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ALGEBRAIC INDEPENDENCE RESULTS FOR A CERTAIN FAMILY OF POWER SERIES, INFINITE PRODUCTS, AND LAMBERT TYPE SERIES

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Abstract

For a certain class of power series, infinite products, and Lambert type series, we establish a necessary and sufficient condition for the infinite set consisting of their values, as well as their derivatives of any order at any algebraic points except their poles and zeroes, to be algebraically independent. As its corollary, we construct an example of an infinite family of entire functions of two variables with the following property: Their values and their partial derivatives of any order at any distinct algebraic points with nonzero components are algebraically independent.

1. Introduction and results

First we consider the algebraic independence of a class of functions

$$\mathcal{F}_m(x;z) := \sum_{k=0}^{\infty} x^k z^{md^k} \quad (m = 1, 2, \ldots), \qquad \mathcal{G}(y;z) := \prod_{k=0}^{\infty}(1 - yz^{d^k}), \tag{1.1}$$

and

$$\mathcal{H}(x, y; z) := \sum_{k=0}^{\infty} \frac{x^k z^{d^k}}{1 - yz^{d^k}}, \tag{1.2}$$

where d is an integer greater than 1. We note that $\mathcal{F}_m(x;z)$ ($m \geq 1$) and $\mathcal{G}(y;z)$ converge at any point $(x, y, z) \in \mathbb{C}^3$ with $|z| < 1$ and $\mathcal{H}(x, y; z)$ converges at any point $(x, y, z) \in \mathbb{C}^3$ with $|z| < 1$ such that $1 - yz^{d^k} \neq 0$ for any $k \geq 0$. In the previous works, the algebraic independence of the values at algebraic numbers of the functions above was treated mainly in the following two cases:

(A) The case where $x = y = 1$ and z runs through a finite set of algebraic numbers.
(B) The case where x, y run through infinite sets of algebraic numbers and z is fixed.

For the case (A), let $f(z) := \mathcal{F}_1(1;z) = \sum_{k=0}^{\infty} z^{d^k}$, $g(z) := \mathcal{G}(1;z) = \prod_{k=0}^{\infty}(1 - z^{d^k})$, and $h(z) := \mathcal{H}(1, 1; z) = \sum_{k=0}^{\infty} z^{d^k}/(1 - z^{d^k})$. Using the functional equations $f(z) = f(z^d) + z$, $g(z) = (1 - z)g(z^d)$, and $h(z) = h(z^d) + z/(1 - z)$, Mahler [5] proved that the values $f(a), g(a)$, and $h(a)$ are transcendental for any algebraic number a with $0 < |a| < 1$. Moreover, applying the theory of Mahler functions of r variables, we can show that, if $a_1, \ldots, a_r$ are multiplicatively independent algebraic numbers with $0 < |a_i| < 1$ $(1 \leq i \leq r)$, then each of the sets $\{f(a_1), \ldots, f(a_r)\}$, $\{g(a_1), \ldots, g(a_r)\}$, and $\{h(a_1), \ldots, h(a_r)\}$ is algebraically independent (cf.

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Nishioka [7, pp. 106–107]). On the other hand, the values of $f(z), g(z)$, and $h(z)$ at multiplicatively dependent algebraic numbers can be algebraically dependent. Let $\overline{\mathbb{Q}}^{\times}$ be the set of nonzero algebraic numbers. By the functional equations above we have $f(a) - f(a^d)$, $g(a)/g(a^d)$, $h(a) - h(a^d) \in \overline{\mathbb{Q}}^{\times}$ for any algebraic number a with $0 < |a| < 1$, which implies that each of the sets $\{f(a), f(a^d)\}$, $\{g(a), g(a^d)\}$, and $\{h(a), h(a^d)\}$ is algebraically dependent. For the function $f(z)$, Loxton and van der Poorten [4, Theorem 3] obtained a necessary and sufficient condition on algebraic numbers $a_1, \ldots, a_r$ for the values $f(a_1), \ldots, f(a_r)$ to be algebraically independent. However, for the functions $g(z)$ and $h(z)$, there are no known results on the algebraic independence of the values at multiplicatively dependent algebraic numbers so far, except for the following two results, involving $y = -1$, on $h(z)$ and $h^-(z) := \mathcal{H}(1, -1; z) = \sum_{k=0}^{\infty} z^{d^k}/(1 + z^{d^k})$ obtained by Bundschuh and Väänänen.

Theorem 1.1 (Bundschuh and Väänänen [1, Theorem 2]). *Let a be an algebraic number with $0 < |a| < 1$ and let $m_1, \ldots, m_r$ be positive integers satisfying*

$$\frac{m_i}{m_j} \notin d^{\mathbb{Z}} \quad (1 \le i < j \le r).$$

Then, for each choice of r signs, the r values $h(\pm a^{m_1}), \ldots, h(\pm a^{m_r})$ are algebraically independent, and the same holds for $h^-(\pm a^{m_1}), \ldots, h^-(\pm a^{m_r})$.

Theorem 1.2 (Bundschuh and Väänänen [1, Theorem 4]). *Suppose $d \ge 3$. Let a be an algebraic number with $0 < |a| < 1$ and let $m_1, \ldots, m_r$ be positive integers satisfying*

$$\frac{m_i}{m_j} \notin 2d^{\mathbb{Z}} \quad (1 \le i < j \le r).$$

Then the $2r$ values $h(a^{m_1}), \ldots, h(a^{m_r}), h^-(a^{m_1}), \ldots, h^-(a^{m_r})$ are algebraically independent.

For the case (B), we fix an algebraic number a with $0 < |a| < 1$. Let m be a positive integer and let $F_m(x) := \mathcal{F}_m(x; a) = \sum_{k=0}^{\infty} a^{md^k} x^k$. In this case, Nishioka [6] proved the following theorem, which deals with the values of the entire function $F_m(x)$ as well as its all successive derivatives at any nonzero distinct algebraic numbers.

Theorem 1.3 (Nishioka [6, Theorem 7]). *For each fixed positive integer m, the infinite set $\{F_m^{(l)}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0\}$ is algebraically independent.*

Remark 1.4. For varying m, the infinite set $\{F_m^{(l)}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0,\ m \ge 1\}$ is algebraically dependent, since $\alpha F_d(\alpha) + a = F_1(\alpha)$ for any nonzero algebraic number α.

In contrast with Theorem 1.3, the values of $G(y) := \prod_{k=0}^{\infty}(1 - a^{d^k} y)$ or $H(x, y) := \sum_{k=0}^{\infty} a^{d^k} x^k/(1 - a^{d^k} y)$ are not always algebraically independent even at distinct algebraic points except the zeroes of $G(y)$ and the poles of $H(x, y)$ as shown below. Let β be a nonzero algebraic number with $\beta \notin \{a^{-d^k} \mid k \ge 0\}$ and let $\gamma_1, \ldots, \gamma_d$ be the d-th roots of β. Then the sets $\{G(\beta), G(\gamma_1), \ldots, G(\gamma_d)\}$ and $\{H(1, \beta), H(1, \gamma_1), \ldots, H(1, \gamma_d)\}$ are respectively algebraically dependent, since $(1 - a\beta) \prod_{i=1}^{d} G(\gamma_i) = G(\beta)$ and $\sum_{i=1}^{d} \gamma_i H(1, \gamma_i) + ad\beta/(1 - a\beta) = d\beta H(1, \beta)$.

If we replace the $\{d^k\}_{k \ge 0}$, appearing in $\mathcal{F}_m(x; z), \mathcal{G}(y; z)$, and $\mathcal{H}(x, y; z)$ at the beginning, with a linear recurrence which is not a geometric progression and satisfies suitable conditions, then the situation on the algebraic independence of the values at algebraic points becomes quite different. Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 0), \tag{1.3}$$

where $n \geq 2$, $R_0, \ldots, R_{n-1}$ are not all zero, and $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. Define the polynomial associated with (1.3) by

$$\Phi(X) := X^n - c_1 X^{n-1} - \cdots - c_n. \tag{1.4}$$

Throughout this paper, we assume the following condition (R) on $\{R_k\}_{k\geq 0}$:

(R) $\Phi(\pm 1) \neq 0$, the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity, and $\{R_k\}_{k\geq 0}$ is not a geometric progression.

