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SUPERSINGULAR LOCI OF LOW DIMENSIONS AND PARAHORIC SUBGROUPS

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Abstract

The theory of polarized supersingular abelian varieties (A, λ) is often essentially related to the theory of quaternion hermitian lattices. In this paper, we add one more such relation by giving an adelic description of supersingular loci of low dimensions. Katsura, Li and Oort have shown that the supersingular locus in the moduli of principally polarized abelian varieties is not irreducible and that the number of its irreducible components is equal to the class number of certain maximal quaternion hermitian lattices. Now the locus has certain natural algebraic subsets consisting of A with fixed *a-numbers* that are defined as the dimensions of embeddings from α_p to A. For low dimensional cases when $\dim A \leq 3$, we describe configuration of these subsets in the locus by intersection properties of some adelic subgroups of quaternion hermitian groups corresponding to parahoric subgroups locally at characteristic p. In particular, we describe which components each superspecial point lies on. This is proved by using certain liftability property of isogenies of abelian varieties, where the isogenies are interpreted to cosets of GL_n and of parahoric subgroups of the quaternion hermitian groups acting on quaternion hermitian matrices.

1. Introduction

To study geometry of supersingular abelian varieties, often a pure arithmetic theory of quaternion hermitian lattices is very powerful, as can be seen for example in [4], [11], [7], [8], [6], [5], [9]. In this paper, from such view point, we consider configurations of the components and certain algebraic subsets with fixed a-numbers in supersingular locus of principally polarized abelian varieties of dimension 2 or 3, explaining geometrical facts by Hecke double cosets of the general linear groups and by adelic cosets or double cosets associated with parahoric subgroups of the quaternion hermitian group of degree 2 or 3. Although many geometrical facts on the supersingular locus have been already known for small dimensions in [11], [12], [10] and also for general dimensions notably in [13], [3], our point here is more arithmetical rather than geometrical, and in fact we add new group theoretical details to such theory in this paper. For example, for small dimensions, we characterize descendable directions of polarizations by cosets of the minimal parahoric subgroups and criterion on the descendability by non-emptiness of intersection of certain adelic double cosets. These results are not trivial interpretation of known geometrical facts at all. Our intention here is to explain everything only by *superspecial* abelian varieties. This way of treatment makes the theory much more arithmetical. The subject is close to Harashita's work in [3] on a-number

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stratification for general dimensional case, and the author believes that the method used in this paper gives a basis for further study in general dimensional case.

Now we explain more details. Let k be an algebraically closed field of characteristic $p > 0$. We denote by $\mathcal{A}_{n,1}$ the moduli of principally polarized abelian varieties (A, λ) defined over k with $\dim A = n$, and by $\mathcal{S}_{n,1}$ the supersingular locus, that is, the locus of those (A, λ) such that A are supersingular abelian varieties. Here being supersingular means that A is isogenous to the product E^n of a supersingular elliptic curve E. The locus $\mathcal{S}_{n,1}$ is not irreducible in general. The number of irreducible components is equal to the class number of the principal genus (resp. the non-principal genus), if $\dim A$ is odd (resp. even) ([12, Theorem 5.7], [13, Theorem 4.9]). Here the genera consist of certain maximal quaternion hermitian lattices over $\mathrm{End}(E)$ whose definitions will be given in later sections. On the other hand, the number of isomorphism classes of principal polarizations on E^n is given by the class number of principal genus. These can be regarded as discrete points in $\mathcal{S}_{n,1}$ ([4, Theorem 2.10]). Here we may pose a problem how to describe in lattice terminology which superspecial points are on which irreducible components of $\mathcal{S}_{n,1}$. We define the a-number of A by $a = \dim \mathrm{Hom}(\alpha_p, A)$. If (A, λ) is generic then $a = 1$, and (A, λ) represents an irreducible component. We have $a = n$ if and only if A is superspecial ([15, Theorem 2]). So the above problem is on relations between some algebraic sets with $a = 1$ and $a = n$. More generally we may ask how to define some natural families W_i in $\mathcal{S}_{n,1}$ of suitably chosen principally polarized supersingular abelian varieties (A, λ) with $a = i$ for some $1 \leq i \leq n$ and to describe inclusion relations between these families W_i. We answer these questions when $n = 2$ and 3 by using adelic cosets or double cosets. Such inclusion relations as above will be interpreted by non-emptiness of intersections of adelic double cosets corresponding to various kinds of genera of quaternion hermitian lattices (See Theorems 5.1, 5.2 in Section 5). Here the genera are not necessarily those of maximal lattices.

