ON THE VALUES OF UNIPOTENT CHARACTERS OF FINITE CHEVALLEY GROUPS OF TYPE E_7 IN CHARACTERISTIC 2

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Abstract

Let G be a finite group of Lie type. In order to determine the character table of G, Lusztig developed the theory of character sheaves. In this framework, one has to find the transformation between two bases for the space of class functions on G, one of them being the irreducible characters of G, the other one consisting of characteristic functions associated to character sheaves. In principle, this has been achieved by Lusztig and Shoji, but the underlying process involves some scalars which are still unknown in many cases. The problem of specifying these scalars can be reduced to considering cuspidal character sheaves. We will deal with the latter for the specific case where $G = E_7(q)$, and q is a power of the bad prime $p = 2$ for E_7.

1. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure $k = \overline{\mathbb{F}}_p$ of the field with p elements (for a prime p). Assume that $\mathbf{G}$ is defined over the finite subfield $\mathbb{F}_q$ of k, where q is a power of p, so the $\mathbb{F}_q$-rational points on $\mathbf{G}$ constitute the corresponding finite group of Lie type $\mathbf{G}(q) = \mathbf{G}(\mathbb{F}_q)$. The theory of character sheaves, due to Lusztig [22]-[26], provides a general procedure by which the problem of determining the character table of $\mathbf{G}(q)$ can be tackled. Character sheaves are certain geometric analogues to the irreducible representations of $\mathbf{G}(q)$, and they give rise to class functions on $\mathbf{G}(q)$, called characteristic functions. Lusztig ([20, 13.7], [26]) conjectured that any such characteristic function co-incides up to multiplication by a non-zero scalar with an appropriate almost character of $\mathbf{G}(q)$, that is, an explicitly known linear combination of the irreducible characters. Then he provides an algorithm to compute the characteristic functions, at least in principle ([26, §24]). While Lusztig's conjecture is still open in general (see, e.g., [33]), it has been proven by Shoji ([40], [41]) under the assumption that the centre of $\mathbf{G}$ is connected. Even in this case, however, the scalars relating characteristic functions of character sheaves and almost characters need to be specified. Following [27, §3], this task can be reduced to considering cuspidal character sheaves, by means of an induction process for these objects which leads to a parametrisation of the character sheaves on $\mathbf{G}$, rather analogous to Harish-Chandra theory for irreducible characters of $\mathbf{G}(q)$.

It therefore seems reasonable to consider the individual quasisimple groups of Lie type separately. In those cases, [21] combined with [22]-[26], see also [40, §7], [41, §5], provides a classification of cuspidal character sheaves. We will be concerned with unipotent character sheaves on $\mathbf{G}$, that is, the subset of the character sheaves on $\mathbf{G}$ corresponding to the unipotent

characters of $\mathbf{G}(q)$ due to Deligne-Lusztig [5].

For classical groups of split type, Shoji [42], [45] has shown that the characteristic functions of cuspidal unipotent character sheaves coincide with the corresponding almost characters. As far as exceptional groups are concerned, this statement is only partially verified. E.g., for groups of type F_4, it has been proven by Marcelo-Shinoda [36] for those cuspidal character sheaves whose support contains unipotent elements, in all characteristics. (In type F_4, any characteristic, every cuspidal character sheaf is unipotent.) The particular case F_4, $p = 2$, is in fact complete, as the respective identities for not unipotently supported cuspidal character sheaves have been verified in [11, §5].

Moreover, by a result of Geck [11, Proposition 3.4], the computation of the scalars relating characteristic functions of unipotent character sheaves and unipotent almost characters can be reduced to the base case $p = q$. This is particularly useful in small characteristics, as one might hope to settle such cases via direct computations, e.g. by applying computer algebra methods. Several concrete examples are considered in [11], such as the above-mentioned F_4, $p = 2$, as well as the split adjoint group of type E_6 in characteristic 2, where the problem is solved for all cuspidal character sheaves (there are two of them, and both are unipotent). Note that, in either of these cases, the character table of $\mathbf{G}(2)$ is known. However, it is also conjectured in [11, 6.6] that an analogous statement will hold for the adjoint groups of type E_6 in good characteristic, even though the character table of $\mathbf{G}(p)$ is not known here. Finally, [16] deals with E_6, $p = 3$ (both split and non-split).

In this paper, we shall investigate the case where $\mathbf{G}$ is a simple Chevalley group of type E_7 over $k = \overline{\mathbb{F}}_2$, defined over $\mathbb{F}_q$, q a power of 2. There are two cuspidal character sheaves on $\mathbf{G}$, and both of them are unipotent [25, Proposition 20.3]. After singling out a specific normalisation of the characteristic functions of these character sheaves, we show that they coincide with the corresponding almost characters of $\mathbf{G}(q)$. In order to achieve this, we will, as in [16], consider the Hecke algebra associated to $\mathbf{G}(q)$ and apply a formula in [10] which gives a relation between the characters of this Hecke algebra and the principal series unipotent characters of $\mathbf{G}(q)$. However, we will need to take into account some more theoretical machinery than in [16]. This identification of the characteristic functions of cuspidal character sheaves with the respective almost characters immediately determines the values at the regular unipotent elements in $\mathbf{G}(q)$ of the two cuspidal unipotent characters and of the two principal series unipotent characters corresponding to the two irreducible characters of the Weyl group of degree 512. More information on character values at unipotent elements is provided by Shoji's and Lusztig's algorithms [39, §5], [26, §24] for the computation of Green functions (as defined in [5]). While these algorithms in general involve some unknown scalars, they have been determined for our case recently in [12].

NOTATION. Whenever $\mathbf{G}$ is a connected reductive algebraic group over $k = \overline{\mathbb{F}}_p$, we assume to have fixed a Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$ as well as a maximal torus $\mathbf{T}$ of $\mathbf{G}$ which is contained in $\mathbf{B}$, and we denote by $\mathbf{W} := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$. Let $\mathbf{U} := R_u(\mathbf{B})$ be the unipotent radical of $\mathbf{B}$. Then $\mathbf{B}$ is the semidirect product of $\mathbf{U}$ and $\mathbf{T}$ (with $\mathbf{U}$ being normal in $\mathbf{B}$).

If, in addition, $\mathbf{G}$ is defined over a finite subfield $\mathbb{F}_q \subseteq k$ (where q is a power of p) with Frobenius map $F \colon \mathbf{G} \to \mathbf{G}$, we write $\mathbf{G}(q) = \mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$ for the corresponding finite group of Lie type. In this case, unless otherwise stated, we will also tacitly assume that

both $\mathbf{B}$ and $\mathbf{T}$ are F-stable. Thus F induces an automorphism of $\mathbf{W}$ which we shall denote again by F.

We will be concerned with characters of $\mathbf{G}^F$ in characteristic 0. As usual in the ordinary representation theory of finite groups of Lie type, we consider representations and characters over $\overline{\mathbb{Q}}_\ell$, an algebraic closure of the field of ℓ-adic numbers, for a fixed prime ℓ different from p. Thus given a finite group Γ, let $\mathrm{CF}(\Gamma)$ be the set of class functions $\Gamma \to \overline{\mathbb{Q}}_\ell$ and denote by

$$\langle f, f'\rangle_\Gamma := |\Gamma|^{-1} \sum_{g\in\Gamma} f(g)\overline{f'(g)} \quad \text{(for } f, f' \in \mathrm{CF}(\Gamma))$$

the standard scalar product on $\mathrm{CF}(\Gamma)$, where bar denotes a field automorphism of $\overline{\mathbb{Q}}_\ell$ which maps roots of unity to their inverses. Let $\mathrm{Irr}(\Gamma) \subseteq \mathrm{CF}(\Gamma)$ be the subset of irreducible characters of Γ. They form an orthonormal basis of $\mathrm{CF}(\Gamma)$ with respect to this scalar product. If $\Gamma = \mathbf{G}^F$, we denote by $\mathrm{Uch}(\mathbf{G}^F) \subseteq \mathrm{Irr}(\mathbf{G}^F)$ the subset of unipotent characters, that is, those $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ which satisfy $\langle \rho, R_w\rangle \neq 0$ for some $w \in \mathbf{W}$. Here, R_w is the virtual character defined by Deligne and Lusztig in [5].

2. Character sheaves

Let $\mathbf{G}$ be a connected reductive algebraic group over $k = \overline{\mathbb{F}}_p$. We begin by very briefly introducing some notions of the theory of character sheaves (only those which are relevant for our purposes), following [22]-[26], [28], [31].

2.1. Denote by $\mathscr{D}\mathbf{G}$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbf{G}$, and by $\mathscr{M}\mathbf{G}$ the full subcategory of perverse sheaves on $\mathbf{G}$ in the sense of [1]. $\mathscr{M}\mathbf{G}$ is an abelian category. The character sheaves on $\mathbf{G}$ are defined ([22, 2.10]) as certain simple objects of $\mathscr{M}\mathbf{G}$ which are equivariant for the conjugation action of $\mathbf{G}$ on itself. We fix a set of representatives $\hat{\mathbf{G}}$ for the isomorphism classes of character sheaves on $\mathbf{G}$.

An important subset of the character sheaves on $\mathbf{G}$ are the so-called unipotent character sheaves, defined as follows. For $w \in \mathbf{W}$, let $K_w := K_w^{\mathscr{L}_0} \in \mathscr{D}\mathbf{G}$ be as defined in [22, 2.4] with respect to the constant local system $\mathscr{L}_0 = \overline{\mathbb{Q}}_\ell$ on $\mathbf{T}$. An element of $\hat{\mathbf{G}}$ is called a unipotent character sheaf if it is a constituent of a perverse cohomology sheaf ${}^pH^i(K_w)$ for some $i \in \mathbb{Z}$, $w \in \mathbf{W}$. We denote by

$$\hat{\mathbf{G}}^{\mathrm{un}} := \{A \in \hat{\mathbf{G}} \mid A \text{ unipotent}\} \subseteq \hat{\mathbf{G}}$$

the unipotent character sheaves in $\hat{\mathbf{G}}$. We also set (see [24, 14.10])

$$R_\phi^{\mathscr{L}_0} := \frac{1}{|\mathbf{W}|} \sum_{w\in\mathbf{W}} \phi(w) \sum_{i\in\mathbb{Z}} (-1)^{i+\dim\mathbf{G}} \, {}^pH^i(K_w) \quad \text{for } \phi \in \mathrm{Irr}(\mathbf{W}),$$

an element of the subgroup of the Grothendieck group of $\mathscr{M}\mathbf{G}$ spanned by the character sheaves. Denote by $(:)$ the symmetric $\overline{\mathbb{Q}}_\ell$-bilinear pairing on this subgroup such that for any two character sheaves A, A' on $\mathbf{G}$ we have

$$(A : A') = \begin{cases} 1 & \text{if} \quad A \cong A', \\ 0 & \text{if} \quad A \not\cong A'. \end{cases}$$