Then we have

$$R_k = c\rho^k + o(\rho^k), \tag{1.5}$$

where $c > 0$ and $\rho > 1$ (see Tanaka [8, Remark 4]). In what follows, we consider the functions

$$\mathcal{F}_m(x; z) := \sum_{k=0}^{\infty} x^k z^{mR_k} \quad (m = 1, 2, \ldots), \qquad \mathcal{G}(y; z) := \prod_{k=0}^{\infty} (1 - yz^{R_k}),$$

and

$$\mathcal{H}(x, y; z) := \sum_{k=0}^{\infty} \frac{x^k z^{R_k}}{1 - yz^{R_k}}$$

given by replacing the geometric progression $\{d^k\}_{k\geq 0}$ in (1.1) and (1.2) with the linear recurrence $\{R_k\}_{k\geq 0}$ defined above. By (1.5), $\mathcal{F}_m(x; z)$ $(m \geq 1)$ and $\mathcal{G}(y; z)$ converge at any point $(x, y, z) \in \mathbb{C}^3$ with $|z| < 1$ and $\mathcal{H}(x, y; z)$ converges at any point $(x, y, z) \in \mathbb{C}^3$ with $|z| < 1$ such that $1 - yz^{R_k} \neq 0$ for any $k \geq 0$. First we introduce the previous studies, including the author's one, on the algebraic independence of the values of these functions.

For the case (A), let

$$f(z) := \mathcal{F}_1(1; z) = \sum_{k=0}^{\infty} z^{R_k}, \qquad g(z) := \mathcal{G}(1; z) = \prod_{k=0}^{\infty} (1 - z^{R_k}),$$

and

$$h(z) := \mathcal{H}(1, 1; z) = \sum_{k=0}^{\infty} \frac{z^{R_k}}{1 - z^{R_k}}.$$

In contrast with the case of geometric progressions, there are no algebraic relations among the values of the functions $f(z), g(z)$, and $h(z)$ at algebraic numbers such as a and a^d with $0 < |a| < 1$ and $d \geq 2$. Tanaka [9] proved the following

Theorem 1.5 (Tanaka [9, Theorem 5]). *Suppose that $\{R_k\}_{k\geq 0}$ satisfies the condition* (R) *with positive initial values $R_0, \ldots, R_{n-1}$. Let $a_1, \ldots, a_r$ be algebraic numbers with $0 < |a_i| < 1$ $(1 \leq i \leq r)$ such that none of a_i/a_j $(1 \leq i < j \leq r)$ is a root of unity. Then the $3r$ values $f(a_i), g(a_i), h(a_i)$ $(1 \leq i \leq r)$ are algebraically independent.*

As shown in Remark 4 of [9], for some $\{R_k\}_{k\geq 0}$, algebraic relations can appear among the values $f(a_i), g(a_i), h(a_i)$ $(1 \le i \le r)$ at distinct $a_1, \dots, a_r$ whose ratios of some pairs are roots of unity. Extending Theorem 1.5, Tanaka [10, Theorem 1] gave a necessary and sufficient condition on $a_1, \dots, a_r$ for the values $f(a_i), g(a_i), h(a_i)$ $(1 \le i \le r)$ to be algebraically independent.

For the case (B), fix an algebraic number a with $0 < |a| < 1$ and let

$$F_m(x) := \mathcal{F}_m(x; a) = \sum_{k=0}^{\infty} a^{mR_k} x^k \quad (m = 1, 2, \dots).$$

In this case, not only the infinite set $\{F_m^{(l)}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0\}$ for each fixed m but also the infinite set $\{F_m^{(l)}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0,\ m \ge 1\}$ is algebraically independent (see Tanaka [9, Theorem 3]).

Moreover, for the functions

$$G(y) := \mathcal{G}(y; a) = \prod_{k=0}^{\infty} (1 - a^{R_k} y)$$

and

$$H(x, y) := \mathcal{H}(x, y; a) = \sum_{k=0}^{\infty} \frac{a^{R_k} x^k}{1 - a^{R_k} y},$$

Tanaka [11, Theorem 2] proved that the infinite set

$$\{G(\beta) \mid \beta \in \mathcal{B}\} \bigcup \left\{ \frac{\partial^m H}{\partial y^m}(1, \beta) \,\middle|\, \beta \in \mathcal{B},\ m \ge 0 \right\}$$

is algebraically independent, where $\mathcal{B}$ denotes the set of nonzero algebraic numbers different from the zeroes of $G(y)$. This implies that the infinite set $\{G^{(m)}(\beta) \mid \beta \in \mathcal{B},\ m \ge 0\}$ is algebraically independent, since the derivatives of $G(y)$ are expressed as polynomials with integral coefficients of $G(y), H(1, y)$, and the partial derivatives of $H(1, y)$ with respect to y (see [11, Theorem 1]).

In what follows, let $a_1, \dots, a_r$ be algebraic numbers with $0 < |a_i| < 1$ $(1 \le i \le r)$. For each i $(1 \le i \le r)$, we define

$$F_{i,m}(x) := \mathcal{F}_m(x; a_i) = \sum_{k=0}^{\infty} a_i^{mR_k} x^k \quad (m = 1, 2, \dots)$$

and

$$G_i(y) := \mathcal{G}(y; a_i) = \prod_{k=0}^{\infty} (1 - a_i^{R_k} y).$$

As a special case of Main Theorem 1.8 of this paper, we can merge and extend the above-mentioned Tanaka's results on the algebraic independence of the values and the derivatives of $F_m(x)$ $(m \ge 1)$ and that of $G(y)$ by considering, in place of a, the r algebraic numbers $a_1, \dots, a_r$ simultaneously.

Theorem 1.6. *Suppose that* $\{R_k\}_{k \ge 0}$ *satisfies the condition* (R). *Assume that* $a_1, \dots, a_r$ *are pairwise multiplicatively independent. Then the infinite set*

$$\left\{F_{i,m}^{(l)}(\alpha)\ \middle|\ 1 \le i \le r,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0,\ m \ge 1\right\}$$
$$\bigcup \left\{G_i^{(m)}(\beta)\ \middle|\ 1 \le i \le r,\ \beta \in \mathcal{B}_i,\ m \ge 0\right\}$$

is algebraically independent, where $\mathcal{B}_i$ $(1 \le i \le r)$ *are defined by*

$$\mathcal{B}_i := \overline{\mathbb{Q}}^{\times} \setminus \{a_i^{-R_k} \mid k \ge 0\} = \{\beta \in \overline{\mathbb{Q}}^{\times} \mid G_i(\beta) \ne 0\}.$$

Furthermore, the author previously treated the algebraic independence of the values at algebraic numbers of a wider class of partial derivatives than Tanaka's results on the functions $F_m(x)$ $(m \ge 1)$, $G(y)$, and $H(x, y)$ mentioned above.

Theorem 1.7 (The case of $p = \infty$ in Theorem 1.11 of Ide [3])**.** *Suppose that* $\{R_k\}_{k\ge 0}$ *satisfies the condition* (R)*. Then the infinite set*

$$\left\{F_m^{(l)}(\alpha)\ \middle|\ \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0,\ m \ge 1\right\} \bigcup \{G(\beta) \mid \beta \in \mathcal{B}\}$$
$$\bigcup \left\{\frac{\partial^{l+m} H}{\partial x^l \partial y^m}(\alpha, \beta)\ \middle|\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \mathcal{B},\ l \ge 0,\ m \ge 0\right\}$$

is algebraically independent.

In this paper we merge the cases (A) and (B) above by extending Theorem 1.7. In addition to the functions $F_{i,m}(x)$ and $G_i(y)$ above, we define

$$H_i(x, y) := \mathcal{H}(x, y; a_i) = \sum_{k=0}^{\infty} \frac{a_i^{R_k} x^k}{1 - a_i^{R_k} y}$$

for each i $(1 \le i \le r)$. Then the infinite set

$$\mathcal{T}_i := \left\{F_{i,m}^{(l)}(\alpha)\ \middle|\ \alpha \in \overline{\mathbb{Q}}^{\times},\ l \ge 0,\ m \ge 1\right\} \bigcup \{G_i(\beta) \mid \beta \in \mathcal{B}_i\}$$
$$\bigcup \left\{\frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha, \beta)\ \middle|\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \mathcal{B}_i,\ l \ge 0,\ m \ge 0\right\}$$

is algebraically independent for each i $(1 \le i \le r)$ by Theorem 1.7. The main aim of this paper is to obtain the necessary and sufficient condition on algebraic numbers $a_1, \ldots, a_r$ for the infinite set $\mathcal{T} := \cup_{i=1}^{r} \mathcal{T}_i$ to be algebraically independent.

