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A DECAY PROPERTY OF SOLUTIONS TO THE K-GENERALIZED KDV EQUATION

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Abstract. We use a Leibniz–rule-type inequality for fractional derivatives to prove conditions under which a solution $u(x,t)$ of the k-generalized KdV equation is in the space $L^2(|x|^{2s}\,dx)$ for $s \in \mathbb{R}_+$.

1. Introduction

The initial value problem for the modified Korteweg–de Vries equation (mKdV),

$$\partial_t u + \partial_x^3 u + \partial_x(u^3) = 0, \qquad u(x,0) = u_0(x), \tag{1.1}$$

has applications to fluid dynamics (for instance, [17], [22]), plasmas (see [21]), and is known to be an example of an integrable system from [5]. It has also been studied for its relation to the KdV equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0$$

via the Miura transformation (see [19]).

Ginibre and Y. Tsutsumi in [6] proved local well-posedness for (1.1) in a weighted L^2 space. In [14], Kenig, Ponce, and Vega proved local well-posedness for u_0 in the Sobolev space H^s, when $s \geq \frac{1}{4}$ by a contraction-mapping argument in mixed L^p_x and L^q_T spaces. For earlier results see [14], and comments and references therein. Christ, Colliander, and Tao in [2] showed that (1.1) was locally well-posed for $u_0 \in H^s$, when $s \geq \frac{1}{4}$, by using a contraction-mapping argument in the Bourgain spaces $X_{s,b}$. Colliander, Keel, Staffilani, Takaoka, and Tao proved global well-posedness for real initial data $u_0 \in H^s$, $s > \frac{1}{4}$ in [3]. Kishimoto in [16] and Guo in [7] proved global well-posedness for real data in the case $s = \frac{1}{4}$.

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The focus of this work will be (1.1), but we will also consider the k-generalized Korteweg–de Vries equation,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^{k+1}) = 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}. \end{cases} \tag{1.2}$$

When $k \geq 4$, local well-posedness was obtained for initial data $u_0 \in H^s$ with $s \geq \frac{k-4}{2k}$ in [14] using a contraction-mapping argument in mixed L_x^p and L_T^q spaces. When $k = 3$, the optimal local well-posedness result was proven by Tao in [24] and Grünrock [8] for $u_0 \in H^s$ with $s \geq -\frac{1}{6}$ by using Bourgain spaces $X_{s,b}$.

We shall be mainly interested in the persistence property of the solution of (1.2) (i.e., if $u_0 \in X$ a function space, then the corresponding solution $u(x,t)$ of (1.2) satisfies $u \in C([0,T] : X))$ in weighted Sobolev spaces $X = H^s(\mathbb{R}) \cap L^2(|x|^{2s})$. Let the H^s-lifespan of a solution $u(x,t)$ to (1.2) be the largest interval $[0, T_*)$ so that $u(x,t) \in H^s$ for $t \in [0, T_*)$. Kato in [12] with energy estimates, and the fact that the operator

$$\Gamma_K \equiv x + 3t\partial_x^2 \tag{1.3}$$

commutes with $\partial_t + \partial_x^3$, was able to prove the following: if $u_0 \in H^{2k} \cap L^2(|x|^{2k}\,dx)$ where $k \in \mathbb{Z}^+$, then the persistence property holds in $H^{2k}(\mathbb{R}) \cap L^2(|x|^{2k}\,dx)$.

Analogous results for the nonlinear Schrödinger equation were first proved by Hayashi, Nakamitsu, and M. Tsutsumi in [9], [10], and [11]. They used the vector field

$$\Gamma_S = x + 2it\nabla,$$

which commutes with the operator $\partial_t - i\Delta$, and a contraction-mapping argument to show that if $u_0 \in L^2(|x|^{2m}\,dx) \cap H^m$, where $m \in \mathbb{N}$, then the solution $u(x,t)$ at any other time in the H^m-lifespan of u is also in the space $L^2(|x|^{2m}\,dx) \cap H^m$. These results were extended to the case when $m \in \mathbb{R}_+$ by the author and Ponce in [20]. The corresponding results for the Benjamin–Ono equation were obtained in [4] by Ponce and Fonseca.

Given these results, one may expect that the work [12] can be extended to noninteger k. The difficulty in the case of fractional decay lies in the lack of an operator Γ that sufficiently describes the relation between initial decay, and properties of the solution at another time (such as (1.3)). This will give us a better description of the L^2-"decay" of the solution which can persist only if such a solution has enough L^2-regularity. Using some standard estimates for solutions to (1.2), we will prove the following theorem that extends this result slightly to $k \in \mathbb{R}_+$.

Theorem 1.1. *Suppose the initial data* u_0 *of* (1.2) *satisfies* $u_0 \in H^{2s+\varepsilon} \cap L^2(|x|^{2s}\,dx)$ *for* $\varepsilon > 0$, *and* $s \geq 0$ *with*

$$2s+\varepsilon \geq \begin{cases} 0 & \text{if } k=1, \\ \frac{1}{4} & \text{if } k=2, \\ 0 & \text{if } k=3, \\ \frac{k-4}{2k} & \text{if } k \geq 4, \end{cases} \tag{1.4}$$

and $[0, T_*)$ *is the lifespan interval of the* $H^{2s+\varepsilon}$*-solution; then* $u \in C([0,T_*) : L^2(|x|^{2s}\,dx))$.

When $s \geq \frac{1}{2}$, *the result holds for* $\varepsilon = 0$. *Namely, if* $u_0 \in H^{2s} \cap L^2(|x|^{2s}\,dx)$, *then* $u \in C([0,T_*) : H^{2s} \cap L^2(|x|^{2s}\,dx))$, *where* $[0,T_*)$ *is the lifespan interval of the* H^{2s}*-solution.*

The constraint on $2s+\varepsilon$ in this theorem comes from the well-posedness results mentioned in the introduction, with $2s+\varepsilon \geq \max\{0, s_k\}$ and s_k the optimal Sobolev exponent for the local well-posedness. With more sophisticated estimates, we improve this further in our main result.

Theorem 1.2. *Let* $u(x,t)$ *be a solution of*

$$\partial_t u + \partial_x^3 u + \partial_x(u^{k+1}) = 0, \qquad u(x,0) = u_0(x),\ x \in \mathbb{R},$$

for $k = 2$, *or* $k \geq 4$ *such that* $u_0 \in H^{2s} \cap L^2(|x|^{2s}\,dx)$. *If*

$$2s \geq \begin{cases} \frac{1}{4} & \text{for } k=2, \\ \frac{k-4}{2k} & \text{for } k \geq 4, \end{cases}$$

then $u \in C([0,T_*] : H^{2s} \cap L^2(|x|^{2s}\,dx)$, *where* $[0,T_*]$ *is the lifespan interval of the* H^{2s}*-solution. Moreover, the map data* $\to$ *solution from* $H^{2s} \cap L^2(|x|^{2s}\,dx)$ *to* $C([0,T] : H^{2s} \cap L^2(|x|^{2s}\,dx)$ *is smooth.*

We only prove this property in the most interesting case, (1.1) when $k = 2$, and $s = \frac{1}{8}$, and remark that the proof for the general case $k = 2, 4, 5, \ldots$ and $s \geq s_k$ follows the same argument. Note that the cases in (1.2) when $k = 1$ or 3 are excluded from Theorem 1.2. We require our technique to be adapted to Bourgain spaces for these nonlinearities, which is an interesting open question.

To illustrate the idea of the proof, we need some notation. If f is a complex-valued function on $\mathbb{R}$, we let $f^\wedge$ (or $\hat{f}$) denote the Fourier transform of f, and $f^\vee$ the inverse Fourier transform. For $\alpha \in \mathbb{R}$, the operator D_x^α is defined as $(D_x^\alpha f(x))^\wedge(\xi) \equiv |\xi|^\alpha f^\wedge(\xi)$. Let $U(t)f$ denote the solution $u(x,t)$

to the linear part of (1.1), with $u(x,0) = f(x)$. Choose $\eta \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\eta) \subset [\frac{1}{2}, 2]$ so that

$$\sum_{N\in\mathbb{Z}} (\eta(\frac{x}{2^N}) + \eta(-\frac{x}{2^N})) = 1 \text{ for } x \neq 0.$$

Define the operator Q_N on a function f as

$$Q_N(f) \equiv ((\eta(\frac{\xi}{2^N}) + \eta(-\frac{\xi}{2^N}))\hat{f}(\xi))^\vee.$$

If $\|\cdot\|_Y$ is a norm on some space of functions, we recall that

$$\|Q_N(f)\|_{Yl^p_N} \equiv \|(\sum_{N\in\mathbb{Z}} |Q_N(f)|^p)^{\frac{1}{p}}\|_Y.$$

Using Duhammel's principle, we can formulate the problem (1.1) as an integral equation:

$$u(x,t) = U(t)u_0 - \int_0^t U(t-t')\partial_x(u^3(x,t'))\,dt'. \tag{1.5}$$

Using a Fourier transform, we can see how to commute an x past $U(t)$,

$$xU(t)f = (-i\partial_\xi(e^{it\xi^3}\hat{f}))^\vee = (3t\xi^2 e^{it\xi^3}\hat{f} - ie^{it\xi^3}\partial_\xi\hat{f})^\vee = U(t)(3t\partial_x^2 f + xf).$$

We would like to use a similar argument with $|x|^{\frac{1}{8}}$ replacing x, but this would require that $D_\xi^{\frac{1}{8}}$ obey a product rule. Inspired by a strategy in [15], we develop an inequality in Lemma 4.2 that is similar enough to the product rule that will allow this argument to work.

With Lemma 4.2, we will require that

$$\Big\|D_\xi^{\frac{1}{8}} Q_N\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L^\infty_\xi l^1_N} < \infty. \tag{1.6}$$

With less sophisticated techniques, we prove Theorem 1.1 in Section 2. We show (1.6) in Section 3, then prove our main result in Section 4. The proof of Lemma 4.2 is almost identical to the proof of a classical Leibniz-rule inequality. Because this proof requires different techniques than the rest of the paper, we present it in Appendix A.

We use the following notation throughout the paper. We let $A \lesssim B$ mean that the quantity A is less than or equal to a fixed constant times the quantity B. Let $\langle x\rangle \equiv (1+x^2)^{\frac{1}{2}}$, and similarly, $\langle D_x\rangle$.

2. Weak Persistence Result

Using some standard estimates, we prove Theorem 1.1, which is a weaker persistence property for low-regularity solutions of the IVP for the gKdV equation, but holds for more values of k in (1.2) than our main result.