Finally, there is the notion of cuspidal character sheaves on a connected reductive group,

see [22, 3.10]. By [22, Proposition 3.12], any cuspidal character sheaf on **G** has the form
$\mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]$, where "IC" stands for the intersection cohomology complex due to
Deligne-Goresky-MacPherson ([15], [1]), Σ is the inverse image of a conjugacy class un-
der the natural map $\mathbf{G} \to \mathbf{G}/\mathbf{Z}(\mathbf{G})^\circ$, and $\mathscr{E}$ is a **G**-equivariant irreducible $\overline{\mathbb{Q}}_\ell$-local system on
Σ, such that $(\Sigma, \mathscr{E})$ is a cuspidal pair for **G** in the sense of [21]. We set

$$\hat{\mathbf{G}}^\circ := \{A \in \hat{\mathbf{G}} \mid A \text{ cuspidal character sheaf}\} \quad \text{and} \quad \hat{\mathbf{G}}^{\circ,\mathrm{un}} := \hat{\mathbf{G}}^{\mathrm{un}} \cap \hat{\mathbf{G}}^\circ.$$

This gives rise to an inductive description of (unipotent) character sheaves, as follows. Let
$\mathbf{L} \subseteq \mathbf{G}$ be a Levi complement of some parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$. To each complex
$K \in \mathscr{M}\mathbf{L}$ which is equivariant for the conjugation action of **L** on itself is associated an
induced complex $\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(K) \in \mathscr{D}\mathbf{G}$, see [22, 4.1]. If $A_0 \in \hat{\mathbf{L}}^\circ$, the complex $\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0)$
is a semisimple perverse sheaf on **G**, all its simple direct summands are character sheaves
on **G**, and any character sheaf on **G** is a simple direct summand of $\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0)$ for some
$\mathbf{L} \subseteq \mathbf{P}$ as above, and some $A_0 \in \hat{\mathbf{L}}^\circ$. (In fact, for the latter statement it is enough to consider
standard Levi subgroups **L** of standard parabolic subgroups **P**. We will use this in 3.5 below.)
Moreover, A_0 is unipotent if and only if some (or equivalently, every) simple summand of
$\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0)$ is unipotent ([22, 4.4, 4.8]). In this way, the character sheaves on **G** correspond to
irreducible modules for the endomorphism algebras (in $\mathscr{M}\mathbf{G}$) of various induced complexes
as above, using the respective Hom functor (see [23, §10]).

Proposition 2.2 ([23, §10], [32, 3.8], [40, 5.9]). *Let $\mathbf{L} \subseteq \mathbf{G}$ be a Levi complement of
the parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$. Assume that $\hat{\mathbf{L}}^{\circ,\mathrm{un}}$ is non-empty and fix some $A_0 \in \hat{\mathbf{L}}^{\circ,\mathrm{un}}$.
Then the endomorphism algebra $\mathrm{End}_{\mathscr{M}\mathbf{G}}(\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0))$ is isomorphic to the group algebra
$\overline{\mathbb{Q}}_\ell[W_{\mathbf{G}}(\mathbf{L})]$, where $W_{\mathbf{G}}(\mathbf{L}) := N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ is the relative Weyl group of **L** in **G**. Hence, we
obtain a bijection*

$$\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})) \xrightarrow{1-1} \left\{ A \in \hat{\mathbf{G}} \mid A \text{ is a simple direct summand of } \mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0) \right\}.$$

2.3. From now until the end of this section, we assume that the connected reductive
group **G** is defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map
$F \colon \mathbf{G} \to \mathbf{G}$. For simplicity, we shall also assume that F induces the identity on **W**.

Given $A \in \mathscr{D}\mathbf{G}$, let $F^*A \in \mathscr{D}\mathbf{G}$ be the inverse image of A under the Frobenius map F.
Suppose that F^*A is isomorphic to A and choose an isomorphism $\varphi \colon F^*A \xrightarrow{\sim} A$. Then φ
induces linear maps $\varphi_{i,g} \colon \mathscr{H}_g^i(A) \to \mathscr{H}_g^i(A)$ for $i \in \mathbb{Z}$ and $g \in \mathbf{G}^F$, where $\mathscr{H}_g^i(A)$ is the
stalk at g of the ith cohomology sheaf of A, a $\overline{\mathbb{Q}}_\ell$-vector space of finite dimension. The
characteristic function $\chi_{A,\varphi} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ associated with A (and φ) is defined by (see [23,
8.4])

$$\chi_{A,\varphi}(g) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Trace}(\varphi_{i,g}, \mathscr{H}_g^i(A)) \quad \text{for } g \in \mathbf{G}^F.$$

This is well-defined since only finitely many of the $\mathscr{H}_g^i(A)$ ($i \in \mathbb{Z}$) are non-zero. In the case
where A is a **G**-equivariant perverse sheaf, we have $\chi_{A,\varphi} \in \mathrm{CF}(\mathbf{G}^F)$. If, in addition, A is a
simple object of $\mathscr{M}\mathbf{G}$, then φ (and, hence, $\chi_{A,\varphi}$) is unique up to a non-zero scalar. Denote
by

$$\hat{\mathbf{G}}^F := \{A \in \hat{\mathbf{G}} \mid F^*A \cong A\} \subseteq \hat{\mathbf{G}}$$

the F-stable character sheaves in $\hat{\mathbf{G}}$. The following is one of the main results of [26], which we can now state without any restriction on the characteristic in view of [31].

Theorem 2.4 (Lusztig [26, §25], [31, 3.10]). *In the setting of* 2.3*, there exist isomorphisms* $\varphi_A \colon F^*A \xrightarrow{\sim} A$ *(for $A \in \hat{\mathbf{G}}^F$) which satisfy the following conditions:*

(a) *The values of the characteristic functions χ_{A,φ_A} are cyclotomic integers;*

(b) $\{\chi_{A,\varphi_A} \mid A \in \hat{\mathbf{G}}^F\}$ *is an orthonormal basis of* $\left(\mathrm{CF}(\mathbf{G}^F), \langle\,,\,\rangle_{\mathbf{G}^F}\right)$.

The required properties for the isomorphisms $\varphi_A \colon F^*A \xrightarrow{\sim} A$ (formulated in [26, 25.1], [24, 13.8]) determine φ_A up to multiplication by a root of unity. Sometimes, for $A \in \hat{\mathbf{G}}^F$, it will be convenient to just write χ_A for a characteristic function associated to the character sheaf A, without referring to a specific isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$. Whenever we do this, we tacitly assume that $\chi_A = \chi_{A,\varphi_A}$ for a(ny) chosen isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$ which satisfies these properties.

3. Parametrisation of unipotent characters and unipotent character sheaves

$\mathbf{G}$ is a connected reductive algebraic group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ with Frobenius map $F \colon \mathbf{G} \to \mathbf{G}$. In this section, we assume in addition that the centre of $\mathbf{G}$ is connected and that the map induced by F on $\mathbf{W}$ is the identity. This assumption ensures that all unipotent character sheaves on $\mathbf{G}$ are F-stable. In 3.5 below, we will further restrict to the case where $\mathbf{G}$ is the simple adjoint group of type E_7. There are two natural ways to give "parallel" parametrisations of unipotent characters of $\mathbf{G}^F$ and of (F-stable) unipotent character sheaves on $\mathbf{G}$. We will see however that they are compatible in the case of E_7.

3.1. According to [20, Main Theorem 4.23], $\mathrm{Uch}(\mathbf{G}^F)$ can be classified in terms of the following data, which only depend on $\mathbf{W}$, and not on p or q. First, Lusztig describes a partition of the irreducible characters of $\mathbf{W}$ into families. To each such family $\mathscr{F} \subseteq \mathrm{Irr}(\mathbf{W})$ is associated a certain finite group $\mathscr{G} = \mathscr{G}_{\mathscr{F}}$ and a set $\mathscr{M}(\mathscr{G})$ consisting of all pairs (g, σ), $g \in \mathscr{G}$, $\sigma \in \mathrm{Irr}(C_{\mathscr{G}}(g))$, modulo the relation $(g, \sigma) \sim (hgh^{-1}, {}^h\sigma)$ for $h \in \mathscr{G}$, where ${}^h\sigma$ is the irreducible character of $C_{\mathscr{G}}(hgh^{-1}) = hC_{\mathscr{G}}(g)h^{-1}$ given by composing σ with conjugation by h^{-1}. $\mathscr{M}(\mathscr{G})$ is equipped with a pairing $\{\,,\,\} \colon \mathscr{M}(\mathscr{G}) \times \mathscr{M}(\mathscr{G}) \to \overline{\mathbb{Q}}_\ell$, defined by

$$\{(g, \sigma), (h, \tau)\} := \frac{1}{|C_{\mathscr{G}}(g)|} \frac{1}{|C_{\mathscr{G}}(h)|} \sum_{\substack{x \in \mathscr{G} \\ gxhx^{-1} = xhx^{-1}g}} \tau(x^{-1}g^{-1}x)\sigma(xhx^{-1}),$$

for $(g, \sigma), (h, \tau) \in \mathscr{M}(\mathscr{G})$. Given a family $\mathscr{F} \subseteq \mathrm{Irr}(\mathbf{W})$, let $\mathscr{F} \hookrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}})$ be the embedding defined in [20, 4.14]. Setting

$$(3.1) \qquad \overline{X}(\mathbf{W}) := \bigsqcup_{\substack{\mathscr{F} \subseteq \mathrm{Irr}(\mathbf{W}) \\ \text{family}}} \mathscr{M}(\mathscr{G}_{\mathscr{F}}),$$

we thus have an embedding

$$(3.2) \qquad \mathrm{Irr}(\mathbf{W}) \hookrightarrow \overline{X}(\mathbf{W}), \quad \phi \mapsto x_\phi.$$

The pairing $\{\,,\,\}$ is extended to $\overline{X}(\mathbf{W})$ such that $\{(g, \sigma), (h, \tau)\} := 0$ whenever $(g, \sigma), (h, \tau) \in \overline{X}(\mathbf{W})$ are not in the same $\mathscr{M}(\mathscr{G}_{\mathscr{F}})$. Then $\overline{X}(\mathbf{W})$ parametrises $\mathrm{Uch}(\mathbf{G}^F)$ as follows. For

$\phi \in \mathrm{Irr}(\mathbf{W})$, let

$$R_\phi := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w) R_w.$$

Then there is a bijection

(3.3) $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \overline{X}(\mathbf{W}), \quad \rho \mapsto \overline{x}_\rho,$

such that for any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ and any $\phi \in \mathrm{Irr}(\mathbf{W})$, we have

(3.4) $\langle \rho, R_\phi \rangle_{\mathbf{G}^F} = \Delta(\overline{x}_\rho)\{\overline{x}_\rho, x_\phi\}.$

Here, $\Delta(\overline{x}_\rho) \in \{\pm 1\}$ is a sign attached to $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, see [20, 4.21]. The bijection (3.3) is not uniquely determined by this property. In the case where $\mathbf{G}$ is of type E_7, we will make a definite choice in Remark 3.6 below. Next, for $x \in \overline{X}(\mathbf{W})$, a unipotent "almost character" $R_x \in \mathrm{CF}(\mathbf{G}^F)$ is defined by

$$R_x := \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \{\overline{x}_\rho, x\}\Delta(\overline{x}_\rho)\rho.$$

We then have $R_\phi = R_{x_\phi}$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$. Since the "Fourier matrix" $\Upsilon := (\{x, x'\})_{x,x' \in \overline{X}(\mathbf{W})}$ is hermitian and Υ^2 is the identity matrix [19, §4], we obtain

$$\langle R_x, R_{x'} \rangle_{\mathbf{G}^F} = \begin{cases} 1 & \text{if } x = x', \\ 0 & \text{if } x \neq x' \end{cases} \quad (\text{for } x, x' \in \overline{X}(\mathbf{W})).$$

It follows that

(3.5) $\rho = \Delta(\overline{x}_\rho) \sum_{x \in \overline{X}(\mathbf{W})} \overline{\{\overline{x}_\rho, x\}} R_x \quad \text{for } \rho \in \mathrm{Uch}(\mathbf{G}^F).$

In particular, knowing the values of the unipotent characters of $\mathbf{G}^F$ is equivalent to knowing the values of the R_x for $x \in \overline{X}(\mathbf{W})$.