Suppose here that a_i and a_j are multiplicatively dependent for some i, j with $1 \le i < j \le r$. Then it is easily seen that the infinite set $\mathcal{T}$ is algebraically dependent. Indeed, assuming $a_1^{d_1} = a_2^{d_2}$ for some positive integers d_1 and d_2, we have $F_{1,d_1}(\alpha) = F_{2,d_2}(\alpha)$ for any nonzero algebraic number α, which implies that the infinite set $\mathcal{T}$ is algebraically dependent. In addition, algebraic relations also appear respectively among the values at several algebraic numbers of $G_i(y)$ $(i = 1, 2)$ and those of $H_i(1, y)$ $(i = 1, 2)$ in this case. To see this, let ζ_i $(i = 1, 2)$ be a primitive d_i-th root of unity. Then we have

$$\prod_{i=0}^{d_1-1} G_1(\zeta_1^i y^{d_2}) = \prod_{k=0}^{\infty}(1 - a_1^{d_1 R_k} y^{d_1 d_2}) = \prod_{k=0}^{\infty}(1 - a_2^{d_2 R_k} y^{d_1 d_2}) = \prod_{j=0}^{d_2-1} G_2(\zeta_2^j y^{d_1}).$$

Taking the logarithmic derivative of this equation and using the relations $H_i(1, y) =$

$-G_i'(y)/G_i(y)$ $(i=1,2)$, we obtain

$$\sum_{i=0}^{d_1-1} d_2\zeta_1^i y^{d_2} H_1(1,\zeta_1^i y^{d_2}) = \sum_{j=0}^{d_2-1} d_1\zeta_2^j y^{d_1} H_2(1,\zeta_2^j y^{d_1}).$$

Hence, if a nonzero algebraic number β satisfies $\zeta_1^i\beta^{d_2}\in\mathcal{B}_1$ $(0\le i\le d_1-1)$ and $\zeta_2^j\beta^{d_1}\in\mathcal{B}_2$ $(0\le j\le d_2-1)$, then the d_1+d_2 elements $G_1(\zeta_1^i\beta^{d_2}), G_2(\zeta_2^j\beta^{d_1})$ $(0\le i\le d_1-1,\ 0\le j\le d_2-1)$ of $\mathcal{T}$ are algebraically dependent and so are the d_1+d_2 elements $H_1(1,\zeta_1^i\beta^{d_2}), H_2(1,\zeta_2^j\beta^{d_1})$ $(0\le i\le d_1-1,\ 0\le j\le d_2-1)$ of $\mathcal{T}$. Therefore the infinite set $\mathcal{T}$ is algebraically independent only if $a_1,\dots,a_r$ are pairwise multiplicatively independent. The main result of this paper is the following

Main Theorem 1.8. *Suppose that* $\{R_k\}_{k\ge0}$ *satisfies the condition* (R). *Then the infinite set*

$$\mathcal{T}=\left\{F_{i,m}^{(l)}(\alpha)\ \middle|\ 1\le i\le r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ l\ge0,\ m\ge1\right\}\cup\{G_i(\beta)\mid 1\le i\le r,\ \beta\in\mathcal{B}_i\}$$
$$\cup\left\{\frac{\partial^{l+m}H_i}{\partial x^l\partial y^m}(\alpha,\beta)\ \middle|\ 1\le i\le r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\mathcal{B}_i,\ l\ge0,\ m\ge0\right\}$$

is algebraically independent if and only if $a_1,\dots,a_r$ *are pairwise multiplicatively independent.*

In Section 3, we prove the *if* part of Main Theorem 1.8, namely the following

Theorem 1.9. *Suppose that* $\{R_k\}_{k\ge0}$ *satisfies the condition* (R). *Assume that* $a_1,\dots,a_r$ *are pairwise multiplicatively independent. Then the infinite set* $\mathcal{T}$ *is algebraically independent.*

Clearly, Theorem 1.9 implies Theorem 1.7. In the previous paper [3, Theorem 1.7], using Theorem 1.7, the author established an algebraic independence result for the values and the partial derivatives of a certain entire function of two variables. Here we shall extend it by applying Theorem 1.9 instead of Theorem 1.7. For each i $(1\le i\le r)$, define

$$\Theta_i(x,y) := G_i(y)H_i(x,y) = \sum_{k=0}^{\infty} a_i^{R_k}x^k \prod_{\substack{k'=0,\\ k'\ne k}}^{\infty}(1-a_i^{R_{k'}}y).$$

Then $\Theta_i(x,y)$ $(1\le i\le r)$ are entire functions on $\mathbb{C}^2$. Let

$$\Xi_i(x,y) := \frac{\partial\Theta_i}{\partial y}(x,y) = G_i(y)\sum_{\substack{k_1,k_2\ge0,\\ k_1\ne k_2}} \frac{-a_i^{R_{k_1}+R_{k_2}}x^{k_1}}{(1-a_i^{R_{k_1}}y)(1-a_i^{R_{k_2}}y)} \quad (1\le i\le r).$$

As a corollary to Theorem 1.12 below, we obtain the following

Theorem 1.10. *Let* $\{R_k\}_{k\ge0}$ *and* $a_1,\dots,a_r$ *be as in* Theorem 1.9. *Assume in addition that* $\{R_k\}_{k\ge0}$ *is strictly increasing. Then the infinite set*

$$\left\{\frac{\partial^{l+m}\Xi_i}{\partial x^l\partial y^m}(\alpha,\beta)\ \middle|\ 1\le i\le r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times},\ l\ge0,\ m\ge0\right\}$$

is algebraically independent.

Using Theorem 1.10, we exhibit a concrete example of an infinite family of entire functions of two variables having the property that the infinite set consisting of their values and their partial derivatives of any order at any distinct algebraic points (α,β) with $\alpha,\beta \neq 0$ is algebraically independent.

Example 1.11. Let $\{F_k\}_{k\geq 0}$ be the Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0).$$

Letting $a_i = 2^{-1}3^{-i}$ and regarding $\{F_{k+2}\}_{k\geq 0}$ as $\{R_k\}_{k\geq 0}$, we define

$$\Xi_i(x,y) = \left(\prod_{k=2}^{\infty}(1-(2\cdot 3^i)^{-F_k}y)\right)\sum_{\substack{k_1,k_2\geq 2,\\ k_1\neq k_2}} \frac{-(2\cdot 3^i)^{-F_{k_1}-F_{k_2}}x^{k_1-2}}{(1-(2\cdot 3^i)^{-F_{k_1}}y)(1-(2\cdot 3^i)^{-F_{k_2}}y)}$$

for any positive integer i. Then by Theorem 1.10 the infinite family $\{\Xi_i(x,y)\}_{i\geq 1}$ has the above-mentioned property, namely the infinite set

$$\left\{\frac{\partial^{l+m}\Xi_i}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\; i\geq 1,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times},\ l\geq 0,\ m\geq 0\right\}$$

is algebraically independent.

For each i $(1\leq i\leq r)$ and for each nonzero algebraic number β, let

$$N_{i,\beta} := \#\{k\geq 0 \mid a_i^{-R_k} = \beta\} = \operatorname*{ord}_{y=\beta} G_i(y).$$

We note that $N_{i,\beta} = 0$ if and only if $\beta\in\mathcal{B}_i$. Applying Theorem 1.9, we prove in Section 3 the following

Theorem 1.12. *Let* $\{R_k\}_{k\geq 0}$ *and* $a_1,\ldots,a_r$ *be as in* Theorem 1.9. *Then the infinite set*

$$\left\{F_{i,m}^{(l)}(\alpha)\;\middle|\;1\leq i\leq r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ l\geq 0,\ m\geq 1\right\}\cup\left\{G_i^{(N_{i,\beta})}(\beta)\;\middle|\;1\leq i\leq r,\ \beta\in\overline{\mathbb{Q}}^{\times}\right\}$$
$$\cup\left\{\frac{\partial^{l+m}\Theta_i}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\;1\leq i\leq r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times},\ l\geq 0,\ m\geq N_{i,\beta}\right\}$$

is algebraically independent.

Theorem 1.10 is an immediate corollary to Theorem 1.12. Indeed, if $\{R_k\}_{k\geq 0}$ is strictly increasing, then $N_{i,\beta}\leq 1$ for any i $(1\leq i\leq r)$ and $\beta\in\overline{\mathbb{Q}}^{\times}$. Thus, if $\{R_k\}_{k\geq 0}$ is strictly increasing, then the infinite set

$$\left\{\frac{\partial^{l+m}\Theta_i}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\;1\leq i\leq r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times},\ l\geq 0,\ m\geq 1\right\}$$
$$=\left\{\frac{\partial^{l+m}\Xi_i}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\;1\leq i\leq r,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times},\ l\geq 0,\ m\geq 0\right\}$$

is a subset of the infinite set treated in Theorem 1.12, and hence it is algebraically independent.

