Barbe, J.
Osaka J. Math.
60 (2023), 777–797

SPECTRAL BOUNDS FOR NON-SMOOTH PERTURBATIONS OF THE LANDAU HAMILTONIAN

Jacques BARBE

(Received April 14, 2022, revised September 1, 2022)

Abstract

In this paper we study spectral estimates for perturbations of the Landau Hamiltonian by a pseudo-differential operator with non-smooth Weyl symbol in a modulation space. We obtain an upper bound for the counting function of the eigenvalues in a spectral gap.

1. Introduction

1.1. Perturbed Landau Hamiltonian. We recall first some classical results. Let H_0 be the self-adjoint realization in $L^2(\mathbf{R}^2)$ of the second order partial differential operator

$$\left(\frac{1}{i}\frac{\partial}{\partial x}+\frac{1}{2}y\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial y}-\frac{1}{2}x\right)^2$$

initially defined on $\mathcal{S}(\mathbf{R}^2)$; this operator is usually called Landau Hamiltonian. The spectrum of H_0 consists of eigenvalues $\Lambda_q := 2q+1$ of infinite multiplicity, the Landau levels (see 4.1).

Let $V = \mathrm{Op}^w(b)$ be a bounded selfadjoint pseudo-differential operator (ΨDO). Under the general assumption that VH_0^{-1} is a compact operator, the Kato-Rellich theorem and the Weyl perturbation theorem imply that the operator $H_0 - V$ is selfadjoint and that

$$\sigma_{ess}(H_0 - V) = \sigma_{ess}(H_0) = \sigma(H_0)\ .$$

Hence the discrete spectrum of $H_0 - V$ consists of eigenvalues of finite multiplicity; these eigenvalues can accumulate only to the Landau levels.

Suppose furthermore that the operator V is non-negative. Fix $q \geq 1$ and let E' be a fixed real number in the gap $(\Lambda_{q-1}, \Lambda_q)$ and E a positive real number such that $\Lambda_{q-1} < E' < \Lambda_q - E < \Lambda_q$. Denote by $N(E', \Lambda_q - E; H_0 - V)$ the number of eigenvalues of the perturbed operator $H_0 - V$ lying in the open interval $(E', \Lambda_q - E)$; for brevity we set

$$N_q(E) := N(E', \Lambda_q - E; H_0 - V). \tag{1.1}$$

Similarly for $q = 0$, we put $N_0(E) = N(-\infty, \Lambda_0 - E; H_0 - V)$. The behaviour of the function N_q has been extensively studied under various assumptions on the perturbation V. The most precise results have been obtained when V is a local smooth potential with derivatives decaying polynomially or exponentially (see [13],[15]). Recently one of the main results proved in [4] concerns the case where the Weyl symbol of V is smooth and belongs to

2020 Mathematics Subject Classification. 47G30, 35P15.

a Hörmander-Shubin class $\Gamma_\rho^m(\mathbf{R}^4)$ of negative order. The authors obtain Weyl's law type asymptotics when $E \downarrow 0^+$.

1.2. Main results. The aim of this paper is to investigate the behaviour of the function N_q defined in (1.1) when the perturbation V is a bounded ΨDO. In particular we are interested in perturbation by an integral operator whose kernel is square-integrable. It is then convenient to consider the scale of Sobolev-Shubin classes $Q^s(\mathbf{R}^4)$ (see 2.5) for $s \geq 0$ as spaces of symbols or kernels. These classes are particular cases of weighted modulation spaces.

Denote by $\mathbb{P}_q$, $q \in \mathbf{Z}_+$, the spectral projection of H_0 corresponding to the eigenvalue Λ_q. A common feature of the papers cited above is the important fact that the spectral properties of the eigenvalues of $H_0 - V$ accumulating near the Landau level Λ_q is governed (in a sense which will be stated more precisely, see section 5) by the compact operator $\mathbb{P}_q V \mathbb{P}_q$; this operator is called the effective Hamiltonian associated with the Landau level Λ_q.

Assume V is a ΨDO with Weyl symbol $b \in L^2(\mathbf{R}^4)$. Evidently $\mathbb{P}_q V \mathbb{P}_q$ is a Hilbert-Schmidt operator. But this property is not sharp. Our first result proves that, under additional assumptions on the symbol of V, the effective Hamiltonian belongs to a Schatten class $\mathbf{S}_p$ with $p \leq 2$. Let us denote by $L^2_s(\mathbf{R}^4)$ the L^2-space with weight $w_s(z, w) = \left(1 + |z|^2 + |w|^2\right)^{s/2}$ and by $\mathcal{F}_2$ the partial Fourier transform on $L^2(\mathbf{R}^{2d})$ (see precise definition in subsection 1.4).

Theorem 1.1. *Let $b \in L^2_s(\mathbf{R}^4)$ ($s \geq 0$). Suppose furthermore that $\mathcal{F}_2 b \in L^2_s(\mathbf{R}^4)$. Then the effective Hamiltonian $\mathbb{P}_q V \mathbb{P}_q$ belongs to the Schatten class $\mathbf{S}_p(L^2(\mathbf{R}^2))$ for $p > 2/(s+1)$. In particular it is a trace-class operator for $s > 1$.*

The second result concerns an upper bound for the counting function N_q defined in (1.1).

Theorem 1.2. *We suppose that b satisfies the same assumptions as in* Theorem 1.1 *and that furthermore $V \geq 0$. Then*

$$N_q(E) = O(E^{-\frac{2}{s+1}}) \quad (E \downarrow 0^+)\,. \tag{1.2}$$

For these two results we use earlier results concerning eigenvalues of ΨDO with symbol in Sobolev-Shubin classes (see [11]).

The paper is organized as follows. Section 2 contains the backgrounds of phase space analysis. We list here some important properties of the Fourier-Wigner and Wigner transforms. Section 3 is devoted to some results of phase space analysis needed in the following section. In Section 4 we apply the results proved in Section 3 to the Landau Hamiltonian and its perturbations. We define in particular the effective Hamiltonian which is the principal object of our work. Section 5 contains the proof of the main theorem on the number of bound states in a gap of the essential spectrum.

1.3. Comments. The technics used in [13] and [4] cannot be used here because there is no symbolic calculus with symbols in Sobolev-Shubin classes. Instead we must use phase-space analysis technics (see Section 3). Similarly the precise asymptotics for ΨDO's with negative order smooth symbols in Hörmander-Shubin classes proved in [5] and used in [13] and [4] cannot be used. Instead we use the results for eigenvalues of ΨDO with symbol in Sobolev-Shubin classes in [11].

Moreover it can be noted that the operator H_0 is unitary equivalent to the operator L selfadjoint realization of the partial differential operator

$$\left(\frac{1}{i}\frac{\partial}{\partial x} - \frac{1}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} + \frac{1}{2}x\right)^2 .$$

This operator is called twisted Laplacian and plays a major role in harmonic analysis on the Heisenberg group. The spectral analysis of the operators L and H_0 is easily performed with the special Hermite functions (see Section 2.4 for the definitions).

1.4. Notations. For $f, g \in L^2(\mathbf{R}^d)$, (f, g) denotes the standard scalar product in $L^2(\mathbf{R}^d)$; it is linear in the first argument.

The standard symplectic form on $\mathbf{R}^{2d}$ is defined by $\sigma(x, \xi; y, \eta) := \xi.y - x.\eta$. Let J be the linear map on $\mathbf{R}^{2d}$ defined by $J(x, \xi) := (\xi, -x)$; J is a symplectic map and $\sigma(x, \xi; y, \eta) = J(x, \xi).(y, \eta)$ where $u.v$ denote the standard scalar product on $\mathbf{R}^d$. For $z = (x, \xi) \in \mathbf{R}^{2d}$ we set $\overline{z} := (x, -\xi)$.