Following an argument by Kato, we multiply (1.2) by $\phi(x)u(x,t)$ for some function $\phi(x)$, and integrating over x and t, we use integration by parts to obtain

$$\int_{\mathbb{R}} \phi(x)u^2(x,T)\,dx - \int_{\mathbb{R}} \phi(x)u^2(x,0)\,dx - 3\int_{[0,T]}\int_{\mathbb{R}} \phi'(x)(\partial_x u)^2\,dx\,dt + \int_{[0,T]}\int_{\mathbb{R}} \phi'''(x)u^2\,dx\,dt + \frac{k+1}{k+2}\int_{[0,T]}\int_{\mathbb{R}} \phi'(x)u^{k+2}\,dx\,dt = 0. \tag{2.1}$$

Equation (2.1), along with the following two interpolation lemmas, are the primary tools for the proof of Theorem 1.1.

Lemma 2.1. *Let $a, b > 0$, and $w(x) > \varepsilon > 0$ be a locally bounded function. Assume that $\langle D_x\rangle^a f \in L^2(\mathbb{R})$ and $w^b(x)f \in L^2(\mathbb{R})$. Then for any $\theta \in (0,1)$,*

$$\|\langle D_x\rangle^{\theta a}(w^{(1-\theta)b}(x)f)\|_2 \lesssim \|w^b(x)f\|_2^{1-\theta}\|\langle D_x\rangle^a f\|_2^{\theta}.$$

Proof. This is an easy consequence of the Three Lines Lemma, and the fact that $\|\langle D_x\rangle^{za}(w^{(1-z)b}(x)f)\|_2$ is an analytic function in z for $\Re z \in (0,1)$, for a dense set of functions in the space $H^a \cap L^2(w^{2b}(x)\,dx)$. □

Lemma 2.2. *For a solution $u = u(x,t)$ of* (1.2)*, where $\varepsilon > 0$ and s obeys* (1.4)*, one has*

$$\|\partial_x u\|_{L_x^{\frac{1}{s+\frac{1}{2}\varepsilon}} L_T^2} \leq c_T\|u_0\|_{H^{2s+\varepsilon}}.$$

Proof. We use an interpolation argument. To this end, let

$$r(z) = (1-z)(1+2s+\varepsilon) + z(2s+\varepsilon), \quad \frac{1}{q(z)} = \frac{z}{2} + (1-z), \quad q = \frac{2}{2-2s-\varepsilon},$$

$$\psi(x,z) = |g(x)|^{q/q(z)}\,\frac{g(x)}{|g(x)|}, \quad \text{with} \quad \|g\|_{L_x^{1/(1-s-\frac{1}{2}\varepsilon)}} = \|f\|_{L^2([0,T])} = 1,$$

and

$$F(z) = \int_{-\infty}^{\infty}\int_0^T D_x^{r(z)}(U(t)u_0)\,\psi(x,z)\,f(t)\,dt\,dx.$$

Clearly, F is analytic for $\Re z \in (0,1)$. Because

$$\|\psi(\cdot, 0+iy)\|_2 = \|\psi(\cdot, 1+iy)\|_1 = 1,$$

one gets from $H^{2s+\varepsilon}$ persistence and the Kato smoothing effect that

$$\begin{aligned}\|\partial_x U(t)u_0\|_{L_x^{\frac{1}{s+\frac{1}{2}\varepsilon}}L_T^2} &\le c\|D_x U(t)u_0\|_{L_x^{\frac{1}{s+\frac{1}{2}\varepsilon}}L_T^2}\\ &\le c\sup_{y\in\mathbb{R}}\|D_x^{1+2s+\varepsilon+iy}U(t)u_0\|_{L_x^\infty L_T^2}^{1-2s-\varepsilon}\sup_{y\in\mathbb{R}}\|D_x^{2s+\varepsilon+iy}U(t)u_0\|_{L_x^2L_T^2}^{2s+\varepsilon}\\ &\le c_T\|D_x^{2s+\varepsilon}U(t)u_0\|_2. \end{aligned}\tag{2.2}$$

Inserting the estimate (2.2) in the proof of local well-posedness for (1.2), the result follows. □

Proof of Theorem 1.1. Let ϕ_N be a smooth function such that

$$\phi_N(x) = \begin{cases} \langle x\rangle^{2s} & \text{if } |x|\le N,\\ (2N)^{2s} & \text{if } |x|>3N.\end{cases}$$

Then from (2.1),

$$\begin{aligned}&\int_{\mathbb{R}}\phi_N(x)u^2(x,T)\,dx - \int_{\mathbb{R}}\phi_N(x)u^2(x,0)\,dx\\ &\quad= 3\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)(\partial_x u)^2\,dx\,dt - \int_{[0,T]}\int_{\mathbb{R}}\phi_N'''(x)u^2\,dx\,dt\\ &\quad- \frac{k+1}{k+2}\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)u^{k+2}\,dx\,dt.\end{aligned}\tag{2.3}$$

We only prove the result in the case where $s<1$ of the KdV equation, when $k=1$. Our main result, Theorem 1.2, is stronger when $k=2$, and $k\ge 4$, and the proof for $s\ge 1$ or $k=3$ is similar. We will use results from [13], which state that the smoothing effects and Strichartz estimates that hold for the linearized KdV and mKdV also hold for the KdV.

The $\phi_N'''(x)u^2$ term on the right-hand side of (2.3) can be bounded by the fact that $\phi_N'''(x)\lesssim 1$ independently of N for $s\le\frac{1}{2}$, and L^2 persistence:

$$\Big|\int_{[0,T]}\int_{\mathbb{R}}\phi_N'''(x)u^2\,dx\,dt\Big| \lesssim T\|u\|_2^2. \tag{2.4}$$

The bounds on the other terms on the right-hand side of (2.3) depend on whether $s<\frac{1}{2}$ or $s\ge\frac{1}{2}$. We first give the proof of the result in the case that $s<\frac{1}{2}$.

Since $|\phi_N'(x)|\lesssim\langle x\rangle^{2s-1}$ independently of N, we can bound the first term on the right-hand side of (2.3) by

$$\Big|\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)(\partial_x u)^2\,dx\,dt\Big| \lesssim \|\langle x\rangle^{s-\frac{1}{2}}\partial_x u\|_{L_x^2L_T^2}^2. \tag{2.5}$$

Using (2.5), Lemma 2.2, and the Hölder inequality,

$$\Big|\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)(\partial_x u)^2\,dx\,dt\Big| \lesssim \|\langle x\rangle^{s-\frac{1}{2}}\|_{\frac{2}{1-2(s+\frac{1}{2}\varepsilon)}}\|D_x u(x,t)\|_{L_x^{\frac{1}{s+\frac{1}{2}\varepsilon}}L_T^2} < \infty. \tag{2.6}$$

For the $\phi_N'(x)u^{k+2}$ term on the right-hand side of (2.3) we use the Hölder inequality,

$$\begin{aligned}\Big|\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)u^3\,dx\,dt\Big| &\lesssim \|\langle x\rangle^{2s-1}|u|^3\|_{L_T^1L_x^1}\\ &\leq \|u\|_{L_T^1L_x^\infty}\|\langle x\rangle^{s-\frac{1}{2}}u\|_{L_T^\infty L_x^2}^2 \leq T^{\frac{5}{6}}\|u\|_{L_T^6L_x^\infty}\|u\|_{L_T^\infty L_x^2}^2.\end{aligned} \tag{2.7}$$

Since $s-\frac{1}{2}<0$, (2.7) is finite by the Strichartz estimates in [13], and L^2 persistence.

It follows from (2.3) that

$$\begin{aligned}\Big|\int_{\mathbb{R}}(\phi_N(x)u^2(x,T))\,dx\Big| \leq &\int_{\mathbb{R}}|\phi_N(x)u^2(x,0)|dx + 3\int_{[0,T]}\int_{\mathbb{R}}|\phi_N'(x)(\partial_x u)^2|dx\,dt\\ &+\frac{2}{3}\int_{[0,T]}\int_{\mathbb{R}}|\phi_N'(x)u^3|\,dx\,dt + \int_{[0,T]}\int_{\mathbb{R}}|\phi_N'''(x)u^2|\,dx\,dt.\end{aligned}$$

By $|x|^s u_0\in L^2$, (2.6), (2.4), and (2.7), the result follows.

We now consider the case that $s\in[\frac{1}{2},1)$. For the first term on the right-hand side of (2.3), we use Lemma 2.1 and H^{2s} persistence to obtain

$$\begin{aligned}\Big|\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)(\partial_x u)^2\,dx\,dt\Big| &\lesssim \|\langle\phi_N'(x)\rangle^{\frac{1}{2}}\partial_x u\|_{L_T^2L_x^2}^2\\ &\lesssim \|\partial_x(\langle\phi_N'(x)\rangle^{\frac{1}{2}}u)\|_{L_T^2L_x^2}^2 + \|(\langle\phi_N'(x)\rangle^{\frac{1}{2}})'u\|_{L_T^2L_x^2}^2\\ &\lesssim \Big\|\frac{\partial_x}{\langle D_x\rangle}\langle D_x\rangle(\langle\phi_N'(x)\rangle^{\frac{1}{2}}u)\Big\|_{L_T^2L_x^2}^2 + \|\langle x\rangle^{s-\frac{3}{2}}u\|_{L_T^2L_x^2}^2\\ &\lesssim \|\langle D_x\rangle(\langle\phi_N'(x)\rangle^{\frac{1}{2}}u)\|_{L_T^2L_x^2}^2 + \|\langle x\rangle^{s-\frac{3}{2}}u\|_{L_T^2L_x^2}^2\\ &\lesssim \|\langle D_x\rangle^{2s}u\|_{L_T^2L_x^2}^{\frac{1}{s}}\|(\langle\phi_N'(x)\rangle^{\frac{s}{2s-1}})u\|_{L_T^2L_x^2}^{2-\frac{1}{s}} + \|\langle x\rangle^{s-\frac{3}{2}}u\|_{L_T^2L_x^2}^2.\end{aligned}$$

Since $\langle\phi_N'(x)\rangle^{\frac{s}{2s-1}}\lesssim\phi_N^{\frac{1}{2}}(x)$, it follows that

$$\Big|\int_{[0,T]}\int_{\mathbb{R}}\phi_N'(x)(\partial_x u)^2\,dx\,dt\Big|$$

$$\lesssim \|\langle D_x\rangle^{2s}u\|_{L_T^2L_x^2}^{\frac{1}{s}}\|\phi_N^{\frac{1}{2}}(x)u\|_{L_T^2L_x^2}^{2-\frac{1}{s}} + \|\langle x\rangle^{s-\frac{3}{2}}u\|_{L_T^2L_x^2}^2. \tag{2.8}$$

For the $\phi_N'(x)u^{k+2}$ term,

$$\begin{aligned}\Big|\int_{[0,T]}\int_{\mathbb{R}} \phi_N'(x)u^3\,dx\,dt\Big| &\lesssim \|\langle x\rangle^{2s-1}|u|^3\|_{L_T^1L_x^1}\\ \le \|u\|_{L_T^1L_x^\infty}\|\langle x\rangle^{s-\frac{1}{2}}u\|_{L_T^\infty L_x^2}^2 &\le T^{\frac{5}{6}}\|u\|_{L_T^6L_x^\infty}\|\langle x\rangle^{s-\frac{1}{2}}u\|_{L_T^\infty L_x^2}^2. \end{aligned}\tag{2.9}$$

The term in (2.9) is finite from the first part of the proof since $s-\frac{1}{2}<\frac{1}{2}$.