In the rest of this section we introduce another result on the algebraic independence of the values and the partial derivatives of $F_{i,m}(x), G_i(y)$, and $H_i(x,y)$. Assume here that none of a_i/a_j $(1 \leq i < j \leq r)$ is a root of unity. In this case, we cannot deduce the algebraic independency of the infinite set $\mathcal{T}$ given in Main Theorem 1.8 itself since $a_1, \ldots, a_r$ are not always pairwise multiplicatively independent. On the other hand, we see by Theorem 1.5 that, if $1 \in \mathcal{B}_i$ $(1 \leq i \leq r)$, then the $3r$ elements $F_{i,1}(1), G_i(1), H_i(1,1)$ $(1 \leq i \leq r)$ of $\mathcal{T}$ are algebraically independent. Here we can extend Theorem 1.5 to the following Theorem 1.13, whose proof will be provided in Section 3.

Theorem 1.13. *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies the condition* (R). *Assume that none of* a_i/a_j $(1 \leq i < j \leq r)$ *is a root of unity. Let* m_0 *be a positive integer. For each* i $(1 \leq i \leq r)$, *let* β_i *and* β_i' *be any elements of* $\mathcal{B}_i$. *Then the infinite subset*

$$\left\{F_{i,m_0}^{(l)}(\alpha) \;\middle|\; 1 \leq i \leq r,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ l \geq 0\right\} \cup \{G_i(\beta_i) \mid 1 \leq i \leq r\}$$
$$\cup \left\{\frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha, \beta_i') \;\middle|\; 1 \leq i \leq r,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ m \geq 0\right\}$$

of $\mathcal{T}$ *is algebraically independent.*

2. Lemmas

In the next section we prove Theorems 1.9, 1.12, and 1.13. In this section we state only the lemmas that are used in the proofs of Theorems 1.9 and 1.13, since the proof of Theorem 1.12 is based on the previous paper of the author [3, Theorem 1.7]. Let $F(z_1, \ldots, z_n)$ and $F[[z_1, \ldots, z_n]]$ denote the field of rational functions and the ring of formal power series in variables $z_1, \ldots, z_n$ with coefficients in a field F, respectively, and $F^{\times}$ the multiplicative group of nonzero elements of F.

Lemma 2.1 (Nishioka [6]). *Let* L *be a subfield of* $\mathbb{C}$ *and let*

$$f(\boldsymbol{z}) \in \mathbb{C}[[z_1, \ldots, z_n]] \cap L(z_1, \ldots, z_n).$$

Then there exist polynomials $A(\boldsymbol{z}), B(\boldsymbol{z}) \in L[z_1, \ldots, z_n]$ *such that*

$$f(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z}), \quad B(\boldsymbol{0}) \neq 0.$$

Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of Ω is itself an eigenvalue of Ω (cf. Gantmacher [2, p. 66]). We define a multiplicative transformation $\Omega \colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega \boldsymbol{z} := \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}}\right) \tag{2.1}$$

for any $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then the iterates $\Omega^k \boldsymbol{z}$ $(k = 0, 1, 2, \ldots)$ are well-defined. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a point with $\alpha_1, \ldots, \alpha_n$ nonzero algebraic numbers. We assume the following four conditions on Ω and $\boldsymbol{\alpha}$.

(I) Ω is nonsingular and none of its eigenvalues is a root of unity, so that $\rho > 1$.

(II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.

(III) If we put $\Omega^k \boldsymbol{\alpha} =: (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$, then

$$\log |\alpha_i^{(k)}| \leq -c\rho^k \quad (1 \leq i \leq n)$$

for all sufficiently large k, where c is a positive constant.

(IV) For any nonzero $f(\boldsymbol{z}) \in \mathbb{C}[[z_1, \ldots, z_n]]$ which converges in some neighborhood of the origin of $\mathbb{C}^n$, there are infinitely many positive integers k such that $f(\Omega^k \boldsymbol{\alpha}) \neq 0$.

Lemma 2.2 (Nishioka [6])**.** *Let C be a field and Ω an $n \times n$ matrix with nonnegative integer entries satisfying the condition* (I). *Then, if an element $f(\boldsymbol{z})$ of the quotient field of $C[[z_1, \ldots, z_n]]$ satisfies the constant coefficient equation*

$$f(\Omega \boldsymbol{z}) = a f(\boldsymbol{z}) + b \qquad (a, b \in C),$$

then $f(\boldsymbol{z}) \in C$.

Lemma 2.3 (The case of $p = \infty$ in Theorem 4.3 of Ide [3])**.** *Let K be a number field and Ω an $n \times n$ matrix with nonnegative integer entries. Let $f_{ij}(\boldsymbol{z}), g_h(\boldsymbol{z}) \in K[[z_1, \ldots, z_n]]$ $(1 \leq i \leq l,\ 1 \leq j \leq n(i),\ 1 \leq h \leq m)$ with $g_h(\boldsymbol{0}) \neq 0$ $(1 \leq h \leq m)$. Suppose that they converge in an n-polydisc U around the origin of $\mathbb{C}^n$ and satisfy the functional equations*

$$\boldsymbol{f}_i(\boldsymbol{z}) = A_i \boldsymbol{f}_i(\Omega \boldsymbol{z}) + \boldsymbol{b}_i(\boldsymbol{z}) \quad (1 \leq i \leq l)$$

and

$$g_h(\boldsymbol{z}) = e_h(\boldsymbol{z}) g_h(\Omega \boldsymbol{z}) \quad (1 \leq h \leq m),$$

where

$$\boldsymbol{f}_i(\boldsymbol{z}) = {}^t(f_{i1}(\boldsymbol{z}), \ldots, f_{in(i)}(\boldsymbol{z})),$$

$$A_i = \begin{pmatrix} a_i & & & \\ a_{21}^{(i)} & a_i & & \mathbf{0} \\ \vdots & \ddots & \ddots & \\ a_{n(i)1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix}, \quad a_i, a_{st}^{(i)} \in \overline{\mathbb{Q}}, \quad a_i \neq 0, \quad a_{s\,s-1}^{(i)} \neq 0,$$

$$\boldsymbol{b}_i(\boldsymbol{z}) = {}^t(b_{i1}(\boldsymbol{z}), \ldots, b_{in(i)}(\boldsymbol{z})) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n)^{n(i)} \quad (1 \leq i \leq l),$$

and $e_h(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n)$ $(1 \leq h \leq m)$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a point in U whose components are nonzero algebraic numbers. Assume that Ω and $\boldsymbol{\alpha}$ satisfy the conditions (I)–(IV). *Assume further that $b_{ij}(\Omega^k \boldsymbol{\alpha})$ $(1 \leq i \leq l,\ 1 \leq j \leq n(i))$ and $e_h(\Omega^k \boldsymbol{\alpha})$ $(1 \leq h \leq m)$ are defined and $e_h(\Omega^k \boldsymbol{\alpha}) \neq 0$ $(1 \leq h \leq m)$ for all $k \geq 0$. Then, if $f_{ij}(\boldsymbol{\alpha})$ $(1 \leq i \leq l,\ 1 \leq j \leq n(i))$ and $g_h(\boldsymbol{\alpha})$ $(1 \leq h \leq m)$ are algebraically dependent, then at least one of the following two conditions holds*:

(i) *There exist a nonempty subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, l\}$ and nonzero algebraic numbers $c_1, \ldots, c_r$ such that*

$$a_{i_1} = \cdots = a_{i_r}$$

and

$$f(\boldsymbol{z}) := c_1 f_{i_1 1}(\boldsymbol{z}) + \cdots + c_r f_{i_r 1}(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n).$$

Moreover, $f(\boldsymbol{z})$ satisfies the functional equation

$$f(\boldsymbol{z}) = a_{i_1} f(\Omega \boldsymbol{z}) + c_1 b_{i_1 1}(\boldsymbol{z}) + \cdots + c_r b_{i_r 1}(\boldsymbol{z}).$$

(ii) *There exist integers* $d_1, \dots, d_m$, *not all zero, and* $g(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1, \dots, z_n)^{\times}$ *such that*

$$g(\boldsymbol{z}) = \left(\prod_{h=1}^{m} e_h(\boldsymbol{z})^{d_h} \right) g(\Omega \boldsymbol{z}).$$

In the rest of this section and in the next section, let

$$\Omega_1 := \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{2.2}$$

where $c_1, \dots, c_n$ are the coefficients of the polynomial $\Phi(X)$ defined by (1.4).