Let us recall some standard notations in spectral theory. For a bounded operator T on a Hilbert space with range $R(T) := T(\mathbf{H})$, rank T is the dimension of $R(T)$, $N(T)$ is the kernel of T, i.e. $\{x \in \mathbf{H} : Tx = 0\}$, $\|T\|$ is the standard norm on $\mathcal{L}(\mathbf{H})$ and T^* is the adjoint of T. Let A be a selfadjoint operator acting in a Hilbert space $\mathbb{H}$. The sets $\rho(A)$, $\sigma(A)$, $\sigma_{ess}(A)$, $\sigma_{disc}(A)$ are respectively the resolvent set, the spectrum, the essential spectrum and the discrete spectrum of A. For $\Omega \in \mathfrak{B}(\mathbf{R})$, the Borel σ-algebra on $\mathbf{R}$, $E_\Omega(A)$ is the spectral projection corresponding to the Borel set Ω. If Ω is a relatively compact Borel subset inside an open gap of the essential spectrum of A, then rank $E_\Omega(A)$ is the (finite) number of eigenvalues, counted with multiplicity, of A lying in Ω. Let T be a compact selfadjoint operator in $\mathbb{H}$, $\lambda_j^+(T)$ be the positive eigenvalues of T, arranged in descending order, counting multiplicity. For $s > 0$ we set

$$n_+(s, T) := \operatorname{card}\{j \in \mathbb{Z}_+ : \lambda_j^+(T) > s\} .$$

For any compact operator T we define the singular values of T by

$$s_j(T) := \lambda_j^+(T^*T)^{1/2}, \ j \geq 1 .$$

For $p \geq 1$ the compact operator T belongs to the Schatten class $\mathbf{S}_p(\mathbf{H})$ if

$$\|T\|_p := \left(\sum_{j=1}^{+\infty} s_j(T)^p\right)^{1/p}$$

is finite.

For f belonging to the Schwartz class $S(\mathbf{R}^d)$, the Fourier transform of f, denoted by $\mathcal{F}f$ or $\widehat{f}$, is defined by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-\frac{d}{2}} \int e^{-i\xi.x} f(x)\, dx .$$

The partial Fourier transforms are defined respectively by

$$\mathcal{F}_1 F(\xi, y) := (2\pi)^{-\frac{d}{2}} \int e^{-i\xi.x} F(x, y) dx$$

and

$$\mathcal{F}_2 F(x, \eta) := (2\pi)^{-\frac{d}{2}} \int e^{-i\eta.y} F(x, y) dy$$

for F in the Schwartz class $S(\mathbf{R}^{2d})$; $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ are the inverse partial Fourier transforms.

For a complex-valued function f defined on $\mathbf{R}^d$, we put $f^*(x) := \overline{f(-x)}$.

2. Fourier-Wigner transform

2.1. Basic properties. Let R be the Schrödinger representation of the Heisenberg group defined by

$$R(z,t) = e^{it}\rho(z)$$

with

$$\rho(z).f(y) := e^{\frac{i}{2}x.\xi}e^{ix.y}f(y+\xi), \qquad z=(x,\xi)\in\mathbf{R}^{2d},\ y\in\mathbf{R}^d,\ f\in L^2(\mathbf{R}^d)\ .$$

The composition law for the projective representation ρ is the following:

$$\rho(z_1)\circ\rho(z_2) = e^{\frac{i}{2}\sigma(z_1,z_2)}\rho(z_1+z_2), \qquad z_1,z_2\in\mathbf{R}^{2d}\ . \tag{2.1}$$

It follows that

$$\rho(z)^* = \rho(z)^{-1} = \rho(-z)\ . \tag{2.2}$$

If $f,g\in L^2(\mathbf{R}^d)$ we define the Fourier-Wigner transform $V(f,g)$ by

$$V(f,g)(z) := (2\pi)^{-\frac{d}{2}}(\rho(z)f,g), \quad z\in\mathbf{R}^{2d}\ .$$

Proposition 2.1. (i) *For all $f,g\in L^2(\mathbf{R}^d)$ we have*

$$V(f,g)(x,\xi) = (2\pi)^{-\frac{d}{2}}\int e^{ix.y}f\left(y+\frac{1}{2}\xi\right)\overline{g\left(y-\frac{1}{2}\xi\right)}dy\ .$$

(ii) *Let U_L be the mixing operator defined on $L^2(\mathbf{R}^{2d})$ by*

$$U_LF(x,\xi) := F\Big(x+\frac{1}{2}\xi, x-\frac{1}{2}\xi\Big)\ .$$

Then, for all $f,g\in L^2(\mathbf{R}^d)$ we have

$$V(f,g) = \overline{\mathcal{F}_1}U_L(f\otimes\bar{g})\ .$$

(iii) *(The Moyal identity) For all $f_1,\ f_2,\ g_1,\ g_2\in L^2(\mathbf{R}^d)$, we have*

$$(V(f_1,g_1),V(f_2,g_2)) = (f_1,f_2)\overline{(g_1,g_2)}\ .$$

(iv)

$$V(g,f) = V(f,g)^* \qquad f,g\in L^2(\mathbf{R}^d)\ .$$

Furthermore if the functions f and g are real-valued, if $z=(x,\xi)\in\mathbf{R}^{2d}$ and $\bar{z}=(x,-\xi)$, then

$$V(g,f)(z) = V(f,g)(\bar{z})\ .$$

Proof. (i) [17] p.10; (ii) follows from (i); (iii) is a consequence of (ii) since U_L and $\mathcal{F}_1$ are unitary operators.

We prove now (iv): by definition and (2.2), we have

$$V(f,g)(-z) = (2\pi)^{-\frac{d}{2}}(\rho(-z)f,g) = (2\pi)^{-\frac{d}{2}}(f,\rho(z)g),$$

hence

$$\overline{V(f,g)(-z)} = (2\pi)^{-\frac{d}{2}}(\rho(z)g,f) = V(g,f)(z),$$

which proves the first part of (iv). Furthermore

$$\overline{V(f,g)(-z)} = (2\pi)^{-\frac{d}{2}}\overline{(\rho(-z)f,g)} = (2\pi)^{-\frac{d}{2}}\int \overline{\rho(-z)f}\; g\,dy\,.$$

A straightforward verification proves that for all $z \in \mathbf{R}^{2d}$ and $f \in L^2(\mathbf{R}^d)$ we have $\overline{\rho(z)f} = \rho(-\bar{z})\bar{f}$. Consequently

$$\overline{V(f,g)(-z)} = (2\pi)^{-\frac{d}{2}}\int \rho(\bar{z})\bar{f}(y)\; g(y)dy\,.$$

If furthermore f and g are real-valued, then

$$\begin{aligned}\overline{V(f,g)(-z)} &= (2\pi)^{-\frac{d}{2}}\int \rho(\bar{z})f(y)\;\bar{g}(y)dy\\ &= (2\pi)^{-\frac{d}{2}}(\rho(\bar{z})f,g))\\ &= V(f,g)(\bar{z})\,.\end{aligned}$$ □

By easy computations taking into account (2.1), we obtain the following

Lemma 2.1. *For all* $f,\ g \in L^2(\mathbf{R}^d)$ *and for all* $z_1,\ z_2 \in \mathbf{R}^{2d}$ *we have*

$$V(\rho(z_1)f,\rho(z_2)g)(z) = e^{\frac{i}{2}\sigma(z_1,z_2)}e^{i\sigma(z,\frac{1}{2}(z_1+z_2))}V(f,g)(z+(z_1-z_2))\,.$$

From this lemma we deduce the following result which is similar to formula (14.34) in [9].

Proposition 2.2.

$$V(V(f,g),V(\phi_1,\phi_2))(z,\zeta) = V(f,\phi_1)\left(Jz+\frac{1}{2}\zeta\right)\overline{V(g,\phi_1)\left(Jz-\frac{1}{2}\zeta\right)}\,.$$

Proof. By definition of the Fourier-Wigner transform we have

$$V(V(f,g),V(\phi_1,\phi_2))(z,\zeta) := (2\pi)^{-d}\,(\rho(z,\zeta).V(f,g),V(\phi_1,\phi_2))_{L^2(\mathbf{R}^{2d})}$$

$$= (2\pi)^{-d}e^{\frac{i}{2}z.\zeta}\int V(f,g)(w+\zeta)\overline{e^{-iz.w}V(\phi_1,\phi_2)(w)}dw\,.$$

As particular cases of Lemma 2.1 we obtain

$$\begin{cases} V(f,g)(w+\zeta) = V\left(\rho\left(\frac{1}{2}\zeta\right)f,\rho\left(-\frac{1}{2}\zeta\right)g\right)(w)\\[2ex] e^{-iz.w}V(\phi_1,\phi_2)(w) = V(\rho(-Jz)\phi_1,\rho(-Jz)\phi_2)(w)\end{cases}$$

By use of the Moyal identity, we deduce

$$V(V(f,g),V(\phi_1,\phi_2))(z,\zeta) = (2\pi)^{-d}e^{\frac{i}{2}z.\zeta}\left(\rho\left(\frac{1}{2}\zeta\right)f,\rho\left(-Jz\right).\phi_1\right)\overline{\left(\rho\left(-\frac{1}{2}\zeta\right)g,\rho(-Jz).\phi_2\right)}$$

$$= (2\pi)^{-d} e^{\frac{i}{2} z.\zeta} \left(\rho(Jz)\rho\left(\frac{1}{2}\zeta\right) f, \phi_1\right) \overline{\left(\rho(Jz)\rho\left(-\frac{1}{2}\zeta\right) g, \phi_2\right)}$$

and by a new application of (2.1) and a simplification of the complex exponents, we obtain the announced equality. □