From (2.3), (2.4), (2.9), (2.8), the fact that $\phi_N(x)\lesssim\langle x\rangle^{2s}$, and our assumption on $u(x,0)$,

$$\begin{aligned}\|\phi_N^{\frac{1}{2}}(x)u^2(x,T)\|_{L_x^2}^2 &\lesssim \|\langle x\rangle^s u^2(x,0)\|_{L_x^2}^2 + \|\langle x\rangle^{s-\frac{3}{2}}u\|_{L_T^2L_x^2}^2\\ &+ \|\langle D_x\rangle^{2s}u\|_{L_T^2L_x^2}^{\frac{1}{s}}\Big(\int_0^T \|\phi_N^{\frac{1}{2}}(x)u(x,t)\|_{L_x^2}^2\,dt\Big)^{1-1/2s}\\ &+ T\|u\|_{L_x^2}^2 + T^{\frac{5}{6}}\|u\|_{L_T^6L_x^\infty}\|\langle x\rangle^{s-\frac{1}{2}}u\|_{L_T^\infty L_x^2}^2. \end{aligned}\tag{2.10}$$

The application of Bihari's inequality (see [1]) to (2.10) yields a bound on $\|\phi_N^{\frac{1}{2}}(x)u(x,T)\|_2$ that is independent of N. By taking N to infinity, the result follows. $\square$

3. Estimating a Derivative

We begin our computation of (1.6). We will show that by scaling out the fractional derivative, it will suffice to bound $\Big|Q_N\big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\big)\Big|$. Since the operator Q_N is convolution with a function whose Fourier transform is very localized, we require estimates on

$$\int_{\mathbb{R}} \varphi_\omega(\xi-z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz, \tag{3.1}$$

where φ_ω is a function whose Fourier transform has support near ω.

We will use a contour integral argument. Because of this, we require estimates on the analytic continuation of φ_ω. These are contained in the following lemma.

Lemma 3.1. *Let $\xi\in\mathbb{R}$, $z=x+yi$ for $x,y\in\mathbb{R}$, $\varphi(\xi)$ be a function so that $\hat{\varphi}(x)$ is a smooth function with support in $[\frac{1}{2},2]$, and for $\omega\in\mathbb{R}\setminus\{0\}$, let*

$\varphi_\omega(\xi)$ be the function with Fourier transform $\hat{\varphi}(\frac{x}{\omega})$. Then φ_ω is an entire function that obeys the following estimates:

$$|\varphi_\omega((\xi - z))| \lesssim \begin{cases} \frac{|e^{2\omega y} - e^{\frac{1}{2}\omega y}|}{\omega^2 y |\xi - z|^2} & \text{if } y \neq 0 \text{ and } x \neq \xi, \\ \\ \frac{1}{|\omega|(\xi - x)^2} & \text{if } y = 0 \text{ and } x \neq \xi. \end{cases}$$

Proof. That φ_ω is entire follows from the Paley–Wiener theorem. Let $y \neq 0$. Since $\hat{\varphi}$ is a smooth function with support in $[\frac{1}{2}, 2]$, we integrate by parts to obtain

$$\begin{aligned} \varphi_\omega(\xi - z) &= \int_{\mathbb{R}} \hat{\varphi}(\frac{\zeta}{\omega}) \frac{1}{i(\xi - z)} \frac{d}{d\zeta} e^{i\zeta(\xi - z)}\, d\zeta \\ &= -\int_{[\frac{\omega}{2}, 2\omega]} \frac{1}{\omega} \hat{\varphi}'(\frac{\zeta}{\omega}) \frac{1}{i(\xi - z)} e^{i\zeta(\xi - z)}\, d\zeta \\ &= \int_{[\frac{\omega}{2}, 2\omega]} \frac{1}{\omega} \hat{\varphi}'(\frac{\zeta}{\omega}) \frac{1}{(\xi - z)^2} \frac{d}{d\zeta} e^{i\zeta(\xi - z)}\, d\zeta \\ &= -\int_{[\frac{\omega}{2}, 2\omega]} \frac{1}{\omega^2} \hat{\varphi}''(\frac{\zeta}{\omega}) \frac{1}{(\xi - z)^2} e^{i\zeta(\xi - z)}\, d\zeta. \end{aligned} \tag{3.2}$$

From (3.2) we conclude that

$$\begin{aligned} |\varphi_\omega(\xi - z)| &\leq \Big| \int_{[\frac{\omega}{2}, 2\omega]} \frac{1}{\omega^2} \hat{\varphi}''(\frac{\zeta}{\omega}) \frac{1}{(\xi - z)^2} e^{i\zeta(\xi - z)}\, d\zeta \Big| \\ &\leq \int_{[\frac{\omega}{2}, 2\omega]} \frac{1}{\omega^2} |\hat{\varphi}''(\frac{\zeta}{\omega})| \frac{1}{|\xi - z|^2} e^{\zeta y}\, d\zeta \leq c_\varphi \frac{|e^{2\omega y} - e^{\frac{1}{2}\omega y}|}{\omega^2 y |\xi - z|^2}. \end{aligned}$$

The case $y = 0$ follows from taking the limit as $y \to 0$ of the first estimate. □

From Lemma 3.1, we can infer the following about the analyticity of the integrand in (3.1).

Corollary 3.1. *For $\xi \in \mathbb{R}$, the function*

$$\varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{1}{8}}} \tag{3.3}$$

is analytic on $\mathbb{C} \setminus \{z : |\Im z| \geq 1,\ \Re z = 0\}$.

The estimate in Lemma 3.1 has good x dependence away from ξ. To estimate (3.1) near $z=\xi$, we use an analytic continuation of the integrand and the Cauchy integral theorem, which we now describe.

The function φ_ω oscillates with frequency near ω. For a fixed $z_0 \in \mathbb{R}$, we think of the function $\exp(itz^3)$ as oscillating with frequency tz_0^2 near the value z_0. For $z=\xi$ where $t\xi^2 \ll \omega$, the function φ_ω oscillates much faster than $\exp(itz^3)$, so Lemma 3.1 shows that analytic continuation of

$$\varphi_\omega(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}$$

changes this rapid oscillation into decay, which yields good ω dependence for (3.1). To formalize this, we make the following definition. Given $t \neq 0$, and $\omega > 0$, we say that $\xi \in \mathbb{R}$ is **near** if $|\xi| \leq \frac{1}{10}\sqrt{\frac{\omega}{|t|}}$. Where the oscillation of $\exp(itz^3)$ is much larger than ω, an analytic continuation of $\exp(itz^3)$ has a similar property. We say that $\xi \in \mathbb{R}$ is **far** if $|\xi| > 10\sqrt{\frac{\omega}{|t|}}$. In the intermediate case where the oscillation of $\exp(itz^3)$ is comparable to ω, analytic continuation does not help. This is where the worst behavior of the estimate occurs. We say that $\xi \in \mathbb{R}$ is **intermediate** if $\frac{1}{10}\sqrt{\frac{\omega}{|t|}} < |\xi| \leq 10\sqrt{\frac{\omega}{|t|}}$. These heuristics are formalized in Lemma 3.3, then used to estimate (1.6) in Lemma 3.4. We require an elementary integral estimate for Lemma 3.3.

One expects that since $\sin\theta \approx \theta$, then

$$\begin{aligned}\int_{[0,\pi]} \frac{e^{a\sin\theta} - e^{b\sin\theta}}{\sin\theta}\,d\theta &\approx \int_{[0,\pi]} \frac{e^{a\theta} - e^{b\theta}}{\theta}\,d\theta \\ &= \int_{[0,\pi]} \frac{e^{a\theta} - e^{b\theta}}{a\theta}a\,d\theta = \int_{[0,\pi]} \frac{e^{\vartheta} - e^{\frac{b}{a}\vartheta}}{\vartheta}\,d\vartheta.\end{aligned}$$

This is the contents of the next lemma, which we state without proof.

Lemma 3.2. *Let $a < b < 0$. Then*

$$\Big|\int_{[0,\pi]} \frac{e^{a\sin\theta} - e^{b\sin\theta}}{\sin\theta}\,d\theta\Big| \lesssim (\pi\frac{a}{b} - 1) + 1 + \frac{b}{\pi a}e^{-\frac{\pi a}{b}}.$$

Lemma 3.3. *Let $\varphi(\xi)$ be a function so that the Fourier transform $\hat{\varphi}(x)$ is a smooth function with support in $[\frac{1}{2}, 2]$, and for $\omega \in \mathbb{R}\setminus[1,-1]$, let $\varphi_\omega(\xi)$*

be the function such that $\hat{\varphi}_\omega = \varphi(\frac{x}{\omega})$. *Then*

$$\Big|\int_{\mathbb{R}} \varphi_\omega(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz\Big| \lesssim \begin{cases} (1+|t|)\omega^{-\frac{1}{8}} & \text{if } \omega > 0 \text{ and } \xi \text{ intermediate,} \\ (1+|t|)|\omega|^{-1} & \text{else.} \end{cases}$$

Proof. We consider separately the four different cases: $\omega < 0$, $\omega > 0$ and ξ near, $\omega > 0$ and ξ intermediate, and $\omega > 0$ and ξ far.