Lemma 2.4 (Tanaka [8, Lemma 4, Proof of Theorem 2])**.** *Suppose that* $\Phi(\pm 1) \neq 0$ *and the ratio of any pair of distinct roots of* $\Phi(X)$ *is not a root of unity. Let* $\gamma_1, \dots, \gamma_s$ *be multiplicatively independent algebraic numbers with* $0 < |\gamma_j| < 1$ $(1 \leq j \leq s)$. *Let p be a positive integer and put*

$$\Omega_2 := \mathrm{diag}(\underbrace{\Omega_1^p, \dots, \Omega_1^p}_{s}).$$

Then the matrix Ω_2 *and the point*

$$\boldsymbol{\gamma} := (\underbrace{1, \dots, 1}_{n-1}, \gamma_1, \dots, \underbrace{1, \dots, 1}_{n-1}, \gamma_s)$$

satisfy the conditions (I)–(IV).

Let $\{R_k\}_{k \geq 0}$ be a linear recurrence of nonnegative integers satisfying (1.3). We define a monomial

$$P(\boldsymbol{z}) := z_1^{R_{n-1}} \cdots z_n^{R_0}, \tag{2.3}$$

which is denoted similarly to (2.1) by

$$P(\boldsymbol{z}) = (R_{n-1}, \dots, R_0)\boldsymbol{z}. \tag{2.4}$$

It follows from (1.3), (2.1), (2.2), and (2.4) that

$$P(\Omega_1^k \boldsymbol{z}) = z_1^{R_{k+n-1}} \cdots z_n^{R_k} \quad (k \geq 0). \tag{2.5}$$

In what follows, let $\overline{C}$ be an algebraically closed field of characteristic 0.

Lemma 2.5 (Tanaka [9])**.** *Suppose that* $\{R_k\}_{k \geq 0}$ *satisfies the condition* (R) *stated in* Section 1. *Assume that* $f(\boldsymbol{z}) \in \overline{C}[[z_1, \dots, z_n]]$ *satisfies the functional equation of the form*

$$f(\boldsymbol{z}) = \alpha f(\Omega_1^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} Q_k(P(\Omega_1^k \boldsymbol{z})),$$

where $\alpha \neq 0$ *is an element of* $\overline{C}$, $p > 0$, $q \geq 0$ *are integers, and* $Q_k(X) \in \overline{C}(X)$ $(q \leq$

$k \le p+q-1$) *are defined at* $X = 0$. *Then, if* $f(\boldsymbol{z}) \in \overline{\mathbf{C}}(z_1,\dots,z_n)$, *then* $f(\boldsymbol{z}) \in \overline{\mathbf{C}}$ *and* $Q_k(X) = Q_k(0)$ $(q \le k \le p+q-1)$.

Lemma 2.6 (Tanaka [9])**.** *Suppose that* $\{R_k\}_{k\ge 0}$ *satisfies the condition* (R) *stated in* Section 1. *Assume that* $g(\boldsymbol{z})$ *is a nonzero element of the quotient field of* $\overline{\mathbf{C}}[[z_1,\dots,z_n]]$ *satisfying the functional equation of the form*

$$g(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} Q_k(P(\Omega_1^k \boldsymbol{z}))\right) g(\Omega_1^p \boldsymbol{z}),$$

where p, q, *and* $Q_k(X)$ *are as in Lemma 2.5. Assume further that* $Q_k(0) \ne 0$. *Then, if* $g(\boldsymbol{z}) \in \overline{\mathbf{C}}(z_1,\dots,z_n)^\times$, *then* $g(\boldsymbol{z}) \in \overline{\mathbf{C}}^\times$ *and* $Q_k(X) = Q_k(0)$ $(q \le k \le p+q-1)$.

3. Proofs of Theorems 1.9, 1.12, and 1.13

In this section we provide all the proofs announced in the first section.

Proof of Theorem 1.9. Let L be any positive integer and $\alpha_1,\dots,\alpha_L$ any nonzero distinct algebraic numbers. For each i $(1 \le i \le r)$, let $\beta_1^{(i)},\dots,\beta_L^{(i)}$ be any distinct elements of $\mathcal{B}_i$. It is enough to show that the finite set

$$(3.1)\qquad \left\{ F_{i,m+1}^{(l)}(\alpha_\lambda) \;\middle|\; 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L \right\}$$
$$\cup \left\{ G_i(\beta_\mu^{(i)}) \;\middle|\; 1 \le i \le r,\ 1 \le \mu \le L \right\}$$
$$\cup \left\{ \frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha_\lambda, \beta_\mu^{(i)}) \;\middle|\; 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda, \mu \le L \right\}$$

is algebraically independent. There exist multiplicatively independent algebraic numbers $\gamma_1,\dots,\gamma_s$ with $0 < |\gamma_j| < 1$ $(1 \le j \le s)$ such that

$$(3.2)\qquad a_i = \zeta_i \prod_{j=1}^{s} \gamma_j^{d_{ij}} \quad (1 \le i \le r),$$

where ζ_i $(1 \le i \le r)$ are roots of unity and d_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ are nonnegative integers (cf. Loxton and van der Poorten [4], Nishioka [7]). Since $a_1,\dots,a_r$ are pairwise multiplicatively independent, we see that, if $1 \le i < j \le r$, then $(d_{i1},\dots,d_{is})$ and $(d_{j1},\dots,d_{js})$ are non-proportional, namely $(d_{i1} : \cdots : d_{is}) \ne (d_{j1} : \cdots : d_{js})$ in $P^{s-1}(\mathbb{Q})$. Take a positive integer N such that $\zeta_i^N = 1$ for any i $(1 \le i \le r)$. We can choose a positive integer p and a nonnegative integer q such that $R_{k+p} \equiv R_k \pmod N$ for any $k \ge q$. Let $y_{j1},\dots,y_{jn}$ $(1 \le j \le s)$ be variables and let $\boldsymbol{y}_j := (y_{j1},\dots,y_{jn})$ $(1 \le j \le s)$, $\boldsymbol{y} := (\boldsymbol{y}_1,\dots,\boldsymbol{y}_s)$. Put $\beta_0^{(i)} := 0$ $(1 \le i \le r)$. We define

$$f_{im\mu}(x;\boldsymbol{y}) := \sum_{k=q}^{\infty} x^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1}$$
$$(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L)$$

and

$$g_{i\mu}(\boldsymbol{y}) := \prod_{k=q}^{\infty}\left(1-\beta_\mu^{(i)}\zeta_i^{R_k}\prod_{j=1}^{s}P(\Omega_1^k\boldsymbol{y}_j)^{d_{ij}}\right)\quad(1\le i\le r,\ 1\le\mu\le L),$$

where Ω_1 and $P(\boldsymbol{z})$ are given by (2.2) and (2.3), respectively. Moreover, define

$$f_{ilm\lambda\mu}(\boldsymbol{y}) := \frac{\partial^l f_{im\mu}}{\partial x^l}(\alpha_\lambda;\boldsymbol{y})\quad(1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda\le L,\ 0\le\mu\le L).$$

Letting

$$\boldsymbol{\gamma} := (\underbrace{1,\dots,1}_{n-1},\gamma_1,\dots,\underbrace{1,\dots,1}_{n-1},\gamma_s),$$

we see by (2.5) and (3.2) that

$$F_{i,m+1}^{(l)}(\alpha_\lambda)-f_{ilm\lambda 0}(\boldsymbol{\gamma})\in\overline{\mathbb{Q}}\quad(1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda\le L),$$
$$\frac{\partial^{l+m}H_i}{\partial x^l\partial y^m}(\alpha_\lambda,\beta_\mu^{(i)})-m!f_{ilm\lambda\mu}(\boldsymbol{\gamma})\in\overline{\mathbb{Q}}\quad(1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda,\mu\le L),$$

and

$$G_i(\beta_\mu^{(i)})/g_{i\mu}(\boldsymbol{\gamma})\in\overline{\mathbb{Q}}^{\times}\quad(1\le i\le r,\ 1\le\mu\le L).$$

Hence the algebraic independency of the finite set (3.1) is equivalent to that of

(3.3) $$\{f_{ilm\lambda\mu}(\boldsymbol{\gamma})\mid 1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda\le L,\ 0\le\mu\le L\}$$
$$\cup\{g_{i\mu}(\boldsymbol{\gamma})\mid 1\le i\le r,\ 1\le\mu\le L\}.$$

