Hayashi, N., Naumkin, P.I. and Sánchez-Suárez, I.
Osaka J. Math.
61 (2024), 163–193

ASYMPTOTICS OF SOLUTIONS TO THE FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION WITH $\alpha > \frac{5}{2}$

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(Received October 5, 2022, revised January 10, 2023)

Abstract

We study the large time asymptotic behavior of solutions to the Cauchy problem for the fractional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \frac{1}{\alpha}\left|\partial_x\right|^{\alpha} u = \lambda \left|u\right|^{\alpha} u,\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}, \end{cases}$$

where $\lambda > 0$, the fractional derivative $\left|\partial_x\right|^{\alpha} = \mathcal{F}^{-1}\left|\xi\right|^{\alpha}\mathcal{F}$, $\alpha > \frac{5}{2}$. This paper is a sequel to our previous papers [17] for $2 < \alpha < \frac{5}{2}$ and [36] for $\alpha = \frac{5}{2}$. We show that solutions decay in time at the rate $t^{-\frac{1}{\alpha}}(\log t)^{-\frac{1}{\alpha}}$, namely that the nonlinearity acts as a dissipative term, when $\lambda > 0$. This phenomena does not occur for the cubic problem

$$\begin{cases} i\partial_t u - \frac{1}{\alpha}\left|\partial_x\right|^{\alpha} u = \lambda \left|u\right|^{2} u,\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}, \end{cases}$$

with $0 < \alpha \leq 2$.

1. Introduction

We study the large time asymptotic behavior of solutions to the Cauchy problem for the fractional nonlinear Schrödinger equation in one space dimension

$$\begin{cases} i\partial_t u - \frac{1}{\alpha}\left|\partial_x\right|^{\alpha} u = \lambda \left|u\right|^{\alpha} u,\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\alpha > \frac{5}{2}$ and $\lambda > 0$, which corresponds to the defocusing nonlinearity. The fractional derivative $\left|\partial_x\right|^{\alpha} = \mathcal{F}^{-1}\left|\xi\right|^{\alpha}\mathcal{F}$, here and below $\mathcal{F}$ stands for the Fourier transformation $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx$, and $\mathcal{F}^{-1}$ is the inverse Fourier transformation $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi$.

Fractional nonlinear Schrödinger equations (1.1) appeared in [32], [33] with applications in quantum mechanics. Later it was derived in various areas such as plasma physics, optimization, finance, free boundary obstacle problems, population dynamics and minimal surfaces. The case of fractional derivative $\left|\partial_x\right|^{\frac{3}{2}}$ has a particular relevance to the two-dimensional water waves with surface tension (see [27], [28]). Recently fractional nonlinear Schrödinger equations attracted much attention of many authors, (see [2], [5], [6], [9], [10], [11], [13], [29], [30], [31] and references cited therein).

The present paper is a sequel to our previous papers [17] for $2 < \alpha < \frac{5}{2}$ and [36] for $\alpha = \frac{5}{2}$.

2020 Mathematics Subject Classification. Primary 35B40; Secondary 35Q92.

Our purpose is to show that $\mathbf{L}^\infty$ - norm of solutions decays in time at the rate $t^{-\frac{1}{\alpha}} (\log t)^{-\frac{1}{\alpha}}$, namely that the nonlinearity acts as a dissipative term, when $\lambda > 0$. As we know, this phenomena does not occur for the cubic problem

$$\begin{cases} i\partial_t u - \frac{1}{\alpha} |\partial_x|^\alpha u = \lambda |u|^2 u, \; t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \; x \in \mathbb{R} \end{cases} \tag{1.2}$$

in the case of $0 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$. The cubic nonlinear problem (1.2) was studied in the previous works [21], [22], [23], [26], [35], [34], where it was shown that the asymptotic behavior of solutions is represented in the form of a solution to the linear problem modified by the logarithmically oscillating term if $\lambda \in \mathbb{R} \setminus \{0\}$, so that the $\mathbf{L}^\infty$ - norm of solutions decays in time at the rate $t^{-\frac{1}{2}}$. The large time asymptotics for solutions of the fractional nonlinear Schrödinger equation (1.2) with $\alpha = \frac{1}{2}$ was obtained in papers [21], [26], where the derivative of order $\frac{3}{4}$ of solutions to the linear problem

$$i\partial_t u - 2 |\partial_x|^{\frac{1}{2}} u = 0$$

was used. More precisely, we have the following estimate for the case of $\alpha = \frac{1}{2}$

$$\left\| e^{2it|\partial_x|^{\frac{1}{2}}} u_0 \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \left\| |\partial_x|^{\frac{3}{4}} u_0 \right\|_{\mathbf{L}^1} \leq Ct^{-\frac{1}{2}} \left(\|x\partial_x u_0\|_{\mathbf{L}^2} + \|u_0\|_{\mathbf{H}^1} \right). \tag{1.3}$$

Estimate (1.3) is valid for $\alpha \in (0,1)$, and the result was extended to $\alpha \in (0,1)$ by the second author in [35]. For a general α, in [19], it was shown that we have

$$\left\| e^{\pm it\frac{1}{\alpha}|\partial_x|^\alpha} u_0 \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1+\beta}{\alpha}} \left\| |\partial_x|^{-\beta} u_0 \right\|_{\mathbf{L}^1}, \tag{1.4}$$

where $0 \leq \beta \leq \frac{\alpha-2}{2}$. This estimate helps us to avoid the use of the estimate of $\|\mathcal{J}u\|_{\mathbf{L}^2}$ itself, when proving the $\mathbf{L}^\infty$ - time decay of small solutions in the case of small α. Indeed, in [21] the key estimate was that of $\left\| |\partial_x|^{\frac{1}{2}} \mathcal{J}u \right\|_{\mathbf{L}^2}$, also it was used the fact that the operator $P = x\partial_x + 2t\partial_t$ works well for the problem, since the right-hand side of (1.3) corresponds to $\|Pu\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^1}$. The case of $\alpha = 2$ (the cubic nonlinear Schrödinger equation itself) was studied previously extensively (see, e.g. [4], [18] and cited therein). Also we note that in the case of $\alpha = 1$ (the so-called half-wave equation) the asymptotics of solutions is unknown up to now.

In the previous paper [15], by using estimate (1.4) we showed a global existence of small solutions to the fractional nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{\alpha} |\partial_x|^\alpha u = \lambda |u|^{p-1} u$$

for a super critical nonlinearity $p > 3$ and $\alpha \in (0,1)$. Therefore the problem with higher-order $\alpha < 2$, which is close to 2 is more difficult than that with $\alpha \in (0,1)$. In the case of $\alpha \in (1,2)$, we need somewhat involved estimates of $\|\mathcal{J}u\|_{\mathbf{L}^2}$. The cases of $\alpha \in \left(1, \frac{3}{2}\right)$, $\alpha = \frac{3}{2}$ and $\alpha \in \left(\frac{3}{2}, 2\right)$ were studied in papers [22], [34] and [23], respectively.

From the works [14] for $\alpha = 3$ and [16] for $\alpha = 4$, we find a sharp contrast between the cases of $\alpha > 2$ and $0 < \alpha < 2$. In these papers, the asymptotic behavior of solutions was obtained for the defocusing case $\lambda > 0$ and it was shown that the nonlinearity of the order $\alpha+1$ is critical from the point of view of the large time asymptotic behavior of solutions. More precisely, it was proved that the solutions decay in time as $t^{-\frac{1}{\alpha}} (\log t)^{-\frac{1}{\alpha}}$. This phenomena tells us that the nonlinearity acts as a dissipative term. These works were extended to the fractional nonlinear Schrödinger equation (1.1) in [17] for $\alpha \in \left(2, \frac{5}{2}\right)$ and in [36] for $\alpha = \frac{5}{2}$, respectively. So, as we stated before, the aim of the present paper is to fill the gap $\alpha \in \left(\frac{5}{2}, \infty\right)$. Our result here is closely related to the previous work [17], where we have used the function space $\widetilde{X}_T$ based on the norm

$$\|u\|_{\widetilde{X}_T} = \sup_{t\in[1,T]} \left(\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} + W^{-1}(t) \left\| \left\langle \widetilde{\xi} \right\rangle^{-\gamma} \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} + P^{-1}(t) \left\| \partial_\xi \widehat{\varphi}(t) \right\|_{\mathbf{L}^2} \right),$$

where

$$\begin{aligned} \widehat{\varphi}(t) &= \mathcal{F}\mathcal{U}(-t)\, u(t), W(t) = \left(1 + \widetilde{\varepsilon}^\alpha \log\langle t\rangle\right)^{-\frac{1}{\alpha}}, \mathcal{U}(t) = e^{it\frac{1}{\alpha}|\partial_x|^\alpha} \\ P(t) &= t^{\frac{1}{4\alpha}} + \varepsilon^\alpha t^{\frac{1}{2\alpha}} W^{\alpha+1}(t), \widetilde{\varepsilon} = C\varepsilon, \widetilde{\xi} = t^{\frac{1}{\alpha}}\xi, \end{aligned}$$

and $\varepsilon > 0, \gamma > 0$ are small enough. In order to get the desired a-priori estimate of the norm $\left\|\partial_\xi \widehat{\varphi}(t)\right\|_{\mathbf{L}^2}$, which requires us to estimate the following norm of the nonlinearity

$$\left\| \partial_\xi \mathcal{Q}^* \left(\left| \mathcal{Q}\widehat{\varphi} \right|^\alpha \mathcal{Q}\widehat{\varphi} \right) \right\|_{\mathbf{L}^2},$$

we have to assume the condition $2 < \alpha < \frac{5}{2}$ (see Lemma 3.2 in [17]), where $\mathcal{Q}$ and its adjoint $\mathcal{Q}^*$ are defined in Section 2 below. Comparing with the previous work [17], in the case of $\alpha > \frac{5}{2}$ we encounter the difficulty of the derivative loss, when proving an optimal estimates for the derivatives of the operators $\mathcal{Q}^*, \mathcal{Q}$. From Lemma 3.2 in [17], it seems difficult to get a-priori estimate of $\left\|\partial_\xi \widehat{\varphi}(t)\right\|_{\mathbf{L}^2}$ in $\widetilde{X}_T$ for $\alpha > \frac{5}{2}$. Therefore we introduce a different function space X_T with the norm

$$\sup_{t\in[1,T]} \left(\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} + W^{-1}(t) \left\| \left\langle \widetilde{\xi} \right\rangle^{-\gamma} \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} + P^{-1}(t) \left\| \left\langle \widetilde{\xi} \right\rangle^{\beta} \partial_\xi \widehat{\varphi}(t) \right\|_{\mathbf{L}^2} \right)$$

with $\beta \in \left(0, \frac{1}{2}\right)$. An additional term $\left\langle \widetilde{\xi} \right\rangle^{\beta}$ corresponds to the regularity of solutions. Therefore we need to change the requirement $\mathbf{H}^{0,1} \cap \mathbf{H}^{\alpha-1,0}$ for the initial data used in [17] to $\mathbf{H}^{1,1} \cap \mathbf{H}^{\alpha,0}$. The main ingredient of the present paper is to show that the contraction mapping principle works well in the function space X_T. Our new point in obtaining the estimate of the norm $\left\| \left\langle \widetilde{\xi} \right\rangle^{\beta} \partial_\xi \widehat{\varphi}(t) \right\|_{\mathbf{L}^2}$ is the $\mathbf{L}^2$ - estimate of the operator $\chi_1\left(\omega t^{\frac{1}{\alpha}} \partial_x\right) \left\langle \omega t^{\frac{1}{\alpha}} \partial_x \right\rangle^{\beta} \partial_x^{-1} \mathcal{P}$, where $\mathcal{P} = \alpha t \partial_t + \partial_x x$, $\beta \in \left(0, \frac{1}{2}\right)$, $\omega > 0$ is small, and the cut off function $\chi_1 \in \mathbf{C}^1(\mathbb{R})$ is such that $\chi_1(x) = 1$ for $|x| \geq 2$ and $\chi_1(x) = 0$ for $|x| \leq 1$. We also use known results on the $\mathbf{L}^2$ - boundedness of the pseudodifferential operators to avoid some complicated computations.

We introduce some notations. Denote by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$ the usual Lebesgue spaces, where the norm is given by $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p\, dx\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbb{R}} |\phi(x)|$. The weighted Sobolev space is defined as follows

$$\mathbf{H}_p^{m,s} = \left\{ \varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \left\| \langle x\rangle^s \langle i\partial_x\rangle^m \phi \right\|_{\mathbf{L}^p} < \infty \right\},$$

where $m, s \in \mathbb{R}$, $1 \leq p \leq \infty$, $\langle x\rangle = \sqrt{1+x^2}$, $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the short notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$. By $\mathbf{C}(\mathbf{I};\mathbf{B})$, we denote the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. By C we denote different positive constants.

Define $G(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi - \frac{i}{\alpha}|\xi|^\alpha} d\xi$ and a constant

$$b = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |G(x)|^{\alpha} G(x)\, dx.$$

Observe that $\operatorname{Im} b < 0$ (see Lemma 3.1 below).

We now present our main result.

Theorem 1.1. *Let* $u_0 \in \mathbf{H}^{1,1} \cap \mathbf{H}^{\alpha,0}$, $\|u_0\|_{\mathbf{H}^{1,1}} \le 100\varepsilon$ *and* $\inf_{|\xi|\le 1} \left|\widehat{u_0}(\xi)\right| \ge 2\varepsilon$. *Then there exists* $\varepsilon_0 > 0$ *such that the Cauchy problem* (1.1) *has a unique solution* $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^{1,1} \cap \mathbf{H}^{\alpha,0}\right)$ *satisfying the time decay estimate* $\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{\alpha}} (\log t)^{-\frac{1}{\alpha}}$ *for any* $\varepsilon \in (0, \varepsilon_0)$. *Moreover the asymptotics*

$$u(t,x) = \widehat{u_0}(0)\, G\left(xt^{-\frac{1}{\alpha}}\right) \times \frac{\exp\left(-i\frac{\operatorname{Re} b}{\alpha \operatorname{Im} b} \log\left(1 - \alpha\lambda \operatorname{Im} b \left|\widehat{u_0}(0)\right|^{\alpha} \log t\right) + iO(\varepsilon)\right)}{t^{\frac{1}{\alpha}}\left(1 - \alpha\lambda \operatorname{Im} b \left|\widehat{u_0}(0)\right|^{\alpha} \log t\right)^{\frac{1}{\alpha}}} + O\left(t^{-\frac{1}{\alpha}} (\log t)^{-\frac{2}{\alpha}}\right) \quad (1.5)$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbb{R}$.

In order to explain our strategy more clearly, we state the plan of the proof of Theorem 1.1 shortly. From (1.4), we have the smoothing property of solutions for $\alpha > 2$ and if $\partial_\xi^j \widehat{u_0}(0) = 0$, $j = 0, 1, ...,$ then the time decay of solutions to the linear problem will be $t^{-\frac{1}{2}}$ for large α. Therefore the order of the nonlinearity $\alpha+1$ is reduced to 3. This fact was used in papers [19], [38]. In [38], the fourth order Schrödinger equation with the cubic nonlinearity

$$i\partial_t u + \frac{1}{4} |\partial_x|^4 u = \lambda |u|^2 u, \lambda \in \mathbf{R}$$

was considered and the modified wave operator was constructed by using the method of [37] under the the final data condition such that

$$\left\|\widehat{u_+}\right\|_{\mathbf{H}^{4,0}} + \sum_{k=0}^{4} \left\||\xi|^{k-14} \partial_\xi^k \widehat{u_+}\right\|_{\mathbf{H}^{4,0}} < \infty.$$

Results of [38] was improved in [19] and the modified wave operator for equation

$$i\partial_t u + \frac{1}{\alpha} |\partial_x|^{\alpha} u = \lambda |u|^2 u$$

was constructed under the vanishing condition for the final data such that

$$\left\|\widehat{u_+}\right\|_{\mathbf{L}^2} + \left\||\xi|^{-\frac{\alpha-2}{2}} \widehat{u_+}\right\|_{\mathbf{L}^\infty} + \left\||\xi|^{-\frac{\alpha-4}{2}} \partial_\xi \widehat{u_+}\right\|_{\mathbf{L}^\infty} < \infty.$$

However due to the vanishing condition, asymptotic behavior of solutions to the initial value problem with cubic nonlinearities is an open problem up to now.

In our result, the non vanishing condition $\widehat{u_0}(0) \neq 0$ implies that the solutions decay with the rate $t^{-\frac{1}{\alpha}}$, and the leading term of the asymptotics of the solutions is represented as $\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}(t,0)$ since

$$\begin{aligned} u(t) &= \mathcal{U}(t)\mathcal{U}(-t)u = \mathcal{U}(t)\mathcal{F}^{-1}\mathcal{F}\mathcal{U}(-t)u \equiv \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}(t,\xi) \\ &= \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}(t,0) + \mathcal{U}(t)\mathcal{F}^{-1}\left(\widehat{\varphi}(t,\xi) - \widehat{\varphi}(t,0)\right) \end{aligned}$$

and the second term is the remainder. We let $m(t) = \widehat{\varphi}(t,0)$, then we have

$$\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}(t,0) = m(t)\mathcal{F}^{-1}e^{-it\frac{1}{\alpha}|\xi|^{\alpha}}$$

which means that the asymptotics of solutions is determined by the behavior of $m(t)$. From (2.1) below we have the equation

$$i\partial_t\widehat{\varphi}(t,\xi) = \lambda t^{-\frac{\alpha}{2}}\mathcal{Q}^*\left(\left|\mathcal{Q}\widehat{\varphi}\right|^{\alpha}\mathcal{Q}\widehat{\varphi}\right)(t,\xi).$$

We put $\xi = 0$, then

$$i\partial_t\widehat{\varphi}(t,0) = \lambda t^{-\frac{\alpha}{2}}\mathcal{Q}^*\left(\left|\mathcal{Q}\widehat{\varphi}\right|^{\alpha}\mathcal{Q}\widehat{\varphi}\right)(t,0).$$

Thus we arrive to the equation for $m(t)$

$$i\partial_t m(t) = \lambda t^{-\frac{\alpha}{2}}|m(t)|^{\alpha}m(t)\mathcal{Q}^*\left(|\mathcal{Q}1|^{\alpha}\mathcal{Q}1\right)(t,0),$$

where the operators $\mathcal{Q}$ and $\mathcal{Q}^*$ will be defined in the next section. Taking into account their definitions and the direct calculation, we find that

$$\mathcal{Q}^*\left(|\mathcal{Q}1|^{\alpha}\mathcal{Q}1\right)(t,0) = t^{\frac{\alpha}{2}-1}b,$$

where b is defined in the theorem (see Section 3). Hence we have

$$i\partial_t m(t) = \lambda b t^{-1}|m(t)|^{\alpha}m(t).$$

From this equation we have the asymptotic behavior of the solutions. Our main task is to get the estimates of the remainder terms in our function space. We note that our result works well for the case of $\mathrm{Im}\lambda b < 0$.