3.2. On the other hand, by [26, Theorem 23.1] combined with the cleanness results in [31], the set $\overline{X}(\mathbf{W})$ in (3.1) also serves as a parameter set for $\hat{\mathbf{G}}^{\mathrm{un}}$. Namely, there is a bijection

(3.6) $\overline{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}, \quad x \mapsto A_x,$

such that for any $x \in \overline{X}(\mathbf{W})$ and any $\phi \in \mathrm{Irr}(\mathbf{W})$, we have

(3.7) $(A_x : R_\phi^{\mathscr{L}_0}) = \hat{\varepsilon}_{A_x}\{x, x_\phi\}.$

Here, $\hat{\varepsilon}_K = (-1)^{\dim \mathbf{G} - \dim \mathrm{supp}\, K}$ for any complex $K \in \mathscr{D}\mathbf{G}$, and $\mathrm{supp}\, K$ is the Zariski closure of $\{g \in \mathbf{G} \mid \mathscr{H}_g^i K \neq 0 \text{ for some } i \in \mathbb{Z}\} \subseteq \mathbf{G}$. Once again, this property does not uniquely determine the bijection (3.6). For $\mathbf{G}$ of type E_7, we will fix a choice in Remark 3.6.

With these notions we can formulate the following theorem of Shoji, which verifies Lusztig's conjecture under the assumption that $\mathbf{G}$ has connected centre. As mentioned in [11, 2.7], this holds without any conditions on p, q, since the cleanness of cuspidal character sheaves is established in full generality ([31]).

Theorem 3.3 (Shoji [41, Theorems 3.2, 4.1]). *Let p be a prime, q a power of p, $\mathbf{G}$ a connected reductive group over $\overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. Assume that $\mathbf{Z}(\mathbf{G})$ is connected and $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ is simple. Then $\hat{\mathbf{G}}^{\mathrm{un}} \subseteq \hat{\mathbf{G}}^F$ and for any $x \in \overline{X}(\mathbf{W})$, R_x and χ_{A_x} coincide up to a non-zero scalar.*

REMARK 3.4. In general, any irreducible character of $\mathbf{G}^F$ has a corresponding almost character, and the set of all these almost characters is an orthonormal basis for $\mathrm{CF}(\mathbf{G}^F)$ ([20, §4]). Shoji formulates and proves Theorem 3.3 for the characteristic function of any F-stable character sheaf and the respective almost character. Determining the character table of $\mathbf{G}^F$ can thus be reformulated to solving the following two problems (see [29]):

(a) Computing the values of the characteristic functions of F-stable character sheaves;
(b) Determining the scalar relating the characteristic function of any F-stable character sheaf with the corresponding almost character.

(b) can be reduced to considering cuspidal (F-stable) character sheaves, via the induction process for cuspidal character sheaves mentioned in Section 2, see [27, §3]. On the other hand, Lusztig [26, §24] shows that (a) can be reduced to the computation of "generalised Green functions" and he provides an algorithm to compute those functions (refining an earlier algorithm of Shoji in [39]), although it involves some unspecified scalars which are still not known in all cases. We will briefly describe this algorithm in the next section.

3.5. We assume here that $\mathbf{G}$ is the simple adjoint group of type E_7 over the field $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$, q a power of p, with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. With respect to this $\mathbf{G}$, $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ and $\mathbf{W}$ are as in the introduction. These choices determine the set of roots $R \subseteq X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, k^\times)$, as well as the positive roots $R^+ \subseteq R$ and the simple roots $\Pi := \{\alpha_1, \dots, \alpha_7\} \subseteq R^+$. We choose the order of $\alpha_1, \dots, \alpha_7$ in such a way that the Dynkin diagram of E_7 is as follows:

$$E_7 \qquad \begin{array}{ccccccc} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \bullet & \!\!-\!\!\bullet & \!\!-\!\!\bullet & \!\!-\!\!\bullet & \!\!-\!\!\bullet & \!\!-\!\!\bullet \\ & & | & & & \\ & & \bullet\, \alpha_2 & & & \end{array}$$

$\mathbf{W}$ can be viewed as a Coxeter group of type E_7 with Coxeter generators $\mathbf{S} = \{s_1, \dots, s_7\}$, where s_i denotes the reflection in the hyperplane orthogonal to α_i. We will describe the parametrisations of $\mathrm{Uch}(\mathbf{G}^F)$ and $\hat{\mathbf{G}}^{\mathrm{un}}$ in terms of Harish-Chandra series, following Lusztig [32, §3]. (Note that all the results in loc. cit. hold whenever $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ is simple or $\{1\}$, and we will implicitly use this below as far as subgroups of $\mathbf{G}$ corresponding to irreducible subgraphs of the Dynkin diagram of type E_7 are concerned.)

To any Coxeter system (W, S) is attached ([32, 3.1]) a certain finite (possibly empty) set $\mathfrak{S}_W^\circ$. If W is irreducible, $\mathfrak{S}_W^\circ$ is contained in the set of roots of unity in $\overline{\mathbb{Q}}_\ell$. Let

$$\mathfrak{S}_W := \left\{ (J, \epsilon, \zeta) \mid J \subseteq S,\ \epsilon \in \mathrm{Irr}(W^{S/J}),\ \zeta \in \mathfrak{S}_{W_J}^\circ \right\},$$

where $W_J := \langle J \rangle \subseteq W$ and $W^{S/J}$ is the subgroup of W generated by the involutions $\sigma_s := w_0^{J \cup \{s\}} w_0^J = w_0^J w_0^{J \cup \{s\}}$ for $s \in S \backslash J$ (for $J' \subseteq S$, we denote by $w_0^{J'}$ the longest element in $W_{J'}$). Then both (W_J, J) and $\left(W^{S/J}, \{\sigma_s \mid s \in S \backslash J\}\right)$ are Coxeter systems, see [17], [18].

(3.8)   From now on we shall fix, once and for all, a square root $\sqrt{p}$ of p in $\overline{\mathbb{Q}}_\ell$.

Furthermore, when $q = p^f (f \geqslant 1)$, we write $\sqrt{q} := \sqrt{p}^f$.

Now consider $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. By [18, 3.25, 3.26], there exist $J \subseteq \mathbf{S}$ and a cuspidal unipotent character $\rho_0 \in \mathrm{Uch}(\mathbf{L}_J^F)$ such that $\langle \rho, R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0) \rangle_{\mathbf{G}^F} \neq 0$. (For $J \subseteq \mathbf{S}$, we denote by $\mathbf{L}_J$ the unique Levi complement of the standard parabolic subgroup $\mathbf{P}_J := \mathbf{B}W_J\mathbf{B} \subseteq \mathbf{G}$ such that $\mathbf{T} \subseteq \mathbf{L}_J$, and $R_{\mathbf{L}_J}^{\mathbf{G}}$ is Harish-Chandra induction.) After having chosen $\sqrt{p}$ in (3.8), there is a natural isomorphism between the endomorphism algebra $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F}(R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0))$ and the group algebra $\overline{\mathbb{Q}}_\ell[\mathbf{W}^{\mathbf{S}/J}]$ (in analogy to 2.2; note that $\mathbf{W}^{\mathbf{S}/J}$ is isomorphic to $W_{\mathbf{G}}(\mathbf{L}_J)$). Hence, among the constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$, ρ is naturally parametrised by some $\epsilon \in \mathrm{Irr}(\mathbf{W}^{\mathbf{S}/J})$ (see also [20, Corollary 8.7]). Moreover, denote by λ_ρ the eigenvalue of Frobenius of ρ (as defined in [20, §11]). Then we have $\lambda_\rho = \lambda_{\rho_0} \in \mathfrak{S}_{\mathbf{W}_J}^\circ$ and the assignment $\rho \mapsto (J, \epsilon, \lambda_\rho)$ gives a well-defined bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{S}_{\mathbf{W}}.$$

In particular, the cuspidal unipotent characters correspond to the elements of $\mathfrak{S}_{\mathbf{W}}^\circ$. On the other hand, let $A \in \hat{\mathbf{G}}^{\mathrm{un}}$. By 2.1, 2.2, there exist $J \subseteq \mathbf{S}$, $A_0 \in \hat{\mathbf{L}}_J^{\circ,\mathrm{un}}$, such that A is a simple summand of $\mathrm{ind}_{\mathbf{L}_J \subseteq \mathbf{P}_J}^{\mathbf{G}}(A_0)$, and A is parametrised by some $\epsilon \in \mathrm{Irr}(\mathbf{W}^{\mathbf{S}/J})$. Furthermore, to A is associated a root of unity λ_A by means of a certain "twisting operator" defined via Shintani descent, see [40, 1.16, Theorem 3.3]. Another definition of λ_A is given in [32, 3.6]. Then we have $\lambda_A = \lambda_{A_0}$ (see [40, 3.6]), and this is an element of $\mathfrak{S}_{\mathbf{W}_J}^\circ$. The assignment $(J, \epsilon, \lambda_A) \mapsto A$ gives a well-defined bijection

$$\mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}.$$

In particular, the unipotent cuspidal character sheaves correspond to the elements of $\mathfrak{S}_{\mathbf{W}}^\circ$.
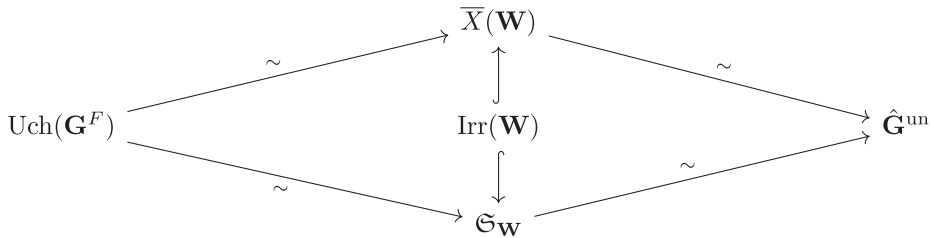
Remark 3.6. We keep the setting of 3.5. In particular, $\mathbf{Z}(\mathbf{G})$ is connected and F acts trivially on $\mathbf{W}$. Recall from 3.1, 3.2, that the bijections $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \overline{X}(\mathbf{W})$ and $\overline{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ are not uniquely determined by the conditions (3.4), (3.7), respectively. We will now make a definite choice following [20, 11.2], see also [6, §6].