Let

$$\Omega_2 := \mathrm{diag}(\underbrace{\Omega_1^p,\dots,\Omega_1^p}_{s}).$$

Noting that $\Omega_2\boldsymbol{y}=(\Omega_1^p\boldsymbol{y}_1,\dots,\Omega_1^p\boldsymbol{y}_s)$, we have

$$f_{im\mu}(x;\boldsymbol{y}) = x^p f_{im\mu}(x;\Omega_2\boldsymbol{y})+b_{im\mu}(x;\boldsymbol{y})\quad(1\le i\le r,\ 0\le m\le L,\ 0\le\mu\le L),$$

where

$$b_{im\mu}(x;\boldsymbol{y}) := \sum_{k=q}^{p+q-1}x^k\left(\frac{\zeta_i^{R_k}\prod_{j=1}^{s}P(\Omega_1^k\boldsymbol{y}_j)^{d_{ij}}}{1-\beta_\mu^{(i)}\zeta_i^{R_k}\prod_{j=1}^{s}P(\Omega_1^k\boldsymbol{y}_j)^{d_{ij}}}\right)^{m+1}$$
$$(1\le i\le r,\ 0\le m\le L,\ 0\le\mu\le L).$$

Hence, for each i,m,λ,μ $(1\le i\le r,\ 0\le m\le L,\ 1\le\lambda\le L,\ 0\le\mu\le L)$, the functions $f_{ilm\lambda\mu}(\boldsymbol{y})$ $(0\le l\le L)$ satisfy the functional equation

(3.4) $$\boldsymbol{f}_{im\lambda\mu}(\boldsymbol{y}) = A_\lambda\boldsymbol{f}_{im\lambda\mu}(\Omega_2\boldsymbol{y})+\boldsymbol{b}_{im\lambda\mu}(\boldsymbol{y}),$$

where

$$\boldsymbol{f}_{im\lambda\mu}(\boldsymbol{y}) := {}^t(f_{i0m\lambda\mu}(\boldsymbol{y}),f_{i1m\lambda\mu}(\boldsymbol{y}),\dots,f_{iLm\lambda\mu}(\boldsymbol{y})),$$
$$\boldsymbol{b}_{im\lambda\mu}(\boldsymbol{y}) := {}^t\!\left(b_{im\mu}(\alpha_\lambda;\boldsymbol{y}),\frac{\partial b_{im\mu}}{\partial x}(\alpha_\lambda;\boldsymbol{y}),\dots,\frac{\partial^L b_{im\mu}}{\partial x^L}(\alpha_\lambda;\boldsymbol{y})\right),$$

and

$$A_\lambda := \begin{pmatrix} \alpha_\lambda^p & & & & 0 \\ p\alpha_\lambda^{p-1} & \alpha_\lambda^p & & & \\ & 2p\alpha_\lambda^{p-1} & \ddots & & \\ & & \ddots & \ddots & \\ * & & & Lp\alpha_\lambda^{p-1} & \alpha_\lambda^p \end{pmatrix}.$$

Moreover, for each i, μ $(1 \le i \le r,\ 1 \le \mu \le L)$, the function $g_{i\mu}(\boldsymbol{y})$ satisfies the functional equation

$$g_{i\mu}(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1} \left(1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right) \right) g_{i\mu}(\Omega_2 \boldsymbol{y}). \tag{3.5}$$

Now we assume on the contrary that the finite set (3.3) is algebraically dependent. Then by Lemmas 2.3 and 2.4 together with the functional equations (3.4) and (3.5), at least one of the following two cases arises:

(i) There exist a nonempty subset $\{\lambda_1, \dots, \lambda_\nu\}$ of $\{1, \dots, L\}$, algebraic numbers $c_{im\mu\sigma}$ $(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L,\ 1 \le \sigma \le \nu)$, not all zero, and $f(\boldsymbol{y}) \in \overline{\mathbb{Q}}[[\boldsymbol{y}]] \cap \overline{\mathbb{Q}}(\boldsymbol{y})$ such that

$$\alpha_{\lambda_1}^p = \cdots = \alpha_{\lambda_\nu}^p \tag{3.6}$$

and

$$f(\boldsymbol{y}) = \alpha_{\lambda_1}^p f(\Omega_2 \boldsymbol{y}) + \sum_{i,m,\mu,\sigma} c_{im\mu\sigma} b_{im\mu}(\alpha_{\lambda_\sigma}; \boldsymbol{y}).$$

(ii) There exist integers $e_{i\mu}$ $(1 \le i \le r,\ 1 \le \mu \le L)$, not all zero, and $g(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y})^\times$ such that

$$g(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1} \prod_{i,\mu} \left(1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right)^{e_{i\mu}} \right) g(\Omega_2 \boldsymbol{y}). \tag{3.7}$$

Suppose first that the case (i) arises. By (3.6) we have $\nu \le p$ since $\alpha_1, \dots, \alpha_L$ are distinct. Changing the indices λ $(1 \le \lambda \le L)$ if necessary, we may assume that $\lambda_\sigma = \sigma$ $(1 \le \sigma \le \nu)$. Then $f(\boldsymbol{y})$ satisfies the functional equation

$$f(\boldsymbol{y}) = \alpha_1^p f(\Omega_2 \boldsymbol{y}) + \sum_{k=q}^{p+q-1} \sum_{i,m,\mu,\sigma} c_{im\mu\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1}. \tag{3.8}$$

Let M be any positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{M^j}, \dots, z_n^{M^j}) \quad (1 \le j \le s).$$

Note that, by Lemma 2.1, the denominator of

$$f^*(\boldsymbol{z}) := f(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s})$$

does not vanish and so $f^*(\boldsymbol{z}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]] \cap \overline{\mathbb{Q}}(\boldsymbol{z})$. Then the functional equation (3.8) is specialized to

$$f^*(z) = \alpha_1^p f^*(\Omega_1^p z) + \sum_{k=q}^{p+q-1} \sum_{i,m,\mu,\sigma} c_{im\mu\sigma}\alpha_\sigma^k \left(\frac{\zeta_i^{R_k} P(\Omega_1^k z)^{D_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} P(\Omega_1^k z)^{D_i}} \right)^{m+1},$$

where $D_i \coloneqq \sum_{j=1}^s d_{ij} M^j > 0$ $(1 \le i \le r)$. Hence, by Lemma 2.5, we see that

$$\sum_{i,m,\mu,\sigma} c_{im\mu\sigma}\alpha_\sigma^k \left(\frac{\zeta_i^{R_k} X^{D_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} X^{D_i}} \right)^{m+1} = 0 \tag{3.9}$$

for any k $(q \le k \le p+q-1)$. For each k $(q \le k \le p+q-1)$, we define

$$Q_k(\boldsymbol{X}) \coloneqq \sum_{i,m,\mu,\sigma} c_{im\mu\sigma}\alpha_\sigma^k \left(\frac{\zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}} \right)^{m+1} \in \overline{\mathbb{Q}}(X_1, \dots, X_s),$$

where $\boldsymbol{X}^{\boldsymbol{d}_i} \coloneqq X_1^{d_{i1}} \cdots X_s^{d_{is}}$ $(1 \le i \le r)$. Then the left-hand side of (3.9) is equal to $Q_k(X^M, \dots, X^{M^s})$. We assert that $Q_k(\boldsymbol{X}) = 0$ for any k $(q \le k \le p+q-1)$. Indeed, if $Q_{k'}(\boldsymbol{X}) \neq 0$ for some k', then there exist nonzero polynomials $A(\boldsymbol{X}), B(\boldsymbol{X}) \in \overline{\mathbb{Q}}[X_1, \dots, X_s]$ with $B(\boldsymbol{0}) = 1$ such that $Q_{k'}(\boldsymbol{X}) = A(\boldsymbol{X})/B(\boldsymbol{X})$. We take M so large that $M > \max_{1 \le j \le s} \deg_{X_j} A(\boldsymbol{X})$. Then, by the uniqueness of the M-ary expression for nonnegative integers, we see that $A(X^M, \dots, X^{M^s}) \neq 0$. Hence $Q_{k'}(X^M, \dots, X^{M^s}) \neq 0$, which contradicts (3.9). For each i $(1 \le i \le r)$ and k $(q \le k \le p+q-1)$, define

$$Q_{ik}(Y) \coloneqq \sum_{m,\mu,\sigma} c_{im\mu\sigma}\alpha_\sigma^k \left(\frac{Y}{1 - \beta_\mu^{(i)} Y} \right)^{m+1}.$$

Then $Q_{ik}(Y) \in Y\overline{\mathbb{Q}}[[Y]]$ $(1 \le i \le r,\ q \le k \le p+q-1)$ and

$$Q_k(\boldsymbol{X}) = \sum_{i=1}^r Q_{ik}(\zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}) = 0 \quad (q \le k \le p+q-1).$$

Since $\boldsymbol{d}_i = (d_{i1}, \dots, d_{is})$ and $\boldsymbol{d}_j = (d_{j1}, \dots, d_{js})$ are non-proportional for any i, j with $1 \le i < j \le r$, we see that $Q_{ik}(\zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}) = 0$ and hence

$$Q_{ik}(Y) = \sum_{m=0}^L \sum_{\mu=0}^L \left(\sum_{\sigma=1}^\nu c_{im\mu\sigma}\alpha_\sigma^k \right) \left(\frac{Y}{1 - \beta_\mu^{(i)} Y} \right)^{m+1} = 0$$

for any i, k $(1 \le i \le r,\ q \le k \le p+q-1)$. Noting that $\beta_\mu^{(i)}$ $(0 \le \mu \le L)$ are distinct for each i $(1 \le i \le r)$, we obtain

$$\sum_{\sigma=1}^\nu c_{im\mu\sigma}\alpha_\sigma^k = 0 \quad (q \le k \le p+q-1)$$

for any i, m, μ $(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L)$. Since $\nu \le p$ and since α_σ $(1 \le \sigma \le \nu)$ are distinct and nonzero, by the non-vanishing of Vandermonde determinant, we see that $c_{im\mu\sigma} = 0$ for any i, m, μ, σ $(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L,\ 1 \le \sigma \le \nu)$, which is a contradiction.