We organize the rest of the paper as follows. In Section 2 we state the factorization techniques for (1.1), and prove $\mathbf{L}^{\infty}$ and $\mathbf{L}^2$ - estimates for the defect operators $\mathcal{Q}$ and $\mathcal{Q}^*$. Section 3 is devoted to the asymptotics of the nonlinearity. In Section 4 we prove a-priori estimates of local solutions u of the Cauchy problem (1.1). Finally we prove Theorem 1.1 in Section 5.

2. Preliminaries

2.1. Factorization techniques. Denote the symbol $\Lambda(\xi) = \frac{1}{\alpha}|\xi|^{\alpha}$, $\alpha > \frac{5}{2}$. The linear evolution group is written as $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\mathcal{F}$. The stationary point $\mu(x) = x|x|^{-\frac{\alpha-2}{\alpha-1}}$ is a root of the equation $\Lambda'(\mu) = x$. Define the dilation operator $\mathcal{D}_t\phi = t^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right)$, the scaling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$, and the defect operator

$$\mathcal{Q}(t)\phi = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi,$$

where the phase function

$$\begin{aligned} S(\xi,\eta) &= \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta) \\ &= \frac{1}{\alpha}\left(|\xi|^{\alpha} - |\eta|^{\alpha}\right) - |\eta|^{\alpha-2}\eta(\xi-\eta). \end{aligned}$$

Then the linear evolution group can be factorized as follows: $\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\phi$, where the multiplication factor $M = e^{-it(\Lambda(\eta)-\eta\Lambda'(\eta))}$. Also, we define the inverse dilation operator $\mathcal{D}_t^{-1}\phi = t^{\frac{1}{2}}\phi(xt)$, the inverse scaling operator $\mathcal{B}^{-1}\phi = \phi(\Lambda'(\eta))$ and the adjoint defect operator

$$\mathcal{Q}^*(t)\phi = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)\,d\eta.$$

Then, the inverse evolution group $\mathcal{U}(-t)$ is factorized as follows: $\mathcal{F}\mathcal{U}(-t)\phi = \mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi$.

We define a new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$. Since $\mathcal{F}\mathcal{U}(-t)\mathcal{L} = i\partial_t\mathcal{F}\mathcal{U}(-t)$ with $\mathcal{L} = i\partial_t - \Lambda(-i\partial_x)$, applying the operator $\mathcal{F}\mathcal{U}(-t)$ to equation (1.1), and substituting $u = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}Mv$, where $v = \mathcal{Q}\widehat{\varphi}$, we get

$$\begin{aligned}(2.1)\qquad i\partial_t\widehat{\varphi} &= \lambda\mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}(|u|^\alpha u) = \lambda\mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}(|\mathcal{D}_t\mathcal{B}Mv|^\alpha\mathcal{D}_t\mathcal{B}Mv)\\ &= \lambda t^{-\frac{\alpha}{2}}\mathcal{Q}^*\overline{M}\mathcal{B}^{-1}(|\mathcal{B}Mv|^\alpha\mathcal{B}Mv) = \lambda t^{-\frac{\alpha}{2}}\mathcal{Q}^*(|v|^\alpha v).\end{aligned}$$

This is our target equation.

We have the identities $\mathcal{A}_1\mathcal{Q} = \mathcal{Q}i\xi$ and $i\xi\mathcal{Q}^* = \mathcal{Q}^*\mathcal{A}_1$, where $\mathcal{A}_1 = \overline{M}\mathcal{A}_0M$, and $\mathcal{A}_0 = \frac{1}{t\Lambda''(\eta)}\partial_\eta$. We mention that the operator $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t) = x - t\Lambda'(-i\partial_x)$, $\Lambda(\xi) = \frac{1}{\alpha}|\xi|^\alpha$ plays an important role in the large time asymptotic estimates. Note that $\mathcal{J}$ commutes with $\mathcal{L}$, i.e. $[\mathcal{J},\mathcal{L}] = 0$. The symbol $\Lambda(\xi) = \frac{1}{\alpha}|\xi|^\alpha$ satisfies the identity $\xi\partial_\xi\Lambda = \alpha\Lambda$. Hence $\left[\widehat{\mathcal{P}}, e^{-it\Lambda(\xi)}\right] = 0$ with $\widehat{\mathcal{P}} = \alpha t\partial_t - \xi\partial_\xi$. So we define the operator $\mathcal{P} = \alpha t\partial_t + \partial_x x$. Using the identity $\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\varphi}$, we get $\mathcal{P}u = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\mathcal{P}}\widehat{\varphi}$. Also we have the identity $\mathcal{P} = -i\alpha t\mathcal{L} + \partial_x\mathcal{J}$ and the commutator $[\mathcal{L},\mathcal{P}] = \alpha\mathcal{L}$.

2.2. Estimates for the operator $\mathcal{Q}$ in the uniform metrics. We consider the preliminary estimates for the integrals in the following lemma. Denote $\widetilde{\eta} = t^{\frac{1}{\alpha}}\eta$.

Lemma 2.1. *Let $\alpha > \frac{5}{2}$. Then the following estimates*

$$\int_{\frac{\eta}{3}}^{3\eta}\frac{(\xi-\eta)^2\,d\xi}{\left(1+t|\eta|^{\alpha-2}(\xi-\eta)^2\right)^2} \le C|\eta|^3\langle\widetilde{\eta}\rangle^{-\frac{3\alpha}{2}},$$

$$\int_{\mathbb{R}}\frac{|\xi|^{2+2j}\,d\xi}{\left(1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)\right)^2} \le Ct^{-\frac{3+2j}{\alpha}}\langle\widetilde{\eta}\rangle^{3+2j-2\alpha}$$

and

$$\int_{\mathbb{R}}\frac{|\xi|^j\,d\xi}{1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)} \le Ct^{-\frac{1+j}{\alpha}}\langle\widetilde{\eta}\rangle^{j+1-\alpha}\log\langle\widetilde{\eta}\rangle$$

are true for all $\eta\in\mathbb{R}$, $t\ge 1$, $j = 0,1$.

Proof. Changing the variable of integration $\xi = \eta z$, we find

$$\begin{aligned}&\left|\int_{\frac{\eta}{3}}^{3\eta}\frac{(\xi-\eta)^2\,d\xi}{\left(1+t|\eta|^{\alpha-2}(\xi-\eta)^2\right)^2}\right|\\ &\le C|\eta|^3\int_{\frac{1}{3}}^{3}\frac{(1-z)^2\,dz}{\left(1+t|\eta|^\alpha(1-z)^2\right)^2} \le C|\eta|^3\int_{-\frac{2}{3}}^{2}\frac{z^2dz}{1+|\widetilde{\eta}|^{2\alpha}z^4} \le C|\eta|^3\langle\widetilde{\eta}\rangle^{-\frac{3\alpha}{2}}.\end{aligned}$$

Changing $\xi = \eta z$ and $\xi t^{\frac{1}{\alpha}} = \widetilde{\xi}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{|\xi|^{2+2j}\, d\xi}{\left(1 + t|\xi|\left(|\xi|^{\alpha-1} + |\eta|^{\alpha-1}\right)\right)^2} \\
&\le C\int_0^{|\eta|} \frac{\xi^{2+2j} d\xi}{\left(1 + t\xi\,|\eta|^{\alpha-1}\right)^2} + C\int_{|\eta|}^{\infty} \frac{\xi^{2+2j} d\xi}{(1 + t\xi^{\alpha})^2} \\
&\le C\,|\eta|^{2j+3}\int_0^1 \frac{|z|^{2+2j}\, dz}{(1 + |\widetilde{\eta}|^{\alpha}\, z)^2} + Ct^{-\frac{3+2j}{\alpha}}\int_{|\widetilde{\eta}|}^{\infty} \left|\widetilde{\xi}\right|^{2+2j} \left\langle\widetilde{\xi}\right\rangle^{-2\alpha} d\widetilde{\xi} \\
&\le C\,|\eta|^{2j+3}\,\langle\widetilde{\eta}\rangle^{-2\alpha} + Ct^{-\frac{3+2j}{\alpha}}\,\langle\widetilde{\eta}\rangle^{3+2j-2\alpha} \le Ct^{-\frac{3+2j}{\alpha}}\,\langle\widetilde{\eta}\rangle^{3+2j-2\alpha},
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|\xi|^{j}\, d\xi}{1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)} &\le C\int_0^{|\eta|} \frac{|\xi|^{j}\, d\xi}{1+t\xi\,|\eta|^{\alpha-1}} + C\int_{|\eta|}^{\infty} \frac{|\xi|^{j}\, d\xi}{1+t\xi^{\alpha}} \\
&\le C\,|\eta|^{j+1}\int_0^1 \frac{|z|^{j}\, dz}{1+|\widetilde{\eta}|^{\alpha} z} + Ct^{-\frac{1+j}{\alpha}}\int_{|\widetilde{\eta}|}^{\infty} \left|\widetilde{\xi}\right|^{j} \left\langle\widetilde{\xi}\right\rangle^{-\alpha} d\widetilde{\xi} \\
&\le C\,|\eta|^{j+1}\,\langle\widetilde{\eta}\rangle^{-\alpha}\log\langle\widetilde{\eta}\rangle + Ct^{-\frac{1+j}{\alpha}}\,\langle\widetilde{\eta}\rangle^{1+j-\alpha} \le Ct^{-\frac{1+j}{\alpha}}\,\langle\widetilde{\eta}\rangle^{j+1-\alpha}\log\langle\widetilde{\eta}\rangle.
\end{aligned}$$

Lemma is proved. □

We prove the estimates for the operator $\mathcal{Q}$ in the $\mathbf{L}^{\infty}$ - norm. Denote $\widetilde{\xi} = \xi t^{\frac{1}{\alpha}}$.

Lemma 2.2. *Let $\alpha > \frac{5}{2}$. Then the following estimate*

$$\left|\mathcal{Q}\xi^{j}\phi\right| \le Ct^{\frac{1}{2}-\frac{1+j}{\alpha}}\,\langle\widetilde{\eta}\rangle^{1+j-\frac{\alpha}{2}+\rho}\log\langle\widetilde{\eta}\rangle\left(|\phi(0)| + \left\|\left\langle\widetilde{\xi}\right\rangle^{-\rho}\phi\right\|_{\mathbf{L}^{\infty}} + t^{-\frac{1}{2\alpha}}\left\|\partial_{\xi}\phi\right\|_{\mathbf{L}^{2}}\right)$$

is true for all $t \ge 1$, where $j = 0, 1, \rho \ge 0$.

Proof. Define the cut off function $\chi_1(x) \in \mathbf{C}^4(\mathbb{R})$, such that $\chi_1(x) = 1$ for $\frac{2}{3} \le x \le 2$ and $\chi_1(x) = 0$ for $x \ge 3$ or $x \le \frac{1}{3}$, also $\chi_2(x) = 1 - \chi_1(x)$. Define the kernel

$$A_j(t,\eta) = \sqrt{\frac{t}{2\pi}}\int_{\frac{1}{3}|\eta|\le|\xi|\le 3|\eta|} e^{-itS(\xi,\eta)}\chi_1\left(\frac{\xi}{\eta}\right)\xi^{j} d\xi$$

for $\eta \ne 0$. Changing $\xi = \eta y$, we get

$$A_j(t,\eta) = |\eta|\,\eta^{j}\sqrt{\frac{t}{2\pi}}\int_{\frac{1}{3}\le|y|\le 3} e^{-i|\widetilde{\eta}|^{\alpha}G(y,1)}y^{j}\chi_1(y)\, dy,$$

where $G(y,1) = \Lambda(y) - \Lambda(1) - \Lambda'(1)(y-1)$. We find the asymptotics of the kernels $A_j(t,\eta)$ by applying the stationary phase method (see [12], p. 110)

$$\int_{\mathbb{R}} e^{irg(y)} f(y)\, dy = e^{irg(y_0)} f(y_0)\sqrt{\frac{2\pi}{r\,|g''(y_0)|}}\, e^{i\frac{\pi}{4}\operatorname{sgn} g''(y_0)} + O\left(r^{-\frac{3}{2}}\right) \tag{2.2}$$

for $r \to +\infty$, where the stationary point y_0 is defined by the equation $g'(y_0) = 0$. Then we find

$$A_j(t,\eta) = \frac{t^{\frac{1}{2}} |\eta| \eta^j}{\sqrt{i(\alpha-1)\langle\widetilde{\eta}\rangle^{\alpha}}} \left(1 + O\left(\langle\widetilde{\eta}\rangle^{-\alpha}\right)\right),$$

as $|\widetilde{\eta}| \to \infty$. In particular we get the estimate $\left|A_j(t,\eta)\right| \le Ct^{\frac{1}{2}} |\eta|^{j+1} \langle\widetilde{\eta}\rangle^{-\frac{\alpha}{2}}$. Next we define the operators

$$Q_l\phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \phi(\xi) \chi_l\left(\frac{\xi}{\eta}\right) d\xi$$

for $l = 1, 2$. Then we write

$$Q_1\xi^j\phi - A_j\phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} (\phi(\xi) - \phi(\eta)) \xi^j \chi_1\left(\frac{\xi}{\eta}\right) d\xi.$$

We use the identity $e^{-itS(\xi,\eta)} = H_1\partial_\xi\left((\xi-\eta) e^{-itS(\xi,\eta)}\right)$ with $H_1 = \left(1 - it(\xi-\eta)\partial_\xi S(\xi,\eta)\right)^{-1}$, and integrate by parts

$$Q_1\xi^j\phi - A_j\phi = Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} (\phi(\xi) - \phi(\eta)) (\xi-\eta) \partial_\xi\left(H_1\xi^j\chi_1\left(\frac{\xi}{\eta}\right)\right) d\xi$$
$$+ Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} (\xi-\eta) \chi_1\left(\frac{\xi}{\eta}\right) H_1\xi^j\partial_\xi\phi(\xi)\, d\xi.$$

Since $\partial_\xi^l\Lambda(\xi) = O\left(|\xi|^{\alpha-l}\right)$, for $l = 0, 1, 2$, and $\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \Lambda'(\eta)$, we have

$$\left|\chi_1\left(\frac{\xi}{\eta}\right) H_1\xi^j\right| + \left|(\xi-\eta)\partial_\xi\left(H_1\xi^j\chi_1\left(\frac{\xi}{\eta}\right)\right)\right| \le \frac{C|\eta|^j}{1 + t|\eta|^{\alpha-2}(\xi-\eta)^2}$$

in the domain $\frac{1}{3}|\eta| \le |\xi| \le 3|\eta|$. Applying the Hardy and the Cauchy-Schwarz inequalities, we find

$$\begin{aligned}
&\left|Q_1\xi^j\phi - A_j\phi\right| \\
&\le Ct^{\frac{1}{2}} |\eta|^j \int_{\frac{1}{3}|\eta|\le|\xi|\le3|\eta|} \left(\frac{|\phi(\xi) - \phi(\eta)|}{|\xi-\eta|} + \left|\partial_\xi\phi(\xi)\right|\right) \frac{|\xi-\eta|\, d\xi}{1 + t|\eta|^{\alpha-2}(\xi-\eta)^2} \\
&\le Ct^{\frac{1}{2}} |\eta|^j \left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} I_1^{\frac{1}{2}}.
\end{aligned}$$

where

$$I_1 = \int_{\frac{1}{3}|\eta|\le|\xi|\le3|\eta|} \frac{(\xi-\eta)^2\, d\xi}{\left(1 + t|\eta|^{\alpha-2}(\xi-\eta)^2\right)^2} \le C|\eta|^3 \langle\widetilde{\eta}\rangle^{-\frac{3}{2}\alpha}$$

by Lemma 2.1. Thus by estimate $\left|A_j(t,\eta)\right| \le Ct^{\frac{1}{2}} |\eta|^{j+1} \langle\widetilde{\eta}\rangle^{-\frac{\alpha}{2}}$ we get

$$\begin{aligned}
\left|Q_1\xi^j\phi\right| &\le \left|A_j\phi\right| + Ct^{\frac{1}{2}} |\eta|^{j+\frac{3}{2}} \langle\widetilde{\eta}\rangle^{-\frac{3}{4}\alpha} \left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} \\
&\le Ct^{\frac{1}{2}} |\eta|^{j+1} \langle\widetilde{\eta}\rangle^{-\frac{\alpha}{2}} |\phi| + Ct^{\frac{1}{2}} |\eta|^{j+\frac{3}{2}} \langle\widetilde{\eta}\rangle^{-\frac{3}{4}\alpha} \left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} \\
&\le Ct^{\frac{1}{2}-\frac{1+j}{\alpha}} \langle\widetilde{\eta}\rangle^{1+j-\frac{\alpha}{2}+\rho} \left\|\left\langle\widetilde{\xi}\right\rangle^{-\rho} \phi\right\|_{\mathbf{L}^\infty} + Ct^{\frac{1}{2}-\frac{j}{\alpha}-\frac{3}{2\alpha}} \langle\widetilde{\eta}\rangle^{\frac{3}{2}+j-\frac{3}{4}\alpha} \left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} \\
&\le Ct^{\frac{1}{2}-\frac{1+j}{\alpha}} \langle\widetilde{\eta}\rangle^{1+j-\frac{\alpha}{2}+\rho} \left(\left\|\left\langle\widetilde{\xi}\right\rangle^{-\rho} \phi\right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2\alpha}} \left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}\right).
\end{aligned}$$

In the term $Q_2\xi^j\phi$, we use the identity $e^{-itS(\xi,\eta)} = H_2\partial_\xi\left(\xi e^{-itS(\xi,\eta)}\right)$ with $H_2 =$