Case $\omega < 0$**:**

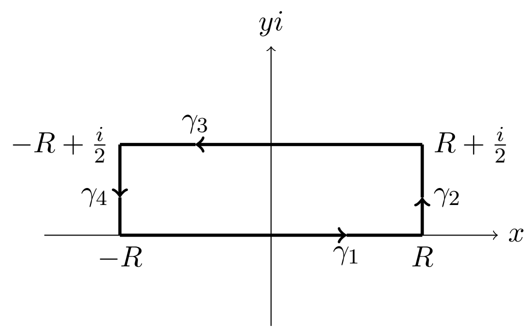

FIGURE 1. The contours used for $\omega < 0$.

Instead of integrating over $\mathbb{R}$ in (3.1), we will compute the integral over the contours γ_1 through γ_4 in Figure 1, taking the limit as R approaches infinity. By Corollary 3.1 and the Cauchy integral theorem,

$$\int_{\gamma_1} \varphi_\omega(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz = -\int_{\gamma_2} \varphi_\omega(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz - \int_{\gamma_3} \cdots - \int_{\gamma_4} \cdots.$$

We will use estimates on the integrals over γ_2, γ_3, and γ_4 to estimate (3.1). Along γ_2,

$$\begin{aligned} &\Big|\int_{[0,\frac{1}{2}]} \varphi_\omega(\xi - R - yi)\frac{e^{it(R+yi)^3}}{(1+(R+yi)^2)^{\frac{1}{8}}}\,i\,dy\Big| \\ &\lesssim \int_{[0,\frac{1}{2}]} \Big|\varphi_\omega(\xi - R - yi)\frac{e^{it(R+yi)^3}}{(1+(R+yi)^2)^{\frac{1}{8}}}i\Big|\,dy \\ &\lesssim \int_{[0,\frac{1}{2}]} \frac{|e^{2\omega y} - e^{\frac{1}{2}\omega y}|}{\omega^2 y|\xi - R - yi|^2}\frac{e^{-t(3R^2-y^2)y}}{(1+R^2)^{\frac{1}{8}}}\,dy. \end{aligned} \tag{3.4}$$

For fixed ω, (3.4) approaches 0 as $R \to \infty$. A similar estimate applies for γ_4. We can estimate the integral along γ_3 using Lemma 3.1:

$$\begin{aligned}
&\Big|\int_{[-R,R]} \varphi_\omega(\xi - x - \frac{i}{2}) \frac{e^{it(x+\frac{i}{2})^3}}{(1+(x+\frac{i}{2})^2)^{\frac{1}{8}}}\, dx\Big| \\
&\lesssim \int_{[-R,R]} \Big|\varphi_\omega(\xi - x - \frac{i}{2}) \frac{e^{it(x+\frac{i}{2})^3}}{(1+(x+\frac{i}{2})^2)^{\frac{1}{8}}}\Big|\, dx \\
&\lesssim \int_{[-R,R]} \frac{|e^{\omega} - e^{\frac{1}{4}\omega}|}{\omega^2((\xi - x)^2+1)} \frac{e^{-t(\frac{3}{2}x^2 - \frac{1}{8})}}{(1+x^2)^{\frac{1}{8}}}\, dx \\
&\lesssim \frac{|e^{\omega} - e^{\frac{1}{4}\omega}|}{\omega^2} \int_{\mathbb{R}} \frac{1}{((\xi - x)^2+1)} \frac{1}{(1+x^2)^{\frac{1}{8}}}\, dx \lesssim \frac{|e^{\omega} - e^{\frac{1}{4}\omega}|}{\omega^2}. \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5) we estimate (3.1):

$$\Big|\int_{\mathbb{R}} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\, dz\Big| \lesssim \frac{|e^{\omega} - e^{\frac{1}{4}\omega}|}{\omega^2} \lesssim (1+|t|)|\omega|^{-1}.$$

End of Case $\omega < 0$.

Let ε be some positive number that will be specified later. For the remaining three cases, we split up the integral (3.1) in the following manner:

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\, dz \\
&= \int_{\mathbb{R}\setminus B_{\frac{1}{10}\varepsilon}(\xi)} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\, dz + \int_{B_{\frac{1}{10}\varepsilon}(\xi)} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\, dz.
\end{aligned}$$

We estimate the integral over $\mathbb{R} \setminus B_{\frac{1}{10}\varepsilon}(\xi)$ using the decay of φ_ω, from Lemma 3.1.

$$\begin{aligned}
&\Big|\int_{\mathbb{R}\setminus B_{\frac{1}{10}\varepsilon}(\xi)} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\, dz\Big| \le \int_{\mathbb{R}\setminus B_{\frac{1}{10}\varepsilon}(\xi)} \Big|\varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\Big|\, dz \\
&\le \int_{\mathbb{R}\setminus B_{\frac{1}{10}\varepsilon}(\xi)} |\varphi_\omega(\xi - z)|\, dz \le \int_{\mathbb{R}\setminus B_{\frac{1}{10}\varepsilon}(\xi)} \frac{1}{\omega(\xi - x)^2}\, dx \lesssim \frac{1}{\omega\varepsilon}. \qquad (3.6)
\end{aligned}$$

In the next three cases we estimate

$$\int_{B_{\frac{1}{10}\varepsilon}(\xi)} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\, dz. \qquad (3.7)$$

Case $\omega > 0$, ξ near:

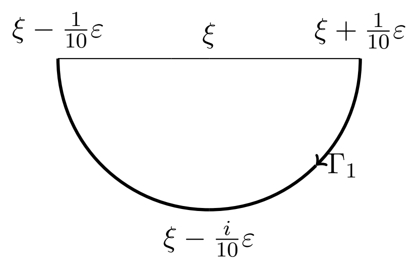

FIGURE 2. The contour used when $\omega > 0$ and $|\xi| \le \frac{1}{10}\sqrt{\frac{\omega}{|t|}}$.

By Corollary 3.1 and the Cauchy integral theorem, we can estimate (3.7) by approximating the integral along the semicircle arc Γ_1 in Figure 2, as long as we avoid the rays where the integrand is not analytic. If $\frac{1}{10}\omega^{\frac{1}{2}}|t|^{-\frac{1}{2}} < 1$, then let $\varepsilon = \omega^{\frac{1}{2}}|t|^{-\frac{1}{2}}$. Otherwise, let $\varepsilon = 1$. We illustrate the estimate only for the case $\varepsilon = 1$, as the other case follows by a similar argument:

$$\Big|\int_{[2\pi,\pi]} \varphi_\omega(-\frac{\varepsilon}{10}e^{i\theta})\frac{1}{(1+(\xi+\frac{\varepsilon}{10}e^{i\theta})^2)^{\frac{1}{8}}}e^{it(\xi+\frac{\varepsilon}{10}e^{i\theta})^3} i\frac{\varepsilon}{10}e^{i\theta}\,d\theta\Big| \tag{3.8}$$
$$\lesssim \int_{[2\pi,\pi]} \Big|\varphi_\omega(-\frac{\varepsilon}{10}e^{i\theta})\frac{1}{(1+(\xi+\frac{\varepsilon}{10}e^{i\theta})^2)^{\frac{1}{8}}}e^{it(\xi+\frac{\varepsilon}{10}e^{i\theta})^3}\Big|\varepsilon\,d\theta$$
$$\lesssim \int_{[2\pi,\pi]} |t|\frac{|e^{\frac{1}{5}\omega\varepsilon\sin\theta} - e^{\frac{1}{20}\omega\varepsilon\sin\theta}|}{\omega^3\varepsilon\sin\theta}e^{-\frac{t}{10}(3(\xi+\frac{1}{10}\varepsilon\cos\theta)^2-\frac{1}{100}\varepsilon^2\sin^2\theta)\varepsilon\sin\theta}\varepsilon\,d\theta.$$

Since $|\xi| \le \frac{1}{10}\sqrt{\frac{\omega}{|t|}}$ and $\varepsilon = 1 \le \sqrt{\frac{\omega}{|t|}}$, it follows that

$$\Big|\frac{t}{10}\Big(3\Big(\xi+\frac{1}{10}\varepsilon\cos\theta\Big)^2 - \frac{1}{100}\varepsilon^2\Big)\Big| \le \frac{|t|}{10}\Big(3\Big(|\xi|+\frac{1}{10}\varepsilon\Big)^2+\frac{1}{100}\varepsilon^2\Big) \le \frac{13}{1000}\omega.$$

Using this and Lemma 3.2, we bound (3.8) with

$$\int_{[2\pi,\pi]} |t|\frac{|e^{\frac{1}{5}\omega\varepsilon\sin\theta} - e^{\frac{1}{20}\omega\varepsilon\sin\theta}|}{\omega^3\sin\theta}e^{-\frac{13}{1000}\omega\varepsilon\sin\theta}\,d\theta$$
$$\lesssim \frac{|t|}{\omega^3}\int_{[2\pi,\pi]} \frac{|e^{0.187\omega\varepsilon\sin\theta} - e^{0.037\omega\varepsilon\sin\theta}|}{\sin\theta}\,d\theta \lesssim \frac{|t|}{\omega}. \tag{3.9}$$

From (3.9) and (3.6), we have the estimate

$$\Big|\int_{\mathbb{R}} \varphi_\omega(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz\Big| \lesssim (1+|t|)\omega^{-1}.$$

End of Case $\omega > 0$, ξ near.

Case $\omega > 0$, ξ intermediate: To estimate (3.7), we use the Young inequality, and the fact that $\|\varphi_\omega\|_1$ is uniformly bounded in ω. Let $\varepsilon = \frac{1}{10}\sqrt{\frac{\omega}{|t|}}$.

$$\Big|\int_{B_{\frac{1}{10}\varepsilon}(\xi)} \varphi(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz\Big| \lesssim \Big|\varphi_\omega * (\chi_{B_{\frac{1}{10}\varepsilon}(\xi)}(z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}})\Big|$$
$$\lesssim \|\varphi_\omega\|_1 \Big\|\chi_{B_{\frac{1}{10}\varepsilon}(\xi)}(z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\Big\|_\infty. \tag{3.10}$$

Since ξ is intermediate and $\varepsilon = \frac{1}{10}\sqrt{\frac{\omega}{|t|}}$, any $z \in B_{\frac{1}{10}\varepsilon}(\xi)$ will obey the estimate $z \approx \sqrt{\frac{\omega}{|t|}}$. This estimate on z allows us to bound the $\|\cdot\|_\infty$ term in (3.10) by

$$\Big\|\chi_{B_{\frac{1}{10}\varepsilon}(\xi)}(z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\Big\|_\infty \lesssim |t|^{\frac{1}{8}}|\omega|^{-\frac{1}{8}}. \tag{3.11}$$

From (3.11), (3.6), and that $|\omega| > 1$ by hypothesis, we have the estimate

$$\Big|\int_{\mathbb{R}} \varphi_\omega(\xi - z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz\Big| \lesssim \frac{\sqrt{|t|}}{\omega^{\frac{3}{2}}} + |t|^{\frac{1}{8}}\omega^{-\frac{1}{8}} \lesssim (1+|t|)\omega^{-\frac{1}{8}}.$$

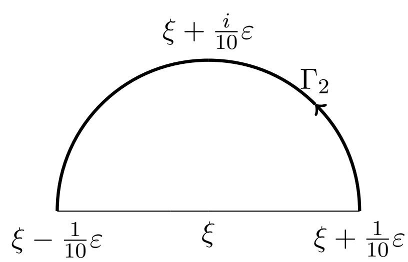

FIGURE 3. The contour used when $\omega > 0$ and $|\xi| > 10\sqrt{\frac{\omega}{|t|}}$.