Given $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, the family $\mathscr{F} \subseteq \mathrm{Irr}(\mathbf{W})$ for which $\mathscr{M}(\mathscr{G}_{\mathscr{F}})$ contains $\overline{x}_\rho$ is independent of a chosen bijection $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \overline{X}(\mathbf{W})$ in (3.3) satisfying (3.4). Consider an element $x = (g, \sigma) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}}) \subseteq \overline{X}(\mathbf{W})$. If $\mathscr{F}$ contains the irreducible characters of $\mathbf{W}$ of degree 512, and if x is not in the image of the embedding $\mathscr{F} \hookrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}})$, we set $\tilde{\lambda}_x := \zeta\sigma(g)/\sigma(1)$, where ζ is a primitive fourth root of unity in $\overline{\mathbb{Q}}_\ell$. In any other case, let $\tilde{\lambda}_x := \sigma(g)/\sigma(1)$. Then it turns out that the bijection (3.3) can be chosen in such a way that $\lambda_\rho = \tilde{\lambda}_{\overline{x}_\rho}$ for any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, and it is almost uniquely determined by this condition along with (3.4), the only ambiguity arises from the two elements of $\mathrm{Irr}(\mathbf{W})$ of degree 512. This can be removed as follows. We have a natural embedding $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_{\mathbf{W}}, \phi \mapsto (\emptyset, \phi, 1)$. Then we require that the embeddings $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \mathrm{Uch}(\mathbf{G}^F)$ and $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \overline{X}(\mathbf{W}) \xrightarrow{\sim} \mathrm{Uch}(\mathbf{G}^F)$ coincide. This does not interfere with our choices above, in view of [20, Proposition 12.6], and it uniquely specifies the bijection $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \overline{X}(\mathbf{W})$.

Now let $A \in \hat{\mathbf{G}}^{\mathrm{un}}$ and let $(J, \epsilon, \lambda_A) \in \mathfrak{S}_{\mathbf{W}}, \rho \in \mathrm{Uch}(\mathbf{G}^F)$ be the elements associated to A in 3.5. By [32, 3.10], we have $\langle \rho, R_\phi \rangle_{\mathbf{G}^F} = (A : R_\phi^{\mathscr{L}_0})$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$. Moreover, $\Delta(\overline{x}_\rho) = \hat{\varepsilon}_A$,

so (3.4) implies $(A : R_\phi^{\mathscr{L}_0}) = \hat{\varepsilon}_A\{\overline{x}_\rho, x_\phi\}$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$. Hence, the assignment $\overline{x}_\rho \mapsto A$ gives rise to a bijection $\overline{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ satisfying (3.7). Thus we can and will from now on always assume that the bijections $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \overline{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ in 3.1, 3.2 are chosen in this way. In particular, the diagram

$$
\begin{array}{ccc}
 & \overline{X}(\mathbf{W}) & \\
 \nearrow {\scriptstyle\sim} & \uparrow & {\scriptstyle\sim} \searrow \\
\mathrm{Uch}(\mathbf{G}^F) & \mathrm{Irr}(\mathbf{W}) & \hat{\mathbf{G}}^{\mathrm{un}} \\
 \searrow {\scriptstyle\sim} & \downarrow & {\scriptstyle\sim} \nearrow \\
 & \mathfrak{S}_{\mathbf{W}} &
\end{array}
$$

is commutative.

4. The generalised Springer correspondence

In this section, $\mathbf{G}$ may be assumed to be an arbitrary connected reductive algebraic group over $\overline{\mathbb{F}}_p$, as in the introduction. We will focus on those character sheaves whose support contains unipotent elements. In [26, §24], Lusztig provides an algorithm for computing the values of their characteristic functions in principle, using intersection cohomology complexes [21]. Further references for the following are [43, 1.1-1.3], [49, §3, §4].

4.1. Denote by $\mathscr{N}_{\mathbf{G}}$ the set of all pairs $(\mathscr{O}, \mathscr{E})$ where $\mathscr{O}$ is a unipotent conjugacy class in $\mathbf{G}$ and $\mathscr{E}$ is the isomorphism class of an irreducible $\mathbf{G}$-equivariant $\overline{\mathbb{Q}}_\ell$-local system on $\mathscr{O}$. For a given $\mathscr{O}$, these local systems on $\mathscr{O}$ naturally correspond to irreducible characters of the group $A(u) := C_{\mathbf{G}}(u)/C_{\mathbf{G}}^\circ(u)$, where u is any fixed element of $\mathscr{O}$. To $(\mathscr{O}, \mathscr{E}) \in \mathscr{N}_{\mathbf{G}}$ is associated ([21, Theorem 6.5]) a triple $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ consisting of a closed subgroup $\mathbf{L} \subseteq \mathbf{G}$ which is the Levi complement of some parabolic subgroup of $\mathbf{G}$, a unipotent conjugacy class $\mathscr{O}_0$ in $\mathbf{L}$ and an $\mathbf{L}$-equivariant $\overline{\mathbb{Q}}_\ell$-local system $\mathscr{E}_0$ on $\mathscr{O}_0$ such that $(\mathbf{Z}(\mathbf{L})^\circ \times \mathscr{O}_0, 1 \boxtimes \mathscr{E}_0)$ is a cuspidal pair for $\mathbf{L}$ in the sense of [21, 2.4]. Denoting by $\mathscr{M}_{\mathbf{G}}$ the set of all those triples up to $\mathbf{G}$-conjugacy, the corresponding map $\tau \colon \mathscr{N}_{\mathbf{G}} \to \mathscr{M}_{\mathbf{G}}$ is surjective, so we have a decomposition

$$
\mathscr{N}_{\mathbf{G}} = \coprod_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) \in \mathscr{M}_{\mathbf{G}}} \tau^{-1}(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)
$$

of $\mathscr{N}_{\mathbf{G}}$ into non-empty subsets. The elements in a given $\tau^{-1}(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ can be distinguished as follows. The triple $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ gives rise to a semisimple perverse sheaf $K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)} \in \mathscr{M}\mathbf{G}$ defined as a certain intersection cohomology complex in [21, §4], see also [23, §8]. Then for any $\iota = (\mathscr{O}_\iota, \mathscr{E}_\iota) \in \tau^{-1}(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ there is (up to isomorphism) exactly one simple direct summand A_ι of $K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)}$ which satisfies

$$(4.1) \qquad A_\iota|_{\mathbf{G}_{\mathrm{uni}}} \cong \mathrm{IC}(\overline{\mathscr{O}_\iota}, \mathscr{E}_\iota)[\dim \mathscr{O}_\iota + \dim \mathbf{Z}(\mathbf{L})^\circ]$$

($\mathbf{G}_{\mathrm{uni}} \subseteq \mathbf{G}$ is the unipotent subvariety of $\mathbf{G}$), and each simple direct summand of $K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)}$ arises in this way from some $\iota \in \tau^{-1}(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$, see [21, 6.5], [26, 24.1]. On the other hand, the endomorphism algebra $\mathscr{A} := \mathrm{End}_{\mathscr{M}\mathbf{G}}(K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)})$ is isomorphic to $\overline{\mathbb{Q}}_\ell[W_{\mathbf{G}}(\mathbf{L})]$ ([21, Theorem 9.2]). Thus the isomorphism classes of simple direct summands of $K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)}$ are naturally parametrised by $\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))$. Hence, we obtain a bijection

(4.2)
$$\mathscr{N}_{\mathbf{G}} \cong \bigsqcup_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) \in \mathscr{M}_{\mathbf{G}}} \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})),$$

which is called the generalised Springer correspondence. The problem of explicitly determining this correspondence can be reduced to the case where $\mathbf{G}$ is almost simple of simply-connected type, and has been solved by Lusztig and Spaltenstein, except for a few minor cases in type E_6 with $p \neq 3$, and in type E_8 with $p = 3$ ([21], [35], [47], [34]). In particular, the generalised Springer correspondence is completely determined in the case of E_7, see [47].

4.2. Let us now bring the $\mathbb{F}_q$-rational structure on $\mathbf{G}$ into the picture, see [26, §24]. So we assume that $\mathbf{G}$ is defined over $\mathbb{F}_q$, with Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. F acts naturally on both $\mathscr{N}_{\mathbf{G}}$ and $\mathscr{M}_{\mathbf{G}}$ via

$$\mathscr{N}_{\mathbf{G}} \to \mathscr{N}_{\mathbf{G}}, \ (\mathscr{O}, \mathscr{E}) \mapsto (F^{-1}(\mathscr{O}), F^*\mathscr{E}),$$

and

$$\mathscr{M}_{\mathbf{G}} \to \mathscr{M}_{\mathbf{G}}, \ (\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) \mapsto (F^{-1}(\mathbf{L}), F^{-1}(\mathscr{O}_0), F^*\mathscr{E}_0),$$

respectively. If $\mathbf{L}$ is F-stable, it is also clear that F acts on $W_{\mathbf{G}}(\mathbf{L})$ and, hence, on $\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))$, by

$$\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})) \to \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})), \ \epsilon \mapsto \epsilon \circ F.$$

Let $\mathscr{N}_{\mathbf{G}}^F \subseteq \mathscr{N}_{\mathbf{G}}$, $\mathscr{M}_{\mathbf{G}}^F \subseteq \mathscr{M}_{\mathbf{G}}$, $\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))^F \subseteq \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))$ be the respective subsets of fixed points under these actions (where, in terms of the local systems, this is only meant up to isomorphism). The generalised Springer correspondence (4.2) is compatible with the above actions, thus it induces a bijection

(4.3)
$$\mathscr{N}_{\mathbf{G}}^F \cong \bigsqcup_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) \in \mathscr{M}_{\mathbf{G}}^F} \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))^F.$$