$\left(1 - it\xi\partial_\xi S(\xi,\eta)\right)^{-1}$, and integrate by parts

$$\mathcal{Q}_2\xi^j\phi = Ct^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\frac{\phi(\xi)-\phi(0)}{\xi}\xi^2\partial_\xi\left(\xi^j\chi_2\left(\frac{\xi}{\eta}\right)H_2\right)d\xi$$
$$+ Ct^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\xi^j\chi_2\left(\frac{\xi}{\eta}\right)H_2\xi\partial_\xi\phi(\xi)\,d\xi$$
$$+ C\phi(0)\,t^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\xi\partial_\xi\left(\xi^j\chi_2\left(\frac{\xi}{\eta}\right)H_2\right)d\xi.$$

Observing that $\partial_\xi^l\Lambda(\xi) = O\left(|\xi|^{\alpha-l}\right)$, for $l = 0,1,2$, and $\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \Lambda'(\eta)$, we have the estimates

$$\left|\xi^j\chi_2\left(\frac{\xi}{\eta}\right)H_2\xi\right| + \left|\xi^2\partial_\xi\left(\xi^j\chi_2\left(\frac{\xi}{\eta}\right)H_2\right)\right| \le C\frac{|\xi|^{1+j}}{1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)},$$

in the domains $|\xi| \le \frac{2}{3}|\eta|$ or $|\xi| \ge 2|\eta|$. Then applying the Cauchy-Schwarz and Hardy inequalities, we find

$$\left|\mathcal{Q}_2\xi^j\phi\right| \le Ct^{\frac{1}{2}}\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}\left(\int_{\mathbb{R}}\frac{|\xi|^{2+2j}\,d\xi}{\left(1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)\right)^2}\right)^{\frac{1}{2}}$$
$$+ Ct^{\frac{1}{2}}|\phi(0)|\int_{\mathbb{R}}\frac{|\xi|^j\,d\xi}{1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)}.$$

Thus, by estimates of Lemma 2.1 we get

$$\left|\mathcal{Q}_2\xi^j\phi\right| \le Ct^{\frac{1}{2}-\frac{3+2j}{2\alpha}}\langle\widetilde{\eta}\rangle^{\frac{3}{2}+j-\alpha}\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} + Ct^{\frac{1}{2}-\frac{1+j}{\alpha}}\langle\widetilde{\eta}\rangle^{j+1-\alpha}\log\langle\widetilde{\eta}\rangle|\phi(0)|$$
$$\le Ct^{\frac{1}{2}-\frac{1+j}{\alpha}}\langle\widetilde{\eta}\rangle^{1+j-\frac{\alpha}{2}}\log\langle\widetilde{\eta}\rangle\left(|\phi(0)| + t^{-\frac{1}{2\alpha}}\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}\right)$$

Hence the estimate of the lemma follows. Lemma 2.2 is proved. □

2.3. Estimates for $\mathcal{Q}^*\phi$ in the uniform norm. Define the kernel

$$A^*(t,\xi) = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\,|\widetilde{\eta}|^{2-\alpha-\rho}\,\Lambda''(\eta)\,d\eta,$$

where $\rho \in [0,1)$. Changing $\widetilde{\eta} = t^{\frac{1}{\alpha}}\eta$, $\widetilde{\xi} = t^{\frac{1}{\alpha}}\xi$, and then $\widetilde{\eta} = \widetilde{\xi}z$, we get

$$A^*(t,\xi) = t^{\frac{1}{\alpha}-\frac{1}{2}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{iS\left(\widetilde{\xi},\widetilde{\eta}\right)}\,|\widetilde{\eta}|^{2-\alpha-\rho}\,\Lambda''(\widetilde{\eta})\,d\widetilde{\eta}$$
$$= t^{\frac{1}{\alpha}-\frac{1}{2}}\left|\widetilde{\xi}\right|^{1-\rho}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\left|\widetilde{\xi}\right|^{\alpha}S(1,z)}\,|z|^{2-\alpha-\rho}\,\Lambda''(z)\,dz.$$

By virtue of the formula (2.2) with $r = \left|\widetilde{\xi}\right|^{\alpha}$, $g(z) = S(1,z)$, $f(z) = |z|^{2-\alpha-\rho}\Lambda''(z)$, $z_0 = 1$, we get

$$A^*(t,\xi) = t^{\frac{1}{\alpha}-\frac{1}{2}}\left|\widetilde{\xi}\right|^{1-\rho-\frac{\alpha}{2}}\sqrt{i(\alpha-1)} + O\left(t^{\frac{1}{\alpha}-\frac{1}{2}}\left|\widetilde{\xi}\right|^{1-\rho-\frac{3}{2}\alpha}\right)$$

for $\left|\widetilde{\xi}\right| \to \infty$. Hence the estimate $|A^*(t,\xi)| \le Ct^{\frac{1}{\alpha}-\frac{1}{2}}\left\langle\widetilde{\xi}\right\rangle^{1-\frac{\alpha}{2}-\rho}$ follows.

In the next lemma, we find the asymptotics of $\mathcal{Q}^* \phi$.

Lemma 2.3. *The estimate*

$$\left\| \mathcal{Q}^* (t) \phi - A^* (t, \xi) \left\langle \xi \right\rangle^{\alpha-2+\rho} \phi (\xi) \right\|_{\mathbf{L}^\infty} \\ \leq C t^{-\frac{1}{2\alpha}+\frac{\rho}{\alpha}} \left(\left\| \sqrt{\Lambda''} |\eta|^{-1+\rho} \phi (\eta) \right\|_{\mathbf{L}^2} + \left\| \sqrt{\Lambda''} |\eta|^{\rho} \partial_\eta \phi (\eta) \right\|_{\mathbf{L}^2} \right).$$

is true for all $t \geq 1$, where $\rho \in [0, 1)$.

Proof. We write

$$\mathcal{Q}^* (t) \phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \phi (\eta) \Lambda'' (\eta) \, d\eta \\ = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it(\Lambda(\xi)-\xi x - G(x))} \left| t^{\frac{1}{\alpha}} \mu (x) \right|^{2-\alpha-\rho} \psi (\mu (x)) \, dx = \mathcal{V}^* \mathcal{B} \psi,$$

where $\psi (\eta) = \langle \eta \rangle^{-2+\alpha+\rho} \phi (\eta)$ and the operator

$$\mathcal{V}^* \mathcal{B} \psi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it(\Lambda(\xi)-\xi x - G(x))} \left| t^{\frac{1}{\alpha}} \mu (x) \right|^{2-\alpha-\rho} (\mathcal{B} \psi) (x) \, dx.$$

We substitute

$$(\mathcal{B} \psi) (x) = \psi (\mu (x)) = \frac{t}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itx\eta} \widehat{\psi_1} (t\eta) \, d\eta = \psi_1 (x),$$

to the above identity to obtain the representation

$$\begin{aligned} \mathcal{V}^* \mathcal{B} \psi &= \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it(\Lambda(\xi)-\xi x - G(x))} \left| t^{\frac{1}{\alpha}} \mu (x) \right|^{2-\alpha-\rho} \psi_1 (x) \, dx \\ &= \frac{t}{\sqrt{2\pi}} \int_{\mathbb{R}} d\eta \widehat{\psi_1} (t\eta) \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it(\Lambda(\xi)-(\xi-\eta)x - G(x))} \left| t^{\frac{1}{\alpha}} \mu (x) \right|^{2-\alpha-\rho} dx \\ &= \frac{t}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it(\Lambda(\xi)-\Lambda(\xi-\eta))} A^* (t, \xi - \eta) \widehat{\psi_1} (t\eta) \, d\eta, \end{aligned}$$

where the kernel

$$\begin{aligned} A^* (t, \xi) &= \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it(\Lambda(\xi)-\xi x - G(x))} \left| t^{\frac{1}{\alpha}} \mu (x) \right|^{2-\alpha-\rho} dx \\ &= \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \langle \eta \rangle^{2-\alpha-\rho} \Lambda'' (\eta) \, d\eta. \end{aligned}$$

Since

$$\psi (\xi) = \psi_1 (\Lambda' (\xi)) = \frac{t}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\Lambda'(\xi)\eta} \widehat{\psi_1} (t\eta) \, d\eta,$$

so we obtain

$$\mathcal{V}^* \mathcal{B} \psi - A^* (t, \xi) \psi \\ = \frac{t}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(e^{it(\Lambda(\xi)-\Lambda(\xi-\eta))} A^* (t, \xi - \eta) - e^{it\eta\Lambda'(\xi)} A^* (t, \xi) \right) \widehat{\psi_1} (t\eta) \, d\eta$$

$$= \frac{t}{\sqrt{2\pi}} \int_{|\eta|\geq\delta} \left(e^{it(\Lambda(\xi)-\Lambda(\xi-\eta))} A^*(t,\xi-\eta) - e^{it\eta\Lambda'(\xi)} A^*(t,\xi) \right) \widehat{\psi}_1(t\eta)\, d\eta$$
$$+ \frac{t}{\sqrt{2\pi}} \int_{|\eta|\leq\delta} \left(e^{it(\Lambda(\xi)-\Lambda(\xi-\eta))} - e^{it\eta\Lambda'(\xi)} \right) A^*(t,\xi-\eta)\, \widehat{\psi}_1(t\eta)\, d\eta$$
$$+ \frac{t}{\sqrt{2\pi}} \int_{|\eta|\leq\delta} e^{it\eta\Lambda'(\xi)} \left(A^*(t,\xi-\eta) - A^*(t,\xi) \right) \widehat{\psi}_1(t\eta)\, d\eta = I_1 + I_2 + I_3,$$

where $\delta = t^{-\frac{1}{\alpha}}$.

By the estimate $|A^*(t,\xi)| \leq Ct^{\frac{1}{\alpha}-\frac{1}{2}} \langle \widetilde{\xi} \rangle^{1-\frac{\alpha}{2}-\rho}$, we get

$$|I_1| \leq Ct \int_{|\eta|\geq\delta} \left(|A^*(t,\xi-\eta)| + |A^*(t,\xi)| \right) \left| \widehat{\psi}_1(t\eta) \right| d\eta$$
$$\leq Ct^{\frac{1}{\alpha}+\frac{1}{2}} \int_{|\eta|\geq\delta} \left| \widehat{\psi}_1(t\eta) \right| d\eta \leq Ct^{\frac{1}{\alpha}-\frac{1}{2}} \left\| \widehat{\psi}_1 \right\|_{\mathbf{L}^1(|\eta|\geq\delta t)}$$
$$\leq Ct^{\frac{1}{\alpha}-\frac{1}{2}} \left\| \eta^{-1} \right\|_{\mathbf{L}^2(|\eta|\geq\delta t)} \left\| \eta \widehat{\psi}_1 \right\|_{\mathbf{L}^2(|\eta|\geq\delta t)}$$
$$\leq Ct^{\frac{1}{\alpha}-\frac{1}{2}+\frac{\rho}{\alpha}} t^{1-\frac{2}{\alpha}} (\delta t)^{-\frac{1}{2}} \left\| \eta \widehat{\psi}_1 \right\|_{\mathbf{L}^2(|\eta|\geq\delta t)}$$
$$\leq Ct^{-\frac{1}{2\alpha}+\frac{\rho}{\alpha}} \left(\left\| \sqrt{\Lambda''}\, |\eta|^{-1+\rho} \phi(\eta) \right\|_{\mathbf{L}^2} + \left\| \sqrt{\Lambda''}\, |\eta|^{\rho}\, \partial_\eta \phi(\eta) \right\|_{\mathbf{L}^2} \right).$$

since

$$\left\| \eta \widehat{\psi}_1(\eta) \right\|_{\mathbf{L}^2}$$
$$= \|\partial_x \mathcal{B}\psi\|_{\mathbf{L}^2} = \left(\int |\partial_x \psi(\mu(x))|^2\, dx \right)^{\frac{1}{2}} = \left(\int \left| \partial_\mu \psi(\mu(x)) \right|^2 \mu'(x)\, d\mu \right)^{\frac{1}{2}}$$
$$= \left(\int \left| \partial_\mu \psi(\mu(x)) \right|^2 \frac{1}{\Lambda''(\mu)} d\mu \right)^{\frac{1}{2}}$$
$$= \left\| \frac{1}{\sqrt{\Lambda''}} \partial_\eta \left(|\widetilde{\eta}|^{\alpha-2+\rho} \phi(\eta) \right) \right\|_{\mathbf{L}^2} = t^{1-\frac{2}{\alpha}+\frac{\rho}{\alpha}} \left\| \frac{1}{\sqrt{\Lambda''}} \partial_\eta \left(|\eta|^{\alpha-2+\rho} \phi(\eta) \right) \right\|_{\mathbf{L}^2}$$
$$\leq Ct^{1-\frac{2}{\alpha}+\frac{\rho}{\alpha}} \left(\left\| \sqrt{\Lambda''}\, |\eta|^{-1+\rho} \phi(\eta) \right\|_{\mathbf{L}^2} + \left\| \sqrt{\Lambda''}\, |\eta|^{\rho}\, \partial_\eta \phi(\eta) \right\|_{\mathbf{L}^2} \right).$$

Next by the Taylor formula we find

$$\Lambda(\xi) - \Lambda(\xi-\eta) - \eta\Lambda'(\xi) = \int_{\xi-\eta}^{\xi} (\xi-\eta-z)\, \Lambda''(z)\, dz = O\left(t^{\frac{2}{\alpha}-1} \eta^2 \left| \widetilde{\xi} \right|^{\alpha-2} \right)$$

in domain $|\eta| \leq 1$. Hence

$$\left| e^{it(\Lambda(\xi)-\Lambda(\xi-\eta))} - e^{it\eta\Lambda'(\xi)} \right| \leq Ct^{\frac{1}{\alpha}} |\eta| \left| \widetilde{\xi} \right|^{\frac{\alpha}{2}-1}$$

for $|\eta| \leq 1$. Then since $|A^*(t,\xi)| \leq Ct^{\frac{1}{\alpha}-\frac{1}{2}} \langle \widetilde{\xi} \rangle^{1-\frac{\alpha}{2}-\rho}$, we obtain

$$|I_2| \leq Ct \int_{|\eta|\leq\delta} \left| e^{it(\Lambda(\xi)-\Lambda(\xi-\eta))} - e^{it\eta\Lambda'(\xi)} \right| |A^*(t,\xi-\eta)| \left| \widehat{\psi}_1(t\eta) \right| d\eta$$
$$\leq Ct^{\frac{2}{\alpha}+\frac{1}{2}} \left| \widetilde{\xi} \right|^{\frac{\alpha}{2}-1} \langle \widetilde{\xi} \rangle^{1-\frac{\alpha}{2}-\rho} \int_{|\eta|\leq\delta} |\eta| \left| \widehat{\psi}_1(\eta t) \right| d\eta$$
$$\leq Ct^{\frac{2}{\alpha}-\frac{3}{2}} \left\| \eta \widehat{\psi}_1(\eta) \right\|_{\mathbf{L}^1(|\eta|\leq\delta t)} \leq Ct^{\frac{2}{\alpha}-1} \delta^{\frac{1}{2}} \left\| \eta \widehat{\psi}_1 \right\|_{\mathbf{L}^2(|\eta|\leq\delta t)}$$

$$\leq Ct^{-\frac{1}{2\alpha}+\frac{\rho}{\alpha}}\left(\left\|\sqrt{\Lambda''}\,|\eta|^{-1+\rho}\,\phi(\eta)\right\|_{\mathbf{L}^2}+\left\|\sqrt{\Lambda''}\,|\eta|^{\rho}\,\partial_\eta\phi(\eta)\right\|_{\mathbf{L}^2}\right).$$

Finally by the estimate $|A^*(t,\xi-\eta)-A^*(t,\xi)|\leq Ct^{\frac{2}{\alpha}-\frac{1}{2}}|\eta|\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}-\rho}$ we get

$$\begin{aligned}|I_3| &\leq Ct\int_{|\eta|\leq\delta}|A^*(t,\xi-\eta)-A^*(t,\xi)|\left|\widehat{\psi}_1(t\eta)\right|d\eta\\ &\leq Ct^{\frac{2}{\alpha}+\frac{1}{2}}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}-\rho}\int_{|\eta|\leq\delta}|\eta|\left|\widehat{\psi}_1(t\eta)\right|d\eta\\ &\leq Ct^{\frac{2}{\alpha}-\frac{3}{2}}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}-\rho}\int_{|\eta|\leq\delta t}|\eta|\left|\widehat{\psi}_1(\eta)\right|d\eta\\ &\leq Ct^{\frac{2}{\alpha}-\frac{3}{2}}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}-\rho}(\delta t)^{\frac{1}{2}}\left\|\eta\widehat{\psi}_1\right\|_{\mathbf{L}^2(|\eta|\leq\delta t)}\leq Ct^{\frac{2}{\alpha}-1}\delta^{\frac{1}{2}}\left\|\eta\widehat{\psi}_1\right\|_{\mathbf{L}^2(|\eta|\leq\delta t)}\\ &\leq Ct^{-\frac{1}{2\alpha}+\frac{\rho}{\alpha}}\left(\left\|\sqrt{\Lambda''}\,|\eta|^{-1+\rho}\,\phi(\eta)\right\|_{\mathbf{L}^2}+\left\|\sqrt{\Lambda''}\,|\eta|^{\rho}\,\partial_\eta\phi(\eta)\right\|_{\mathbf{L}^2}\right).\end{aligned}$$

Lemma 2.3 is proved. □

2.4. Estimates for pseudodifferential operators. An extensive literature is devoted to the theory of pseudodifferential operators and to their $\mathbf{L}^2$ - bounds (see, e.g. [3], [7], [8], [25]). We consider the following time dependent pseudodifferential operator

$$\mathbf{a}(t,x,\mathbf{D})\phi\equiv\int_{\mathbb{R}}e^{ix\xi}\mathbf{a}(t,x,\xi)\widehat{\phi}(\xi)\,d\xi$$

defined by the symbol $\mathbf{a}(t,x,\xi)$. We will use the following result (see papers [1], [35]). Denote $\langle x\rangle=\sqrt{1+x^2}$, $\{x\}=\frac{|x|}{\langle x\rangle}$.