End of Case $\omega > 0$, ξ intermediate.

Case $\omega > 0$, ξ far: Let $\varepsilon = \sqrt{\frac{\omega}{|t|}}$. We use an argument similar to the near case, integrating along the semicircle arc Γ_2 in Figure 3:

$$\begin{aligned}
&\Big|\int_{[0,\pi]} \varphi_\omega(-\frac{1}{10}\varepsilon e^{i\theta})\frac{1}{(1+(\xi+\frac{1}{10}\varepsilon e^{i\theta})^2)^{\frac{1}{8}}}e^{it(\xi+\frac{1}{10}\varepsilon e^{i\theta})^3}\frac{i}{10}\varepsilon e^{i\theta}\,d\theta\Big| \qquad (3.12)\\
&\lesssim \int_{[0,\pi]} \Big|\varphi_\omega(-\frac{1}{10}\varepsilon e^{i\theta})\frac{1}{(1+(\xi+\frac{1}{10}\varepsilon e^{i\theta})^2)^{\frac{1}{8}}}e^{it(\xi+\frac{1}{10}\varepsilon e^{i\theta})^3}\Big|\varepsilon\,d\theta\\
&\lesssim \int_{[0,\pi]} |t|\frac{|e^{\frac{1}{5}\omega\varepsilon\sin\theta}-e^{\frac{1}{20}\omega\varepsilon\sin\theta}|}{\omega^3\varepsilon\sin\theta}e^{-\frac{t}{10}(3(\xi+\frac{1}{10}\varepsilon\cos\theta)^2-\frac{1}{100}\varepsilon^2\sin^2\theta)\varepsilon\sin\theta}\varepsilon\,d\theta.
\end{aligned}$$

Since $\xi > 10\sqrt{\frac{\omega}{|t|}}$,

$$\begin{aligned}
-29.402\omega &\le -\frac{|t|}{10}\Big(3\Big(10\sqrt{\frac{\omega}{|t|}}-\frac{1}{10}\sqrt{\frac{\omega}{|t|}}\Big)^2-\frac{1}{100}\sqrt{\frac{\omega}{|t|}}\Big)\\
&\le -\frac{|t|}{10}\Big(3\Big(\xi+\frac{1}{10}\varepsilon\cos\theta\Big)^2-\frac{1}{100}\varepsilon^2\sin^2\theta\Big).
\end{aligned}$$

We use this with Lemma 3.2 to bound (3.12) by

$$\begin{aligned}
&\int_{[0,\pi]} |t|\frac{|e^{\frac{1}{5}\omega\varepsilon\sin\theta}-e^{\frac{1}{20}\omega\varepsilon\sin\theta}|}{\omega^3\sin\theta}e^{-29.402\omega\varepsilon\sin\theta}\,d\theta\\
&\lesssim \frac{|t|}{\omega^3}\int_{[0,\pi]}\frac{|e^{-29.202\omega\varepsilon\sin\theta}-e^{-29.352\omega\varepsilon\sin\theta}|}{\sin\theta}\,d\theta \lesssim \frac{|t|}{\omega^3}. \qquad (3.13)
\end{aligned}$$

From (3.13) and (3.6), we have the estimate

$$\Big|\int_{\mathbb{R}}\varphi_\omega(\xi-z)\frac{e^{itz^3}}{(1+z^2)^{\frac{1}{8}}}\,dz\Big| \lesssim \frac{\sqrt{|t|}}{\omega^{\frac{3}{2}}}+\frac{|t|}{\omega^3}\lesssim(1+|t|)\omega^{-\frac{3}{2}}\lesssim(1+|t|)\omega^{-1}.$$

End of Case $\omega > 0$, ξ far. □

Lemma 3.4.

$$\Big\|D_\xi^{\frac{1}{8}}Q_N\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L_\xi^\infty l_N^1}\lesssim 1+|t|.$$

Proof. The operator Q_N^5 (see also Appendix A) is defined by

$$Q_N^5 f \equiv \big(\big|\frac{x}{2^N}\big|^{\frac{1}{8}}\big(\eta(\frac{x}{2^N})+\eta(\frac{-x}{2^N})\big)\hat{f}(x)\big)^\vee.$$

Since Q_N is just convolution against the Fourier transform of a scaled smooth function, by rescaling we obtain

$$\Big\|D_\xi^{\frac{1}{8}} Q_N\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L_\xi^\infty l_N^1} = \Big\|2^{\frac{N}{8}} Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L_\xi^\infty l_N^1}.$$

We can estimate the low-frequency part using the Young inequality in the following manner:

$$\begin{aligned}\Big\|2^{\frac{N}{8}} Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L_\xi^\infty l_{N\le 0}^1} &\le \sum_{N\le 0} 2^{\frac{N}{8}}\Big\|Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L_\xi^\infty} \\ \lesssim \sum_{N\le 0} 2^{\frac{N}{8}}\Big\|\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big\|_{L_\xi^\infty} \lesssim \sum_{N\le 0} 2^{\frac{N}{8}} \lesssim 1.\end{aligned}$$

We use Lemma 3.3, noting that if t is fixed, for each $|\xi|$, there is a unique dyadic 2^N so that ξ is intermediate. We use this to bound the remaining frequencies:

$$\begin{aligned}\Big\|2^{\frac{N}{8}} Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L_\xi^\infty l_{N>1}^1} &= \Big\|\sum_{N=1}^{\infty} 2^{\frac{N}{8}}\Big|Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big|\Big\|_{L_\xi^\infty} \\ \lesssim \Big\|\sum_{2^N|\ \xi \text{ not intermediate}} 2^{\frac{N}{8}}\Big|Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big|\Big\|_{L_\xi^\infty} \\ + \Big\|\sum_{2^N|\ \xi \text{ intermediate}} 2^{\frac{N}{8}}\Big|Q_N^5\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big|\Big\|_{L_\xi^\infty} \\ \lesssim \Big(\sum_{N=1}^{\infty} 2^{\frac{N}{8}} 2^{-N} + 1\Big)(1+|t|). \qquad \square\end{aligned}$$

4. DECAY ESTIMATES FOR MKDV SOLUTIONS

With our bound from Lemma 3.4, we will show that our main result follows. This will come from the fact that for $\alpha \in (0,1)$,

$$\|D_x^\alpha(fg) - gD_x^\alpha f\|_2 \lesssim \|Q_N D_x^\alpha g\|_{L_x^\infty l_N^1}\|f\|_2. \tag{4.1}$$

A classical Leibniz-type inequality for fractional derivatives is the following (see [14]).

Lemma 4.1. *Let $0 < \alpha, \alpha_1, \alpha_2 < 1$, $\alpha = \alpha_1 + \alpha_2$, $1 < p, p_1, p_2 < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. In addition, the case $\alpha_1 = \alpha$, $p = p_2$, and $p_1 = \infty$ is allowed. Then the following holds for functions f, g on $\mathbb{R}^n$:*

$$\|D_x^\alpha(fg) - D_x^\alpha(f)g - fD_x^\alpha(g)\|_p \lesssim \|D_x^{\alpha_1} g\|_{p_1} \|D_x^{\alpha_2} f\|_{p_2}.$$

The proof uses the Littlewood–Paley theorem (see [23]), which states that for any function f, if $1 < p < \infty$, then

$$\|Q_N(f)\|_{L_x^p l_N^2} \lesssim \|f\|_p \lesssim \|Q_N(f)\|_{L_x^p l_N^2}. \tag{4.2}$$

Lemma 4.1 is not sufficient for our argument in the previous section, since we need to put the *derivative* term in the infinity norm. A product rule like this can be obtained by following the proof of Lemma 4.1 line for line. The only difference is that since (4.2) fails for $p = \infty$, $\|Q_N(D_x^\alpha g)\|_{L_x^\infty l_N^2}$ is not equivalent to $\|D_x^\alpha g\|_\infty$. This idea was inspired by [15], where the authors use $\|Q_N \cdot\|_{l_N^2 L_x^4 L_T^\infty}$ in an estimate where the $\|\cdot\|_{L_x^4 L_T^\infty}$ norm may fail.

Lemma 4.2. *Let $0 < \alpha < 1$ and $1 < p < \infty$. For functions f and g,*

$$\|D_x^\alpha(fg) - gD_x^\alpha f - fD_x^\alpha g\|_p \lesssim (\|Q_N D_x^\alpha g\|_{L_x^\infty l_N^2} + \|D_x^\alpha g\|_{L_x^\infty})\|f\|_p.$$

In particular,

$$\|D_x^\alpha(fg) - gD_x^\alpha f\|_2 \lesssim \|Q_N D_x^\alpha g\|_{L_x^\infty l_N^1} \|f\|_2.$$

The proof is in Appendix A.

For a number $1 \le p \le \infty$, let p' denote the conjugate exponent. We recall the following properties of the operator $U(t)$:

$$\left\|\partial_x \int_0^t U(t-t')f(x,t')\,dt'\right\|_{L_x^2} \lesssim \|f\|_{L_x^1 L_T^2}, \tag{4.3}$$

$$\left\|\int_0^t U(t-t')f(t')\,dt'\right\|_{L_x^2} \lesssim \|f\|_{L_T^{q'} L_x^{p'}}, \tag{4.4}$$

where $p \ge 2$ and q satisfy $\frac{1}{q} = \frac{1}{6} - \frac{1}{3p}$.

The proof of (4.3) can be found in [18] or [14]. Inequality (4.4) follows from the fact that $U(t)$ is an L_x^2 isometry, along with the dual of the homogenous Strichartz estimate for $U(t)$ (see [6], page 1392).