For any $\iota = (\mathscr{O}, \mathscr{E}) \in \mathscr{N}_{\mathbf{G}}^F$, the complex A_ι is F-stable. More precisely, let $\tau(\iota) = (\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) \in \mathscr{M}_{\mathbf{G}}^F$. We may choose an isomorphism $F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Such a choice induces an isomorphism $F^*K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)} \xrightarrow{\sim} K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)}$, which in turn gives rise to an isomorphism $\varphi_{A_\iota}\colon F^*A_\iota \xrightarrow{\sim} A_\iota$, as described in [26, 24.2]. Once φ_{A_ι} is fixed, it determines an isomorphism $\psi_\iota\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ (via [26, (24.2.2)]), and for any $g \in \mathscr{O}^F$, the induced map $\psi_{\iota,g}\colon \mathscr{E}_g \to \mathscr{E}_g$ on the stalk of $\mathscr{E}$ at g is of finite order. Now define two functions $Y_\iota, X_\iota\colon \mathbf{G}_{\mathrm{uni}}^F \to \overline{\mathbb{Q}}_\ell$ by

$$Y_\iota(g) := \begin{cases} \mathrm{Trace}(\psi_{\iota,g}, \mathscr{E}_g) & \text{if } g \in \mathscr{O}^F, \\ 0 & \text{if } g \notin \mathscr{O}^F \end{cases}$$

and

$$X_\iota(g) := (-1)^{\dim \mathscr{O} + \dim \mathbf{Z}(\mathbf{L})^\circ} q^{-d_\iota} \chi_{A_\iota, \varphi_{A_\iota}}(g)$$

for $g \in \mathbf{G}_{\mathrm{uni}}^F$, where $d_\iota = \frac{1}{2}(\dim \mathrm{supp}\, A_\iota - \dim \mathscr{O} - \dim \mathbf{Z}(\mathbf{L})^\circ)$. Then both Y_ι and X_ι are invariant under the conjugation action of $\mathbf{G}^F$ on $\mathbf{G}_{\mathrm{uni}}^F$.

Theorem 4.3 (Lusztig [26, §24]). *In the setting of 4.2, the following hold.*

(a) *The functions Y_ι, $\iota \in \mathcal{N}_{\mathbf{G}}^F$, form a basis for the vector space consisting of all functions $\mathbf{G}_{\mathrm{uni}}^F \to \overline{\mathbb{Q}}_\ell$ which are invariant under $\mathbf{G}^F$-conjugacy.*

(b) *There is a system of equations*

$$X_\iota = \sum_{\iota' \in \mathcal{N}_{\mathbf{G}}^F} p_{\iota',\iota} Y_{\iota'} \quad (\iota \in \mathcal{N}_{\mathbf{G}}^F),$$

for some uniquely determined $p_{\iota',\iota} \in \mathbb{Z}$.

Note that the restrictions (23.0.1) on the characteristic of k can be removed since the cleanness of cuspidal character sheaves has been established in complete generality [31].

REMARK 4.4. In the setting of Theorem 4.3, if we define a total order $\leqslant$ on $\mathcal{N}_{\mathbf{G}}^F$ in such a way that for $\iota = (\mathcal{O}, \mathcal{E})$, $\iota' = (\mathcal{O}', \mathcal{E}') \in \mathcal{N}_{\mathbf{G}}^F$ we have $\iota' \leqslant \iota$ whenever $\mathcal{O}' \subseteq \overline{\mathcal{O}}$ (the latter condition defines a partial order on the set of unipotent classes of $\mathbf{G}$), the matrix $(p_{\iota',\iota})_{\iota',\iota \in \mathcal{N}_{\mathbf{G}}^F}$ has upper unitriangular shape, as follows immediately from the definitions. Lusztig [26, 24.4] provides an algorithm for computing this matrix $(p_{\iota',\iota})$, which entirely relies on combinatorial data. This algorithm is implemented in CHEVIE [37] and is accessible via the functions UnipotentClasses and ICCTable. Hence, the computation of the characteristic functions $\chi_{A_\iota,\varphi_{A_\iota}}$ ($\iota \in \mathcal{N}_{\mathbf{G}}^F$) at unipotent elements is reduced to the computation of the functions Y_ι, $\iota \in \mathcal{N}_{\mathbf{G}}^F$. It should be noted, however, that the latter is still a non-trivial task since it is difficult to describe the isomorphisms φ_{A_ι} (and thus the ψ_ι) explicitly. This has been accomplished for classical groups (in any characteristic) by Shoji, [43], [44], [46]. As far as exceptional groups are concerned, this problem is not yet solved.

REMARK 4.5. In the setting of 4.1, let us fix a triple $(\mathbf{L}, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_{\mathbf{G}}$ and consider the complex $K_{(\mathbf{L}, \mathcal{O}_0, \mathcal{E}_0)} \in \mathcal{M}\mathbf{G}$. By [21, 4.5], we have a canonical isomorphism

$$K_{(\mathbf{L}, \mathcal{O}_0, \mathcal{E}_0)} \cong \mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(\mathrm{IC}(\overline{\Sigma}_0, \mathcal{E}_0)[\dim \Sigma_0]), \quad \text{where } \Sigma_0 = \mathbf{Z}(\mathbf{L})^\circ \mathcal{O}_0.$$

Here, $A_0 := \mathrm{IC}(\overline{\Sigma}_0, \mathcal{E}_0)[\dim \Sigma_0]$ is a cuspidal character sheaf on $\mathbf{L}$. In particular, since the definition of $K_{(\mathbf{L}, \mathcal{O}_0, \mathcal{E}_0)}$ does not involve the choice of a parabolic subgroup with Levi complement $\mathbf{L}$, this shows that $\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0)$ is independent of $\mathbf{P}$, and we can just write $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$ instead. Next, as the elements of $\mathcal{M}_{\mathbf{G}}$ are taken up to $\mathbf{G}$-conjugacy, we may assume that $\mathbf{L}$ is a standard Levi subgroup of some standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ (compare 3.5), that is, there exists some $J \subseteq \mathbf{S}$ such that $\mathbf{P} = \mathbf{P}_J = \mathbf{B}\mathbf{W}_J\mathbf{B}$, and $\mathbf{L} = \mathbf{L}_J$ is the unique Levi complement of $\mathbf{P}_J$ containing $\mathbf{T}$.

5. The scalars ξ_x for cuspidal A_x in E_7, $p = 2$

The notation and assumptions are as in 3.5, and in addition we set $p := 2$. In particular, $\mathbf{G}$ has trivial centre and we can apply Theorem 3.3. So there are scalars $\xi_x \in \overline{\mathbb{Q}}_\ell$ such that

$$(5.1) \qquad\qquad R_x = \xi_x \chi_{A_x} \quad \text{for } x \in \overline{X}(\mathbf{W}).$$

Since $\langle R_x, R_x \rangle = 1 = \langle \chi_A, \chi_A \rangle$ for any $x \in \overline{X}(\mathbf{W})$, $A \in \hat{\mathbf{G}}^{\mathrm{un}}$, we know that $\xi_x \overline{\xi}_x = 1$ for any $x \in \overline{X}(\mathbf{W})$. In this section, we determine the scalars ξ_x for the cuspidal (unipotent) character sheaves A_x, after having chosen specific isomorphisms $F^* A_x \xrightarrow{\sim} A_x$ for the corresponding

$x \in \overline{X}(\mathbf{W})$.

5.1. By [25, Proposition 20.3] there are two cuspidal character sheaves A_1, A_2 for $\mathbf{G}$, and both of them lie in $\hat{\mathbf{G}}^{\mathrm{un}}$. Their support is the unipotent variety $\mathbf{G}_{\mathrm{uni}} \subseteq \mathbf{G}$ consisting of all unipotent elements in $\mathbf{G}$. This variety is the (Zariski) closure of the regular unipotent conjugacy class $\mathscr{O}_{\mathrm{reg}}$, which is the unique class of all unipotent $x \in \mathbf{G}$ with the property $\dim C_{\mathbf{G}}(x) = \operatorname{rank} \mathbf{G} = 7$. In particular, $\mathscr{O}_{\mathrm{reg}}$ is F-stable. Denote by $u_i = u_{\alpha_i}$ ($1 \leqslant i \leqslant 7$) the closed embedding $\mathbf{k}^+ \to \mathbf{G}$ whose image is the root subgroup $\mathbf{U}_{\alpha_i} \subseteq \mathbf{U}$. We set

$$(5.2) \qquad u_0 := u_1(1) \cdot u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_6(1) \cdot u_7(1) \in \mathscr{O}_{\mathrm{reg}}^F.$$

Let a_0 be the image of u_0 in $A(u_0) = C_{\mathbf{G}}(u_0)/C_{\mathbf{G}}^\circ(u_0)$ under the natural map $C_{\mathbf{G}}(u_0) \to A(u_0)$. Then $A(u_0)$ is a cyclic group of order 4 generated by a_0 (see [38, §4] and [7, 14.15, 14.18]), and the automorphism of $A(u_0)$ induced by F is the identity. Hence, the elements of $A(u_0)$ correspond to the $\mathbf{G}^F$-conjugacy classes contained in $\mathscr{O}_{\mathrm{reg}}^F$ (see, for instance, [8, 4.3.6]). In particular, there are 4 such classes and every irreducible $\mathbf{G}$-equivariant $\overline{\mathbb{Q}}_\ell$-local system on $\mathscr{O}_{\mathrm{reg}}$ is one-dimensional. For $a \in A(u_0)$, we choose some $x \in \mathbf{G}$ such that $x^{-1}F(x) \in C_{\mathbf{G}}(u_0)$ has image $a \in A(u_0)$. Then $u_a := xu_0x^{-1} \in \mathscr{O}_{\mathrm{reg}}^F$ (defined only up to $\mathbf{G}^F$-conjugacy) represents the $\mathbf{G}^F$-conjugacy class corresponding to a. If a is the trivial element of $A(u_0)$, we will always assume that the representative for the corresponding $\mathbf{G}^F$-conjugacy class is u_0. Thus $\{u_{a_0^i} \mid 0 \leqslant i \leqslant 3\}$ is a set of representatives for the $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}_{\mathrm{reg}}^F$. Singling out the element $u_0 \in \mathscr{O}_{\mathrm{reg}}^F$ as above (or, more precisely, the $\mathbf{G}^F$-conjugacy class of u_0) gives rise to specific isomorphisms $\varphi_{A_i} \colon F^*A_i \xrightarrow{\sim} A_i$ with respect to the two cuspidal character sheaves $A_i \in \hat{\mathbf{G}}^{\mathrm{un}}$ ($i = 1, 2$), as follows. Let $\mathscr{E}_i$ be the F-stable irreducible $\mathbf{G}$-equivariant $\overline{\mathbb{Q}}_\ell$-local system on $\mathscr{O}_{\mathrm{reg}}$ such that $A_i \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_i)[\dim \mathscr{O}_{\mathrm{reg}}]$. By [26, 24.1], we have

$$\mathscr{H}^s(A_i)|_{\mathbf{G}_{\mathrm{uni}}} \cong \begin{cases} \mathscr{E}_i & \text{if } s = -d, \\ 0 & \text{if } s \neq -d, \end{cases}$$

where $d = \dim \mathscr{O}_{\mathrm{reg}}$. Hence, for $u \in \mathscr{O}_{\mathrm{reg}}^F$, the stalk $\mathscr{H}_u^{-d}(A_i) \cong \mathscr{E}_{i,u}$ is one-dimensional. Using [26, 25.1], the φ_{A_i} may thus be chosen in such a way that, in particular, for any $u \in \mathscr{O}_{\mathrm{reg}}^F$, the induced map $\mathscr{E}_{i,u} \to \mathscr{E}_{i,u}$ is given by scalar multiplication with $\sqrt{q}^7$ times a root of unity. We can therefore modify the φ_{A_i} by a root of unity, if necessary, such that the induced map on the stalk $\mathscr{E}_{i,u_0}$ is just scalar multiplication with $\sqrt{q}^7$, and this uniquely specifies our choices for $\varphi_{A_i} \colon F^*A_i \xrightarrow{\sim} A_i$ ($i = 1, 2$). Denoting by $\varphi_{\mathscr{E}_i} \colon F^*\mathscr{E}_i \xrightarrow{\sim} \mathscr{E}_i$ the restriction of φ_{A_i}, we then simply have

$$\chi_{A_i, \varphi_{A_i}}(u) = \chi_{\mathscr{E}_i, \varphi_{\mathscr{E}_i}}(u) \quad \text{for } u \in \mathscr{O}_{\mathrm{reg}}^F, \ i = 1, 2.$$