Lemma 2.4. *Let $\nu\in(0,1)$, and assume that the symbol $\mathbf{a}(t,x,\xi)$ satisfies*

$$\sup_{x,\xi\in\mathbb{R},t\geq1}\left|\{x\}^{-\nu}\langle x\rangle^{\nu}(x\partial_x)^k\,\mathbf{a}(t,x,\xi)\right|\leq C_\nu$$

for $k=0,1,2$. Then there exists a positive constant C_ν such that $\|\mathbf{a}(t,x,\mathbf{D})\phi\|_{\mathbf{L}^2_x}\leq C_\nu\|\phi\|_{\mathbf{L}^2}$ for all $t\geq1$.

We use Lemma 2.4 to estimate the $\mathbf{L}^2$ - norm of the following weighted defect operator

$$\mathcal{V}_h\phi=\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}h(t,\xi,\eta)\,\phi(\xi)\,d\xi.$$

Lemma 2.5. *Let $\nu\in(0,1)$ and $h(t,\xi,\eta)$ satisfy the estimate*

$$\sup_{\xi,\eta\in\mathbb{R},t\geq1}\left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^k h(t,\xi,\eta)\right|\leq C_\nu$$

for $k=0,1,2$, where $\widetilde{\eta}=t^{\frac{1}{\alpha}}\eta$. Then there exists a positive constant C_ν such that the inequality $\left\|\sqrt{\Lambda''}\mathcal{V}_h\phi\right\|_{\mathbf{L}^2}\leq C_\nu\|\phi\|_{\mathbf{L}^2}$ holds.

In the same manner, we get the $\mathbf{L}^2$ - estimate for the adjoint weighted defect operator

$$\mathcal{V}_h^*\phi=\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{itS(\xi,\eta)}h(t,\xi,\eta)\,\phi(\eta)\,\Lambda''(\eta)\,d\eta.$$

Lemma 2.6. *Let $\nu \in (0,1)$ and $h(t,\xi,\eta)$ satisfy the estimate*

$$\sup_{\xi,\eta\in\mathbb{R},t\geq 1}\left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_{\eta}\right)^{k}h(t,\xi,\eta)\right|\leq C_{\nu}$$

for $k=0,1,2$, where $\widetilde{\eta}=t^{\frac{1}{\alpha}}\eta$. Then there exists a positive constant C_{ν} such that the inequality $\left\|\mathcal{V}_{h}^{*}\phi\right\|_{\mathbf{L}^{2}}\leq C_{\nu}\left\|\sqrt{\Lambda''}\phi\right\|_{\mathbf{L}^{2}}$ holds.

2.5. Estimates for the derivatives of Q. In order to get the desired estimate of $\left\|\sqrt{\Lambda''}\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\partial_{\eta}Q\phi\right\|_{\mathbf{L}^{2}}$, we first estimate the term $\mathcal{V}_{q}1$, with $q(\xi,\eta)=\partial_{\xi}\left(\frac{\partial_{\eta}S(\xi,\eta)}{\partial_{\xi}S(\xi,\eta)}\right)$.

Lemma 2.7. *The estimate $\left\|\sqrt{\Lambda''}\mathcal{V}_{q}1\right\|_{\mathbf{L}^{2}}\leq Ct^{\frac{1}{2\alpha}}$ is valid for all $t\geq 1$.*

Proof. Define the cut of function $\chi_1\in\mathbf{C}^4(\mathbb{R})$ such that $\chi_1(x)=1$ for $\frac{2}{3}\leq x\leq 2$ and $\chi_1(x)=0$ for $x\leq\frac{1}{3}$ or $x\geq 3$, also $\chi_2(x)=1-\chi_1(x)$. Then we represent

$$\mathcal{V}_{q}1=\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q(\xi,\eta)\chi_1\left(\frac{\xi}{\eta}\right)d\xi+\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q(\xi,\eta)\chi_2\left(\frac{\xi}{\eta}\right)d\xi=I_1+I_2.$$

In the first term I_1 we use the identity $e^{-itS(\xi,\eta)}=H_1\partial_{\xi}\left((\xi-\eta)e^{-itS(\xi,\eta)}\right)$ with $H_1=\left(1-it(\xi-\eta)\partial_{\xi}S(\xi,\eta)\right)^{-1}$, and integrate by parts

$$\begin{aligned}I_1&=\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q(\xi,\eta)\chi_1\left(\frac{\xi}{\eta}\right)d\xi\\&=-\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}(\xi-\eta)\partial_{\xi}\left(H_1q(\xi,\eta)\chi_1\left(\frac{\xi}{\eta}\right)\right)d\xi.\end{aligned}$$

Since

$$\begin{aligned}q(\xi,\eta)&=\partial_{\xi}\left(\frac{\partial_{\eta}S(\xi,\eta)}{\partial_{\xi}S(\xi,\eta)}\right)=\partial_{\xi}\left(\frac{\Lambda''(\eta)}{\int_0^1\Lambda''(\xi+z(\eta-\xi))\,dz}\right)\\&=O\left(\frac{\Lambda''(\eta)\int_0^1\Lambda'''(\xi+z(\eta-\xi))(1-z)\,dz}{\left(\int_0^1\Lambda''(\xi+z(\eta-\xi))\,dz\right)^2}\right)=O\left(\frac{1}{\eta}\right)\end{aligned}$$

in the domain $\frac{1}{3}\leq\frac{\xi}{\eta}\leq 3$, then we have

$$\left|(\xi-\eta)\partial_{\xi}\left(H_1q(\xi,\eta)\chi_1\left(\frac{\xi}{\eta}\right)\right)\right|=O\left((\xi-\eta)\partial_{\xi}\left(H_1O\left(\frac{1}{\eta}\right)\chi_1\left(\frac{\xi}{\eta}\right)\right)\right)\leq\frac{C|\eta|^{-1}}{1+t|\eta|^{\alpha-2}(\xi-\eta)^2}$$

in the domain $\frac{1}{3}\leq\frac{\xi}{\eta}\leq 3$. Thus, changing $\xi=\eta y$, we obtain

$$\begin{aligned}|I_1|&=\left|\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}(\xi-\eta)\partial_{\xi}\left(H_1q(\xi,\eta)\chi_1\left(\frac{\xi}{\eta}\right)\right)d\xi\right|\\&\leq Ct^{\frac{1}{2}}|\eta|^{-1}\int_{\frac{1}{3}\leq\frac{\xi}{\eta}\leq 3}\frac{d\xi}{1+t|\eta|^{\alpha-2}(\xi-\eta)^2}\\&\leq Ct^{\frac{1}{2}}\int_{\frac{1}{3}\leq y\leq 3}\frac{dy}{1+|\widetilde{\eta}|^{\alpha}(y-1)^2}\leq Ct^{\frac{1}{2}}\langle\widetilde{\eta}\rangle^{-\frac{\alpha}{2}}.\end{aligned}$$

Therefore, we get

$$\left\|\sqrt{\Lambda''}I_1\right\|_{\mathbf{L}^2} \leq Ct^{\frac{1}{2}} \left\||\eta|^{\frac{\alpha-2}{2}} \langle\widetilde{\eta}\rangle^{-\frac{\alpha}{2}}\right\|_{\mathbf{L}^2} \leq Ct^{\frac{1}{2\alpha}} \left(\int_0^\infty \widetilde{\eta}^{\alpha-2} \langle\widetilde{\eta}\rangle^{-\alpha} d\widetilde{\eta}\right)^{\frac{1}{2}} \leq Ct^{\frac{1}{2\alpha}}.$$

In the second integral I_2 we use the identity $e^{-itS(\xi,\eta)} = H_2\partial_\xi\left(\xi e^{-itS(\xi,\eta)}\right)$ with $H_2 = \left(1 - it\xi\partial_\xi S(\xi,\eta)\right)^{-1}$, integrating by parts, we find

$$I_2 = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q(\xi,\eta)\chi_2\left(\frac{\xi}{\eta}\right) d\xi = -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \xi\partial_\xi\left(H_2 q(\xi,\eta)\chi_2\left(\frac{\xi}{\eta}\right)\right) d\xi.$$

We have

$$\partial_\eta S(\xi,\eta) = -\Lambda''(\eta)(\xi-\eta) = -(\alpha-1)|\eta|^{\alpha-2}(\xi-\eta)$$

and

$$\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \Lambda'(\eta) = |\xi|^{\alpha-2}\xi - |\eta|^{\alpha-2}\eta,$$

then

$$q(\xi,\eta) = \partial_\xi\left(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right) = O\left(\partial_\xi\left(\frac{|\eta|^{\alpha-2}(\xi-\eta)}{|\xi|^{\alpha-2}\xi - |\eta|^{\alpha-2}\eta}\right)\right) = O\left(\frac{1}{|\xi|+|\eta|}\right)$$

for $\frac{\xi}{\eta} \leq \frac{2}{3}$, or $\frac{\xi}{\eta} \geq 2$. Hence

$$\left|\xi\partial_\xi\left(H_2 q(\xi,\eta)\chi_2\left(\frac{\xi}{\eta}\right)\right)\right| \leq \frac{C(|\xi|+|\eta|)^{-1}}{1+t|\xi|\left(|\xi|^{\alpha-1}+|\eta|^{\alpha-1}\right)} \leq \frac{Ct^{\frac{1}{\alpha}}\left(\left|\widetilde{\xi}\right|+|\widetilde{\eta}|\right)^{-1}}{1+\left|\widetilde{\xi}\right|\left(\left|\widetilde{\xi}\right|^{\alpha-1}+|\widetilde{\eta}|^{\alpha-1}\right)}.$$

Thus, changing $\widetilde{\xi} = \widetilde{\eta}y$, we obtain

$$|I_2| \leq Ct^{\frac{1}{2}} \int_{\mathbb{R}} \frac{\left(\left|\widetilde{\xi}\right|+|\widetilde{\eta}|\right)^{-1} d\widetilde{\xi}}{1+\left|\widetilde{\xi}\right|\left(\left|\widetilde{\xi}\right|^{\alpha-1}+|\widetilde{\eta}|^{\alpha-1}\right)} \leq Ct^{\frac{1}{2}} \int_{\mathbb{R}} \frac{\langle y\rangle^{-1} dy}{1+|\widetilde{\eta}|^\alpha |y| \langle y\rangle^{\alpha-1}} \leq Ct^{\frac{1}{2}} \{\widetilde{\eta}\}^{-\mu} \langle\widetilde{\eta}\rangle^{\mu-\alpha}$$

with small $\mu > 0$. Therefore, we get

$$\left\|\sqrt{\Lambda''}I_2\right\|_{\mathbf{L}^2} \leq Ct^{\frac{1}{2}} \left\||\eta|^{\frac{\alpha-2}{2}} \{\widetilde{\eta}\}^{-\mu} \langle\widetilde{\eta}\rangle^{\mu-\alpha}\right\|_{\mathbf{L}^2} \leq Ct^{\frac{1}{2\alpha}} \left(\int_0^\infty \{\widetilde{\eta}\}^{\alpha-2-2\mu} \langle\widetilde{\eta}\rangle^{2\mu-\alpha-2} d\widetilde{\eta}\right)^{\frac{1}{2}} \leq Ct^{\frac{1}{2\alpha}}.$$

Lemma 2.7 is proved. □

We estimate the derivative of Q.

Lemma 2.8. *The estimate*

$$\left\|\sqrt{\Lambda''} \{\widetilde{\eta}\}^\nu \langle\widetilde{\eta}\rangle^{-\nu} \partial_\eta Q\phi\right\|_{\mathbf{L}^2} \leq C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} + Ct^{\frac{1}{2\alpha}} |\phi(0)|$$

is valid for all $t \geq 1$*, where* $0 < \nu < 1$.

Proof. We integrate by parts

$$\begin{aligned}\partial_\eta \mathcal{Q}\phi &= -it\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\eta S(\xi,\eta)\,\phi(\xi)\,d\xi \\ &= -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\partial_\xi\phi(\xi)\,d\xi \\ &\quad -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\frac{\phi(\xi)-\phi(0)}{\xi}\left(\xi\partial_\xi\left(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right)\right)d\xi \\ &\quad -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\phi(0)\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\xi\left(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right)d\xi.\end{aligned}$$

Hence

$$\begin{aligned}\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\partial_\eta\mathcal{Q}\phi &= \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_1(t,\xi,\eta)\,\partial_\xi\phi(\xi)\,d\xi \\ &\quad +\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\frac{\phi(\xi)-\phi(0)}{\xi}q_2(t,\xi,\eta)\,d\xi \\ &\quad -\phi(0)\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(\xi,\eta)\,d\xi, \\ &= \mathcal{V}_{q_1}\partial_\xi\phi+\mathcal{V}_{q_2}\frac{\phi(\xi)-\phi(0)}{\xi}+\phi(0)\,\mathcal{V}_{q_3},\end{aligned}$$

where

$$\begin{aligned}q_1(t,\xi,\eta) &= -\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}, \\ q_2(t,\xi,\eta) &= -\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\left(\xi\partial_\xi\left(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right)\right), \\ q_3(t,\xi,\eta) &= \partial_\xi\left(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right).\end{aligned}$$

Since

$$\partial_\xi S(\xi,\eta)=\Lambda'(\xi)-\Lambda'(\eta),\ \partial_\eta S(\xi,\eta)=\Lambda''(\eta)(\eta-\xi),\ \partial_\xi^j\Lambda(\xi)=O\left(|\xi|^{\alpha-j}\right),$$

we find

$$q_1(t,\xi,\eta)=-\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\frac{\Lambda''(\eta)(\eta-\xi)}{\Lambda'(\xi)-\Lambda'(\eta)}=O\left(\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\right)$$

and

$$q_2(t,\xi,\eta)=-\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\xi\partial_\xi\frac{\Lambda''(\eta)(\eta-\xi)}{\Lambda'(\xi)-\Lambda'(\eta)}=O\left(\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\right).$$

Thus, we have the estimates

$$\begin{aligned}&\left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^k q_1(t,\xi,\eta)\right|+\left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^k q_2(t,\xi,\eta)\right| \\ &= O\left(\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^k\{\widetilde{\eta}\}^{\nu}\langle\widetilde{\eta}\rangle^{-\nu}\right)\le C.\end{aligned}$$

Therefore, using Lemma 2.5 we get $\left\|\sqrt{\Lambda''}\mathcal{V}_{q_1}\partial_\xi\phi\right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}$ and by the Hardy inequality

$$\left\|\sqrt{\Lambda''}\mathcal{V}_{q_2}\frac{\phi(\xi)-\phi(0)}{\xi}\right\|_{\mathbf{L}^2} \le \left\|\frac{\phi(\xi)-\phi(0)}{\xi}\right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}.$$

Also by Lemma 2.7 we obtain $\left\|\sqrt{\Lambda''}\mathcal{V}_{q_3}1\right\|_{\mathbf{L}^2} \le Ct^{\frac{1}{2\alpha}}$. Lemma 2.8 is proved. □

2.6. Estimates for the derivative of $\mathcal{Q}^*$. We now estimate the derivative of the adjoint defect operator in the domain $\left|\widetilde{\xi}\right| \le N = \frac{2}{\omega}$, with $\omega > 0$ (we will choose sufficiently small $\omega > 0$ below in the proof of Lemma 4.1).

Lemma 2.9. *The estimate*

$$\left\|\partial_\xi\mathcal{Q}^*\phi\right\|_{\mathbf{L}^2_\xi\left(\left|\widetilde{\xi}\right|\le N\right)} \le C\left\|\sqrt{\Lambda''}\left\langle\widetilde{\eta}\right\rangle^\nu\partial_\eta\phi\right\|_{\mathbf{L}^2_\eta\left(\left|\widetilde{\eta}\right|\ge 2N\right)} + C\left\|\sqrt{\Lambda''}\left\langle\widetilde{\eta}\right\rangle^\nu\frac{1}{\eta}\phi\right\|_{\mathbf{L}^2_\eta\left(\left|\widetilde{\eta}\right|\ge 2N\right)} + Ct^{\frac{3}{2\alpha}-\frac{1}{2}}\left\|\phi\right\|_{\mathbf{L}^\infty_\eta\left(\left|\widetilde{\eta}\right|\le 3N\right)}$$

is true for all $t \ge 1$, where $0 < \nu < 1$.