We define the Wigner transform of $f,\ g \in L^2(\mathbf{R}^d)$ by:

$$W(f,g)(x,\xi) := (2\pi)^{-\frac{d}{2}} \int e^{-i\xi.y} f\left(x + \frac{1}{2}y\right) \overline{g\left(x - \frac{1}{2}y\right)} dy\ .$$

Proposition 2.3. *For all $f,\ g \in L^2(\mathbf{R}^d)$, we have*

$$W(f,g) = \mathcal{F}_2 U_L(f \otimes \bar{g}) = \mathcal{F}(V(f,g))\ .$$

Proof. For the first equality, see [9]. The second equality results of Proposition 2.1 since $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$. □

Proposition 2.4.

$$W(V(f,g), V(\phi_1,\phi_2))(z,\zeta) = W(f,\phi_1)\left(\zeta + \frac{1}{2}Jz\right) \overline{W(g,\phi_2)\left(\zeta - \frac{1}{2}Jz\right)}$$

Proof. By the last proposition we get

$$W(V(f,g), V(\phi_1,\phi_2))(z,\zeta) = (2\pi)^{-2d} \int e^{-i(z,\zeta).(v,w)} V(V(f,g), V(\phi_1,\phi_2))(v,w) dv dw$$

$$= (2\pi)^{-2d} \int e^{-i(z.v+\zeta.w)} V(f,\phi_1)\left(Jv + \frac{1}{2}w\right) \overline{V(g,\phi_1)\left(Jv - \frac{1}{2}w\right)} dv dw\ .$$

We consider the change of variables defined by

$$\begin{cases} Jv + \frac{1}{2}w &= v' \\ Jv - \frac{1}{2}w &= w' \end{cases}$$

Then

$$z.v + \zeta.w = (\zeta + \frac{1}{2}Jz).v' + (-\zeta + \frac{1}{2}Jz).w'.$$

We deduce that

$$W(V(f,g), V(\phi_1,\phi_2))(z,\zeta) =$$

$$(2\pi)^{-d} \int e^{-i(\zeta+\frac{1}{2}Jz).v'} V(f,\phi_1)(v') dv'\ (2\pi)^{-d} \int e^{i(\zeta-\frac{1}{2}Jz).w'} V(g,\phi_2)(w') dw'$$

$$= W(f,\phi_1)\left(\zeta + \frac{1}{2}Jz\right) \overline{W(g,\phi_2)\left(\zeta - \frac{1}{2}Jz\right)}\ .$$ □

2.2. Weyl transform. We summarize the main properties of the Weyl quantization we will use. We omit certain proofs for classical results (see [17]).

Theorem 2.1. *There exists a unique bounded operator* $W : L^2(\mathbf{R}^{2d}) \mapsto \mathcal{L}(L^2(\mathbf{R}^d))$ *with the following properties :*
(i) *Denoting* $W(a) = Op^w(a)$, *then for all* $a \in L^2(\mathbf{R}^{2d})$, *for all* $f, g \in L^2(\mathbf{R}^d)$ *we have*

$$(Op^w(a)f, g) = (2\pi)^{-d} \int \widehat{a}(z)\, V(f, g)(z)dz$$

and

$$\|Op^w(a)\| \le (2\pi)^{-\frac{d}{2}} \|a\| \ .$$

(ii) *Furthermore*

$$(Op^w(a)f, g) = (2\pi)^{-\frac{d}{2}} \int a(z)\, W(f, g)(z)dz = (2\pi)^{-\frac{d}{2}} (a, W(g, f))$$

and

$$(Op^w(\widehat{a})f, g) = (2\pi)^{-\frac{d}{2}} (a, V(g, f))$$

(iii) *For all* $a \in L^2(\mathbf{R}^{2d})$ *the operator* $Op^w(a)$ *is a Hilbert-Schmidt operator. Let k be the kernel of this operator; then*

$$a = (2\pi)^{\frac{d}{2}} \mathcal{F}_2 U_L\, k \ .$$

In particular

$$\|Op^w(a)\|_{S_2} = \|k\|_{L^2} = (2\pi)^{-\frac{d}{2}} \|a\|_{L^2} \ .$$

Remark Let $\phi, \psi \in L^2(\mathbf{R}^d)$ and $f \in L^2(\mathbf{R}^d)$. We put

$$\Pi_{\psi,\phi} f := (f, \phi)\psi \ .$$

Then $\Pi_{\psi,\phi} = S_k = \mathrm{Op}^w(a)$ with $a = W(\psi, \phi)$ and $k = \psi \otimes \bar{\phi}$.

The following proposition will be particularly useful in the next section.

Proposition 2.5. *Let* $a \in L^2(\mathbf{R}^{2d})$ *and* $f, g \in L^2(\mathbf{R}^d)$. *Then*

$$V(Op^w(a)f, g) = (2\pi)^{\frac{d}{2}} Op^w(\widetilde{a}).V(f, g)$$

where $\widetilde{a}(x, \xi) := a(\xi + \frac{1}{2}Jx)$.

Proof. The subspace generated by the Fourier-Wigner transforms $V(\phi, \psi)$ when ϕ, $\psi \in L^2(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^{2d})$. Thus it suffices to prove that for all $\phi, \psi \in L^2(\mathbf{R}^d)$

$$(V(\mathrm{Op}^w(a)f, g), V(\phi, \psi)) = (2\pi)^{\frac{d}{2}} \left(\mathrm{Op}^w(\widetilde{a}).V(f, g), V(\phi, \psi)\right) \ .$$

By Theorem 2.1 (ii), Proposition 2.4 and a linear change of variables, we obtain

$$(\mathrm{Op}^w(\widetilde{a}).V(f, g), V(\phi, \psi)) = (2\pi)^{-d} \left(\widetilde{a}, W(V(\phi, \psi), V(f, g))\right)$$

$$= (2\pi)^{-d} \iint a\Big(\xi + \frac{1}{2}Jx\Big) \overline{W(\phi, f)\Big(\xi + \frac{1}{2}Jx\Big)} W(\psi, g)\Big(\xi - \frac{1}{2}Jx\Big) dx d\xi$$

$$= (2\pi)^{-d} \iint a(u)\overline{W(\phi, f)(u)} W(\psi, g)(v) du dv = (2\pi)^{-d} (a, W(\phi, f)) \int W(\psi, g)(v) dv .$$

But

$$\int W(\psi, g)(v) dv = \int 1(v) W(\psi, g)(v) dv = (\psi, g) = \overline{(g, \psi)} .$$

Therefore by Proposition 2.1(ii) and the Moyal identity we get

$$\begin{aligned} (\mathrm{Op}^w(\widetilde{a}) V(f,g), V(\phi,\psi)) &= (2\pi)^{-d} (a, W(\phi, f)) \overline{(g,\psi)} \\ &= (2\pi)^{-\frac{d}{2}} (\mathrm{Op}^w(a) f, \phi) \overline{(g,\psi)} \\ &= \left((2\pi)^{-\frac{d}{2}} V(\mathrm{Op}^w(a) f, g), V(\phi,\psi)\right) . \end{aligned}$$

□

2.3. Twisted convolution. Let F and G be measurable functions defined on $\mathbf{R}^{2d}$.We define the twisted convolution $F \underset{\sigma}{*} G$ of F and G by

$$(F \underset{\sigma}{*} G)(z) := \int F(z-w) G(w) e^{\frac{i}{2}\sigma(z,w)} dw, \qquad z \in \mathbf{R}^{2d} \tag{2.3}$$

$$= \int F(w) G(z-w) e^{-\frac{i}{2}\sigma(z,w)} dw \tag{2.4}$$

whenever the function of w in the integral is integrable. The notation $F\natural G$ is also used. The following property is classical ([6]).

Proposition 2.6. *If $F, G \in L^2(\mathbf{R}^{2d})$, then so is $F \underset{\sigma}{*} G$ and*

$$\|F \underset{\sigma}{*} G\|_{L^2(\mathbf{R}^{2d})} \le \|F\|_{L^2(\mathbf{R}^{2d})} \, \|G\|_{L^2(\mathbf{R}^{2d})} .$$

Before stating the next proposition we must recall some classical notations. Let $\omega : \mathbf{R}^{2d} \to (0, \infty)$ be a submultiplicative weight defined on $\mathbf{R}^{2d}$, which means that for all $z, w \in \mathbf{R}^{2d}$

$$\omega(z+w) \le \omega(z)\, \omega(w) .$$

Let $s \in \mathbf{R}$; the standard weight ω_s is defined on $\mathbf{R}^{2d}$ by

$$\omega_s(x, \xi) = \omega_s(z) = (1 + |z|)^s$$

is submultiplicative for $s \ge 0$.

