\mathbb{R})\}$ (see [6] and [15]). Even if the PNLS equation defines a semigroup in $X^1(\mathbb{R})$, we do not know if there exists a global attractor in this framework.

We also have proved that if γ converges towards 0 fast enough at infinity, then the global attractor is finite dimensional. For weakly damped NLS equations with a forcing term, it is classical to assume such decay on the forcing term to obtain that the fractal or Hausdorff dimension is finite. We also point out that in our computations, we did not use information on each eigenvalue (or singular value) of the linearized operator, using instead information on the sum of these since the inverse of the linearized operator belongs to some Hilbert–Schmidt class.

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