Proof. Define the cut-off function $\chi_3 \in \mathbf{C}^1(\mathbb{R})$ such that $\chi_3(x) = 1$ for $|x| \le 2N$ and $\chi_3(x) = 0$ for $|x| \ge 3N$, also $\chi_4(x) = 1 - \chi_3(x)$. We write

$$\partial_\xi\mathcal{Q}^*\phi = \partial_\xi\mathcal{Q}^*\phi\chi_3\left(\widetilde{\eta}\right) + \partial_\xi\mathcal{Q}^*\phi\chi_4\left(\widetilde{\eta}\right)$$

We estimate the first summand by using the inequality $\left\|\mathcal{Q}^*\phi\right\|_{\mathbf{L}^2} = \left\|\sqrt{\Lambda''}\phi\right\|_{\mathbf{L}^2}$. We get

$$\begin{aligned}
\left\|\partial_\xi\mathcal{Q}^*\phi\chi_3\left(\widetilde{\eta}\right)\right\|_{\mathbf{L}^2\left(\left|\widetilde{\xi}\right|\le N\right)} &\le Ct\left\||\xi|^{\alpha-1}\mathcal{Q}^*\phi\chi_3\left(\widetilde{\eta}\right)\right\|_{\mathbf{L}^2\left(\left|\widetilde{\xi}\right|\le N\right)} \\
&\quad + Ct\left\|\mathcal{Q}^*|\eta|^{\alpha-1}\phi\chi_3\left(\widetilde{\eta}\right)\right\|_{\mathbf{L}^2\left(\left|\widetilde{\xi}\right|\le N\right)} \\
&\le Ct^{\frac{1}{\alpha}}\left\|\mathcal{Q}^*\phi\chi_3\left(\widetilde{\eta}\right)\right\|_{\mathbf{L}^2} + Ct\left\|\mathcal{Q}^*|\eta|^{\alpha-1}\phi\chi_3\left(\widetilde{\eta}\right)\right\|_{\mathbf{L}^2} \\
&\le Ct^{\frac{1}{\alpha}}\left\|\sqrt{\Lambda''}\phi\right\|_{\mathbf{L}^2\left(\left|\widetilde{\eta}\right|\le 3N\right)} + Ct\left\|\sqrt{\Lambda''}|\eta|^{\alpha-1}\phi\right\|_{\mathbf{L}^2\left(\left|\widetilde{\eta}\right|\le 3N\right)} \\
&\le Ct^{\frac{3}{2\alpha}-\frac{1}{2}}\left\|\phi\right\|_{\mathbf{L}^\infty\left(\left|\widetilde{\eta}\right|\le 3N\right)}.
\end{aligned}$$

In the second one we integrate by parts

$$\begin{aligned}
&\chi_3\left(2\widetilde{\xi}\right)\partial_\xi\mathcal{Q}^*\phi\chi_4\left(\widetilde{\eta}\right) \\
&= \frac{t^{\frac{1}{2}}\chi_3\left(2\widetilde{\xi}\right)}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}it\partial_\xi S(\xi,\eta)\chi_4\left(\widetilde{\eta}\right)\phi(\eta)\Lambda''(\eta)\,d\eta \\
&= -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\left\langle\widetilde{\eta}\right\rangle^{-\nu}\chi_3\left(2\widetilde{\xi}\right)\frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)}\chi_4\left(\widetilde{\eta}\right)\left\langle\widetilde{\eta}\right\rangle^\nu\partial_\eta\phi(\eta)\Lambda''(\eta)\,d\eta \\
&\quad - \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\frac{\left\langle\widetilde{\eta}\right\rangle^{-\nu}\chi_3\left(2\widetilde{\xi}\right)}{\Lambda''(\eta)}\eta\partial_\eta\left(\frac{\partial_\xi S(\xi,\eta)\Lambda''(\eta)}{\partial_\eta S(\xi,\eta)}\chi_4\left(\widetilde{\eta}\right)\right)\left\langle\widetilde{\eta}\right\rangle^\nu\frac{\phi(\eta)}{\eta}\Lambda''(\eta)\,d\eta \\
&= \mathcal{V}^*_{h_3}\widetilde{\chi}\left(\widetilde{\eta}\right)\left\langle\widetilde{\eta}\right\rangle^\nu\partial_\eta\phi + \mathcal{V}^*_{h_4}\widetilde{\chi}\left(\widetilde{\eta}\right)\left\langle\widetilde{\eta}\right\rangle^\nu\frac{\phi(\eta)}{\eta},
\end{aligned}$$

where

$$h_3(t,\xi,\eta) = -\chi_3\left(2\widetilde{\xi}\right)\left\langle\widetilde{\eta}\right\rangle^{-\nu}\frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)}\chi_4\left(\widetilde{\eta}\right),$$

$$h_4(t,\xi,\eta) = -\chi_3\left(2\widetilde{\xi}\right)\frac{\langle\widetilde{\eta}\rangle^{-\nu}}{\Lambda''(\eta)}\eta\partial_\eta\left(\frac{\partial_\xi S(\xi,\eta)\Lambda''(\eta)}{\partial_\eta S(\xi,\eta)}\chi_4(\widetilde{\eta})\right),$$

and $\widetilde{\chi}(x) = 0$ for $|x| < 2N, \widetilde{\chi}(x) = 1$ for $|x| \geq 2N$.

Note that $h_k(t,\xi,\eta), k = 3,4$ satisfy the estimates

$$\begin{aligned}
&\left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^j h_3(t,\xi,\eta)\right| \\
&= \left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^j\langle\widetilde{\eta}\rangle^{-\nu}\frac{\chi_3\left(2\widetilde{\xi}\right)\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)}\chi_4(\widetilde{\eta})\right| \\
&= O\left(\frac{|\eta|^{\alpha-1}}{|\eta|^{\alpha-1}}\chi_1\left(\frac{\xi}{\eta}\right)\chi_4(\widetilde{\eta})\right) \leq C,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^j h_4(t,\xi,\eta)\right| \\
&\leq \left|\{\widetilde{\eta}\}^{-\nu}\langle\widetilde{\eta}\rangle^{\nu}\left(\eta\partial_\eta\right)^j\frac{\langle\widetilde{\eta}\rangle^{-\nu}\chi_3\left(2\widetilde{\xi}\right)}{\Lambda''(\eta)}\eta\partial_\eta\left(\frac{\partial_\xi S(\xi,\eta)\Lambda''(\eta)}{\partial_\eta S(\xi,\eta)}\chi_4(\widetilde{\eta})\right)\right| \\
&= O\left(\frac{|\eta|^{\alpha-1}}{|\eta|^{\alpha-1}}\chi_3\left(2\widetilde{\xi}\right)\chi_4(\widetilde{\eta})\right) \leq C,
\end{aligned}$$

for all $\xi,\eta \in \mathbb{R}, t \geq 1, j = 0,1,2,$ where $\nu \in (0,1)$. Then by Lemma 2.6 we get

$$\left\|\mathcal{V}_{h_3}^*\widetilde{\chi}(\widetilde{\eta})\langle\widetilde{\eta}\rangle^{\nu}\partial_\eta\phi\right\|_{\mathbf{L}^2} \leq C\left\|\left|\Lambda''\right|^{\frac{1}{2}}\widetilde{\chi}(\widetilde{\eta})\langle\widetilde{\eta}\rangle^{\nu}\partial_\eta\phi\right\|_{\mathbf{L}^2} \leq C\left\|\left|\Lambda''\right|^{\frac{1}{2}}\langle\widetilde{\eta}\rangle^{\nu}\partial_\eta\phi\right\|_{\mathbf{L}^2(|\widetilde{\eta}|\geq 2N)}$$

and

$$\left\|\mathcal{V}_{h_4}^*\widetilde{\chi}(\widetilde{\eta})\langle\widetilde{\eta}\rangle^{\nu}\frac{\phi(\eta)}{\eta}\right\|_{\mathbf{L}^2} \leq C\left\|\left|\Lambda''\right|^{\frac{1}{2}}\widetilde{\chi}(\widetilde{\eta})\langle\widetilde{\eta}\rangle^{\nu}\frac{\phi(\eta)}{\eta}\right\|_{\mathbf{L}^2} \leq C\left\|\left|\Lambda''\right|^{\frac{1}{2}}\langle\widetilde{\eta}\rangle^{\nu}\frac{\phi(\eta)}{\eta}\right\|_{\mathbf{L}^2(|\widetilde{\eta}|\geq 2N)}.$$

Lemma 2.9 is proved. □

3. Asymptotics of the nonlinearity

Define the norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]}\left(\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} + W^{-1}\left\|\left\langle\widetilde{\xi}\right\rangle^{-\gamma}\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} + P^{-1}(t)\left\|\left\langle\widetilde{\xi}\right\rangle^{\beta}\partial_\xi\widehat{\varphi}(t)\right\|_{\mathbf{L}^2}\right),$$

where

$$\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t), W(t) = \left(1+\widetilde{\varepsilon}^{\alpha}\log\langle t\rangle\right)^{-\frac{1}{\alpha}}, P(t) = t^{\frac{1}{4\alpha}} + \varepsilon^{\alpha}t^{\frac{1}{2\alpha}}W^{\alpha+1}(t),$$

$$\widetilde{\varepsilon} = C\varepsilon, \varepsilon > 0, \widetilde{\xi} = \xi t^{\frac{1}{\alpha}}, \gamma > 0, \beta \in \left[0,\frac{1}{2}\right).$$

Denote

$$B(t,\xi) = t^{1-\frac{\alpha}{2}}\mathcal{Q}^*\left(|\mathcal{Q}1|^{\alpha}\mathcal{Q}1\right),$$

where $\mathcal{Q}1 = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}d\xi, b = B(t,0)$. Changing $\widetilde{\xi} = \xi t^{\frac{1}{\alpha}}, \widetilde{\eta} = \eta t^{\frac{1}{\alpha}}$ we get

$$Q1 = t^{\frac{1}{2}-\frac{1}{\alpha}} \frac{1}{\sqrt{2\pi}} e^{-i\left(1-\frac{1}{\alpha}\right)|\widetilde{\eta}|^{\alpha}} \int_{\mathbb{R}} e^{i|\widetilde{\eta}|^{\alpha-2}\widetilde{\eta}\xi - \frac{i}{\alpha}|\xi|^{\alpha}} d\xi,$$

and changing $\widetilde{\eta} = \mu\left(\widehat{x}\right), \widehat{x} = xt^{1-\frac{1}{\alpha}}$ we get

$$B(t,\xi) = B\left(\widetilde{\xi}\right) = \frac{1}{\sqrt{2\pi}} e^{\frac{i}{\alpha}\left|\widetilde{\xi}\right|^{\alpha}} \int_{\mathbb{R}} e^{-i\widetilde{\xi}\widehat{x}} \left|G\left(\widehat{x}\right)\right|^{\alpha} G\left(\widehat{x}\right) d\widehat{x},$$

where $G\left(\widehat{x}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi\widehat{x} - \frac{i}{\alpha}|\xi|^{\alpha}} d\xi$ is an even function. Hence we have

$$B\left(\widetilde{\xi}\right) = \frac{2}{\sqrt{2\pi}} e^{\frac{i}{\alpha}\left|\widetilde{\xi}\right|^{\alpha}} \int_0^{\infty} \cos\left(\widetilde{\xi}x\right) |G(x)|^{\alpha} G(x)\, dx.$$

As in [17], we estimate B in the next lemma. For convenience of the reader we give the proof.

Lemma 3.1. *The estimate* $\|B\|_{\mathbf{L}^{\infty}} \le C$ *holds and numeric computations say that if we let* $b = b(\alpha)$, *then*

$$\operatorname{Im} b(\alpha) < 0 \textit{ for any } \alpha > 2, \lim_{\alpha\to\infty} \operatorname{Im} b(\alpha) = 0$$

$$\operatorname{Re} b(\alpha) > 0 \textit{ for any } \alpha > 2, \lim_{\alpha\to\infty} \operatorname{Re} b(\alpha) = 0$$

are true.

Proof. By Lemma 2.3 with $\rho = \frac{1}{4}$ we get

$$\begin{aligned} |Q^{*}(|Q1|^{\alpha} Q1)| &\le Ct^{\frac{1}{\alpha}-\frac{1}{2}} \left|\widetilde{\xi}\right|^{\frac{\alpha}{2}-1} |Q1|^{\alpha+1} \\ &\quad + Ct^{-\frac{1}{2\alpha}+\frac{\rho}{\alpha}} \left(\left\||\eta|^{\frac{\alpha}{2}-2+\rho} (|Q1|^{\alpha} Q1)\right\|_{\mathbf{L}^2} + \left\||\eta|^{\frac{\alpha}{2}-1+\rho} \partial_{\eta} (|Q1|^{\alpha} Q1)\right\|_{\mathbf{L}^2}\right) \\ &\le Ct^{\frac{1}{\alpha}-\frac{1}{2}} \left\|\langle\widetilde{\eta}\rangle^{\frac{\alpha}{2}-1} |Q1|^{\alpha+1}\right\|_{\mathbf{L}^{\infty}} \\ &\quad + Ct^{\frac{2}{\alpha}-\frac{1}{2}-\frac{1}{2\alpha}} \left\||\widetilde{\eta}|^{\frac{\alpha}{2}-2+\frac{1}{4}} |Q1|^{\alpha+1}\right\|_{\mathbf{L}^2} \\ &\quad + Ct^{-\frac{1}{2\alpha}} \left\|\langle\widetilde{\eta}\rangle^{\frac{1}{2}} |Q1|^{\alpha}\right\|_{\mathbf{L}^{\infty}} \left\|\sqrt{\Lambda''} \{\widetilde{\eta}\}^{\frac{1}{4}} \langle\widetilde{\eta}\rangle^{-\frac{1}{4}} \partial_{\eta} Q1\right\|_{\mathbf{L}^2} \\ &\le Ct^{\frac{1}{\alpha}-\frac{1}{2}} \left\|\langle\widetilde{\eta}\rangle^{\frac{\alpha}{2}-1} |Q1|^{\alpha+1}\right\|_{\mathbf{L}^{\infty}} \\ &\quad + Ct^{\frac{2}{\alpha}-\frac{1}{2}-\frac{1}{2\alpha}} \left\|\langle\widetilde{\eta}\rangle^{\frac{\alpha}{2}-1+\frac{1}{4}} |Q1|^{\alpha+1}\right\|_{\mathbf{L}^{\infty}} \left\|\{\widetilde{\eta}\}^{\frac{\alpha}{2}-2+\frac{1}{4}} \langle\widetilde{\eta}\rangle^{-1}\right\|_{\mathbf{L}^2} \\ &\quad + Ct^{-\frac{1}{2\alpha}} \left\|\langle\widetilde{\eta}\rangle^{\frac{1}{2}} |Q1|^{\alpha}\right\|_{\mathbf{L}^{\infty}} \left\|\sqrt{\Lambda''} \{\widetilde{\eta}\}^{\frac{1}{4}} \langle\widetilde{\eta}\rangle^{-\frac{1}{4}} \partial_{\eta} Q1\right\|_{\mathbf{L}^2}. \end{aligned}$$

By Lemma 2.2 with $j = 0$ we have $|Q1| \le Ct^{\frac{1}{2}-\frac{1}{\alpha}} \langle\widetilde{\eta}\rangle^{1-\frac{\alpha}{2}}$ and by virtue of Lemma 2.8

$$\left\|\sqrt{\Lambda''} \{\widetilde{\eta}\}^{\frac{1}{4}} \langle\widetilde{\eta}\rangle^{-\frac{1}{4}} \partial_{\eta} Q1\right\|_{\mathbf{L}^2} \le Ct^{\frac{1}{2\alpha}}.$$

Hence

$$|Q^{*}(|Q1|^{\alpha} Q1)| \le Ct^{-1+\frac{\alpha}{2}}$$

which implies $\|B\|_{\mathbf{L}^{\infty}} \le C$.

Next we consider

$$B\left(\widetilde{\xi}\right) = \frac{2}{\sqrt{2\pi}} e^{\frac{i}{\alpha}\left|\widetilde{\xi}\right|^{\alpha}} \int_0^{\infty} \cos\left(\widetilde{\xi}x\right) |G(x)|^{\alpha} G(x)\, dx.$$

We see that the integral $B\left(\widetilde{\xi}\right)$ is a continuous function. Therefore it is sufficient to estimate $b = b(\alpha) = B(0) = \frac{2}{\sqrt{2\pi}} \int_0^\infty |G(x)|^\alpha G(x)\,dx$. For $0 < x \le 10$ we rotate the contour of integration $\xi = ye^{-\frac{\pi}{2\alpha}i}$, then

$$G(x) \approx G_1(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-i\frac{\pi}{2\alpha}} \int_0^\infty \left(e^{xye^{i\frac{\pi}{2}-\frac{\pi}{2\alpha}i}} + e^{-xye^{i\frac{\pi}{2}-\frac{\pi}{2\alpha}i}}\right) e^{-\frac{1}{\alpha}y^\alpha}\,dy,$$

where $\approx$ denotes an approximation with some remainder. For $x \ge 10$ we substitute the asymptotics $G(x) \approx G_2(x) \equiv \frac{|x|^{\frac{2-\alpha}{2\alpha-2}}}{\sqrt{(\alpha-1)i}} e^{i\frac{\alpha-1}{\alpha}|x|^{\frac{\alpha}{\alpha-1}}}$. Hence we find

$$b(\alpha) \approx \frac{2}{\sqrt{2\pi}} \int_0^{10} |G_1(x)|^\alpha G_1(x)\,dx + \frac{2}{\sqrt{2\pi}} \int_{10}^{100} |G_2(x)|^\alpha G_2(x)\,dx.$$

It is easy to compute $\mathrm{Im}b(2) = 0$ and $\mathrm{Re}b(2) > 0.25 > 0$. By numeric computations we obtain Lemma 3.1. □

In the next lemma we prove the asymptotics of the nonlinearity in the equation (2.1).

Lemma 3.2. *Suppose that* $\|u\|_{\mathbf{X}_T} \le C\varepsilon$. *Then the estimates*

$$|\mathcal{Q}^* |v|^\alpha v| \le Ct^{\frac{\alpha}{2}-1}\varepsilon^{\alpha+1}W^{\alpha+1}$$

and

$$\left|(\mathcal{Q}^* |v|^\alpha v)(0) - t^{\frac{\alpha}{2}-1}B(0)|m|^\alpha m\right| \le Ct^{\frac{\alpha}{2}-1}\varepsilon^{\alpha+1+\nu}W^{\alpha+1+\nu}$$

are true for all $t \in [1,T]$, *where* $v = \mathcal{Q}\widehat{\varphi}$, $m = \widehat{\varphi}(t,0)$, $W(t) = \left(1 + \widetilde{\varepsilon}^\alpha \log\langle t\rangle\right)^{-\frac{1}{\alpha}}$, *and* $0 < \nu < \frac{3}{16}$.