The existence theorem for solutions to (1.1) is proved by a contraction-mapping argument, which can also be found in [18].

Theorem 4.1. *Let $\|\cdot\|_{Y_T}$ denote the norm such that*

$$\|f\|_{Y_T} \equiv \|f\|_{L_x^4 L_T^\infty} + \|D_x^s \partial_x f\|_{L_x^\infty L_T^2}$$

$$+\|f\|_{L^\infty_T H^s} + \|D_x^{s-\frac14}\partial_x f\|_{L^{20}_x L^{\frac52}_T} + \|D^s_x f\|_{L^5_x L^{10}_T},$$

$$Y_T \equiv \{f:\ \|f\|_{Y_T} < \infty\}.$$

Let $u_0 \in L^2$, *and* Φ *be the map from* Y_T *to* Y_T *such that*

$$\Phi(u) \equiv U(t)u_0 - \int_0^t U(t-t')\partial_x(u^3(t'))\,dt'.$$

Then

$$\|\Phi(u)\|_{Y_T} \lesssim \|u_0\|_{H^{\frac14}} + T^{\frac12}\|u\|^3_{Y_T}. \tag{4.5}$$

This implies by the contraction-mapping theorem that there exist

$$T = c\|D_x^{\frac14}u_0\|_2^{-4}$$

and a unique solution $u(x,t)$ *of the integral equation* (1.5)*.*

The proof requires a Leibniz–rule-type inequality for $L^p_x L^q_T$ norms, which we need as well.

Lemma 4.3. *Let* $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$ *with* $\alpha = \alpha_1 + \alpha_2$. *Let* $p, q, p_1, p_2, q_2 \in (1,\infty)$, $q_1 \in (1,\infty]$ *be such that* $\frac1p = \frac1{p_1} + \frac1{p_2}$ *and* $\frac1q = \frac1{q_1} + \frac1{q_2}$. *Then*

$$\|D^\alpha_x(fg) - fD^\alpha_x g - gD^\alpha_x f\|_{L^p_x L^q_T} \lesssim \|D^{\alpha_1}_x f\|_{L^{p_1}_x L^{q_1}_T}\|D^{\alpha_2}_x f\|_{L^{p_2}_x L^{q_2}_T}.$$

Moreover, for $\alpha_1 = 0$, *the value* $q_1 = \infty$ *is allowed.*

We will need an estimate on the Fourier transform $k(x)$ of $(1+\xi^2)^{-\frac18}$. We expect k to have good decay properties since it is the inverse Fourier transform of a smooth function. Since

$$|\hat{\xi}|^{-\frac14} = c_0|x|^{-\frac34}, \tag{4.6}$$

we expect that $k(x) \approx |x|^{-\frac34}$ for small x. This is formalized in the following lemma, which is a simple consequence of integrating by parts appropriately.

Lemma 4.4. *Let* $k(x)$ *denote the Fourier transform of the function* $(1+\xi^2)^{-\frac18}$. *Then for any* $n \in \mathbb{N}$,

$$|k(x)| \lesssim \frac{1}{|x|^{\frac34}(1+x^{2n})}. \tag{4.7}$$

In particular,

$$\int_{\mathbb{R}} |x|^{\frac18}|k(x)| < \infty.$$

Before proving Theorem 1.2, we prove the corresponding decay result for solutions to the linear part of (1.1). This is necessary for the proof of Theorem 1.2, and it is also a simpler case that illustrates the main idea of our proof of Theorem 1.2. We note that it is also possible to prove this result using an argument like Lemma 2 in [20], but this proof does not generalize to solutions of (1.1) as discussed in the introduction.

Lemma 4.5. *For $u_0 \in C_0^\infty(\mathbb{R})$ and $s < 1$,*

$$\||x|^s U(t)u_0(x)\|_2 \lesssim (1+|t|)\|u_0\|_{H^{2s}} + \||x|^s u_0\|_2.$$

Proof. For concreteness, it will suffice to prove the result in the case $s = \frac{1}{8}$. By the definition of $U(t)$ and the triangle inequality,

$$\begin{aligned}
\||x|^{\frac{1}{8}}U(t)u_0\|_2 &= \|D_\xi^{\frac{1}{8}}\left(e^{it\xi^3}\hat{u}_0\right)\|_{L^2_\xi} = \left\|D_\xi^{\frac{1}{8}}\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}(1+\xi^2)^{\frac{1}{8}}\hat{u}_0\Big)\right\|_2 \\
&\lesssim \left\|D_\xi^{\frac{1}{8}}\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}(1+\xi^2)^{\frac{1}{8}}\hat{u}_0\Big) - \frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}D_\xi^{\frac{1}{8}}((1+\xi^2)^{\frac{1}{8}}\hat{u}_0)\right\|_2 \\
&\quad + \|\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}D_\xi^{\frac{1}{8}}((1+\xi^2)^{\frac{1}{8}}\hat{u}_0)\|_2 \equiv \mathcal{I} + \mathcal{II}.
\end{aligned}$$

We can write term $\mathcal{II}$ as

$$\begin{aligned}
\mathcal{II} &= \|(1+\xi^2)^{-\frac{1}{8}}D_\xi^{\frac{1}{8}}((1+\xi^2)^{\frac{1}{8}}\hat{u}_0)\|_2 \\
&= \|[(1+\xi^2)^{-\frac{1}{8}}, D_\xi^{\frac{1}{8}}]((1+\xi^2)^{\frac{1}{8}}\hat{u}_0) + D_\xi^{\frac{1}{8}}\hat{u}_0\|_2 \\
&\leq \|[(1+\xi^2)^{-\frac{1}{8}}, D_\xi^{\frac{1}{8}}]((1+\xi^2)^{\frac{1}{8}}\hat{u}_0)\|_2 + \|D_\xi^{\frac{1}{8}}\hat{u}_0\|_2. \qquad (4.8)
\end{aligned}$$

We need to bound the commutator term in $\mathcal{II}$. For any function h, we use the Plancherel theorem, the Young inequality, and Lemma 4.4 to obtain

$$\begin{aligned}
\|[(1+\xi^2)^{-\frac{1}{8}}, D_\xi^{\frac{1}{8}}]h\|_{L^2_\xi} &= \left\|\int_{\mathbb{R}}(|x|^{\frac{1}{8}} - |y|^{\frac{1}{8}})k(x-y)\hat{h}(y)\,dy\right\|_{L^2_x} \\
&\lesssim \left\|\int_{\mathbb{R}}|x-y|^{\frac{1}{8}}|k(x-y)||\hat{h}(y)|\,dy\right\|_{L^2_x} \lesssim c_k\|\hat{h}\|_{L^2_x} = c_k\|h\|_{L^2_\xi}.
\end{aligned}$$

We apply this to (4.8):

$$|\mathcal{II}| \lesssim c_k\|u_0\|_{H^{\frac{1}{4}}} + \||x|^{\frac{1}{8}}u_0\|_2.$$

For term $\mathcal{I}$, we use Lemma 4.2 with Lemma 3.4:

$$|\mathcal{I}| \lesssim \Big\| Q_N D_\xi^{\frac{1}{8}} \Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}} \Big) \Big\|_{L_x^\infty l_N^1} \|u_0\|_2 \lesssim (1+t)\|u_0\|_2.$$

Combining our estimates for $\mathcal{I}$ and $\mathcal{II}$, the result follows. □

Proof of Theorem 1.2. For concreteness, we prove the result in the most interesting case when $k = 2$, $s = s' = \frac{1}{8}$, and $t > 0$. We use a contraction-mapping argument to prove our decay estimate. The resolution space is

$$\|f\|_{Z_T} \equiv \||x|^{\frac{1}{8}} f\|_{L_T^\infty L_x^2} + \|f\|_{Y_T}, \qquad Z_T \equiv \{f | \ \|f\|_{Z_T} < \infty\},$$

where $\||x|^{\frac{1}{8}} f\|_{L_T^\infty L_x^2}$ more specifically means $\||x|^{\frac{1}{8}} f\|_{C_t^0([0,T))L_x^2}$.

Let $f(t) \equiv \partial_x(u^3(t))$ for convenience, and consider

$$\Phi(u)(x,t) = U(t)u(x,0) - \int_0^t U(t-t')f(t')\,dt'. \tag{4.9}$$

Multiply (4.9) by $|x|^{\frac{1}{8}}$. The $|x|^{\frac{1}{8}} U(t)u(x,0)$ term is bounded by Lemma 4.5. We concentrate on the nonlinear term:

$$\begin{aligned}
&\Big\| |x|^{\frac{1}{8}} \int_0^t U(t-t')f(t')\,dt' \Big\|_{L_T^\infty L_x^2} = \Big\| D_\xi^{\frac{1}{8}} \Big(\int_0^t e^{i(t-t')\xi^3} f^\wedge(t')\,dt' \Big) \Big\|_{L_T^\infty L_\xi^2} \\
&= \Big\| D_\xi^{\frac{1}{8}} \Big(\int_0^t \frac{e^{i(t-t')\xi^3}}{(1+\xi^2)^{\frac{1}{8}}} (1+\xi^2)^{\frac{1}{8}} f^\wedge(t') \Big)\,dt' \Big\|_{L_T^\infty L_\xi^2} \\
&\lesssim \| D_\xi^{\frac{1}{8}} \Big(\int_0^t \frac{e^{i(t-t')\xi^3}}{(1+\xi^2)^{\frac{1}{8}}} (1+\xi^2)^{\frac{1}{8}} f^\wedge(t')\,dt' \Big) \\
&\quad - \int_0^t \frac{e^{i(t-t')\xi^3}}{(1+\xi^2)^{\frac{1}{8}}} D_\xi^{\frac{1}{8}} ((1+\xi^2)^{\frac{1}{8}} f^\wedge(t'))\,dt' \|_{L_T^\infty L_\xi^2} \\
&\quad + \| \int_0^t \frac{e^{i(t-t')\xi^3}}{(1+\xi^2)^{\frac{1}{8}}} D_\xi^{\frac{1}{8}} ((1+\xi^2)^{\frac{1}{8}} f^\wedge(t'))\,dt' \|_{L_T^\infty L_\xi^2} \\
&\equiv I + II.
\end{aligned} \tag{4.10}$$