The latter function can be computed explicitly, as described in [30, 19.7]. Namely, assigning to a local system $\mathscr{E}$ on $\mathscr{O}_{\mathrm{reg}}$ its stalk $\mathscr{E}_{u_0}$ at u_0 gives a bijection between the isomorphism classes of F-stable $\mathbf{G}$-equivariant irreducible local systems $\mathscr{E}$ on $\mathscr{O}_{\mathrm{reg}}$ and F-invariant simple $\overline{\mathbb{Q}}_\ell[A(u_0)]$-modules. For $i = 1, 2$, let $\sigma_i \in \mathrm{Irr}(A(u_0))^F$ be the character corresponding to the local system $\mathscr{E}_i$ on $\mathscr{O}_{\mathrm{reg}}$. The σ_i are the two faithful irreducible characters of $A(u_0)$. We number the cuspidal character sheaves A_i (and, correspondingly, the $\mathscr{E}_i, \sigma_i$) in such a way that $\sigma_1(a_0) = \zeta, \sigma_2(a_0) = -\zeta$, where ζ is the primitive fourth root of unity in $\overline{\mathbb{Q}}_\ell$ chosen in

Remark 3.6. Using the cleanness results in [31], $\chi_{A_i,\varphi_{A_i}}$ vanishes outside $\mathscr{O}_{\text{reg}}^F$. Hence, for $g \in \mathbf{G}^F$, we obtain the following table, where $q^{7/2} = \sqrt{q}^7$.

	$g \notin \mathscr{O}_{\text{reg}}^F$	$g = u_0$	$g = u_{a_0}$	$g = u_{a_0^2}$	$g = u_{a_0^3}$
$\chi_{A_1,\varphi_{A_1}}(g)$	0	$q^{7/2}$	$q^{7/2}\zeta$	$-q^{7/2}$	$-q^{7/2}\zeta$
$\chi_{A_2,\varphi_{A_2}}(g)$	0	$q^{7/2}$	$-q^{7/2}\zeta$	$-q^{7/2}$	$q^{7/2}\zeta$

We can now formulate the result.

Proposition 5.2. *In the setting of* 5.1, *let* x_1, x_2 *be the elements of* $\overline{X}(\mathbf{W})$ *such that* $A_i = A_{x_i}$, $i = 1, 2$. (A_{x_i} *is given by the parametrisation* (3.6), *which is uniquely determined by the requirements in* Remark 3.6.) *Then we have*

$$R_{x_i} = \chi_{A_i,\varphi_{A_i}} \quad \text{for } i = 1, 2.$$

In other words, with the choices for $\varphi_{A_i} \colon F^*A_i \xrightarrow{\sim} A_i$ *made in* 5.1 ($i = 1, 2$), *the scalars* ξ_{x_1}, ξ_{x_2} *in* (5.1) *are both* 1.

The proof will be given in 5.4–5.9 below. We start with the following simple observation.

Lemma 5.3. *Consider the element* $u_0 = u_1(1) \cdot u_2(1) \cdots u_7(1) \in \mathscr{O}_{\text{reg}}^F$ *defined in* (5.2). *For any permutation* π *of* $\{1, 2, \ldots, 7\}$, *there is an element* $u \in \mathbf{U}^F$ *such that*

$$u \cdot u_0 \cdot u^{-1} = u_{\pi(1)}(1) \cdot u_{\pi(2)}(1) \cdots u_{\pi(7)}(1).$$

In particular, u_0 *is conjugate to* u_0^{-1} *in* $\mathbf{U}^F \subseteq \mathbf{G}^F$.

Proof. First consider the Weyl group $\mathbf{W}$ of $\mathbf{G}$, viewed as the irreducible finite Coxeter group of type E_7 with simple reflections $\mathbf{S} = \{s_1, s_2, \ldots, s_7\}$ (see 3.5). It is well-known that any two Coxeter elements of $(\mathbf{W}, \mathbf{S})$ are conjugate in $\mathbf{W}$, that is, for any permutation π of $\{1, 2, \ldots, 7\}$ there exists some $w \in \mathbf{W}$ such that

$$w \cdot (s_1 \cdot s_2 \cdots s_7) \cdot w^{-1} = s_{\pi(1)} \cdot s_{\pi(2)} \cdots s_{\pi(7)}.$$

More precisely, [3, §1] provides an algorithm to compute such an element w which is only based on the facts that an element s_i ($1 \leqslant i \leqslant 7$) in the first (respectively, last) position of a given Coxeter word can be shifted to the last (respectively, first) position by means of conjugation with s_i, and that s_i, s_j ($1 \leqslant i, j \leqslant 7$, $i \neq j$) commute if and only if they are not linked in the Coxeter graph of $(\mathbf{W}, \mathbf{S})$. But the analogous statements hold for the $u_i(1)$, $1 \leqslant i \leqslant 7$, so we can just mimic the proof of [3, Lemma 1.4] to obtain an element $u \in \mathbf{U}^F$ (a product of certain $u_i(1)$, $1 \leqslant i \leqslant 7$; note that $u_i(1) = u_i(-1)$ since k has characteristic 2) such that

$$u \cdot u_0 \cdot u^{-1} = u_{\pi(1)}(1) \cdot u_{\pi(2)}(1) \cdots u_{\pi(7)}(1).$$

In particular, since $u_0^{-1} = u_7(1) \cdot u_6(1) \cdots u_1(1)$, u_0 is conjugate to u_0^{-1} in $\mathbf{U}^F \subseteq \mathbf{G}^F$. $\quad\square$

5.4. Let us denote by $\rho_x \in \text{Uch}(\mathbf{G}^F)$ the unipotent character corresponding to $x \in \overline{X}(\mathbf{W})$ in (3.3). The discussion in Remark 3.6 together with [40, 3.8] (or [32, 3.6]) shows that $\rho_{x_1} = E_7[\zeta]$, $\rho_{x_2} = E_7[-\zeta]$ are the two cuspidal unipotent characters in Carter's table [2, pp. 482-483]. We have

$$R_{x_1} = \sum_{x \in \overline{X}(\mathbf{W})} \{x, x_1\} \Delta(x) \rho_x = -\frac{1}{2}\rho_{x_1} + \frac{1}{2}\rho_{x_2} + \sum_{x \in \overline{X}(\mathbf{W}) \setminus \{x_1, x_2\}} \{x, x_1\} \rho_x,$$

$$R_{x_2} = \sum_{x \in \overline{X}(\mathbf{W})} \{x, x_2\} \Delta(x) \rho_x = -\frac{1}{2}\rho_{x_2} + \frac{1}{2}\rho_{x_1} + \sum_{x \in \overline{X}(\mathbf{W}) \setminus \{x_1, x_2\}} \{x, x_2\} \rho_x.$$

(Note that $\Delta(x_1) = \Delta(x_2) = -1$, see [20, 4.14].) Now consider the character $\overline{\rho}_{x_1}$, defined by

$$\overline{\rho}_{x_1}(g) := \overline{\rho_{x_1}(g)} \quad \text{for } g \in \mathbf{G}^F.$$

$\overline{\rho}_{x_1}$ is again a cuspidal unipotent (irreducible) character of $\mathbf{G}^F$. Since the character field of the two cuspidal unipotent characters of groups of type E_7 contains non-real elements (see [9, 5.4, Table 1]), we conclude $\overline{\rho}_{x_1} = \rho_{x_2}$. In view of Lemma 5.3 and since $\{x, x_1\} = \{x, x_2\}$ for any $x \in \overline{X}(\mathbf{W}) \setminus \{x_1, x_2\}$, we obtain

$$\xi_{x_1} q^{7/2} = \xi_{x_1} \chi_1(u_0) = R_{x_1}(u_0) = R_{x_2}(u_0) = \xi_{x_2} \chi_2(u_0) = \xi_{x_2} q^{7/2},$$

which also equals $\overline{R_{x_2}(u_0)}$ (using once again Lemma 5.3). We deduce $\xi_{x_1} = \xi_{x_2} = \overline{\xi}_{x_2}$ and thus $\xi_{x_1} = \xi_{x_2} \in \{\pm 1\}$, since $\xi_{x_2}\overline{\xi}_{x_2} = 1$.