A weight v is said to be ω-moderate if for all $z \in \mathbf{R}^{2d}$

$$v(z+w) \le C v(z) \omega(w) .$$

We frequently use the weight v_s defined by

$$v_s(z) := (1 + |z|^2)^{s/2} .$$

Let $1 \le p, q \le +\infty$. The weighted mixed Lebesgue space $L^{p,q}_\omega(\mathbf{R}^{2d})$ is the set of all measurable functions F for which

$$\|F\|_{L^{p,q}_\omega(\mathbf{R}^{2d})} := \left(\int \left(\int |F(x,\xi)|^p \omega(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q} .$$

is finite. We note L^p_ω in the case where $p = q$.

The following property is known ([9]); we recall it for further explicit references.

Proposition 2.7. *Let p satisfying $1 \leq p < +\infty$, v a submultiplicative weight and ω a v-moderate weight. Let $F \in L^p_\omega(\mathbf{R}^{2d})$, $G \in L^1_v(\mathbf{R}^{2d})$ and $z, w \in \mathbf{R}^d$. Denote by T_z the translation operator defined by $T_z F(w) := F(w - z)$ for $F \in L^2(\mathbf{R}^d)$. Then*
(i) *$T_z(F) \in L^p_\omega(\mathbf{R}^{2d})$ and*

$$\|T_z(F)\|_{L^p_\omega(\mathbf{R}^{2d})} \leq Cv(z)\|F\|_{L^p_\omega(\mathbf{R}^{2d})} .$$

(ii) *The function $F * G$ belongs to $L^p_\omega(\mathbf{R}^{2d})$ and*

$$\|F * G\|_{L^p_\omega(\mathbf{R}^{2d})} \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}\|G\|_{L^1_v(\mathbf{R}^{2d})} .$$

(iii) *The same result is true when convolution is replaced by twisted convolution in* (ii)*: the function $F \underset{\sigma}{*} G$ belongs to $L^p_\omega(\mathbf{R}^{2d})$ and*

$$\|F \underset{\sigma}{*} G\|_{L^p_\omega(\mathbf{R}^{2d})} \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}\|G\|_{L^1_v(\mathbf{R}^{2d})} .$$

Proof. (i) Since the weight ω is v-moderate

$$\begin{aligned}\int |T_z(F)|^p \omega(w)^p dw &= \int |F(w - z)|^p \omega(w)^p dw \\ &\leq \int |F(w)|^p \omega(z + w)^p dw \\ &\leq C^p v(z)^p \int |F(w)|^p \omega(w)^p dw .\end{aligned}$$

(ii) Since $L^p_\omega(\mathbf{R}^{2d})$ is invariant by translation, we can define a vector-valued map $\phi : \mathbf{R}^{2d} \mapsto L^p_\omega(\mathbf{R}^{2d})$ by setting

$$\phi(w) := T_w(F)G(w) .$$

An application of (i) yields to the following inequality

$$\|\phi(w)\|_{L^p_\omega(\mathbf{R}^{2d})} \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}|G(w)|v(w) .$$

Hence the function ϕ is Bochner integrable and

$$\Big\| \int \phi(w)dw \Big\|_{L^p_\omega(\mathbf{R}^{2d})} \leq \int \|\phi(w)\|_{L^p_\omega(\mathbf{R}^{2d})} dw \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}\|G\|_{L^1_v(\mathbf{R}^{2d})} .$$

Since the convolution can be rewrited as the Bochner integral

$$F * G = \int \phi(w)dw$$

we obtain (ii).

(iii) For all z, $w \in \mathbf{R}^{2d}$ we have

$$|(F \underset{\sigma}{*} G)(z)| \leq (|F| * |G|)(z) .$$

Again with (ii) we obtain that $F \underset{\sigma}{*} G \in L^p_\omega(\mathbf{R}^{2d})$ and

$$\|F \underset{\sigma}{*} G\|_{L^p_\omega(\mathbf{R}^{2d})} \leq \||F| * |G|\|_{L^p_\omega(\mathbf{R}^{2d})}$$

which proves (iii). □

By use of the twisted convolution, we get a result similar to Proposition 2.5.

Proposition 2.8. *Let $a \in L^2(\mathbf{R}^d)$ and $f, g \in L^2(\mathbf{R}^d)$. Then*

$$V(Op^w(\widehat{a})f, g) = (2\pi)^{-d}\; a \underset{\sigma}{*} V(f, g)\,.$$

Proof. By Proposition 2.1 (ii), we have

$$\begin{aligned} V(\mathrm{Op}^w(\widehat{a})f, g)(z) &= (2\pi)^{-\frac{d}{2}}(\rho(z)\mathrm{Op}^w(\widehat{a})f, g) \\ &= (2\pi)^{-\frac{d}{2}}(\mathrm{Op}^w(\widehat{a})f, \rho(-z)g) \\ &= (2\pi)^{-d}(a, V(\rho(-z)g, f))\,. \end{aligned}$$

But by Lemma 2.1

$$V(\rho(-z)g, f)(w) = e^{-\frac{i}{2}\sigma(w,z)}V(g, f)(w - z) = e^{\frac{i}{2}\sigma(z,w)}\overline{V(f, g)(z - w)},$$

therefore

$$V(\mathrm{Op}^w(\widehat{a})f, g) = (2\pi)^{-d}\int a(w)e^{-\frac{i}{2}\sigma(z,w)}V(f, g)(z - w)dw = (2\pi)^{-d}a \underset{\sigma}{*} V(f, g)\,.$$ □

2.4. Special Hermite functions. Let e_n be the Hermite function of order n defined by

$$e_n(x) = \pi^{-\frac{1}{4}}2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}H_n(x)e^{-\frac{x^2}{2}}$$

where H_n is the Hermite polynomial of order n.

We define the special Hermite functions by setting

$$e_{i,j} := V(e_i, e_j)$$

for $i,\ j \in \mathbf{Z}_+$.

Proposition 2.9. (i) *The system $(e_{i,j})_{i,j}$ is an orthonormal basis of $L^2(\mathbf{R}^2)$.*
(ii) *Let be $q \in \mathbb{N}$. Then we have*

$$e_{q,q}(z) = (2\pi)^{-\frac{1}{2}}L_q\left(\frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2}$$

where L_q is the Laguerre polynomial of degree q and order 0.
(iii) *For all $z \in \mathbf{R}^2$ and $j, k \in \mathbf{Z}_+$*

$$e_{k,j}(z) = e_{j,k}(\bar{z})\,.$$

Proof. For (i) and (ii), see [17] and (iii) results of Proposition 2.1 (iv). □

2.5. Modulation spaces. As explained in the introduction, we will be concerned with pseudo-differential operators with symbols in modulation spaces rather than in Hörmander-Shubin classes. We state the definitions and results we need in the following.

First at all, we recall the definition of the Short Time Fourier Transform (STFT). Let $g \in S(\mathbf{R}^d) \setminus \{0\}$. For $f \in L^2(\mathbf{R}^d)$ we define the STFT $V_g f$ by

$$V_g f(x,\xi) := (2\pi)^{-d/2} \int f(t)\overline{g(t-x)} e^{-it.\xi} dt \ .$$

The modulation space $M^{p,q}_\omega(\mathbf{R}^d)$ is the set of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ for which the short-time Fourier transform $V_g f \in L^{p,q}_\omega(\mathbf{R}^{2d})$. This space equipped with the norm

$$\|f\|_{M^{p,q}_\omega(\mathbf{R}^d)} := \left(\int \left(\int |V_g f(x,\xi)|^p \omega(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q}$$

is a Banach space. We note L^p_ω, resp. M^p_ω, in the case where $p = q$. The definition of $M^{p,q}_\omega(\mathbf{R}^d)$ is independent of the choice of the window $g \in S(\mathbf{R}^d) \setminus \{0\}$ and the norms corresponding to different choices of the window are equivalent. In particular we will use the following spaces

$$L^{2,2}_{v_s}(\mathbf{R}^{2d}),\ M^{2,2}_{v_s}(\mathbf{R}^d),\ L^{1,1}_{\omega_s}(\mathbf{R}^{2d})$$

abbreviated respectively by L^2_s, M^2_s and L^1_s (the weights v_s and ω_s are defined in section 2.3).