Suppose next that the case (ii) arises. By Lemma 2.2 and the functional equations (3.5) and (3.7), we see that $g(\boldsymbol{y}) / \prod_{i,\mu} g_{i\mu}(\boldsymbol{y})^{e_{i\mu}} \in \overline{\mathbb{Q}}^\times$. Then $g(\boldsymbol{y}), g(\boldsymbol{y})^{-1} \in \overline{\mathbb{Q}}[[\boldsymbol{y}]]$ and hence, by Lemma 2.1,

$$g^*(\boldsymbol{z}) := g(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \overline{\mathbb{Q}}(\boldsymbol{z})^\times$$

for any positive integer M. Letting $\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{M^j}, \dots, z_n^{M^j})$ $(1 \le j \le s)$ in (3.7), we have

$$g^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \prod_{i,\mu} \left(1 - \beta_\mu^{(i)} \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}\right)^{e_{i\mu}} \right) g^*(\Omega_1^p \boldsymbol{z}),$$

where $D_i > 0$ $(1 \le i \le r)$ are as in the case (i) above. Hence, by Lemma 2.6, we see in particular that

$$\prod_{i,\mu} (1 - \beta_\mu^{(i)} \zeta_i^{R_q} X^{D_i})^{e_{i\mu}} = 1.$$

Taking the logarithmic derivative of this equation and then multiplying both sides by $-X$, we get

$$\sum_{i,\mu} e_{i\mu} \frac{D_i \beta_\mu^{(i)} \zeta_i^{R_q} X^{D_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_q} X^{D_i}} = 0.$$

Let

$$R(\boldsymbol{X}) := \sum_{i,\mu} e_{i\mu} \frac{D_i \beta_\mu^{(i)} \zeta_i^{R_q} \boldsymbol{X}^{\boldsymbol{d}_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_q} \boldsymbol{X}^{\boldsymbol{d}_i}} \in \overline{\mathbb{Q}}(X_1, \dots, X_s).$$

Although D_i $(1 \le i \le r)$ depend on M, the maximum of the partial degrees of the numerator of $R(\boldsymbol{X})$ is bounded by a constant independent of M. Hence, similarly to the case (i), we see that $R(\boldsymbol{X}) = 0$ for any sufficiently large M. Using the fact that $\boldsymbol{d}_i = (d_{i1}, \dots, d_{is})$ and $\boldsymbol{d}_j = (d_{j1}, \dots, d_{js})$ are non-proportional for any i, j with $1 \le i < j \le r$, we obtain

$$\sum_{\mu=1}^{L} e_{i\mu} \frac{D_i \beta_\mu^{(i)} Y}{1 - \beta_\mu^{(i)} Y} = 0$$

for any i $(1 \le i \le r)$. Since $\beta_\mu^{(i)}$ $(1 \le \mu \le L)$ are distinct and nonzero for each i $(1 \le i \le r)$, we see that $e_{i\mu} = 0$ for any i, μ $(1 \le i \le r,\ 1 \le \mu \le L)$, which is a contradiction. This concludes the proof of Theorem 1.9. □

Next, using Theorem 1.9, we prove Theorem 1.12.

Proof of Theorem 1.12. Let L be any positive integer and $\alpha_1, \dots, \alpha_L$ any nonzero distinct algebraic numbers with $\alpha_1 = 1$. Let $\beta_1, \dots, \beta_L$ be any nonzero distinct algebraic numbers. To simplify our notation, we write $N_{i,\mu} := N_{i,\beta_\mu}$ $(1 \le i \le r,\ 1 \le \mu \le L)$. It is enough to show that the finite set

$$\left\{ F_{i,m+1}^{(l)}(\alpha_\lambda) \,\middle|\, 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L \right\} \tag{3.10}$$
$$\cup \left\{ G_i^{(N_{i,\mu})}(\beta_\mu) \,\middle|\, 1 \le i \le r,\ 1 \le \mu \le L \right\}$$
$$\cup \left\{ \frac{\partial^{l+m+N_{i,\mu}} \Theta_i}{\partial x^l \partial y^{m+N_{i,\mu}}}(\alpha_\lambda, \beta_\mu) \,\middle|\, 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda, \mu \le L \right\}$$

is algebraically independent. Since $R_k \to \infty$ as $k \to \infty$ by (1.5), there exists a sufficiently large integer k_0 such that $1 - a_i^{R_k}\beta_\mu \neq 0$ $(1 \le i \le r,\ 1 \le \mu \le L)$ for all $k \ge k_0$. Let $\widetilde{R}_k := R_{k+k_0}$ $(k \ge 0)$. Using the linear recurrence $\{\widetilde{R}_k\}_{\ge 0}$, we define the functions $\widetilde{F}_{i,m+1}(x)$, $\widetilde{G}_i(y)$, $\widetilde{\Theta}_i(x,y)$, and $\widetilde{H}_i(x,y)$ from those without tilde replacing R_k with $\widetilde{R}_k$ for $k \ge 0$. Then by Theorem 2.1 of [3], the algebraic independency of the finite set (3.10) is equivalent to that of the set corresponding to (3.10) having tilde for the functions involved. Moreover, by the equations (3.1) and (3.2) in [3], we see that the algebraic independency of this finite set is equivalent to that of the set corresponding to the set (3.1) having tilde for the functions involved. This concludes the proof since Theorem 1.9 for the linear recurrence $\{\widetilde{R}_k\}_{k\ge 0}$ asserts that the last finite set is algebraically independent. □

Finally, we prove Theorem 1.13, using the method of Tanaka [9, Proof of Theorem 5].

Proof of Theorem 1.13. Let L be any positive integer and $\alpha_1, \dots, \alpha_L$ any nonzero distinct algebraic numbers. It is enough to show that the finite set

$$\text{(3.11)}\qquad \left\{ F_{i,m_0}^{(l)}(\alpha_\lambda) \,\middle|\, 1 \le i \le r,\ 0 \le l \le L,\ 1 \le \lambda \le L \right\} \cup \{G_i(\beta_i) \mid 1 \le i \le r\}$$
$$\cup \left\{ \frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha_\lambda, \beta_i') \,\middle|\, 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L \right\}$$

is algebraically independent. Let ζ_i, γ_j, and d_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ be as in (3.2). Then the s-tuples $(d_{i1}, \dots, d_{is})$ $(1 \le i \le r)$ are distinct since none of a_i/a_j $(1 \le i < j \le r)$ is a root of unity. In what follows, let $N, p, q, P(\boldsymbol{z}), \Omega_1, \Omega_2$, and γ be as in the proof of Theorem 1.9. Define

$$f_i(x; \boldsymbol{y}) := \sum_{k=q}^{\infty} x^k \left(\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right)^{m_0} \quad (1 \le i \le r),$$

$$h_{im}(x; \boldsymbol{y}) := \sum_{k=q}^{\infty} x^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_i' \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1} \quad (1 \le i \le r,\ 0 \le m \le L),$$

and

$$g_i(\boldsymbol{y}) := \prod_{k=q}^{\infty} \left(1 - \beta_i \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right) \quad (1 \le i \le r).$$

Then the algebraic independency of the finite set (3.11) is equivalent to that of

$$\left\{ \frac{\partial^l f_i}{\partial x^l}(\alpha_\lambda; \gamma) \,\middle|\, 1 \le i \le r,\ 0 \le l \le L,\ 1 \le \lambda \le L \right\}$$
$$\cup \left\{ \frac{\partial^l h_{im}}{\partial x^l}(\alpha_\lambda; \gamma) \,\middle|\, 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L \right\} \cup \{g_i(\gamma) \mid 1 \le i \le r\}.$$