Proof. It follows from Lemma 2.3 with $\phi = |v|^\alpha v, \rho = \frac{1}{4}$ that

$$\begin{aligned}|\mathcal{Q}^* |v|^\alpha v| &\le Ct^{\frac{1}{\alpha}-\frac{1}{2}} \left\|\langle\widehat{\eta}\rangle^{\frac{\alpha}{2}-1} |v|^{\alpha+1}\right\|_{\mathbf{L}^\infty} \\ &+ Ct^{\frac{1}{\alpha}-\frac{1}{2}+\frac{1}{2\alpha}} \left\|\langle\widehat{\eta}\rangle^{\frac{\alpha}{2}-1+\frac{1}{4}} |v|^{\alpha+1}\right\|_{\mathbf{L}^\infty} \left\|\{\widehat{\eta}\}^{\frac{\alpha}{2}-2+\frac{1}{4}} \langle\widehat{\eta}\rangle^{-1}\right\|_{\mathbf{L}^2} \\ &+ Ct^{-\frac{1}{2\alpha}} \left\|\langle\widehat{\eta}\rangle^{\frac{1}{2}} |v|^\alpha\right\|_{\mathbf{L}^\infty} \left\|\sqrt{\Lambda''}\{\widehat{\eta}\}^{\frac{1}{4}} \langle\widehat{\eta}\rangle^{-\frac{1}{4}} \partial_\eta v\right\|_{\mathbf{L}^2}.\end{aligned}$$

By Lemma 2.8 we have

$$\begin{aligned}\left\|\sqrt{\Lambda''}\{\widehat{\eta}\}^{\frac{1}{4}} \langle\widehat{\eta}\rangle^{-\frac{1}{4}} \partial_\eta v\right\|_{\mathbf{L}^2} &= \left\|\sqrt{\Lambda''}\{\widehat{\eta}\}^{\frac{1}{4}} \langle\widehat{\eta}\rangle^{-\frac{1}{4}} \partial_\eta \mathcal{Q}\widehat{\varphi}\right\|_{\mathbf{L}^2} \\ &\le C\left\|\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} + Ct^{\frac{1}{2\alpha}}\left|\widehat{\varphi}(0)\right| \le C\left\|\left\langle\widetilde{\xi}\right\rangle^\beta \partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} + Ct^{\frac{1}{2\alpha}}\left|\widehat{\varphi}(0)\right| \\ &\le C\varepsilon P(t) + Ct^{\frac{1}{2\alpha}}\varepsilon W \le Ct^{\frac{1}{2\alpha}}\varepsilon W,\end{aligned}$$

where $P(t) = t^{\frac{1}{4\alpha}} + \varepsilon^\alpha t^{\frac{1}{2\alpha}} W^{\alpha+1}(t)$. We again use Lemma 2.2 with $j = 0$ $\gamma > 0$ small, to get

$$\begin{aligned}\langle\widehat{\eta}\rangle^{\frac{\alpha}{2}-1+\frac{1}{4}} |v|^{\alpha+1} &= \langle\widehat{\eta}\rangle^{\frac{\alpha}{2}-1+\frac{1}{4}} \left|\mathcal{Q}\widehat{\varphi}\right|^{\alpha+1} \\ &\le Ct^{\frac{\alpha}{2}-\frac{1}{\alpha}-\frac{1}{2}} \langle\widehat{\eta}\rangle^{\alpha-\frac{\alpha^2}{2}+(\alpha+1)\gamma+\frac{1}{4}} \left(\left|\widehat{\varphi}(0)\right| + \left\|\left\langle\widetilde{\xi}\right\rangle^{-\gamma}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2\alpha}}\left\|\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}\right)^{\alpha+1} \\ &\le Ct^{\frac{\alpha}{2}-\frac{1}{\alpha}-\frac{1}{2}}\varepsilon^{\alpha+1}W^{\alpha+1}\end{aligned}$$

which implies

$$\left\| \langle \widehat{\eta} \rangle^{\frac{\alpha}{2}-1+\frac{1}{4}} |v|^{\alpha+1} \right\|_{\mathbf{L}^\infty} \le C t^{\frac{\alpha}{2}-\frac{1}{\alpha}-\frac{1}{2}} \varepsilon^{\alpha+1} W^{\alpha+1}$$

and similarly

$$\left\| \langle \widehat{\eta} \rangle^{\frac{\alpha}{2}-1} |v|^{\alpha+1} \right\|_{\mathbf{L}^\infty} \le C t^{\frac{\alpha}{2}-\frac{1}{\alpha}-\frac{1}{2}} \varepsilon^{\alpha+1} W^{\alpha+1}, \left\| \langle \widehat{\eta} \rangle^{\frac{1}{2}} |v|^{\alpha} \right\|_{\mathbf{L}^\infty} \le C t^{\frac{\alpha}{2}-1} \varepsilon^{\alpha} W^{\alpha}.$$

We also have

$$\left\| \{\widehat{\eta}\}^{\frac{\alpha}{2}-2+\frac{1}{4}} \langle \widehat{\eta} \rangle^{-1} \right\|_{\mathbf{L}^2} \le C t^{-\frac{1}{2\alpha}}.$$

Therefore

$$|\mathcal{Q}^* |v|^\alpha v| \le C t^{\frac{\alpha}{2}-1} \varepsilon^{\alpha+1} W^{\alpha+1}.$$

Thus the first estimate of the lemma is true.

Next, by Lemma 2.3 with $\phi = |v|^\alpha v - |\mathcal{Q}m|^\alpha \mathcal{Q}m$, since $\phi(0) = 0$, we get

$$\begin{aligned}
&\left| (\mathcal{Q}^* |v|^\alpha v)(0) - t^{\frac{\alpha}{2}-1} B(0) |m|^\alpha m \right| = |(\mathcal{Q}^* (|v|^\alpha v - |\mathcal{Q}m|^\alpha \mathcal{Q}m))(0)| \\
&\le C t^{\frac{1}{\alpha}-\frac{1}{2}+\frac{1}{2\alpha}} \left\| \langle \widehat{\eta} \rangle^{\frac{\alpha}{2}-1+\nu} (|v|^\alpha v - |\mathcal{Q}m|^\alpha \mathcal{Q}m) \right\|_{\mathbf{L}^\infty} \left\| \{\widehat{\eta}\}^{\frac{\alpha}{2}-2+\nu} \langle \widehat{\eta} \rangle^{-1} \right\|_{\mathbf{L}^2} \\
&\quad + C t^{-\frac{1}{2\alpha}} \left\| \langle \widehat{\eta} \rangle^{2\nu} (|v|^\alpha + |\mathcal{Q}m|^\alpha) \right\|_{\mathbf{L}^\infty} \left\| \sqrt{\Lambda''} \{\widehat{\eta}\}^{\nu} \langle \widehat{\eta} \rangle^{-\nu} \partial_\eta (v - \mathcal{Q}m) \right\|_{\mathbf{L}^2} \\
&\quad + C t^{-\frac{1}{2\alpha}} \left\| \langle \widehat{\eta} \rangle^{2\nu} \left(|v|^{\alpha-1} + |\mathcal{Q}m|^{\alpha-1} \right) \right\|_{\mathbf{L}^\infty} \left\| \langle \widehat{\eta} \rangle^{-\nu} (v - \mathcal{Q}m) \right\|_{\mathbf{L}^\infty} \\
&\quad \times \left\| \sqrt{\Lambda''} \{\widehat{\eta}\}^{\nu} \langle \widehat{\eta} \rangle^{-\nu} \partial_\eta (v + \mathcal{Q}m) \right\|_{\mathbf{L}^2} = \sum_{k=1}^{3} I_k.
\end{aligned}$$

Since

$$\left| \widehat{\varphi} - m \right|^{\frac{1}{\nu}} = \int_0^\xi \partial_\xi \left| \widehat{\varphi}(\xi) - m \right|^{\frac{1}{\nu}} d\xi \le C \left| \left\| \widehat{\varphi} \right\|_{\mathbf{L}^\infty} + |m| \right|^{\frac{1}{\nu}-1} |\xi|^{\frac{1}{2}} \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2}^{\frac{1}{2}}$$

we find

$$\left| \widehat{\varphi} - m \right| \le C \left| \left\| \widehat{\varphi} \right\|_{\mathbf{L}^\infty} + |m| \right|^{1-\nu} \left(|\xi|^{\frac{1}{2}} \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} \right)^{\nu}. \tag{3.1}$$

By Lemma 2.2 we have the estimate

$$\left| \mathcal{Q}(t) |\xi|^j \phi \right| \le C t^{\frac{1}{2}-\frac{j+1}{\alpha}} \langle \widehat{\eta} \rangle^{1+j-\frac{\alpha}{2}+\rho} \left(|\phi(0)| + \left\| \left\langle \widetilde{\xi} \right\rangle^{-\rho} \phi \right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2\alpha}} \left\| \partial_\xi \phi \right\|_{\mathbf{L}^2} \right)$$

is true for all $t \ge 1$, where $j = 0, 1, \rho \ge 0$. Then taking $j = 0, \phi = (\widehat{\varphi} - m)$ we get

$$\left| \langle \widehat{\eta} \rangle^{-\nu} \mathcal{Q}(\widehat{\varphi} - m) \right| \le C t^{\frac{1}{2}-\frac{1}{\alpha}} \langle \widehat{\eta} \rangle^{1-\frac{\alpha}{2}+\rho-\nu} \left(\left\| \left\langle \widetilde{\xi} \right\rangle^{-\rho} (\widehat{\varphi} - m) \right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2\alpha}} \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} \right)$$

By (3.1) and

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in [1,T]} \left(\left\| \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} + W^{-1}(t) \left\| \left\langle \widetilde{\xi} \right\rangle^{-\gamma} \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} + P^{-1}(t) \left\| \left\langle \widetilde{\xi} \right\rangle^{\beta} \partial_\xi \widehat{\varphi}(t) \right\|_{\mathbf{L}^2} \right),$$

where

$$\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t) u(t), W(t) = \left(1 + \widetilde{\varepsilon}^\alpha \log \langle t \rangle\right)^{-\frac{1}{\alpha}}, P(t) = t^{\frac{1}{4\alpha}} + \varepsilon^\alpha t^{\frac{1}{2\alpha}} W^{\alpha+1}(t),$$

$$\widetilde{\varepsilon} = C\varepsilon, \varepsilon > 0, \widetilde{\xi} = \xi t^{\frac{1}{\alpha}}, \gamma > 0, \beta \in \left[0, \frac{1}{2}\right),$$

we have with $\rho = \frac{\nu}{2}$

$$\begin{aligned}\left\| \left\langle \widetilde{\xi} \right\rangle^{-\frac{\nu}{2}} (\widehat{\varphi} - m) \right\|_{\mathbf{L}^\infty} &\leq C \left\| \|\widehat{\varphi}\|_{\mathbf{L}^\infty} + |m| \right\|^{1-\nu} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\nu}{2}} |\xi|^{\frac{\nu}{2}} \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2}^{\nu} \\ &\leq C\varepsilon W^{1-\nu} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\nu}{2}} |\xi|^{\frac{\nu}{2}} \left(t^{\frac{1}{4\alpha}} + \varepsilon^\alpha t^{\frac{1}{2\alpha}} W^{\alpha+1}(t) \right)^{\nu} \\ &\leq C\varepsilon W^{1-\nu} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\nu}{2}} \left| \widetilde{\xi} \right|^{\frac{\nu}{2}} t^{-\frac{\nu}{2\alpha}} \left(t^{\frac{1}{4\alpha}} + \varepsilon^\alpha t^{\frac{1}{2\alpha}} W^{\alpha+1}(t) \right)^{\nu} \\ &\leq C\varepsilon W^{1-\nu} \left(t^{-\frac{1}{4\alpha}} + \varepsilon^\alpha W^{\alpha+1}(t) \right)^{\nu}\end{aligned}$$

Then

$$\left| \langle \widetilde{\eta} \rangle^{-\nu} \mathcal{Q} (\widehat{\varphi} - m) \right| \leq C\varepsilon t^{\frac{1}{2}-\frac{1}{\alpha}} \left(W^{1-\nu} t^{-\frac{\nu}{4\alpha}} + \varepsilon^{\alpha\nu} W^{\alpha\nu+\nu} W^{1-\nu} \right) \leq C t^{\frac{1}{2}-\frac{1}{\alpha}} \varepsilon^{1+\alpha\nu} W^{1+\alpha\nu}$$

since

$$t^{-\frac{\nu}{4\alpha}} \leq C (\log \langle t \rangle)^{-\frac{\alpha\nu+\nu}{\alpha}} \leq C \widetilde{\varepsilon}^{\alpha\nu+\nu} (\widetilde{\varepsilon}^{\alpha} \log \langle t \rangle)^{-\frac{\alpha\nu+\nu}{\alpha}} \leq C\varepsilon^{\alpha\nu} (1 + \widetilde{\varepsilon}^{\alpha} \log \langle t \rangle)^{-\frac{\alpha\nu+\nu}{\alpha}} \leq C\varepsilon^{\alpha\nu} W^{\alpha\nu+\nu}$$

for large t. Therefore

$$\left\| \langle \widetilde{\eta} \rangle^{-\nu} (v - \mathcal{Q}m) \right\|_{\mathbf{L}^\infty} = \left\| \langle \widetilde{\eta} \rangle^{-\nu} \mathcal{Q} (\widehat{\varphi} - m) \right\|_{\mathbf{L}^\infty} \leq C t^{\frac{1}{2}-\frac{1}{\alpha}} \varepsilon^{1+\alpha\nu} W^{1+\alpha\nu}. \tag{3.2}$$

We have

$$\begin{aligned} I_1 &\leq C t^{\frac{1}{\alpha}-\frac{1}{2}} \left\| \langle \widetilde{\eta} \rangle^{\frac{\alpha}{2}-1+\nu} \left(|v|^\alpha v - |\mathcal{Q}m|^\alpha \mathcal{Q}m \right) \right\|_{\mathbf{L}^\infty} \\ &\leq C t^{\frac{1}{\alpha}-\frac{1}{2}} \left\| \langle \widetilde{\eta} \rangle^{\frac{\alpha}{2}-1+2\nu} \left(|v|^\alpha + |\mathcal{Q}m|^\alpha \right) \right\|_{\mathbf{L}^\infty} \left\| \langle \widetilde{\eta} \rangle^{-\nu} (v - \mathcal{Q}m) \right\|_{\mathbf{L}^\infty}. \end{aligned}$$

By Lemma 2.2 with $j = 0, \rho$ small and (3.2), we find the above is

$$\begin{aligned} I_1 &\leq C\varepsilon^{1+\alpha\nu} W^{1+\alpha\nu} \left\| \langle \widetilde{\eta} \rangle^{\frac{\alpha}{2}-1+2\nu} \left(|v|^\alpha + |\mathcal{Q}m|^\alpha \right) \right\|_{\mathbf{L}^\infty} \\ &\leq C\varepsilon^{1+\alpha+\alpha\nu} t^{-1+\frac{\alpha}{2}} W^{1+\alpha+\alpha\nu} \left\| \langle \widetilde{\eta} \rangle^{\frac{\alpha}{2}-1+2\nu+\left(1-\frac{\alpha}{2}+\rho\right)\alpha} \right\|_{\mathbf{L}^\infty} \\ &\leq C t^{-1+\frac{\alpha}{2}} \varepsilon^{1+\alpha+\alpha\nu} W^{1+\alpha+\alpha\nu} \end{aligned}$$

if $0 < \nu < \frac{1}{2}\left(\frac{\alpha}{2} - 1\right)(\alpha - 1) - \alpha\rho$. We consider I_2. By Lemma 2.8

$$\left\| \sqrt{\Lambda''} \{\widetilde{\eta}\}^{\nu} \langle \widetilde{\eta} \rangle^{-\nu} \partial_\eta (v - \mathcal{Q}m) \right\|_{\mathbf{L}^2} \leq C \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} \leq C t^{\frac{1}{2\alpha}} \varepsilon^\alpha W^{\alpha+1}$$

and

$$\left\| \sqrt{\Lambda''} \{\widetilde{\eta}\}^{\nu} \langle \widetilde{\eta} \rangle^{-\nu} \partial_\eta (v + \mathcal{Q}m) \right\|_{\mathbf{L}^2} \leq C t^{\frac{1}{2\alpha}} \varepsilon W.$$

Applying Lemma 2.2 with $\sigma = 0$, we have

$$\left\| \langle \widetilde{\eta} \rangle^{2\nu} \left(|v|^\alpha + |\mathcal{Q}m|^\alpha \right) \right\|_{\mathbf{L}^\infty} \leq C t^{\frac{\alpha}{2}-1} \varepsilon^\alpha W^\alpha$$

and

$$\left\| \langle \widetilde{\eta} \rangle^{2\nu} \left(|v|^{\alpha-1} + |\mathcal{Q}m|^{\alpha-1} \right) \right\|_{\mathbf{L}^\infty} \leq C t^{\frac{\alpha}{2}+\frac{1}{\alpha}-\frac{3}{2}} \varepsilon^{\alpha-1} W^{\alpha-1}.$$

Hence we get

$$I_2 \leq C t^{-1+\frac{\alpha}{2}} \varepsilon^{2\alpha} W^{2\alpha+1} \leq C t^{-1+\frac{\alpha}{2}} \varepsilon^{1+\alpha+\nu} W^{1+\alpha+\nu}$$

Similarly,

$$I_3 \leq Ct^{-1+\frac{\alpha}{2}} \varepsilon^{1+\alpha+\alpha\nu} W^{1+\alpha+\alpha\nu}$$

Thus, we get the second estimate of the lemma. Lemma 3.2 is proved. □

We now prove the asymptotics for the solutions.

Lemma 3.3. *Suppose that $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon$. Then the asymptotics*

$$\begin{aligned} u(t,x) &= m(1) G\left(xt^{-\frac{1}{\alpha}}\right) \\ &\times \frac{\exp\left(-i \frac{\operatorname{Re} b}{\alpha \operatorname{Im} b} \log\left(1 - \alpha\lambda \operatorname{Im} b \,|m(1)|^{\alpha} \log t\right)\right)}{t^{\frac{1}{\alpha}} \left(1 - \alpha\lambda \operatorname{Im} b \,|m(1)|^{\alpha} \log t\right)^{\frac{1}{\alpha}}} \\ &+ O\left(t^{-\frac{1}{\alpha}} (\log t)^{-1-\frac{1}{\alpha}}\right) \end{aligned}$$

is valid for large t uniformly with respect to $x \in \mathbb{R}$.