5.5. In order to determine the sign $\xi := \xi_{x_1} = \xi_{x_2} \in \{\pm 1\}$, we want to apply a formula in [10, 3.6(b)], as in [16]. However, we will need more theoretical ingredients than in [16], most notably the generalised Springer correspondence (see Section 4). We consider the Hecke algebra of the group $\mathbf{G}^F$ with its BN-pair $(\mathbf{B}^F, N_{\mathbf{G}}(\mathbf{T})^F)$, that is, the endomorphism algebra

$$\mathcal{H} := \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F}(\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}^F])^{\mathrm{opp}}$$

("opp" stands for the opposite algebra). $\mathcal{H}$ has a $\overline{\mathbb{Q}}_\ell$-basis $\{T_w \mid w \in \mathbf{W}\}$, where

$$T_w \colon \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}^F] \to \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}^F], \quad x\mathbf{B}^F \mapsto \sum_{\substack{y\mathbf{B}^F \in \mathbf{G}^F/\mathbf{B}^F \\ x^{-1}y \in \mathbf{B}^F \dot{w}\mathbf{B}^F}} y\mathbf{B}^F,$$

for $w \in \mathbf{W}$. Recall from 3.5 that $\mathbf{S} = \{s_1, \ldots, s_7\}$ is a set of Coxeter generators for $\mathbf{W}$. Denote by $\ell \colon \mathbf{W} \to \mathbb{Z}_{\geqslant 0}$ the length function of $\mathbf{W}$ with respect to $\mathbf{S}$. Then the multiplication in $\mathcal{H}$ is determined by the following equations.

$$T_s \cdot T_w = \begin{cases} T_{sw} & \text{if} \quad \ell(sw) = \ell(w) + 1 \\ qT_{sw} + (q-1)T_w & \text{if} \quad \ell(sw) = \ell(w) - 1 \end{cases} \quad (\text{for } s \in \mathbf{S}, \, w \in \mathbf{W}).$$

The irreducible characters of $\mathbf{W}$ naturally parametrise the isomorphism classes of irreducible modules of $\mathcal{H}$, see [4, §11D]. Given $\phi \in \mathrm{Irr}(\mathbf{W})$, let V_ϕ be the corresponding module of $\mathcal{H}$, and let $\rho_\phi \in \mathrm{Uch}(\mathbf{G}^F)$ be the image of ϕ under the map $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \mathrm{Uch}(\mathbf{G}^F)$, see Remark 3.6. By [10, 3.6] and [14, §8.4], we have

$$(5.3) \qquad \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \rho_\phi(g) \, \mathrm{Tr}(T_w, V_\phi) = \frac{|O_g \cap \mathbf{B}^F \dot{w}\mathbf{B}^F| \cdot |C_{\mathbf{G}^F}(g)|}{|\mathbf{B}^F|}$$

for any $g \in \mathbf{G}^F$ and $w \in \mathbf{W}$, where $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})^F$ is a representative of w, $O_g \subseteq \mathbf{G}^F$ denotes

the $\mathbf{G}^F$-conjugacy class of g and $\mathrm{Tr}(T_w, V_\phi)$ is the trace of the linear map on V_ϕ defined by T_w. The character table of $\mathcal{H}$ is contained in CHEVIE [13], so the numbers $\mathrm{Tr}(T_w, V_\phi)$ are known. We want to apply (5.3) with $g = u_0$ to get hold of the sign ξ. In order to achieve this, we thus need some information about the values $\rho_\phi(u_0)$ for $\phi \in \mathrm{Irr}(\mathbf{W})$. By (3.5) and (5.1), we have

$$\rho_\phi = \Delta(x_\phi) \sum_{x \in \overline{X}(\mathbf{W})} \{x_\phi, x\} R_x = \sum_{x \in \overline{X}(\mathbf{W})} \{x_\phi, x\} \xi_x \chi_{A_x}.$$

Lemma 5.6. *Let $x \in \overline{X}(\mathbf{W})$ be such that, via the generalised Springer correspondence* (4.3), *A_x arises from a pair $\iota = (\mathcal{O}, \mathscr{E}) \in \mathscr{N}_{\mathbf{G}}^F$ where $\mathcal{O} \neq \mathcal{O}_{\mathrm{reg}}$. Then*

$$\chi_{A_x}(u_0) = R_x(u_0) = 0.$$

Proof. By (4.1), the support of $A_x|_{\mathbf{G}_{\mathrm{uni}}}$ is $\overline{\mathcal{O}}$ which is a union of conjugacy classes of $\mathbf{G}$ and therefore does not contain any regular unipotent elements in case $\mathcal{O} \neq \mathcal{O}_{\mathrm{reg}}$, as $\mathcal{O}_{\mathrm{reg}}$ is the unique class of $\mathbf{G}$ whose closure is $\mathbf{G}_{\mathrm{uni}}$. So we have $\chi_{A_x}(u_0) = 0$ and then also $R_x(u_0) = 0$ in view of Theorem 3.3. $\qquad\square$

5.7. Let us now describe the images of the pairs $\iota = (\mathcal{O}_{\mathrm{reg}}, \mathscr{E}) \in \mathscr{N}_{\mathbf{G}}^F$ under the map (4.3), following Spaltenstein [47, p. 331]. Since F acts trivially on $A(u_0) \cong \mathbb{Z}/4\mathbb{Z}$, there are (up to isomorphism) four F-stable irreducible $\mathbf{G}$-equivariant local systems on the regular unipotent class $\mathcal{O}_{\mathrm{reg}}$. We will write $\mathscr{E}^{\zeta^i}$ ($1 \leqslant i \leqslant 4$) for the local system corresponding to the irreducible character of $A(u_0)$ which takes the value ζ^i at $a_0 = u_0 C_{\mathbf{G}^\circ}(u_0) \in A(u_0)$, where ζ is the primitive fourth root of unity in $\overline{\mathbb{Q}}_\ell$ specified in 5.1, 3.6. First consider $\iota = (\mathcal{O}_{\mathrm{reg}}, \mathscr{E}^1) \in \mathscr{N}_{\mathbf{G}}^F$. The corresponding element of $\mathscr{M}_{\mathbf{G}}^F$ is $(\mathbf{T}, \{1\}, \overline{\mathbb{Q}}_\ell)$, and among the simple summands of $K_{(\mathbf{T},\{1\},\overline{\mathbb{Q}}_\ell)}$, the character sheaf $A_x \cong A_{(\mathcal{O}_{\mathrm{reg}}, 1)}$ is parametrised by the trivial character $1_{\mathbf{W}}$ of $\mathbf{W} = W_{\mathbf{G}}(\mathbf{T})$, whence $R_x = R_{1_{\mathbf{W}}} = \rho_{1_{\mathbf{W}}} = 1_{\mathbf{G}^F}$. Next, the pairs $(\mathcal{O}_{\mathrm{reg}}, \mathscr{E}^{\pm\zeta})$ correspond to $(\mathbf{G}, \mathcal{O}_{\mathrm{reg}}, \pm\zeta)$, so they give rise to the two cuspidal (unipotent) character sheaves $A_{(\mathcal{O}_{\mathrm{reg}}, \pm\zeta)}$. With the arrangements in 5.1, 5.2, we have $A_{x_1} \cong A_{(\mathcal{O}_{\mathrm{reg}}, \zeta)}$ and $A_{x_2} \cong A_{(\mathcal{O}_{\mathrm{reg}}, -\zeta)}$. Finally, to the pair $(\mathcal{O}_{\mathrm{reg}}, \mathscr{E}^{-1}) \in \mathscr{N}_{\mathbf{G}}^F$ is associated the triple $(\mathbf{L}, \mathcal{O}_0, \mathscr{E}_0) \in \mathscr{M}_{\mathbf{G}}^F$ where $\mathbf{L} \subseteq \mathbf{G}$ is the Levi complement of some parabolic subgroup of $\mathbf{G}$ such that $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ is simple of type D_4, and $(\mathcal{O}_0, \mathscr{E}_0)$ is the unique cuspidal pair for $\mathbf{L}$, see [21, 15.2]. Among the simple summands of $K_{(\mathbf{L}, \mathcal{O}_0, \mathscr{E}_0)}$, the character sheaf $A_{(\mathcal{O}_{\mathrm{reg}}, \mathscr{E}^{-1})}$ is described by the trivial character of $W_{\mathbf{G}}(\mathbf{L}) \cong W(B_3)$. Let $x_0 \in \overline{X}(\mathbf{W})$ be such that $A_{x_0} \cong A_{(\mathcal{O}_{\mathrm{reg}}, \mathscr{E}^{-1})}$, and let $\mathscr{F} \subseteq \mathrm{Irr}(\mathbf{W})$ be the family for which $x_0 \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$. Then, for $\phi \in \mathscr{F}$, we have

$$\rho_\phi(u_0) = \sum_{x \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})} \{x_\phi, x\} R_x(u_0) = \{x_\phi, x_0\} R_{x_0}(u_0).$$

Using the notation of [2, p. 482], we obtain

$$\rho_{\phi_{56,3}}(u_0) = -\rho_{\phi_{35,4}}(u_0) = -\rho_{\phi_{21,6}}(u_0) = \frac{1}{2} R_{x_0}(u_0).$$

Now, considering the two characters $\phi_{512,11}, \phi_{512,12} \in \mathrm{Irr}(\mathbf{W})$ of degree 512, we see that

(5.4)
$$\rho_{\phi_{512,11}}(u_0) = \{x_{\phi_{512,11}}, x_1\} \xi \chi_{A_{x_1}}(u_0) + \{x_{\phi_{512,11}}, x_2\} \xi \chi_{A_{x_2}}(u_0) = \xi q^{7/2},$$
$$\rho_{\phi_{512,12}}(u_0) = \{x_{\phi_{512,12}}, x_1\} \xi \chi_{A_{x_1}}(u_0) + \{x_{\phi_{512,12}}, x_2\} \xi \chi_{A_{x_2}}(u_0) = -\xi q^{7/2}.$$

For any $\phi \in \mathrm{Irr}(\mathbf{W})$ which is not one of $1_{\mathbf{W}} = \phi_{1,0}, \phi_{56,3}, \phi_{35,4}, \phi_{21,6}, \phi_{512,11}, \phi_{512,12}$, we have $\rho_\phi(u_0) = 0$. We evaluate the left side of (5.3) with $g = u_0$ and

$$w = w_c := s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \in \mathbf{W},$$

a Coxeter element of $\mathbf{W}$. (In fact any other Coxeter element of $\mathbf{W}$ would lead to the same result, see [14, 8.2.6].) We obtain

$$q^7 + \frac{1}{2} R_{x_0}(u_0) \left(\mathrm{Tr}(T_{w_c}, V_{\phi_{56,3}}) - \mathrm{Tr}(T_{w_c}, V_{\phi_{35,4}}) - \mathrm{Tr}(T_{w_c}, V_{\phi_{21,6}}) \right) + 2\xi q^7.$$

Here, $\mathrm{Tr}(T_{w_c}, V_{\phi_{56,3}}) - \mathrm{Tr}(T_{w_c}, V_{\phi_{35,4}}) - \mathrm{Tr}(T_{w_c}, V_{\phi_{21,6}}) = 2q^5$, so we have

$$(5.5) \qquad \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \rho_\phi(u_0) \, \mathrm{Tr}(T_{w_c}, V_\phi) = q^7(1 + 2\xi) + q^5 R_{x_0}(u_0).$$