Lemma 2.2. *$f \in M^2_s$ if and only if $V(f,g) \in L^2_s$ and for all $f \in M^2_s$ we have*

$$\|f\|_{M^2_s} = \|V(f,g)\|_{L^2_s} \ .$$

Proof. The Fourier-Wigner $V(f,g)$ and the STFT $V_g f$ are connected by the relation

$$V(f,g)(x,\xi) = e^{-ix.\xi} V_g f(\xi,-x) \ .$$

Furthermore $v_s(\xi,-x) = v_s(x,\xi)$; then by using the (symplectic) change of variables $(x,\xi) \mapsto (\xi,-x)$ we obtain

$$\|f\|^2_{M^2_s} = \|V_g f\|^2_{L^2_s} = \iint |V_g f(\xi,-x)|^2 v_s(\xi,-x)^2 dx d\xi$$

$$= \iint |V_g f(x,\xi)|^2 v_s(x,\xi)^2 dx d\xi = \|V(f,g)\|^2_{L^2_s} \ . \qquad \square$$

It happens that the modulation space $M^2_s(\mathbf{R}^d)$ coincide with the Sobolev-Shubin space $Q^s(\mathbf{R}^d)$ initially defined in [16] as a Sobolev space corresponding to pseudo-differential calculus with symbols in Hörmander-Shubin classes (see the remark below). This space is defined for $s \geq 0$ by

$$Q^s(\mathbf{R}^d) := \{ f \in L^2(\mathbf{R}^d) : \mathrm{Op}^{aw}_{\psi_0}(v_s) f \in L^2(\mathbf{R}^d) \}$$

where $\mathrm{Op}^{aw}_{\psi_0}(v_s)$ is the Anti-Wick operator with symbol v_s and Gaussian window defined by

$$\psi_0(x) := \pi^{-d/4} \exp\Big(-\frac{|x|^2}{2} \Big) \ .$$

Lemma 2.3 (2, lemma 2.3)**.** *For all $s \geq 0$ $M^2_s(\mathbf{R}^d)$ coincide with $Q^s(\mathbf{R}^d)$ and the norms are equivalent.*

Combining the two preceding lemma, we obtain immediatly the following result.

Proposition 2.10. *Let $f \in L^2(\mathbf{R}^d)$ and $s \geq 0$. Then $f \in Q^s(\mathbf{R}^d)$ if and only if $V(f,g) \in L^2_s(\mathbf{R}^{2d})$. Furthermore, there exists $C_s > 0$ such that for all $f \in Q^s(\mathbf{R}^d)$*

$$C_s^{-1}\|V(f,g)\|_{L_s^2} \le \|f\|_{Q^s(\mathbf{R}^d)} \le C_s\|V(f,g)\|_{L_s^2} \ .$$

Remark For $s = m \in \mathbb{Z}_+$, the Sobolev-Shubin space is defined by

$$Q^m(\mathbf{R}^d) = \{u \in \mathcal{S}'(\mathbf{R}^d);\ x^\alpha D^\beta u \in L^2(\mathbf{R}^d),\ |\alpha| + |\beta| \le m\} \ .$$

Another explicit characterization of $M_s^2(\mathbf{R}^d)$ is (see [9], [11])

$$M_s^2(\mathbf{R}^d) = L_s^2(\mathbf{R}^d) \cap H^s(\mathbf{R}^d) \ .$$

3. Some results of phase-space analysis

Let U be an isometry from a Hilbert space $\mathbb{H}_1$ into an other $\mathbb{H}_2$; denote by G its range $G := U(\mathbb{H}_1)$. It is well known that G is a closed subspace of $\mathbb{H}_2$ and that, if P is the orthogonal projection on G, then $U^*U = I_{\mathbb{H}_1}$ and $UU^* = P$.

Let $g \in S(\mathbf{R}^d)$ satisfying $\|g\|_2 = 1$. For $f \in L^2(\mathbf{R}^d)$ define $\mathcal{V}_g(f) := V(f,g)$.

Proposition 3.1. *The map $\mathcal{V}_g$ is an isometry from $L^2(\mathbf{R}^d)$ on a closed subspace of $L^2(\mathbf{R}^{2d})$ and if P_g is the orthogonal projection on this closed subspace, then $\mathcal{V}_g^*\mathcal{V}_g = I_{L^2(\mathbf{R}^d)}$ and $\mathcal{V}_g\mathcal{V}_g^* = P_g$.*

Proof. $\mathcal{V}_g$ is an isometry by the Moyal identity (Proposition 2.1 (iii)) and the assumption on g. We apply then the result above to the isometry $U = \mathcal{V}_g$. □

The following results are inspired by [7] and [8] Chapter 18; the isometry $\mathcal{V}_g$ plays the role of the windowed wavepacket transform W_ϕ in [8] p. 299.

Proposition 3.2. *Let be $F \in L^2(\mathbf{R}^{2d})$ and $g \in L^2(\mathbf{R}^d)$. Then*

$$\mathcal{V}_g^*(F) = (2\pi)^{\frac{d}{2}} Op^w(\mathcal{F}F).g \ .$$

Proof. Let $h \in L^2(\mathbf{R}^d)$. By (ii) of Theorem 2.1 and Proposition 2.3 we get

$$(\mathrm{Op}^w(\mathcal{F}F).g, h) = (2\pi)^{-\frac{d}{2}}(\mathcal{F}F, W(h,g)) = (2\pi)^{-\frac{d}{2}}(F, V(h,g)) \ .$$

On the other hand

$$(\mathcal{V}_g^*(F), h) = (F, \mathcal{V}_g(h)) = (F, V(h,g)) \ . \qquad \square$$

Corollary 3.1. *Under the same assumption on F and g, we have $\mathcal{V}_g^*(F) = S_k\, g$ where S_k is the Hilbert-Schmidt operator with kernel*

$$k(x,y) := (2\pi)^{-\frac{d}{2}}\mathcal{F}_1 F\left(\frac{1}{2}(x+y), x-y\right) \ .$$

Proof. We apply Theorem 2.1 (iii) :

$$\mathrm{Op}^w(\mathcal{F}F) = S_k$$

with

$$k = (2\pi)^{-\frac{d}{2}} U_L^{-1}\overline{\mathcal{F}_2}(\mathcal{F}F) = (2\pi)^{-\frac{d}{2}} U_L^{-1}(\mathcal{F}_1(F))$$

and

$$U_L^{-1}(\mathcal{F}_1(F))(x,y) = \mathcal{F}_1 F\left(\frac{1}{2}(x+y), x-y\right) . \qquad \square$$

We deduce in particular

Corollary 3.2. *Under the same assumptions as in* Corollary 3.1 *we have*

$$(2\pi)^{-\frac{d}{2}} \int F\left(\frac{1}{2}(x+y), x-y\right) g(y)dy = \mathcal{V}_g^*(\overline{\mathcal{F}_1} F)(x) \quad (a.e.\ in\ \mathbf{R}^d) .$$

Proposition 3.3. *Let be $F \in L_s^2(\mathbf{R}^{2d})$ and $g \in \mathcal{S}(\mathbf{R}^d)$. Then $\mathcal{V}_g^*(F)$ belongs to $Q^s(\mathbf{R}^d)$ and there is $C_s > 0$ such that*

$$\|\mathcal{V}_g^*(F)\|_{Q^s(\mathbf{R}^d)} \le (2\pi)^{-\frac{d}{2}} C_s \|V(g,g)\|_{L_s^1} \|F\|_{L_s^2} .$$

Proof. Put $f := \mathcal{V}_g^*(F) \in L^2(\mathbf{R}^d)$. Applying Proposition 3.2 and Proposition 2.8, we obtain

$$\begin{aligned} \mathcal{V}_g(f) &= \mathcal{V}_g(\mathcal{V}_g^*(F)) = (2\pi)^{\frac{d}{2}} \mathcal{V}_g(\mathrm{Op}^w(\mathcal{F}F).g) = (2\pi)^{\frac{d}{2}} V(\mathrm{Op}^w(\mathcal{F}F).g, g) \\ &= (2\pi)^{-\frac{d}{2}} F \underset{\sigma}{*} V(g,g) . \end{aligned}$$

And now we conclude from Proposition 2.7 (iii) that $\mathcal{V}_g(f) \in L_s^2(\mathbf{R}^{2d})$ and

$$\|\mathcal{V}_g(f)\|_{L_s^2} \le (2\pi)^{-\frac{d}{2}} \|V(g,g)\|_{L_s^1} \|F\|_{L_s^2},$$

or equivalently by Proposition 2.10 that $f \in Q^s(\mathbf{R}^d)$ and

$$\|f\|_{Q^s(\mathbf{R}^d)} \le C_s \|\mathcal{V}_g(f)\|_{L_s^2} \le (2\pi)^{-\frac{d}{2}} C_s \|V(g,g)\|_{L_s^1} \|F\|_{L_s^2} . \qquad \square$$

The next result is a consequence of Proposition 2.5; we use the same notations.