Proof. By Lemma 2.2 we find

$$\left\|\mathcal{D}_t \mathcal{B} M \mathcal{Q} (\widehat{\varphi} - m)\right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{3}{2\alpha}} \left\|\partial_\xi \widehat{\varphi}\right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{\alpha}} \varepsilon (\log\langle t\rangle)^{-1-\frac{1}{\alpha}}.$$

Therefore, we have

$$u(t) = m \mathcal{D}_t \mathcal{B} M \mathcal{Q} 1 + O\left(\varepsilon t^{-\frac{1}{\alpha}} (\log\langle t\rangle)^{-1-\frac{1}{\alpha}}\right)$$

for $t \to \infty$. We see that

$$\mathcal{D}_t \mathcal{B} M \mathcal{Q} 1 = t^{-\frac{1}{\alpha}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\frac{\widehat{x}}{t}\xi - \frac{i}{\alpha}|\xi|^{\alpha}} d\xi = t^{-\frac{1}{\alpha}} G\left(\widehat{x}t^{-1}\right) = t^{-\frac{1}{\alpha}} G\left(xt^{-\frac{1}{\alpha}}\right),$$

where $G(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x - \frac{i}{\alpha}|\xi|^{\alpha}} d\xi$. By Lemma 3.2, we get from equation (2.1) at $\xi = 0$

$$im' = \lambda t^{-1} b |m|^{\alpha} m + O\left(t^{-1} \varepsilon^{\alpha+1+\nu} W^{\alpha+1+\nu}\right).$$

Thus

$$\partial_t |m|^{-\alpha} = -\alpha\lambda t^{-1} \operatorname{Im} b + O\left(t^{-1} \varepsilon^{1+\nu} W^{1+\nu}\right).$$

Observe that $\operatorname{Im} b < 0$. Integrating with respect to time we find

$$|m(t)|^{-\alpha} = |m(1)|^{-\alpha} - \alpha\lambda \operatorname{Im} b \log t + O\left(\varepsilon^{1+\nu} (\log\langle t\rangle)^{1-\frac{1+\nu}{\alpha}}\right)$$

from which it follows that

$$|m(t)|^{\alpha} = \frac{|m(1)|^{\alpha}}{\left(1 - \alpha\lambda \operatorname{Im} b \,|m(1)|^{\alpha} \log t\right)\left(1 + O\left(\varepsilon^{1+\nu} (\log\langle t\rangle)^{-\frac{1+\nu}{\alpha}}\right)\right)}.$$

Namely we have

$$|m(t)| = \frac{|m(1)|}{\left(1 - \alpha\lambda \operatorname{Im} b \,|m(1)|^{\alpha} \log t\right)^{\frac{1}{\alpha}}} \left(1 + O\left(\varepsilon^{1+\nu} (\log\langle t\rangle)^{-\frac{1+\nu}{\alpha}}\right)\right).$$

Also

$$\partial_t \arg m = -\lambda t^{-1} |m|^{\alpha} \operatorname{Re} b + O\left(t^{-1} \varepsilon^{\alpha+\nu} W^{\alpha+\nu}\right)$$

$$= -\lambda t^{-1}\frac{|m(1)|^{\alpha}\,\mathrm{Re}b}{(1-\alpha\lambda\mathrm{Im}b\,|m(1)|^{\alpha}\log t)\left(1+O\left(\varepsilon^{1+\nu}(\log\langle t\rangle)^{-\frac{1+\nu}{\alpha}}\right)\right)}$$
$$+O\left(t^{-1}\varepsilon^{\alpha+\nu}W^{\alpha+\nu}\right)$$

Integration with respect to the time yields

$$\arg m(t) = \arg m(1) - \frac{\mathrm{Re}b}{\alpha\mathrm{Im}b}\log(1-\alpha\lambda\mathrm{Im}b\,|m(1)|^{\alpha}\log t) + O\left(\varepsilon^{1+\nu}\right).$$

Therefore, we get

$$u(t) = m\mathcal{D}_t\mathcal{B}M\mathcal{Q}1 + O\left(t^{-\frac{1}{\alpha}}(\log\langle t\rangle)^{-1-\frac{1}{\alpha}}\right)$$
$$= m(1)\,G\left(xt^{-\frac{1}{\alpha}}\right)\frac{\exp\left(-i\frac{\mathrm{Re}b}{\alpha\mathrm{Im}b}\log(1-\alpha\lambda\mathrm{Im}b\,|m(1)|^{\alpha}\log t)+iO(\varepsilon)\right)}{t^{\frac{1}{\alpha}}(1-\alpha\lambda\mathrm{Im}b\,|m(1)|^{\alpha}\log t)^{\frac{1}{\alpha}}}$$
$$+ O\left(t^{-\frac{1}{\alpha}}(\log\langle t\rangle)^{-\frac{2}{\alpha}}\right)$$

for large t. Lemma 3.3 is proved. □

4. A priori estimates

First, we state the local existence of solutions to the Cauchy problem (1.1) in the functional space $\mathbf{H}^{1,1}\cap\mathbf{H}^{\alpha,0}$ (see [17]).

Theorem 4.1. *We assume that the initial data $u_0\in\mathbf{H}^{1,1}\cap\mathbf{H}^{\alpha,0}$ are such that $\|u_0\|_{\mathbf{H}^{1,1}}\le C\varepsilon$, and $\left|\widehat{u_0}(0)\right|\ge 2\varepsilon$. Then there exists a time T such that the estimates $\|u\|_{\mathbf{X}_T}\le C\varepsilon$ and $\inf_{t\in[0,T]}\left|\widehat{\varphi}(t,0)\right|\ge\varepsilon$ are true. Furthermore if ε is small, then we can take $T>1$.*

In order to prove the global result, we need to obtain a priori estimates of solutions uniformly with respect to $T\ge 1$ in the norm $\|u\|_{\mathbf{X}_T}$.

Lemma 4.1. *There exist $\varepsilon_0>0$, and a constant $C>0$ such that if the initial data $u_0\in\mathbf{H}^{1,1}\cap\mathbf{H}^{\alpha,0}$ are such that $\|u_0\|_{\mathbf{H}^{1,1}}\le C\varepsilon$ and $\inf_{|\xi|\le 1}\left|\widehat{u_0}(0)\right|\ge 2\varepsilon$ with $\varepsilon\in(0,\varepsilon_0)$, then the estimate $\|u\|_{\mathbf{X}_T}<C\varepsilon$ is true for all $T>1$.*

Proof. We proceed by a contradiction. By continuity of the norm $\|u\|_{\mathbf{X}_T}$ with respect to T, we can find the maximal time $T>1$ such that $\|u\|_{\mathbf{X}_T}=C\varepsilon$. Let us estimate the norm

$$\left\|\left\langle\xi\right\rangle^{\beta}\partial_{\xi}\widehat{\varphi}(t)\right\|_{\mathbf{L}^2} = \left\|\left\langle t^{\frac{1}{\alpha}}\partial_x\right\rangle^{\beta}\mathcal{J}u\right\|_{\mathbf{L}^2}.$$

Define $\chi_1\in\mathbf{C}^1(\mathbb{R})$ such that $\chi_1(x)=1$ for $|x|\ge 2$ and $\chi_1(x)=0$ for $|x|\le 1$, also $\chi_0(x)=1-\chi_1(x)$. Then we consider the operators $\mathbf{\Omega}_j=\chi_j\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\left\langle\omega t^{\frac{1}{\alpha}}\partial_x\right\rangle^{\beta}$. By the identity $\mathcal{P}=-i\alpha t\mathcal{L}+\partial_x\mathcal{J}$, where $\mathcal{P}=\alpha t\partial_t+\partial_x x$, we have

$$\left\|\left\langle t^{\frac{1}{\alpha}}\partial_x\right\rangle^{\beta}\mathcal{J}u\right\|_{\mathbf{L}^2} \le \|\mathbf{\Omega}_0\mathcal{J}u\|_{\mathbf{L}^2}+\|\mathbf{\Omega}_1\mathcal{J}u\|_{\mathbf{L}^2} \tag{4.1}$$
$$\le \|\mathbf{\Omega}_0\mathcal{J}u\|_{\mathbf{L}^2}+\alpha t\left\|\mathbf{\Omega}_1\partial_x^{-1}\mathcal{L}u\right\|_{\mathbf{L}^2}+\left\|\mathbf{\Omega}_1\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2}.$$

Let us estimate the first summand

$$\|\mathbf{\Omega}_0 \mathcal{J} u\|_{\mathbf{L}^2} = \left\| \chi_0\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta} \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} \leq \left\| \left\langle\omega\widetilde{\xi}\right\rangle^{\beta} \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2\left(\left|\widetilde{\xi}\right| \leq \frac{2}{\omega}\right)}.$$

Differentiating (2.1) with respect to ξ, multiplying both sides of (2.1) by $\chi_0\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta}$, we get

$$i\partial_t\left(\chi_0\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta}\partial_\xi\widehat{\varphi}\right) = i\left(\partial_\xi\widehat{\varphi}\right)\partial_t\left(\chi_0\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta}\right) + \lambda t^{-\frac{\alpha}{2}}\chi_0\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta}\partial_\xi Q^*\left(|v|^\alpha v\right).$$

By Lemma 2.9 we have

$$\text{(4.2)} \quad \left\|\partial_\xi Q^*\left(|v|^\alpha v\right)\right\|_{\mathbf{L}^2\left(\left|\widetilde{\xi}\right|\leq N\right)} \leq C\left\|\sqrt{\Lambda''}\left\langle\widetilde{\eta}\right\rangle^{\nu}|v|^\alpha\partial_\eta v\right\|_{\mathbf{L}^2(|\widetilde{\eta}|\geq 2N)} + C\left\|\sqrt{\Lambda''}\left\langle\widetilde{\eta}\right\rangle^{\nu}\frac{1}{\eta}|v|^{\alpha+1}\right\|_{\mathbf{L}^2(|\widetilde{\eta}|\geq 2N)} + Ct^{\frac{3}{2\alpha}-\frac{1}{2}}\left\||v|^{\alpha+1}\right\|_{\mathbf{L}^\infty(|\widetilde{\eta}|\leq 3N)}.$$

Using Lemma 2.2 we get

$$\left\langle\widetilde{\eta}\right\rangle^{\frac{\alpha}{2}-1-2\gamma}|v| \leq Ct^{\frac{1}{2}-\frac{\alpha}{2}}\left(\left\|\left\langle\widetilde{\xi}\right\rangle^{-\gamma}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2\alpha}}\left\|\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}\right) \leq Ct^{\frac{1}{2}-\frac{1}{\alpha}}\varepsilon W$$

We take $0 < \nu < \min\left(\frac{\alpha}{2}-1, 1\right)$ to find that

$$\text{(4.3)} \quad \left\langle\widetilde{\eta}\right\rangle^{\nu}|v| \leq Ct^{\frac{1}{2}-\frac{1}{\alpha}}\varepsilon W.$$

and by Lemma 2.8

$$\left\|\sqrt{\Lambda''}\left\{\widetilde{\eta}\right\}^{\nu}\left\langle\widetilde{\eta}\right\rangle^{-\nu}\partial_\eta v\right\|_{\mathbf{L}^2} \leq C\varepsilon t^{\frac{1}{2\alpha}}W.$$

Hence

$$\text{(4.4)} \quad \left\|\sqrt{\Lambda''}\left\langle\widetilde{\eta}\right\rangle^{\nu}|v|^\alpha\partial_\eta v\right\|_{\mathbf{L}^2(|\widetilde{\eta}|\geq 2N)} \leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{\frac{\alpha}{2}-1+\frac{1}{2\alpha}}.$$

We now consider the second term of (4.2) to have

$$\text{(4.5)} \quad \begin{aligned} \left\|\sqrt{\Lambda''}\left\langle\widetilde{\eta}\right\rangle^{\nu}\frac{1}{\eta}|v|^{\alpha+1}\right\|_{\mathbf{L}^2_\eta(|\widetilde{\eta}|\geq 2N)} &= \left\|\eta^{\frac{\alpha}{2}-2}\left\langle\widetilde{\eta}\right\rangle^{\nu}|v|^{\alpha+1}\right\|_{\mathbf{L}^2_\eta(|\widetilde{\eta}|\geq 2N)} \\ &\leq Ct^{-\frac{1}{2}+\frac{2}{\alpha}-\frac{1}{2\alpha}}\left\|\left\langle\widetilde{\eta}\right\rangle^{\frac{\alpha}{2}-2+\nu}|v|^{\alpha+1}\right\|_{\mathbf{L}^2_\eta} \\ &\leq Ct^{-\frac{1}{2}+\frac{2}{\alpha}+\left(\frac{1}{2}-\frac{1}{\alpha}\right)(\alpha+1)-\frac{1}{2\alpha}}\varepsilon^{\alpha+1}W^{\alpha+1}\left\|\left\langle\widetilde{\eta}\right\rangle^{\frac{\alpha}{2}-2+\nu+\left(1-\frac{\alpha}{2}+2\gamma\right)(\alpha+1)}\right\|_{\mathbf{L}^2_\eta} \\ &= Ct^{-1+\frac{\alpha}{2}+\frac{1}{2\alpha}}\varepsilon^{\alpha+1}W^{\alpha+1}\left\|\left\langle\widetilde{\eta}\right\rangle^{\frac{\alpha}{2}-2+\nu+\left(1-\frac{\alpha}{2}+2\gamma\right)(\alpha+1)}\right\|_{\mathbf{L}^2_\eta} \\ &\leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{\frac{\alpha}{2}-1+\frac{1}{2\alpha}} \end{aligned}$$

since

$$\frac{\alpha}{2}-2+\nu+\left(1-\frac{\alpha}{2}+2\gamma\right)(\alpha+1) < -\frac{1}{2}$$

for $\nu < 1$. We have from (4.3)

$$\text{(4.6)} \quad Ct^{\frac{3}{2\alpha}-\frac{1}{2}}\left\||v|^{\alpha+1}\right\|_{\mathbf{L}^\infty(|\widetilde{\eta}|\leq 3N)} \leq Ct^{\frac{\alpha}{2}-1+\frac{1}{2\alpha}}\varepsilon^{\alpha+1}W^{\alpha+1}.$$

We apply (4.4)-(4.6) to (4.2) to get

$$\left\|\partial_{\xi}\mathcal{Q}^{*}\left(|v|^{\alpha}v\right)\right\|_{\mathbf{L}^{2}\left(\left|\widetilde{\xi}\right|\leq\frac{2}{\omega}\right)}\leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{\frac{\alpha}{2}-1+\frac{1}{2\alpha}}.$$

Therefore we find

$$\begin{aligned}\frac{d}{dt}\left\|\chi_{0}\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta}\partial_{\xi}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}&\leq C\omega t^{-1}\left\|\left\langle\widetilde{\xi}\right\rangle^{\beta}\partial_{\xi}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}+C\varepsilon^{\alpha+1}W^{\alpha+1}t^{-1+\frac{1}{2\alpha}}\\&\leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{-1+\frac{1}{2\alpha}},\end{aligned}$$

where we used the fact that $\omega>0$ is sufficiently small. Integrating in time we obtain

$$\begin{aligned}\|\mathbf{\Omega}_{0}\mathcal{J}u\|_{\mathbf{L}^{2}}&=\left\|\chi_{0}\left(\omega\widetilde{\xi}\right)\left\langle\omega\widetilde{\xi}\right\rangle^{\beta}\partial_{\xi}\widehat{\varphi}\right\|_{\mathbf{L}^{2}}\\&\leq C\varepsilon+C\varepsilon^{\alpha+1}\int_{0}^{t}\tau^{\frac{1}{2\alpha}-1}W^{\alpha+1}(\tau)\,d\tau\\&\leq C\varepsilon+C\varepsilon^{\alpha+1}\int_{0}^{\sqrt{t}}\tau^{\frac{1}{2\alpha}-1}d\tau+C\varepsilon^{\alpha+1}W^{\alpha+1}\left(t^{\frac{1}{2}}\right)\int_{\sqrt{t}}^{t}\tau^{\frac{1}{2\alpha}-1}d\tau\\&\leq C\varepsilon+C\varepsilon^{\alpha+1}t^{\frac{1}{4\alpha}}+C\varepsilon^{\alpha+1}t^{\frac{1}{2\alpha}}W^{\alpha+1}(t)<C\varepsilon P(t),\end{aligned}$$

for all $t\in[1,T]$, if $\varepsilon>0$ is sufficiently small.

Next by (2.1) we have

$$\begin{aligned}t\left\|\mathbf{\Omega}_{1}\partial_{x}^{-1}\mathcal{L}u\right\|_{\mathbf{L}^{2}}&\leq Ct\left\|\mathbf{\Omega}_{1}\partial_{x}^{-1}|u|^{\alpha}u\right\|_{\mathbf{L}^{2}}\\&\leq Ct\left\|\chi_{1}\left(\omega t^{\frac{1}{\alpha}}\partial_{x}\right)\left\langle\omega t^{\frac{1}{\alpha}}\partial_{x}\right\rangle^{\beta}\partial_{x}^{-1}|u|^{\alpha}u\right\|_{\mathbf{L}^{2}}\\&=Ct\left\|\chi_{1}\left(\omega t^{\frac{1}{\alpha}}\xi\right)\left\langle\omega t^{\frac{1}{\alpha}}\xi\right\rangle^{\beta}\xi^{-1}\mathcal{F}|u|^{\alpha}u\right\|_{\mathbf{L}^{2}}\\&\leq Ct^{1+\frac{1}{\alpha}}\||u|^{\alpha}u\|_{\mathbf{L}^{2}}=Ct^{1+\frac{1}{\alpha}}\|\mathcal{F}\mathcal{U}(-t)|u|^{\alpha}u\|_{\mathbf{L}^{2}}\\&=Ct^{1+\frac{1}{\alpha}-\frac{\alpha}{2}}\|\mathcal{Q}^{*}(|v|^{\alpha}v)\|_{\mathbf{L}^{2}}\leq t^{1+\frac{1}{\alpha}-\frac{\alpha}{2}}\left\|\sqrt{\Lambda''}|v|^{\alpha+1}\right\|_{\mathbf{L}^{2}}\end{aligned}$$

where we have used $\mathcal{F}\mathcal{U}(-t)|u|^{\alpha}u=\lambda t^{-\frac{\alpha}{2}}\mathcal{Q}^{*}(|v|^{\alpha}v)$. By Lemma 2.2 with $\sigma=0$ we have the estimate $|v|\leq C\varepsilon t^{\frac{1}{2}-\frac{1}{\alpha}}\langle\widetilde{\eta}\rangle^{1-\frac{\alpha}{2}+\gamma}W$. Also $\left(1-\frac{\alpha}{2}\right)(\alpha+1)+\frac{\alpha-2}{2}<-\frac{1}{2}$ for $\alpha>\frac{5}{2}$. Therefore

$$\begin{aligned}&t^{1+\frac{1}{\alpha}-\frac{\alpha}{2}}\left\|\sqrt{\Lambda''}|v|^{\alpha+1}\right\|_{\mathbf{L}^{2}}\\&\leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{\frac{1}{2}}\left\|\sqrt{\Lambda''}\langle\widetilde{\eta}\rangle^{\left(1-\frac{\alpha}{2}+\gamma\right)(\alpha+1)}\right\|_{\mathbf{L}^{2}}\\&\leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{\frac{1}{2\alpha}}\left\||\widetilde{\eta}|^{\frac{\alpha-2}{2}}\langle\widetilde{\eta}\rangle^{\left(1-\frac{\alpha}{2}+\gamma\right)(\alpha+1)}\right\|_{\mathbf{L}^{2}_{\eta}}\\&\leq C\varepsilon^{\alpha+1}W^{\alpha+1}t^{\frac{1}{2\alpha}}<C\varepsilon P(t),\end{aligned}$$

which implies

$$t\left\|\mathbf{\Omega}_{1}\partial_{x}^{-1}\mathcal{L}u\right\|_{\mathbf{L}^{2}}\leq Ct^{1+\frac{1}{\alpha}}\||u|^{\alpha}u\|_{\mathbf{L}^{2}}<C\varepsilon P(t).\tag{4.7}$$