We bound term II in a similar fashion to term $\mathcal{II}$ in Lemma 4.5:

$$\begin{aligned}
II \lesssim &\Big\| \int_0^t e^{i(t-t')\xi^3} [\frac{1}{(1+\xi^2)^{\frac{1}{8}}}, D_\xi^{\frac{1}{8}}] ((1+\xi^2)^{\frac{1}{8}} f^\wedge(t'))\,dt' \Big\|_{L_T^\infty L_\xi^2} \\
&+ \Big\| \int_0^t e^{i(t-t')\xi^3} D_\xi^{\frac{1}{8}} (f^\wedge(t'))\,dt' \Big\|_{L_T^\infty L_\xi^2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \Big\|[\frac{1}{(1+\xi^2)^{\frac18}}, D_\xi^{\frac18}]((1+\xi^2)^{\frac18} f^\wedge(t'))\Big\|_{L^1_T L^2_\xi} \\
&\quad + \Big\|\int_0^t e^{i(t-t')\xi^3} D_\xi^{\frac18}(f^\wedge(t'))\,dt'\Big\|_{L^\infty_T L^2_\xi} \\
&\lesssim \Big\|((1+\xi^2)^{\frac18} f^\wedge(t'))\Big\|_{L^1_T L^2_\xi} + \Big\|\int_0^t e^{i(t-t')\xi^3} D_\xi^{\frac18}(f^\wedge(t'))\,dt'\Big\|_{L^\infty_T L^2_\xi} \\
&\lesssim \Big\|((1+D_x^2)^{\frac18} f(t'))\Big\|_{L^1_T L^2_x} + \Big\|\int_0^t U(t-t')|x|^{\frac18} f(t')\,dt'\Big\|_{L^\infty_T L^2_x} \\
&\equiv II.1 + II.2.
\end{aligned}
$$

Specializing to the case of the mKdV, $f(t') = \partial_x(u^3(t'))$, we bound $II.1$ using Theorem 4.1:

$$
\begin{aligned}
II.1 &\lesssim \|\partial_x(u^3)\|_{L^1_T L^2_x} + \|D_x^{\frac14}\partial_x(u^3)\|_{L^1_T L^2_x} \\
&\lesssim T^{\frac12}\|\partial_x(u^3)\|_{L^2_T L^2_x} + T^{\frac12}\|D_x^{\frac14}\partial_x(u^3)\|_{L^2_T L^2_x} \\
&\lesssim T^{\frac12}\|u\|^2_{L^4_x L^\infty_T}\|\partial_x u\|_{L^\infty_x L^2_T} + T^{\frac12}\|u^2\|_{L^2_x L^\infty_T}\|D_x^{\frac14}\partial_x u\|_{L^\infty_x L^2_T} \\
&\quad + T^{\frac12}\|D_x^{\frac14}(u^2)\|_{L_x^{\frac{20}{9}} L_T^{10}}\|\partial_x u\|_{L_x^{20} L_T^{\frac52}} \\
&\lesssim T^{\frac12}\|u\|^2_{L^4_x L^\infty_T}\|\partial_x u\|_{L^\infty_x L^2_T} + T^{\frac12}\|u\|^2_{L^4_x L^\infty_T}\|D_x^{\frac14}\partial_x u\|_{L^\infty_x L^2_T} \\
&\quad + T^{\frac12}\|u\|_{L^4_x L^\infty_T}\|D_x^{\frac14} u\|_{L^5_x L^{10}_T}\|\partial_x u\|_{L_x^{20} L_T^{\frac52}} \lesssim T^{\frac12}\|u\|^3_{Z_T}.
\end{aligned}
$$

Let $\phi(x) \in C_0^\infty(\mathbb{R})$ have the property that $\phi(x) = 1$ for $x \in (-1,1)$. We handle $II.2$ with the following argument:

$$
\begin{aligned}
&\Big\|\int_0^t U(t-t')|x|^{\frac18} f(t')\,dt'\Big\|_{L^\infty_T L^2_x} \lesssim \Big\|\int_0^t U(t-t')|x|^{\frac18}\phi(x) f(t')\,dt'\Big\|_{L^\infty_T L^2_x} \\
&\quad + \Big\|\int_0^t U(t-t')|x|^{\frac18}(1-\phi(x)) f(t')\,dt'\Big\|_{L^\infty_T L^2_x} \\
&\lesssim \Big\|\int_0^t U(t-t')[|x|^{\frac18}(1-\phi(x)), \partial_x] u^3(t')\,dt'\Big\|_{L^\infty_T L^2_x} \\
&\quad + \Big\|\int_0^t U(t-t')\partial_x((|x|^{\frac18}(1-\phi(x)))u^3(t'))\,dt'\Big\|_{L^\infty_T L^2_x}
\end{aligned}
$$

$$+\Big\|\int_0^t U(t-t')|x|^{\frac{1}{8}}\phi(x)\partial_x(u^3(t'))\,dt'\Big\|_{L^\infty_T L^2_x}$$
$$\equiv II.2.a + II.2.b + II.2.c.$$

For $II.2.a$, we use (4.4), and that for any function h, and $p \geq 1$,

$$\|[|x|^{\frac{1}{8}}(1-\phi(x)), \partial_x]h\|_p \lesssim \Big\|\frac{\partial}{\partial x}(|x|^{\frac{1}{8}}(1-\phi(x)))\Big\|_\infty \|h\|_p,$$

along with the Sobolev inequality to obtain the bound

$$II.2.a \lesssim \|u^3\|_{L_T^{\frac{12}{11}}L_x^{\frac{4}{3}}} = \|u\|^3_{L_T^{\frac{36}{11}}L_x^4} \lesssim T^{\frac{11}{12}}\|u\|^3_{L_T^\infty H^{\frac{1}{4}}} \lesssim T^{\frac{11}{12}}\|u\|^3_{Z_T}.$$

We use (4.3) to estimate $II.2.b$:

$$\begin{aligned} II.2.b &\lesssim \||x|^{\frac{1}{8}}(1-\phi(x))u^3\|_{L^1_x L^2_T} \lesssim \|u^2\|_{L^2_x L^\infty_T}\||x|^{\frac{1}{8}}(1-\phi(x))u\|_{L^2_x L^2_T} \\ &\lesssim \|u\|^2_{L^4_x L^\infty_T}(\||x|^{\frac{1}{8}}u\|_{L^2_T L^2_x} + \|u\|_{L^2_T L^2_x}) \\ &\lesssim T^{\frac{1}{2}}\|u\|^2_{L^4_x L^\infty_T}(\||x|^{\frac{1}{8}}u\|_{L^\infty_T L^2_x} + \|u\|_{L^\infty_T L^2_x}) \lesssim T^{\frac{1}{2}}\|u\|^3_{Z_T}. \end{aligned}$$

We use Theorem 4.1 and the fact that ϕ has compact support to control $II.2.c$:

$$\begin{aligned} II.2.c &\lesssim \||x|^{\frac{1}{8}}\phi(x)\partial_x(u^3)\|_{L^1_T L^2_x} \lesssim \|\partial_x(u^3)\|_{L^1_T L^2_x} \\ &\lesssim T^{\frac{1}{2}}\|u\|^2_{L^4_x L^\infty_T}\|\partial_x u\|_{L^\infty_x L^2_T} \lesssim T^{\frac{1}{2}}\|u\|^3_{Z_T}. \end{aligned}$$

Term I from (4.10) can be controlled using Theorem 4.1, and the same argument as the bound for $II.1$:

$$\begin{aligned} I &\lesssim \Big\|Q_N D_\xi^{\frac{1}{8}}\Big(\frac{e^{it\xi^3}}{(1+\xi^2)^{\frac{1}{8}}}\Big)\Big\|_{L^2_T L^\infty_\xi l^1_N}\|(1+D_x^2)^{\frac{1}{8}}(u^2\partial_x u)\|_{L^2_T L^2_x} \\ &\lesssim (1+T^{\frac{3}{2}})(\|u^2\partial_x u\|_{L^2_T L^2_x} + \|D_x^{\frac{1}{4}}(u^2\partial_x u)\|_{L^2_T L^2_x}) \\ &\lesssim T^{\frac{1}{2}}(1+T)(\|u\|^2_{L^4_x L^\infty_T}\|\partial_x u\|_{L^\infty_x L^2_T} + \|u\|^2_{L^4_x L^\infty_T}\|D_x^{\frac{1}{4}}\partial_x u\|_{L^\infty_x L^2_T} \\ &\quad + \|u\|_{L^4_x L^\infty_T}\|D_x^{\frac{1}{4}}u\|_{L^5_x L^{10}_T}\|\partial_x u\|_{L^{20}_x L^{\frac{5}{2}}_T}) \\ &\lesssim T^{\frac{1}{2}}(1+T)\|u\|^3_{Z_T} \lesssim T^{\frac{1}{2}}(1+T)\|u\|^3_{Z_T}. \end{aligned}$$

Putting these estimates together,

$$\||x|^{\frac{1}{8}}u\|_{L^\infty_T L^2_x} \lesssim \||x|^{\frac{1}{8}}U(t)u_0\|_{L^\infty_T L^2_x} + I + II.1 + II.2.a + II.2.b + II.2.c$$

$$\lesssim \||x|^{\frac{1}{8}}u_0\|_{L_x^2} + (1+T)\|u_0\|_{H^{\frac{1}{4}}} + T^{\frac{1}{2}}(1+T^{\frac{5}{12}}+T)\|u\|_{Z_T}^3. \qquad (4.11)$$

In order to get a contraction, we need to bound $\|u\|_{Y_T}$ in terms of $\|u\|_{Z_T}$. This follows from estimate (4.5) in Theorem 4.1. By combining this with (4.11), we obtain a contraction by taking T small enough,

$$\|u\|_{Z_T} \lesssim \||x|^{\frac{1}{8}}u_0\|_{L_x^2} + (1+T)\|u_0\|_{H^{\frac{1}{4}}} + T^{\frac{1}{2}}(1+T^{\frac{5}{12}}+T)\|u\|_{Z_T}^3. \qquad \square$$

5. Appendix A

For our proof of Lemma 4.1, we closely follow the proof of Theorem A.8 in [14]. This requires more notation. Let $\alpha_1 = 0$ and $\alpha_2 = \alpha \in [0,1]$. For a function f, let

$$P_N f \equiv \sum_{j\le N-3} Q_j f.$$

Define $p(x)$ to be the function so that

$$(P_N f)^\wedge = p(2^{-N}x)\hat{f}.$$

Let $\tilde{p} \in C_0^\infty(\mathbb{R})$, with $\tilde{p}(x) = 1$ for $x \in [-100, 100]$, and let

$$(\tilde{P}_N f)^\wedge(x) = \tilde{p}(2^{-N}x)\hat{f}.$$

Let $\tilde{\eta} \in C_0^\infty(\mathbb{R})$ with $\tilde{\eta}(x) = 1$ for $x \in [\frac{1}{4}, 4]$, and $\operatorname{supp}\tilde{\eta} \in [\frac{1}{8}, 8]$. Then define $(\tilde{Q}_k f)^\wedge(x) = \tilde{\eta}(2^{-k}x)\hat{f}$. Let

$$\Psi^i(x) = |x|^{\alpha_j}p(x),\ \eta^j(x) = \frac{\eta(x)}{|x|^{\alpha_j}},$$

$$(\Psi_j^k)^\wedge(x) = \Psi_j(2^{-k})\hat{f}(x), \text{ and } (Q_k^j f)^\wedge(x) = \eta^j(2^{-k}x)\hat{f}(x).$$

Similarly, with $\eta^3(x) = |x|^\alpha \tilde{p}(x)$, $\eta^4(x) = |x|^{\alpha_1}\eta(x)$, and $\eta^5(x) = |x|^{\alpha_2}\eta(x)$ we define Q_k^3, Q_k^4, and Q_k^5. Let

$$\eta^{\nu,j}(x) = \exp(i\nu x)\eta^j(x), \qquad \eta^{\mu,j}(x) = \exp(i\mu x)x|x|^{-\alpha_j}p(x),$$

with $j = 1, 2$ and $Q_k^{\nu,j}$ and $Q_k^{\mu,j}$ the corresponding operators.