Proposition 3.4. *We suppose that the symbol a is real-valued and therefore that the operator $Op^w(a)$ is self-adjoint. The operators $Op^w(a)$ and $Op^w(\widetilde{a})$ have the same eigenvalues. If f is an eigenfunction of $Op^w(a)$ corresponding to the eigenvalue λ, then for all $g \in S(\mathbf{R}^d)$ the function $F := \mathcal{V}_g(f)$ is an eigenfunction of $Op^w(\widetilde{a})$ corresponding to the same eigenvalue.*

Proof. By Proposition 2.5 we have

$$\mathcal{V}_g \circ \mathrm{Op}^w(a) = \mathrm{Op}^w(\widetilde{a}) \circ \mathcal{V}_g . \tag{3.1}$$

Let λ be an eigenvalue of $\mathrm{Op}^w(a)$, let f be in $L^2(\mathbf{R}) \setminus \{0\}$ such that $\mathrm{Op}^w(a)f = \lambda f$ and $F := \mathcal{V}_g(f)$. We have $\mathrm{Op}^w(\widetilde{a}).F = \lambda F$, and since $\|F\| = \|f\| > 0$ we deduce that λ is an eigenvalue of $\mathrm{Op}^w(\widetilde{a})$ and F is an eigenfunction of $\mathrm{Op}^w(\widetilde{a})$.

Let now λ be an eigenvalue of $\mathrm{Op}^w(\widetilde{a}))$ and let $F \neq 0 \in L^2(\mathbf{R}^d)$ be an eigenfunction. Taking adjoint operators in (3.1), it results that $\mathcal{V}_g^* \circ \mathrm{Op}^w(\widetilde{a}) = \mathrm{Op}^w(a) \circ \mathcal{V}_g^*$. Since $\mathrm{Op}^w(\widetilde{a})F = \lambda F$, we get that $\mathrm{Op}^w(a)\mathcal{V}_g^* F = \lambda \mathcal{V}_g^* F$ for all $g \in L^2(\mathbf{R}^d)$. Suppose that $\mathcal{V}_g^* F = 0$ for all $g \in L^2(\mathbf{R}^d)$; then $F \in \mathrm{N}(\mathcal{V}_g^*) = \mathrm{R}(\mathcal{V}_g)^\perp$, hence $(F, V(f,g)) = 0$ for all $f, g \in L^2(\mathbf{R}^d)$; in particular $(F, V(e_j, e_k)) = (F, e_{j,k}) = 0$. Since $(e_{j,k})$ is an orthonormal basis of $L^2(\mathbf{R}^d)$, we deduce that $F = 0$ which is not possible. Therefore there exists $g \in L^2(\mathbf{R}^d)$ such that $\mathcal{V}_g^*(F)$ is not the null function and this proves that λ is an eigenvalue of $\mathrm{Op}^w(a)$ and that F is a corresponding eigenfunction. $\square$

4. Pseudo-differential perturbation: the Hilbert-Schmidt case

We will now apply the results proved in the previous section to the Landau Hamiltonian perturbed by a Hilbert-Schmidt ΨDO operator.

4.1. The Landau Hamiltonian. We suppose $d = 1$ and $a(q,p) = q^2 + p^2$ for $(q,p) \in \mathbf{R}^2$. The DO $\mathrm{Op}^w(a)$, initially defined on $S(\mathbf{R})$, is essentially self-adjoint and its unique realization as unbounded operator on $L^2(\mathbf{R})$ is the harmonic oscillator h_0. It is well known that $\sigma(h_0) = \{2q+1; q \in \mathbb{N}\}$, $h_0(e_q) = (2q+1)e_q$ and $\mathrm{Ker}(h_0 - (2q+1)I) = \mathrm{Vect}\,(e_q)$, where e_q is the Hermite function of order q.

With the notations of section 3, the symbol $\widetilde{a}$ associated to a is defined by $\widetilde{a}(x,\xi) = a(\xi + \frac{1}{2}Jx)$; more precisely here

$$\widetilde{a}(x_1,x_2,\xi_1,\xi_2) = a\left(\xi_1 + \frac{1}{2}x_2,\, \xi_2 - \frac{1}{2}x_1\right) = \left(\xi_1 + \frac{1}{2}x_2\right)^2 + \left(\xi_2 - \frac{1}{2}x_1\right)^2 .$$

Therefore

$$\mathrm{Op}^w(\widetilde{a}) = \left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{1}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{1}{2}x\right)^2 .$$

We deduce from Proposition (3.4) that

$$\sigma(H_0) = \sigma(h_0) = \{2q+1; q \in \mathbb{N}\} .$$

Let E_q^0 be the eigenspace of H_0 corresponding to the eigenvalue $\Lambda_q = 2q+1$:

$$E_q^0 := \mathrm{Ker}(H_0 - \Lambda_q I) .$$

We know again from Proposition 3.4 that, since e_q is an eigenfunction of h_0 with respect to the eigenvalue Λ_q, then $\mathcal{V}_{e_j}(e_q) = e_{q,j}$ is an eigenfunction of H_0 for the same eigenvalue:

$$H_0\, e_{q,j} = (2q+1)e_{q,j}, \qquad j = 0,1,\ldots .$$

Proposition 4.1. *Let $q \in \mathbf{Z}_+$ be fixed and let $\mathrm{Vect}(\{e_{q,j}; j \in \mathbf{Z}_+\})$ be the subspace of $L^2(\mathbf{R}^2)$ spanned by the special Hermite functions $e_{q,j}$ for $j \in \mathbf{Z}_+$. Then*

$$E_q^0 = \overline{\mathrm{Vect}(\{e_{q,j}; j \in \mathbf{Z}_+\})} .$$

Proof. It is sufficient to prove the inclusion $E_q^0 \subseteq \overline{\mathrm{Vect}(\{e_{q,j}; j \in \mathbb{N}\})}$. Let be $F \in E_q^0$; we have $\mathcal{V}_g^* \circ H_0 = h_0 \circ \mathcal{V}_g^*$ and $H_0 F = (2q+1)F$, from which we deduce $(2q+1)\mathcal{V}_g^* F = h_0 \mathcal{V}_g^* F$, thereby $\mathcal{V}_g^* F \in \mathrm{Ker}(h_0 - (2q+1)I) = \mathrm{Vect}\,(e_q)$. In particular $\mathcal{V}_{e_j}^* F \in \mathrm{Vect}\,(e_q)$. But for all $k \in \mathbb{N}$

$$(\mathcal{V}_{e_j}^* F, e_k) = (F, V(e_k, e_j)) = (F, e_{k,j}) .$$

Since $\mathcal{V}_{e_j}^*(F) \in \mathrm{Vect}\,(e_q)$, we deduce that $(F, e_{k,j}) = 0$ for $k \neq q$ and then

$$F = \sum_k \sum_j (F, e_{k,j}) e_{k,j} = \sum_j (F, e_{k,j}) e_{k,j}$$

which proves that $F \in \mathrm{Vect}(\{e_{q,j}; j \in \mathbb{N}\})$. □

Next we wish to express E_q^0 with $\mathcal{V}_{e_q}$ and factorise $\mathbb{P}_q$ the orthogonal projection on E_q^0.

Recall that $R(\mathcal{V}_{e_q}) = \text{Vect}(\{e_{j,q}; j \in \mathbb{N}\}) := E_q$. Let be U the unitary involutive operator defined on $L^2(\mathbf{R}^2)$ by

$$U\,F(z) := F(\bar{z}), \qquad F \in L^2(\mathbf{R}^2),\ z \in \mathbf{R}^2$$

or equivalently $U\,F(x_1, x_2) = F(x_1, -x_2)$.

By Proposition 2.1, for all $j, k \in \mathbf{Z}_+$ we have $e_{j,k}(\bar{z}) = e_{k,j}(z)$ or $U(e_{j,q}) = e_{q,j}$. We deduce that $U(E_q) = E_q^0$. Let P_q be the orthogonal projection on E_q; we have $P_q = \mathcal{V}_{e_q}\mathcal{V}_{e_q}^*$; therefore $UP_qU^* = \mathbb{P}_q$. Define now

$$\mathbb{V}_{e_q} := U\,\mathcal{V}_{e_q}\ . \tag{4.1}$$

Then

$$\mathbb{P}_q = \mathbb{V}_{e_q}\mathbb{V}_{e_q}^*\ . \tag{4.2}$$

4.2. Hilbert-Schmidt perturbation. Let be $b \in L^2(\mathbf{R}^4)$ a real-valued symbol and $V := \text{Op}^w(b)$ the ΨDO with Weyl symbol b. Since b is square-integrable, V is an Hilbert-Schmidt operator. Hence the operators VH_0^{-1} and

$$T_q := \mathbb{P}_q\,V\,\mathbb{P}_q$$

are also Hilbert-Schmidt operators on $L^2(\mathbf{R}^2)$. The last operator is the effective Hamiltonian corresponding to the perturbed operator $H_0 - V$ and to the Landau level Λ_q as we will see in the next section.