Next we estimate the third term $\left\|\mathbf{\Omega}_{1}\partial_{x}^{-1}\mathcal{P}u\right\|_{\mathbf{L}^{2}}$. Using the commutator $\mathcal{L}\mathcal{P}=(\alpha+\mathcal{P})\mathcal{L}$, we get from (1.1)

$$\begin{aligned}\mathcal{L}\mathbf{\Omega}_{1}\partial_{x}^{-1}\mathcal{P}u&=\mathbf{\Omega}_{1}\partial_{x}^{-1}(\alpha+\mathcal{P})\mathcal{L}u+i\left(\partial_{t}\mathbf{\Omega}_{1}\right)\partial_{x}^{-1}\mathcal{P}u\\&=\alpha\lambda\mathbf{\Omega}_{1}\partial_{x}^{-1}|u|^{\alpha}u+i\partial_{t}\left(\chi_{1}\left(\omega t^{\frac{1}{\alpha}}\partial_{x}\right)\left\langle\omega t^{\frac{1}{\alpha}}\partial_{x}\right\rangle^{\beta}\right)\left(\partial_{x}^{-1}\mathcal{P}u\right)\end{aligned}$$

$$+ \lambda \mathbf{\Omega}_1 \partial_x^{-1} \mathcal{P} |u|^{\alpha} u.$$

As above we have $\left\| \mathbf{\Omega}_1 \partial_x^{-1} |u|^{\alpha} u \right\|_{\mathbf{L}^2} < C\varepsilon t^{-1} P(t)$ and

$$\left\| \partial_t \left(\chi_1 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \left\langle \omega t^{\frac{1}{\alpha}} \partial_x \right\rangle^{\beta} \right) \left(\partial_x^{-1} \mathcal{P} u \right) \right\|_{\mathbf{L}^2} \leq C\omega\varepsilon t^{-1} P(t).$$

Consider the third term

$$\begin{aligned} &\mathbf{\Omega}_1 \partial_x^{-1} \mathcal{P} |u|^{\alpha} u \\ &= C\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \mathcal{P}u \right) + C\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha-2} u^2 \overline{\mathcal{P}u} \right) \\ &= C\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{P}u \right) + C\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha-2} u^2 \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \overline{\mathcal{P}u} \right) \\ &\quad + C\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \chi_1 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{P}u \right) + C\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha-2} u^2 \chi_1 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \overline{\mathcal{P}u} \right). \end{aligned}$$

Then we use the identity $\mathcal{P} = \alpha t \mathcal{L} + \partial_x \mathcal{J}$ and

$$\left\| \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{J} u \right\|_{\mathbf{L}^2} = \left\| \chi_0 \left(\omega \widetilde{\xi} \right) \mathcal{F} \mathcal{U}(-t) \mathcal{J} u \right\|_{\mathbf{L}^2} \leq \left\| \chi_0 \left(\omega \widetilde{\xi} \right) \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} \leq C\varepsilon P(t),$$

to get with (4.7)

$$\begin{aligned} &\left\| \mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{P}u \right) \right\|_{\mathbf{L}^2} \leq C t^{\frac{1}{\alpha}} \left\| |u|^{\alpha} \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{P}u \right\|_{\mathbf{L}^2} \\ &\leq C t^{\frac{1}{\alpha}} \|u\|_{\mathbf{L}^\infty}^{\alpha} \left(t \left\| \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) |u|^{\alpha} u \right\|_{\mathbf{L}^2} + \left\| \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \partial_x \mathcal{J} u \right\|_{\mathbf{L}^2} \right) \\ &\leq C \|u\|_{\mathbf{L}^\infty}^{\alpha} \left(t^{1+\frac{1}{\alpha}} \| |u|^{\alpha} u \|_{\mathbf{L}^2} + \left\| \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{J} u \right\|_{\mathbf{L}^2} \right) \\ &\leq C \|u\|_{\mathbf{L}^\infty}^{\alpha} \left(t^{1+\frac{1}{\alpha}} \| |u|^{\alpha} u \|_{\mathbf{L}^2} + \varepsilon P(t) \right). \end{aligned}$$

By Lemma 2.2 we have

$$\|u\|_{\mathbf{L}^\infty} = C t^{-\frac{1}{2}} \|\mathcal{D}_t \mathcal{B} M v\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{2}} \|\mathcal{B} M v\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{2}} \|v\|_{\mathbf{L}^\infty}.$$

Then we get

$$\|u\|_{\mathbf{L}^\infty}^{\alpha} \leq C t^{-\frac{\alpha}{2}} \|v\|_{\mathbf{L}^\infty}^{\alpha} \leq C t^{-1} \varepsilon^{\alpha} W^{\alpha},$$

and in the same way as in the proof of (4.7) we find

$$t^{1+\frac{1}{\alpha}} \| |u|^{\alpha} u \|_{\mathbf{L}^2} \leq C t^{\frac{1}{\alpha}} \varepsilon^{\alpha+1} W^{\alpha} \leq C t^{\frac{1}{2\alpha}} \varepsilon W^{-1} P.$$

Therefore

$$\left\| \mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{P}u \right) \right\|_{\mathbf{L}^2} \leq C \varepsilon^{\alpha+1} t^{-1} W^{\alpha} P(t).$$

In the same manner we find

$$\left\| \mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha-2} u^2 \chi_0 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \overline{\mathcal{P}u} \right) \right\|_{\mathbf{L}^2} \leq C \varepsilon^{\alpha+1} t^{-1} W^{\alpha} P(t).$$

We represent

$$\begin{aligned} &\mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \chi_1 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \mathcal{P}u \right) \\ &= \mathbf{\Omega}_1 \partial_x^{-1} \left(|u|^{\alpha} \partial_x^{1-\beta} \chi_1 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \partial_x^{\beta-1} \mathcal{P}u \right) \\ &= \mathbf{\Omega}_1 \partial_x^{-\beta} \left(|u|^{\alpha} \chi_1 \left(\omega t^{\frac{1}{\alpha}} \partial_x \right) \partial_x^{\beta-1} \mathcal{P}u \right) \end{aligned}$$

$$-\mathbf{\Omega}_1\partial_x^{-1}\left(\left[\partial_x^{1-\beta},|u|^{\alpha}\right]\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u\right),$$

where the commutator

$$\left[\partial_x^{1-\beta},\phi\right]f=C\int_{\mathbb{R}}\frac{\phi(x)-\phi(x-y)}{y^{2-\beta}}f(x-y)\,dy.$$

Then in the same way as in the proof of (4.7) we obtain

$$\begin{aligned}&\left\|\mathbf{\Omega}_1\partial_x^{-\beta}\left(|u|^{\alpha}\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u\right)\right\|_{\mathbf{L}^2}\\ &\le Ct^{\frac{\beta}{\alpha}}\|u\|_{\mathbf{L}^{\infty}}^{\alpha}\left\|\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u\right\|_{\mathbf{L}^2}\\ &\le C\|u\|_{\mathbf{L}^{\infty}}^{\alpha}\left\|\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\left\langle\omega t^{\frac{1}{\alpha}}\partial_x\right\rangle^{\beta}\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2}\le C\varepsilon^{\alpha+1}t^{-1}W^{\alpha}P(t).\end{aligned}$$

Next we have for the commutator

$$\begin{aligned}\left\|\left[\partial_x^{1-\beta},|u|^{\alpha}\right]f\right\|_{\mathbf{L}^2}&=C\left\|\int_{\mathbb{R}}\frac{(|u(x)|^{\alpha}-|u(x-y)|^{\alpha})}{|y|^{2-\beta}}|f(x-y)|\,dy\right\|_{\mathbf{L}^2}\\ &\le C\left\|\int_{|y|\le t^{\frac{1}{\alpha}}}\frac{(|u(x)|^{\alpha}-|u(x-y)|^{\alpha})}{|y|^{2-\beta}}|f(x-y)|\,dy\right\|_{\mathbf{L}^2}\\ &\quad+C\left\|\int_{|y|\ge t^{\frac{1}{\alpha}}}\frac{(|u(x)|^{\alpha}-|u(x-y)|^{\alpha})}{|y|^{2-\beta}}|f(x-y)|\,dy\right\|_{\mathbf{L}^2}=I_1+I_2\end{aligned}$$

with $f=\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u$. The first summand

$$\begin{aligned}I_1\ \le\ & C\sup_{|y|\le t^{\frac{1}{\alpha}}}\sup_x\left(|u(x)|^{\alpha-1+\beta_1}\left|\frac{u(x)-u(x-y)}{y}\right|^{1-\beta_1}\right)\\ &\times\left\|\int_{|y|\le t^{\frac{1}{\alpha}}}\frac{1}{|y|^{1-\beta+\beta_1}}|f(x-y)|\,dy\right\|_{\mathbf{L}^2},\end{aligned}$$

where $0<\beta_1<\beta<\frac{1}{2}$. By the estimate of Lemma 2.2 we have

$$\left|\partial_x^{\sigma}u(x)\right|\le C\varepsilon Wt^{-\frac{1+\sigma}{\alpha}}\left\langle t^{\frac{1}{\alpha}}|x|^{\frac{1}{\alpha-1}}\right\rangle^{1+\sigma-\frac{\alpha}{2}+\gamma}.$$

Hence

$$\begin{aligned}&|u(x)|^{\alpha-1+\beta_1}\left|\frac{u(x)-u(x-y)}{y}\right|^{1-\beta_1}\le C|u(x)|^{\alpha-1+\beta_1}\left|\int_0^1u_x(x-yz)\,dz\right|^{1-\beta_1}\\ &\le Ct^{-1-\frac{1-\beta_1}{\alpha}}\varepsilon^{\alpha}W^{\alpha}\left\langle\left|t^{1-\frac{1}{\alpha}}x\right|^{\frac{1}{\alpha-1}}\right\rangle^{\left(1-\frac{\alpha}{2}+\gamma\right)(\alpha-1+\beta_1)}\\ &\times\left(\int_0^1\left\langle\left|t^{1-\frac{1}{\alpha}}x-t^{1-\frac{1}{\alpha}}yz\right|^{\frac{1}{\alpha-1}}\right\rangle^{2-\frac{\alpha}{2}+\gamma}dz\right)^{1-\beta_1}\le Ct^{-1-\frac{1-\beta_1}{\alpha}}\varepsilon^{\alpha}W^{\alpha}.\end{aligned}$$

Also

$$\begin{aligned}&\left\|\int_{|y|\le t^{\frac{1}{\alpha}}}\frac{1}{|y|^{1-\beta+\beta_1}}|f(x-y)|\,dy\right\|_{\mathbf{L}^2}\\ &\le C\|f\|_{\mathbf{L}^2}\int_{|y|\le t^{\frac{1}{\alpha}}}\frac{1}{|y|^{1-\beta+\beta_1}}dy\le Ct^{\frac{1}{\alpha}(\beta-\beta_1)}\|f\|_{\mathbf{L}^2}.\end{aligned}$$

Therefore

$$\begin{aligned} I_1 &\le Ct^{-1-\frac{1}{\alpha}}\varepsilon^{\alpha}W^{\alpha}t^{\frac{\beta}{\alpha}}\left\|\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \\ &\le Ct^{-1-\frac{1}{\alpha}}\varepsilon^{\alpha}W^{\alpha}\left\|\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\left\langle\omega t^{\frac{1}{\alpha}}\partial_x\right\rangle^{\beta}\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \\ &\le C\varepsilon^{\alpha+1}t^{-1-\frac{1}{\alpha}}W^{\alpha}P(t). \end{aligned}$$

For the second summand we get

$$\begin{aligned} I_2 &\le C\|u\|_{\mathbf{L}^\infty}^{\alpha}\left\|\int_{|y|\ge t^{\frac{1}{\alpha}}}|f(x-y)|\frac{dy}{|y|^{2-\beta}}\right\|_{\mathbf{L}^2} \\ &\le C\|u\|_{\mathbf{L}^\infty}^{\alpha}\|f\|_{\mathbf{L}^2}\int_{|y|\ge t^{\frac{1}{\alpha}}}\frac{dy}{|y|^{2-\beta}} \le Ct^{\frac{\beta-1}{\alpha}}\|u\|_{\mathbf{L}^\infty}^{\alpha}\|f\|_{\mathbf{L}^2} \\ &\le Ct^{\frac{\beta-1}{\alpha}}\|u\|_{\mathbf{L}^\infty}^{\alpha}\left\|\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \\ &\le Ct^{-1-\frac{1}{\alpha}}\varepsilon^{\alpha}W^{\alpha}\left\|\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\left\langle\omega t^{\frac{1}{\alpha}}\partial_x\right\rangle^{\beta}\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \\ &\le C\varepsilon^{\alpha+1}t^{-1-\frac{1}{\alpha}}W^{\alpha}P(t). \end{aligned}$$

Hence

$$\left\|\mathbf{\Omega}_1\partial_x^{-1}\left(\left[\partial_x^{1-\beta},|u|^{\alpha}\right]\chi_1\left(\omega t^{\frac{1}{\alpha}}\partial_x\right)\partial_x^{\beta-1}\mathcal{P}u\right)\right\|_{\mathbf{L}^2} \le C\varepsilon^{\alpha+1}t^{-1}W^{\alpha}P(t).$$

Thus we obtain

$$\frac{d}{dt}\left\|\mathbf{\Omega}_1\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \le C\varepsilon^{\alpha+1}W^{\alpha+1}t^{-1+\frac{1}{2\alpha}}.$$

Integrating in time, we get $\left\|\mathbf{\Omega}_1\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} < C\varepsilon P(t)$ for all $t \in [1,T]$, if $\varepsilon > 0$ is sufficiently small. Hence $\left\|\left\langle\xi\right\rangle^{\beta}\partial_{\xi}\widehat{\varphi}(t)\right\|_{\mathbf{L}^2} < C\varepsilon P(t)$.

Next taking equation (2.1) at $\xi = 0$ and applying Lemma 3.2, we get for $m(t) = \widehat{\varphi}(t,0)$

$$im' = \lambda t^{-1}b|m|^{\alpha}m + O\left(t^{-1}\varepsilon^{\alpha+1+\nu}W^{\alpha+1+\nu}\right).$$

Hence $\partial_t|m|^{-\alpha} = -\alpha\lambda t^{-1}\mathrm{Im}b + O\left(t^{-1}\varepsilon^{1+\nu}W^{1+\nu}\right)$. Using $\mathrm{Im}b < 0$ and integrating with respect to time, we obtain

$$|m(t)|^{-\alpha} = |m(1)|^{-\alpha}\left(1-\alpha\lambda\mathrm{Im}b|m(1)|^{\alpha}\log t + C|m(1)|^{\alpha}(\log\langle t\rangle)^{1-\frac{1+\nu}{\alpha}}\right)$$

which implies that

$$|m(t)| = |m(1)|\left(1-\alpha\lambda\mathrm{Im}b|m(1)|^{\alpha}\log t + C|m(1)|^{\alpha}(\log\langle t\rangle)^{1-\frac{1+\nu}{\alpha}}\right)^{-\frac{1}{\alpha}}.$$

We may assume that $|m(1)| \ge \varepsilon$ if

$$\inf_{|\xi|\le 1}\left|\widehat{\varphi}(0,\xi)\right| = \inf_{|\xi|\le 1}|u_0(\xi)| \ge 2\varepsilon.$$

Therefore we obtain $\varepsilon W(t) \le |m(t)| = \left|\widehat{\varphi}(t,0)\right| < C\varepsilon W(t)$, if $\varepsilon > 0$ is sufficiently small. By (3.1) it follows that

$$\left\|\left\langle\xi\right\rangle^{-\gamma}\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty}$$

$$\leq C\varepsilon W(t) + C(\varepsilon W(t))^{1-\nu} |\xi|^{\frac{1}{2}\nu} \langle \xi \rangle^{-\gamma} \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2}^{\nu}$$
$$\leq C\varepsilon W(t) + C(\varepsilon W(t))^{1-\nu} t^{-\frac{1}{2\alpha}\nu} (\varepsilon P(t))^{\nu} < C\varepsilon W(t),$$

if we take $\gamma \geq \frac{1}{2}\nu$. Using Lemma 3.2 we have $|\mathcal{Q}^* |v|^{\alpha} v| \leq Ct^{\frac{\alpha}{2}-1}\varepsilon^{\alpha+1}W^{\alpha+1}$. Then, by (2.1) we get $\left|\widehat{\varphi}_t\right| \leq Ct^{-1}\varepsilon^{\alpha+1}W^{\alpha+1}$ for all $t \in [1, T]$. Integrating in time we deduce that

$$\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} \leq \varepsilon + C\varepsilon^{\alpha+1} \int_1^t \frac{d\tau}{(1 + \widetilde{\varepsilon}^{\alpha} \log\langle\tau\rangle)^{1+\frac{1}{\alpha}} \tau}.$$

Thus, we obtain $\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} < C\varepsilon$. Therefore, $\|u\|_{\mathbf{X}_T} < C\varepsilon$, which contradicts the assumption $\|u\|_{\mathbf{X}_T} = C\varepsilon$. Lemma 4.1 is proved. □

5. Proof of Theorem 1.1

The global existence of solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^{1,1} \cap \mathbf{H}^{\alpha,0}\right)$ to the Cauchy problem (1.1) satisfying a priori estimate $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon$ follows from Lemma 4.1 and the local existence Theorem 4.1 by a standard continuation argument. Finally the asymptotics (1.5) follows from Lemma 3.3. Thus the proof of Theorem 1.1 is complete.

Acknowledgements. We would like to thank the referee for useful suggestions on the draft. The work of N.H. is partially supported by JSPS KAKENHI Grant Numbers JP20K03680, JP19H05597. The work of P.I.N. is partially supported by CONACYT and PAPIIT project IN103221.

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