Assume on the contrary that this finite set is algebraically dependent. Similarly to the proof of Theorem 1.9, changing the indices λ $(1 \le \lambda \le L)$ if necessary, we see that at least one of the following two cases arises:

(i) There exist a positive integer ν with $\nu \le L$, algebraic numbers $b_{i\sigma}$ $(1 \le i \le r,\ 1 \le \sigma \le \nu)$, $c_{im\sigma}$ $(1 \le i \le r,\ 0 \le m \le L,\ 1 \le \sigma \le \nu)$, not all zero, and $f(\boldsymbol{y}) \in$

$\overline{\mathbb{Q}}[[\boldsymbol{y}]] \cap \overline{\mathbb{Q}}(\boldsymbol{y})$ such that

$$\alpha_1^p = \cdots = \alpha_\nu^p$$

and

$$f(\boldsymbol{y}) = \alpha_1^p f(\Omega_2 \boldsymbol{y}) + \sum_{k=q}^{p+q-1} \sum_{i,\sigma} b_{i\sigma} \alpha_\sigma^k \Big(\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \Big)^{m_0} \tag{3.12}$$
$$+ \sum_{k=q}^{p+q-1} \sum_{i,m,\sigma} c_{im\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_i' \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1}.$$

(ii) There exist integers e_i $(1 \le i \le r)$, not all zero, and $g(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y})^\times$ such that

$$g(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1} \prod_{i=1}^{r} \Big(1 - \beta_i \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \Big)^{e_i} \right) g(\Omega_2 \boldsymbol{y}). \tag{3.13}$$

Let M be a positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{M^j}, \dots, z_n^{M^j}) \quad (1 \le j \le s).$$

Since $(d_{i1}, \dots, d_{is})$ $(1 \le i \le r)$ are distinct, we can take M so large that the following two properties are both satisfied:

(A) $D_i := \sum_{j=1}^{s} d_{ij} M^j$ $(1 \le i \le r)$ are distinct positive integers.
(B) $D_1, \dots, D_r \ge m_0$ and $D_1 \cdots D_r \ge L$.

Then by (3.12), (3.13), and Lemma 2.1, at least one of the following two conditions holds:

(i) $f^*(\boldsymbol{z}) := f(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]] \cap \overline{\mathbb{Q}}(\boldsymbol{z})$ satisfies

$$f^*(\boldsymbol{z}) = \alpha_1^p f^*(\Omega_1^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} \sum_{i,\sigma} b_{i\sigma} \alpha_\sigma^k \Big(\zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i} \Big)^{m_0}$$
$$+ \sum_{k=q}^{p+q-1} \sum_{i,m,\sigma} c_{im\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}}{1 - \beta_i' \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}} \right)^{m+1}.$$

(ii) $g^*(\boldsymbol{z}) := g(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \overline{\mathbb{Q}}(\boldsymbol{z})^\times$ satisfies

$$g^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \prod_{i=1}^{r} \Big(1 - \beta_i \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i} \Big)^{e_i} \right) g^*(\Omega_1^p \boldsymbol{z}).$$

Hence by Lemmas 2.5 and 2.6, at least one of the following two properties is satisfied:

(i) For any k $(q \le k \le p + q - 1)$,

$$\sum_{i=1}^{r} \left(B_i(k) (\zeta_i^{R_k} X^{D_i})^{m_0} + \sum_{m=0}^{L} C_{im}(k) \left(\frac{\zeta_i^{R_k} X^{D_i}}{1 - \beta_i' \zeta_i^{R_k} X^{D_i}} \right)^{m+1} \right) \tag{3.14}$$
$$= \sum_{i=1}^{r} \left(B_i(k) (\zeta_i^{R_k} X^{D_i})^{m_0} + \sum_{m=0}^{L} C_{im}(k) \sum_{h=0}^{\infty} \binom{h+m}{m} \beta_i'^h (\zeta_i^{R_k} X^{D_i})^{h+m+1} \right)$$

$$= \sum_{i=1}^{r}\left(B_i(k)(\zeta_i^{R_k}X^{D_i})^{m_0} + \sum_{h'=0}^{\infty}\left(\sum_{m=0}^{\min\{L,h'\}} C_{im}(k)\binom{h'}{m}\beta_i'^{h'-m}\right)(\zeta_i^{R_k}X^{D_i})^{h'+1}\right)$$
$$= 0,$$

where $B_i(k) := \sum_{\sigma=1}^{\nu} b_{i\sigma}\alpha_\sigma^k$ $(1 \le i \le r)$ and $C_{im}(k) := \sum_{\sigma=1}^{\nu} c_{im\sigma}\alpha_\sigma^k$ $(1 \le i \le r,\ 0 \le m \le L)$.

(ii) For any k $(q \le k \le p+q-1)$,

$$\prod_{i=1}^{r}(1-\beta_i\zeta_i^{R_k}X^{D_i})^{e_i} = 1. \tag{3.15}$$

Suppose first that (i) is satisfied. We show that $C_{im}(k) = 0$ for any i $(1 \le i \le r)$, m $(0 \le m \le L)$, and k $(q \le k \le p+q-1)$. Assume on the contrary that $C_{im}(k')$ $(1 \le i \le r,\ 0 \le m \le L)$ are not all zero for some k'. Let $S := \{i \in \{1,\dots,r\} \mid C_{im}(k')\ (0 \le m \le L)$ are not all zero$\}$ and let $i' \in S$ be the index such that $D_{i'} < D_i$ for any $i \in S\backslash\{i'\}$. Note that $C_{i'm}(k')$ $(0 \le m \le L)$ are determined independently of M. Hence, replacing M if necessary, we may assume that the following property (C) is satisfied in addition to the properties (A) and (B) above.

(C) $\sum_{m=0}^{L} C_{i'm}(k')\binom{D_1\cdots D_r}{m}\beta_{i'}'^{D_1\cdots D_r-m} \neq 0$.

Indeed, let m' be the maximum of $m \in \{0,\dots,L\}$ such that $C_{i'm}(k') \neq 0$. Since

$$\binom{x}{m} = \frac{x^m}{m!} + o(x^m) \qquad (\mathbb{Z} \ni x \to \infty)$$

for each $m \in \{0,\dots,m'\}$, we see that

$$\sum_{m=0}^{L} C_{i'm}(k')\binom{x}{m}\beta_{i'}'^{x-m} = C_{i'm'}(k')\binom{x}{m'}\beta_{i'}'^{x-m'} + \sum_{m=0}^{m'-1} C_{i'm}(k')\binom{x}{m}\beta_{i'}'^{x-m}$$
$$= \frac{C_{i'm'}(k')}{m'!\beta_{i'}'^{m'}}x^{m'}\beta_{i'}'^{x} + o(x^{m'}\beta_{i'}'^{x}) \qquad (\mathbb{Z} \ni x \to \infty).$$

Thus the property (C) is satisfied if M is sufficiently large. Noting the fact that $(D_1\cdots D_r + 1)D_{i'}$ is not divided by any D_i with $i \in S \setminus \{i'\}$, we see by the properties (B) and (C) that the term

$$\left(\sum_{m=0}^{L} C_{i'm}(k')\binom{D_1\cdots D_r}{m}\beta_{i'}'^{D_1\cdots D_r-m}\right)(\zeta_{i'}^{R_{k'}}X^{D_{i'}})^{D_1\cdots D_r+1}$$

does not cancel in (3.14), which is a contradiction. Hence $C_{im}(k) = 0$ $(1 \le i \le r,\ 0 \le m \le L,\ q \le k \le p+q-1)$. Then, since $D_1,\dots,D_r$ are distinct by the property (A), we have $B_i(k) = 0$ $(1 \le i \le r,\ q \le k \le p+q-1)$ by (3.14). Therefore, noting that $\nu \le p$, we see that $b_{i\sigma} = 0$ $(1 \le i \le r,\ 1 \le \sigma \le \nu)$ and $c_{im\sigma} = 0$ $(1 \le i \le r,\ 0 \le m \le L,\ 1 \le \sigma \le \nu)$, which is also a contradiction.

Next suppose that (ii) is satisfied. Taking the logarithmic derivative of (3.15) and then multiplying both sides by $-X$, we see in particular that

$$\sum_{i=1}^{r} e_i\frac{D_i\beta_i\zeta_i^{R_q}X^{D_i}}{1-\beta_i\zeta_i^{R_q}X^{D_i}} = 0.$$

This is a contradiction since $\mathrm{ord}_{X=0} D_i\beta_i\zeta_i^{R_q}X^{D_i}/(1-\beta_i\zeta_i^{R_q}X^{D_i}) = D_i$ $(1 \le i \le r)$, and the theorem is proved. □

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