Our first goal is to show that the operator T_q has the same spectrum as a ΨDO operator S_q on $L^2(\mathbf{R})$ easier to study. We can first give an abstract result.

Proposition 4.2. *Let $U : \mathbb{H}_1 \mapsto \mathbb{H}_2$ be an isometry, $S = S^* \in \mathcal{L}(\mathbb{H}_1)$ and $T = T^* \in \mathcal{L}(\mathbb{H}_2)$. We suppose that*

$$T := U\ S\ U^*\ .$$

Then the operators S and T have the same non-zero eigenvalues; more precisely for all $\lambda \neq 0$ and for all $u \in \mathbb{H}_1$, u is an eigenvector of S corresponding to the eigenvalue λ iff Uu is an eigenvector of T corresponding to the same eigenvalue.

Proof. Let be $u \in \mathbb{H}_1 \setminus \{0\}$ and $\lambda \neq 0$ such that $Su = \lambda u$. We set $v := Uu$:

$$Tv = (U\ S\ U^*)Uu = USu = \lambda Uu = \lambda v$$

and since v and u have the same norm, then v is non null, and λ is an eigenvalue and v an eigenvecteur of T.

Conversely, suppose $Tv = \lambda v$ with λ and v non null and let be $u := U^*v$. Then, composing on the left hand side by U^*, we obtain that $Su = \lambda u$. Furthermore, if $u = 0$, then $Tv = (US)(u) = 0$ and since $Tv = \lambda v$, we deduce $\lambda v = 0$, hence $v = 0$ since $\lambda \neq 0$, which contradicts the assumption $v \neq 0$. □

We will apply this result to our problem by considering the operator

$$S_q := \mathbb{V}_{e_q}^*\,V\,\mathbb{V}_{e_q}\ .$$

The operator T_q defined above can be rewrited since

$$T_q = \mathbb{P}_q V \mathbb{P}_q = \mathbb{V}_{e_q}\mathbb{V}_{e_q}^* V\, \mathbb{V}_{e_q}\mathbb{V}_{e_q}^* = \mathbb{V}_{e_q} S_q \mathbb{V}_{e_q}^* \,.$$

Furthermore $\mathbb{V}_{e_q} = U\mathcal{V}_{e_q}$ is also an isometry. Applying the latest proposition, we obtain that the operators S_q and T_q have the same non-zero eigenvalues.

We will now prove that the operator S_q is a ΨDO and determine its Weyl-symbol. By definition of S_q and by Theorem 2.1 (ii) we have

$$(S_q f, g) = \frac{1}{2\pi}\iint b(x,y;\xi,\eta) W(V(f,e_q), V(g,e_q))(x,-y;\xi,-\eta)\,dxdyd\xi d\eta \,.$$

By Proposition (2.4) we get

$$W(V(f,e_q),V(g,e_q))(x,-y,\xi,-\eta) = W(f,g)\left(\xi - \frac{1}{2}y, -\eta - \frac{1}{2}x\right) W(e_q,e_q)\left(\xi + \frac{1}{2}y, -\eta + \frac{1}{2}x\right) \,.$$

Making use of a change of variables, we deduce

$$(S_q f, g) =$$

$$\frac{1}{2\pi}\iint \Big[\iint b(\xi-\eta, x-y, \frac{1}{2}(x+y), -\frac{1}{2}(\xi+\eta)) W(e_q,e_q)(x,\xi)\,dxd\xi\Big] W(f,g)(y,\eta)\,dyd\eta$$

therefore S_q is a ΨDO with Weyl symbol γ_q defined by

$$\gamma_q(y,\eta) = (2\pi)^{-\frac{1}{2}} \iint b\left(\xi-\eta, x-y, \frac{1}{2}(x+y), -\frac{1}{2}(\xi+\eta)\right) W(e_q,e_q)(x,\xi)\,dxd\xi. \tag{4.3}$$

Let us define $F \in L^2(\mathbf{R}^4)$ by

$$F(u_1,u_2,v_1,v_2) := b(-v_2,-v_1,u_1,-u_2) \tag{4.4}$$

We put $w = (x,\xi)$ et $z = (y,\eta)$. Then we have

$$F\left(\frac{1}{2}(z+w), z-w\right) = F\left(\frac{1}{2}(x+y), \frac{1}{2}(\xi+\eta), -(x-y), -(\xi-\eta)\right)$$

$$= b\left(\xi-\eta, x-y, \frac{1}{2}(x+y), -\frac{1}{2}(\xi+\eta)\right),$$

hence by (4.3)

$$\gamma_q(z) = (2\pi)^{-\frac{1}{2}} \iint F\left(\frac{1}{2}(z+w), z-w\right) W(e_q,e_q)(w)\,dw$$

and by Corollary 3.2

$$\gamma_q = \mathcal{V}^*_{W(e_q,e_q)}(\overline{\mathcal{F}_1} F) \,.$$

If we set $\Lambda(u_1,u_2,v_1,v_2) := (-v_2,-v_1,-u_1,u_2)$, we obtain easily

$$\gamma_q = \mathcal{V}^*_{W(e_q,e_q)}(\mathcal{F}_2 b \circ \Lambda) \,.$$

Suppose now that $\mathcal{F}_2 b \in L^2_s(\mathbf{R}^4)$. The function space $L^2_s(\mathbf{R}^4)$ is invariant by linear change of variables; thus $\mathcal{F}_2 b \circ \Lambda$ is also in $L^2_s(\mathbf{R}^4)$. According to Proposition 3.3 we deduce that

$$\gamma_q = \mathcal{V}^*_{W(e_q,e_q)}(\mathcal{F}_2 b \circ \Lambda) \tag{4.5}$$

belongs to $Q^s(\mathbf{R}^2)$. We have therefore prove the following result.

Proposition 4.3. *Let $b \in L^2_s(\mathbf{R}^4)$. Suppose that $\mathcal{F}_2 b$ belongs to $L^2_s(\mathbf{R}^4)$. Then $S_q = Op^w(\gamma_q)$ with γ_q, defined by* (4.5)*, belonging to $Q^s(\mathbf{R}^2)$.*

We can now achieve the proof of Theorem 1.1. We recall first the following result about Schatten class properties for ΨDO with symbols in Shubin-Sobolev classes obtained by C. Heil in [11]. Let $s \geq 0$ be given and let $a \in L^2(\mathbf{R}^2)$. Define the operator $L := \mathrm{Op}^w(a)$. By Theorem 2.1 we know that L is a Hilbert-Schmidt operator; let $s_j(L)$ be the singular values of L, arranged in descending order, counting multiplicity.

Proposition 4.4. *If the symbol a lies in $Q^s(\mathbf{R}^2)$, then*

$$s_j(L) = O\left(j^{-\frac{s+1}{2}}\right) .$$

Consequently $L \in S_p(L^2(\mathbf{R}))$ for $p > 2/(s+1)$. In particular L is trace-class if $s > 1$.

From this result we deduce that, with the same hypothesis as in Proposition 4.3, the singular values of $S_q = \mathrm{Op}(\gamma_q)$ verify

$$s_j(S_q) = O\left(j^{-\frac{s+1}{2}}\right) .$$

Therefore we obtain the same estimates for the operator T_q since the two operators S_q and T_q have the same non-zero eigenvalues by Proposition 4.2.

5. Proof of Theorem 1.2

5.1. Reduction to an effective Hamiltonian. The first aim of this section is to prove that the operator $T_q = \mathbb{P}_q V \mathbb{P}_q$ is the effective Hamiltonian for estimating the number of eigenvalues of the perturbed Landau Hamiltonian near the Landau level Λ_q. We follow [15] but some modifications must be precised.

We recall the classical Weyl inequality ([2], chap. 9).

Lemma 5.1. *Let T_1 and T_2 be linear self-adjoint compact operators in a Hilbert space. Then for each $s_1 > 0$ and $s_2 > 0$*

$$n_+(s_1 + s_2, T_1 + T_2) \leq n_+(s_1, T_1) + n_+(s_2, T_2)$$

holds true.

Let H_0 and V be as in Section 1 and let Λ_q be a fixed Landau level, $\mathbb{P}_q$ be the corresponding spectral projection and $\mathbb{Q}_q := I - \mathbb{P}_q$. For $\lambda \in \rho(H_0)$ we set

$$T(\lambda) := V^{1/2}(H_0 - \lambda)^{-1} V^{1/2} .$$

This operator is selfadjoint and compact.