Lemma 5.8. *As in 5.7 (and with the notations there), let $x_0 \in \overline{X}(\mathbf{W})$ be such that $A_{x_0} \cong A_{(\mathscr{O}_{\mathrm{reg}}, \mathscr{E}^{-1})}$. Then there is a sign $\delta \in \{\pm 1\}$ such that*

$$R_{x_0}(u_0) = \delta q^2.$$

Proof. We keep the notation of 5.7 and set $\iota := (\mathscr{O}_{\mathrm{reg}}, \mathscr{E}^{-1}) \in \mathscr{N}_{\mathbf{G}}^F$. The character sheaf $A_{x_0} \cong A_\iota \in \hat{\mathbf{G}}^{\mathrm{un}}$ is an irreducible direct summand of $K := K_{(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)}$, where $\mathbf{L} \subseteq \mathbf{G}$ is the Levi complement of some parabolic subgroup of $\mathbf{G}$ such that $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ is simple of type D_4, and $(\mathscr{O}_0, \mathscr{E}_0)$ is the unique cuspidal pair for $\mathbf{L}$. Among the simple summands of K, A_ι is uniquely characterised by the property

$$A_\iota|_{\mathbf{G}_{\mathrm{uni}}} \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}^{-1})[\dim \mathscr{O}_{\mathrm{reg}} + \dim \mathbf{Z}(\mathbf{L})^\circ],$$

see (4.1). Now consider the functions X_ι, Y_ι defined in 4.2. Using Theorem 4.3 and Remark 4.4, we see that

$$\chi_{A_\iota, \varphi_{A_\iota}}(u_0) = -X_\iota(u_0) = -Y_\iota(u_0) = -\mathrm{Trace}(\psi_{\iota, u_0}, \mathscr{E}_{u_0}^{-1}).$$

In order to meet the requirements of Theorem 2.4 (see [26, 25.1]), we need to modify the isomorphism $\varphi_{A_\iota} \colon F^* A_\iota \xrightarrow{\sim} A_\iota$, as follows. Let $\Sigma_0 := \mathbf{Z}(\mathbf{L})^\circ \mathscr{O}_0$. By [25, 19.3], the image of Σ_0 (or $\mathscr{O}_0$) under the canonical map $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ is the regular unipotent class of $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$. As in [11, 4.4], we may single out a specific element $g_1 \in (\Sigma_0/\mathbf{Z}(\mathbf{L})^\circ)^F$. Following [27, 3.2], the choice of g_1 then gives rise to an isomorphism $\varphi_0 \colon F^* \mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ by requiring that it induces on the stalk of $\mathscr{E}_0$ at g_1 (or, more precisely, at the preimage of g_1 under $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$) the map given by scalar multiplication with $q^{(\dim(\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ) - \dim \mathscr{O}_0)/2} = q^2$. Once φ_0 is fixed, it naturally induces an isomorphism $F^* K \xrightarrow{\sim} K$, which in turn determines an isomorphism $\varphi_{A_{x_0}} \colon F^* A_{x_0} \xrightarrow{\sim} A_{x_0}$, see again [27, 3.2]. Then $\varphi_{A_{x_0}}$ satisfies the requirements formulated in [26, 25.1], so it is a valid choice for Theorem 2.4. In particular, we have $\langle \chi_{A_{x_0}, \varphi_{A_{x_0}}}, \chi_{A_{x_0}, \varphi_{A_{x_0}}} \rangle = 1$. On the other hand, recall from 4.2 that the definition of $\varphi_{A_\iota} \colon F^* A_\iota \xrightarrow{\sim} A_\iota$ is based on the choice of an isomorphism $F^* \mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order on the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Modifying this isomorphism by multiplication with a root of unity, if necessary, we may thus achieve that $\varphi_{A_{x_0}} = q^2 \cdot \varphi_{A_\iota}$. Now, by Theorem 3.3, we have

$$R_{x_0} = \xi_{x_0} \chi_{A_{x_0}, \varphi_{A_{x_0}}}$$

for some $\xi_{x_0} \in \overline{\mathbb{Q}}_\ell$ such that $\xi_{x_0}\overline{\xi}_{x_0} = 1$. So we obtain

$$R_{x_0}(u_0) = \xi_{x_0}\chi_{A_{x_0},\varphi_{A_{x_0}}}(u_0) = \xi_{x_0}q^2 \cdot \chi_{A_\iota,\varphi_{A_\iota}}(u_0) = -\xi_{x_0}q^2 \cdot \mathrm{Trace}(\psi_{\iota,u_0}, \mathscr{E}_{u_0}^{-1})$$
$$= \delta q^2,$$

where $\delta := -\xi_{x_0}\mathrm{Trace}(\psi_{\iota,u_0}, \mathscr{E}_{u_0}^{-1}) \in \overline{\mathbb{Q}}_\ell$. Since the local system $\mathscr{E}^{-1}$ is one-dimensional and $\psi_{\iota,u_0}: \mathscr{E}_{u_0}^{-1} \to \mathscr{E}_{u_0}^{-1}$ has finite order, the number $\mathrm{Trace}(\psi_{\iota,u_0}, \mathscr{E}_{u_0}^{-1})$ is a root of unity in $\overline{\mathbb{Q}}_\ell$, so δ satisfies $\delta\overline{\delta} = 1$, that is, $\overline{\delta} = \delta^{-1}$. Using Lemma 5.3, we conclude

$$\delta q^2 = R_{x_0}(u_0) = R_{x_0}(u_0^{-1}) = \overline{\delta}q^2,$$

so we must have $\delta = \overline{\delta} = \delta^{-1} \in \{\pm 1\}$, as desired. $\qquad\square$

5.9. In view of Lemma 5.8, (5.3) and (5.5), we have

$$(5.6) \qquad \frac{|O_{u_0} \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F| \cdot |C_{\mathbf{G}^F}(u_0)|}{|\mathbf{B}^F|} = q^7(1 + 2\xi + \delta),$$

where $\xi \in \{\pm 1\}$ is as in 5.5, $\delta \in \{\pm 1\}$ is as in Lemma 5.8, and $\dot{w}_c \in N_{\mathbf{G}}(\mathbf{T})^F$ is a representative of $w_c \in \mathbf{W}$. We claim that $O_{u_0} \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F \neq \emptyset$. Indeed, with the notation of 3.5, consider the longest element w_0 of $\mathbf{W}$ with respect to the usual length function on $\mathbf{W}$ determined by the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_7\} \subseteq R^+$. $\mathbf{W}$ is a Coxeter group with generators $\{s_1, \ldots, s_7\}$, and there is a natural action of $\mathbf{W}$ on the roots R. Using some standard properties of finite Coxeter groups (see, e.g., [48, Appendix]), the element w_0 is then characterised by the property $w_0(R^+) = -R^+$, so we have $-w_0(\Pi) = \Pi$. More precisely, $-w_0$ defines a graph automorphism of the Dynkin diagram of E_7, but the only such automorphism is the identity, so $-w_0(\alpha_i) = \alpha_i$ for $1 \leqslant i \leqslant 7$. Let us choose a representative $\dot{w}_0 \in N_{\mathbf{G}}(\mathbf{T})^F$ of $w_0 \in \mathbf{W}$. For $1 \leqslant i \leqslant 7$, we have $\dot{w}_0 \mathbf{U}_{\alpha_i}\dot{w}_0^{-1} = \mathbf{U}_{-\alpha_i}$, so $\dot{w}_0 u_i(1)\dot{w}_0^{-1} \in \mathbf{U}_{-\alpha_i}$. Now

$$\mathbf{U}_{-\alpha_i} \subseteq \mathbf{L}_{\{s_i\}} \subseteq \mathbf{P}_{\{s_i\}} = \mathbf{B} \cup \mathbf{B}s_i\mathbf{B}$$

(see, e.g., [2, 2.6.2]). Since $\mathbf{U}_{-\alpha_i} \cap \mathbf{B} = \{1\}$, we must have $\dot{w}_0 u_i(1)\dot{w}_0^{-1} \in \mathbf{B}s_i\mathbf{B}$. As $w_c = s_1 \cdots s_7$ is a reduced expression for w_c, we know that

$$\mathbf{B}s_1\mathbf{B} \cdot \mathbf{B}s_2\mathbf{B} \cdots \mathbf{B}s_7\mathbf{B} = \mathbf{B}w_c\mathbf{B},$$

and so

$$\dot{w}_0 \cdot u_0 \cdot \dot{w}_0^{-1} \in O_{u_0} \cap \mathbf{B}w_c\mathbf{B} = O_{u_0} \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F.$$

Hence, the left side of (5.6) is strictly positive. Thus (5.6) implies that $\xi \neq -1$, so we must have $\xi = +1$. This proves Proposition 5.2.

REMARK 5.10. As described in 5.1, the isomorphisms $\varphi_{A_i}: F^*A_i \xrightarrow{\sim} A_i$ $(i = 1, 2)$ are determined by choosing one of the four $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}_{\mathrm{reg}}^F$. The proof of Proposition 5.2 relies on the following two properties of the $\mathbf{G}^F$-conjugacy class O_{u_0} of u_0:

(1) Any element of O_{u_0} is $\mathbf{G}^F$-conjugate to its inverse, that is, $O_{u_0} = O_{u_0}^{-1}$;
(2) We have $O_{u_0} \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F \neq \emptyset$.

Once ξ is specified, a similar computation as in 5.9 shows that $O_{u_{a_0^2}} \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F = \emptyset$. Hence,

among the four $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$, the class O_{u_0} is uniquely determined by (1) and (2) above.

5.11. We conclude by giving the values of the (unipotent) characters in the family

$$\mathscr{F}_0 = \left\{\rho_{\phi_{512,11}}, \rho_{\phi_{512,12}}, E_7[\zeta], E_7[-\zeta]\right\} \subseteq \overline{X}(\mathbf{W})$$

at the regular unipotent elements. Using (3.5), 5.6 and 5.2, we have for $u \in \mathscr{O}_{\mathrm{reg}}^F$:

$$\rho_{\phi_{512,11}}(u) = -\rho_{\phi_{512,12}}(u) = (\chi_{A_1,\varphi_{A_1}}(u) + \chi_{A_2,\varphi_{A_2}}(u))/2;$$
$$E_7[\zeta](u) = -E_7[-\zeta](u) = -(\chi_{A_1,\varphi_{A_1}}(u) - \chi_{A_2,\varphi_{A_2}}(u))/2.$$

The values of the $\chi_{A_i,\varphi_{A_i}}$ are given at the end of 5.1. We thus get the following table, where $u \sim u_a$ means that u lies in the $\mathbf{G}^F$-conjugacy class corresponding to $a \in A(u_0)$ as described in 5.1.

	$u \sim u_0$	$u \sim u_{a_0}$	$u \sim u_{a_0^2}$	$u \sim u_{a_0^3}$
$\rho_{\phi_{512,11}}(u)$	$q^{7/2}$	0	$-q^{7/2}$	0
$\rho_{\phi_{512,12}}(u)$	$-q^{7/2}$	0	$q^{7/2}$	0
$E_7[\zeta](u)$	0	$-q^{7/2}\zeta$	0	$q^{7/2}\zeta$
$E_7[-\zeta](u)$	0	$q^{7/2}\zeta$	0	$-q^{7/2}\zeta$

Note added in proof. In the meantime, the remaining open problems for type F_4 and E_6, as well as E_7 in characteristic $p > 2$, have been solved in a preprint by Geck, see arXiv:2105.00722

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