The following is Proposition A.2 from [14].

Lemma 5.1.

$$D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f = \sum_{|j|<2} 2^{j\alpha_2}\sum_k Q_k^3(Q_k^1(D^{\alpha_1}f)Q_{k-j}^2(D^{\alpha_2}g))$$

$$+ \sum_k \tilde{Q}_k(\Psi_k^1(D^{\alpha_2}g)Q_k^1(D^{\alpha_1}f)) + \sum_k \tilde{Q}_k(Q_k^2(D^{\alpha_2}g)\Psi_k^2(D^{\alpha_1}f))$$

$$+\sum_{|j|\le 2}2^{j\alpha_2}\sum_k Q_k^1(D^{\alpha_1}f)Q_{k-j}^4(D^{\alpha_2}g)+\sum_{|j|\le 2}2^{j\alpha_2}\sum_k Q_{k-j}^2(D^{\alpha_2}g)Q_k^5(D^{\alpha_1}f)$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\sum_k \tilde{Q}_k(Q_k^{\nu,1}(D^{\alpha_1}f)Q_k^{\mu,2}(D^{\alpha_2}g))\Big]r_1(\mu,\nu)\,d\nu\,d\mu$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\sum_k \tilde{Q}_k(Q_k^{\nu,2}(D^{\alpha_2}g)Q_k^{\mu,1}(D^{\alpha_1}f))\Big]r_2(\mu,\nu)\,d\nu\,d\mu,$$

where $r_1, r_2 \in \mathcal{S}(\mathbb{R}^2)$.

Proof of Lemma 4.2. From Lemma 5.1, we need to bound four types of terms:

(1) $\sum_{-\infty}^{\infty} Q_k(Q_k(f)Q_k(D_x^\alpha g))$
(2) $\sum_{-\infty}^{\infty} Q_k(\Psi_k(f)Q_k(D_x^\alpha g))$
(3) $\sum_{-\infty}^{\infty} Q_k(Q_k(f)\Psi_k(D_x^\alpha g))$
(4) $\sum_{-\infty}^{\infty} Q_k(f)Q_k(D_x^\alpha g)$

Let $\mathcal{M}h$ denote the Hardy maximal operator applied to the function h. We control the first term using duality:

$$\Big|\int_{\mathbb{R}}\sum_{-\infty}^{\infty} Q_k(Q_k(f)Q_k(D_x^\alpha g))h\,dx\Big| = \Big|\int_{\mathbb{R}}\sum_{-\infty}^{\infty} Q_k(f)Q_k(D_x^\alpha g)Q_k(h)\,dx\Big|$$
$$\lesssim \int_{\mathbb{R}}\sqrt{\sum_{-\infty}^{\infty}|Q_k(f)|^2|Q_k(D_x^\alpha g)|^2}\sqrt{\sum_{-\infty}^{\infty}|Q_n(h)|^2}\,dx$$
$$\lesssim \|Q_k(f)Q_k(D_x^\alpha g)\|_{L_x^p l_k^2}\|Q_n(h)\|_{L_x^{p'} l_n^2}$$
$$\lesssim \|\mathcal{M}(f)\|_{L_x^p}\|Q_k(D_x^\alpha g)\|_{L_x^\infty l_k^2}\|Q_n(h)\|_{L_x^{p'} l_n^2}$$
$$\lesssim \|f\|_{p'}\|Q_k(D_x^\alpha g)\|_{L_x^\infty l_k^2}\|Q_n(h)\|_{L_x^{p'} l_n^2}$$
$$\lesssim \|f\|_{p}\|Q_k(D_x^\alpha g)\|_{L_x^\infty l_k^2}\|h\|_{L_x^{p'}}.$$

The second item is treated as the first, with $\Psi_k(f)$ replacing $Q_k(f)$. A similar argument is used on the third term, with $\Psi_k(D_x^\alpha g)$ replacing $\Psi_k(f)$, and the fact that

$$\|\mathcal{M}(D_x^\alpha g)\|_{L_x^\infty} \lesssim \|D_x^\alpha g\|_{L_x^\infty},$$

because $\mathcal{M}$ is a bounded operator from L^∞ to L^∞.

The last term is treated with Cauchy–Schwartz:

$$\|\sum_{-\infty}^{\infty} Q_k(f)Q_k(D_x^\alpha g)\|_p \lesssim \|\|Q_n(f)\|_{l_n^2}\|Q_k(D_x^\alpha g)\|_{l_k^2}\|_p$$

$$\lesssim \|Q_n(f)\|_{L_x^p l_k^2} \|Q_k(D_x^\alpha g)\|_{L_x^\infty l_k^2}.$$

This proves the first part of the lemma.

The second part follows from

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_p \geq \|D_x^\alpha(fg) - gD_x^\alpha f\|_p - \|fD_x^\alpha g\|_p,$$

along with the observation that

$$|D^\alpha g| \leq \sum_N |Q_N(D^\alpha g)|,$$

and for arbitrary functions φ_N,

$$\|\varphi_N\|_{l_N^2} \leq \|\varphi_N\|_{l_N^\infty}^{\frac{1}{2}} \|\varphi_N\|_{l_N^1}^{\frac{1}{2}} \leq \|\varphi_N\|_{l_N^1}. \qquad \square$$

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References

[1] I. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations,* Acta Math. Acad. Sci. Hungar., 7 (1956), 81–94.

[2] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations,* Amer. J. Math., 125 (2003), 1235–1293.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$*,* J. Amer. Math. Soc., 16 (2003), 705–749 (electronic).

[4] G. Fonseca and G. Ponce, *The IVP for the Benjamin–Ono equation in weighted Sobolev spaces,* J. Funct. Anal., 260 (2011), 436–459.

[5] C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, *Method for solving the Korteweg–de Vries equation,* Phys. Rev. Lett., 19 (1967), 1095–1097.

[6] J. Ginibre and Y. Tsutsumi, *Uniqueness of solutions for the generalized Korteweg–de Vries equation,* SIAM J. Math. Anal., 20 (1989), 1388–1425.

[7] Z. Guo, *Global well-posedness of Korteweg–de Vries equation in* $H^{-3/4}(\mathbb{R})$*,* J. Math. Pures Appl. 91 (2009), 583–597.

[8] A. Grünrock, *A bilinear Airy-estimate with application to gKdV-3,* Differential Integral Equations, 18 (2005), 1333–1339.

[9] N. Hayashi, K. Nakamitsu, and M. Tsutsumi, *On solutions of the initial value problem for the nonlinear Schrödinger equations in one space dimension,* Math. Z., 192 (1986), 637–650.

[10] N. Hayashi, K. Nakamitsu, and M. Tsutsumi, *On solutions of the initial value problem for the nonlinear Schrödinger equations,* J. Funct. Anal., 71 (1987), 218–245.

[11] N. Hayashi, K. Nakamitsu, and M. Tsutsumi, *Nonlinear Schrödinger equations in weighted Sobolev spaces,* Funkcial. Ekvac., 31 (1988), 363–381.

[12] T. Kato, *On the Cauchy problem for the (generalized) Korteweg–de Vries equation,* In "Studies in Applied Mathematics," volume 8 of *Adv. Math. Suppl. Stud.*, pages 93–128. Academic Press, New York, 1983.

[13] C.E. Kenig, G. Ponce, and L. Vega, *The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces of negative indices,* Duke Math. J., 71 (1993), 1–21.

[14] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle,* Comm. Pure Appl. Math., 46 (1993), 527–620.

[15] C.E. Kenig and H. Takaoka, *Global wellposedness of the modified Benjamin–Ono equation with initial data in $H^{1/2}$,* Int. Math. Res. Not., pages Art. ID 95702, 44, 2006.

[16] N. Kishimoto, *Well-posedness of the Cauchy problem for the Korteweg–de Vries equation at the critical regularity,* Diff. Integ. Eqns., 22 (2009), 447–464.

[17] Z.-L. Li, *Application of higher-order KdV–mKdV model with higher-degree nonlinear terms to gravity waves in atmosphere,* Chinese Physics B, 18 (2009), 4074–4082.

[18] F. Linares and G. Ponce, "Introduction to Nonlinear Dispersive Equations," Universitext, Springer, New York, 2009.

[19] R.M. Miura, *Korteweg–de Vries equation and generalizations. I, A remarkable explicit nonlinear transformation,* J. Mathematical Phys., 9 (1968), 1202–1204.

[20] J. Nahas and G. Ponce, *On the persistent properties of solutions to semi-linear Schrödinger equation,* Comm. Partial Diff. Eqs., 34 (2009), 1208–1227

[21] V.V. Prudskikh, *Ion-acoustic solitons in bi-ion dusty plasma,* Plasma Physics Reports, 34 (2008), 955–962.

[22] E.A. Ralph and L. Pratt, *Predicting eddy detachment for an equivalent barotropic thin jet,* Journal of NonLinear Science, 4 (1994), 355–374.

[23] E.M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[24] T. Tao, *Scattering for the quartic generalised Korteweg–de Vries equation,* J. Diff. Eqns., 232 (2007), 623–651.