Proposition 5.1. *Assume that the interval $[\lambda_1, \lambda_2]$, $\lambda_1 < \lambda_2$ belongs to the gap $(\Lambda_{q-1}, \Lambda_q)$, then*

$$\mathrm{rank}\ E_{[\lambda_1, \lambda_2)}(H_0 - V) = n_+(1, T(\lambda_2)) - n_+(1, T(\lambda_1)) .$$

For the proof of this result, we refer to [14, Sections 1 and 3], and to the earlier article [1, Proposition 1.6].

Lemma 5.2. *Let E', E be positive real numbers satisfying $\Lambda_{q-1} < E' < 2q$ and $0 < E < 1$. Then*

$$N(E', \Lambda_q - E;\ H_0 - V) = n_+\left(1, V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1}V^{\frac{1}{2}}\right) + O(1), \quad E \downarrow 0^+ .$$

Proof. With these assumptions, the interval $[E', \Lambda_q - E]$ is included in the gap $(\Lambda_{q-1}, \Lambda_q)$. Therefore we can apply Proposition 5.1 :

$$\operatorname{rank} E_{[E',\Lambda_q - E)}(H_0 - V) = n_+(1, T(\Lambda_q - E)) - n_+(1, T(E'))$$

or equivalently with the notations of section 1 :

$$N(E', \Lambda_q - E; H_0 - V) = n_+(1, T(\Lambda_q - E)) - n_+(1, T(E')) - \dim[\operatorname{Ker}(H_0 - V - E')] .$$

But the last two terms in the right-hand side are independent of E. □

For brevity, we set

$$T_q(E) := T(\Lambda_q - E) = V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1}V^{\frac{1}{2}} .$$

We then write $T_q(E) = T_{1,q}(E) + T_{q,2}(E)$ with

$$\begin{cases} T_{1,q}(E) & := & V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1}\mathbb{P}_q V^{\frac{1}{2}} \\ T_{2,q}(E) & := & V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1}\mathbb{Q}_q V^{\frac{1}{2}} . \end{cases}$$

First we remark that

$$(H_0 - \Lambda_q + E)^{-1}\mathbb{P}_q = \sum_{l=0}^{+\infty}(\Lambda_l - \Lambda_q + E)^{-1}\mathbb{P}_l\mathbb{P}_q = E^{-1}\mathbb{P}_q,$$

and so

$$T_{1,q}(E) = E^{-1}V^{\frac{1}{2}}\mathbb{P}_q V^{\frac{1}{2}} .$$

The operator $T_{1,q}(E)$ is compact, selfadjoint and positive. The operator $T_{2,q}(E)$ can be rewritten as

$$T_{2,q}(E) = \sum_{l\neq q}(\Lambda_l - \Lambda_q + E)^{-1}V^{\frac{1}{2}}P_l V^{\frac{1}{2}} .$$

Proposition 5.2. *For all $s > 0$ we have*

$$n_+(s, T_{2,q}(E)) \leq 4\Lambda_q^2 s^{-2}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2}^2 .$$

Proof. The operator $T_{2,q}(E)$ is compact, selfadjoint but not positive. We are led to define

$$\begin{cases} T_{2,q}^+(E) & := & \sum_{l>q}(\Lambda_l - \Lambda_q + E)^{-1}V^{\frac{1}{2}}\mathbb{P}_l V^{\frac{1}{2}} \\ T_{2,q}^-(E) & := & -\sum_{l<q}(\Lambda_l - \Lambda_q + E)^{-1}V^{\frac{1}{2}}\mathbb{P}_l V^{\frac{1}{2}} . \end{cases}$$

Since $0 < E < 1$, we have $\Lambda_l - \Lambda_q + E < -1$ if $l < q$, and $\Lambda_l - \Lambda_q + E > 2$ if $l > q$. Consequently the operators $T_{2,q}^+(E)$ and $T_{2,q}^-(E)$ are selfadjoint and positive and

$$T_{2,q}(E) = T_{2,q}^+(E) - T_{2,q}^-(E) .$$

By straightforward inequalities we get

$$0 < (\Lambda_l - \Lambda_q + E)^{-1} \leq \Lambda_{q+1}\Lambda_l^{-1} \quad , \; l > q \; .$$

Thereby it follows that

$$\begin{aligned}(T^+_{2,q}(E)u,u) &\leq \Lambda_{q+1}\sum_{l>q}\Lambda_l^{-1}\left(V^{\frac{1}{2}}P_lV^{\frac{1}{2}}u,u\right) \leq \Lambda_{q+1}\left((V^{\frac{1}{2}}(\sum_{l\neq q}\Lambda_l^{-1}P_l)V^{\frac{1}{2}})u,u\right)\\ &\leq \Lambda_{q+1}\left((V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}})u,u\right) .\end{aligned}$$

Similarly we have

$$(T^-_{2,q}(E)u,u) \leq \Lambda_{q-1}\left((V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}})u,u\right) .$$

By assumption the operator V is a Hilbert-Schmidt operator. Since the operators $V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}$ and $H_0^{-\frac{1}{2}}VH_0^{-\frac{1}{2}}$ have the same non-zero eigenvalues, we deduce that $V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. As a consequence of the preceding inequalities, we obtain that $T^+_{2,q}(E)$ and $T^-_{2,q}(E)$ are Hilbert-Schmidt operators and

$$\left\{\begin{aligned}\|T^+_{2,q}(E)\|_{\mathbf{S}_2} &\leq \Lambda_{q+1}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2}\\ \|T^-_{2,q}(E)\|_{\mathbf{S}_2} &\leq \Lambda_{q-1}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2} .\end{aligned}\right.$$

Since $T_{2,q}(E) = T^+_{2,q}(E) - T^-_{2,q}(E)$, it follows that $T_{2,q}(E) \in \mathbf{S}_2$ and

$$\|T_{2,q}(E)\|_{\mathbf{S}_2} \leq (\Lambda_{q-1} + \Lambda_{q+1})\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2} \leq 2\Lambda_q\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2},$$

and

$$n_+(s, T_{2,q}(E)) \leq 4\Lambda_q^2 s^{-2}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|^2_{\mathbf{S}_2} . \qquad \square$$

5.2. End of the proof of Theorem 1.2. For $0 < \varepsilon < 1$ we deduce from the Weyl inequality that

$$n_+(1, T_q(E)) \leq n_+(1-\varepsilon, T_{1,q}(E)) + n_+(\varepsilon, T_{2,q}(E)) .$$

For the first term of the right-hand side, we have

$$n_+(1-\varepsilon, T_{1,q}(E)) = n_+(1-\varepsilon, E^{-1}V^{\frac{1}{2}}\mathbb{P}_qV^{\frac{1}{2}}) = n_+((1-\varepsilon)E, \mathbb{P}_qV\mathbb{P}_q) .$$

But by Proposition 4.4

$$\lambda_j(\mathbb{P}_qV\mathbb{P}_q) = O\left(j^{-\frac{s+1}{2}}\right)$$

or equivalently there is $C_q > 0$ independent of ε and E such that

$$n_+((1-\varepsilon)E, \mathbb{P}_qV\mathbb{P}_q) \leq C_q(1-\varepsilon)^{-\frac{2}{s+1}}E^{-\frac{2}{s+1}} .$$

For the second term of the right-hand side, we have by Proposition 5.2

$$n_+(\varepsilon, T_{2,q}(E)) \leq 4\Lambda_q^2\varepsilon^{-2}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|^2_{\mathbf{S}_2} .$$

Finally we deduce from the preceding inequalities that

$$E^{\frac{2}{s+1}} n_+(1, T_q(E)) \leq C_q(1-\varepsilon)^{-\frac{2}{s+1}} + 4\Lambda_q^2 \varepsilon^{-2} E^{\frac{2}{s+1}} \|V^{\frac{1}{2}} H_0^{-1} V^{\frac{1}{2}}\|_{\mathbf{S}_2}^2 .$$

Consequently

$$\limsup_{E \to 0^+} E^{\frac{2}{s+1}} n_+(1, T_q(E)) \leq C_q(1-\varepsilon)^{-\frac{2}{s+1}},$$

and letting ε tend to 0^+ we deduce

$$n_+(1, T_q(E)) \leq C_q E^{-\frac{2}{s+1}}$$

or equivalently

$$N(E', \Lambda_q - E;\, H_0 - V) \leq C_q E^{-\frac{2}{s+1}} .$$

Acknowledgements. The author is thankful to the referee for the numerous corrections and helpful suggestions given for the improvement of the present paper.

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8 allée des Favrières
44240, La Chapelle-sur-Erdre
France
e-mail:
js.barbe2@wanadoo.fr