The results mentioned above are derived from characterization of polarized flag type quotients (pftq for short) in [11], [13]. Roughly speaking a pftq is a certain sequence of polarized abelian varieties, and our point here is how to describe a polarization and an isogeny of E^n to E^n such that the polarization has a descent along that isogeny. When $n = 2$, these isogenies appear as good directions and very good directions in [12, p. 119]. So we will treat a generalization and characterization of these directions. For two kinds of *directions* we will treat, we give bijective correspondences to cosets of the minimal parahoric subgroup in two different maximal parahoric subgroups when $n = 2$ and 3. Then we describe configuration by intersection properties of certain adelic double cosets. Our description is new even when $n = 2$. Our ingredient of the proof is essentially a pure arithmetic theory of integral quaternion hermitian matrices.

The paper is organized as follows. In the next section we review well-known facts on geometry from [14], [12], [11], [13], [3], [10] mainly for the three dimensional case. In section 3, we review Hecke double cosets of GL_n and general arithmetic theory of quaternion hermitian lattices and the quaternion hermitian groups. In section 4, first we define quaternion hermitian matrices and lattices corresponding to polarizations we need. Then we define a *direction* to be a coset $GL_n(O)g$ for $g \in M_n(O)$ for $O = \mathrm{End}(E)$ and we give characterization when a quaternion hermitian matrix in question has a descent by which directions. This amounts to give descendability of polarizations by some specified isogenies. We establish

here not only bijection between *descendable* directions and cosets of the minimal parahoric subgroup in maximal parahoric subgroups, but also a bijection between the set of orbits of both by the action of lattice automorphisms. As a corollary, we see that existence of a part of a pftq sequence is equivalent to a non-emptiness of the intersection of the double cosets corresponding to neighbouring polarizations. This section is an essence of the whole proof. A geometrical interpretation of the results in this section is almost clear, but in Section 5, we state some main conclusions obtained from Section 4.

2. Geometric terminology and known facts

As for most of the terminology or results in this section, see [14], [4], [12], [11], [13]. Let k be as before. We say that an elliptic curve E is supersingular if $\mathrm{End}(E)$ is a maximal order O of the definite quaternion algebra $B = O \otimes_{\mathbb{Z}} \mathbb{Q}$ which ramifies at p and ∞. We say that A is supersingular if it is isogenous to a product of supersingular elliptic curves, or equivalently, to E^n for a supersingular elliptic curve E. By the way, if $n \geq 2$, then any product of n supersingular elliptic curves are isomorphic (Deligne, Ogus, Shioda). We take E such that it is defined over $\mathbb{F}_p$ and hence $\mathrm{End}(E)$ contains an element π with $\pi^2 = -p$. We define an effective divisor X of E^n by

$$X = \{0\} \times E^{n-1} + E \times \{0\} \times E^{n-2} + \cdots + E^{n-1} \times \{0\}$$

and define an isomorphism ϕ_X of E^n to the dual $(E^n)^t$ of E^n by $\phi_X(t) = Cl(X_t - X)$ where X_t is the translation of X by t and Cl is the linear equivalence class. This gives a principal polarization of E^n. We denote by $\mathrm{Herm}_n(B)$ the set of quaternion hermitian matrices in $M_n(B)$ and by $\mathrm{Herm}_n^+(B)$ the subset of $\mathrm{Herm}_n(B)$ of positive definite elements. We put $\mathrm{Herm}_n(O) = \mathrm{Herm}_n(B) \cap M_n(O)$ and $\mathrm{Herm}_n^+(O) = \mathrm{Herm}_n^+(B) \cap M_n(O)$. Then by the mapping

$$\mathrm{Hom}(E^n, (E^n)^t) \ni \lambda \to \phi_X^{-1} \circ \lambda \in M_n(O),$$

we can identify the Neron Severi group $\mathrm{NS}(E^n)$ with $\mathrm{Herm}_n(O)$ and the polarizations of E^n with $\mathrm{Herm}_n^+(O)$. We say that elements H_1, $H_2 \in \mathrm{Herm}_n(B)$ are equivalent if $H_2 = \epsilon H_1 \epsilon^*$ for some $\epsilon \in GL_n(O) = M_n(O)^\times$, where for $\epsilon = (e_{ij})$, we write $\epsilon^* = (\overline{e_{ji}})$, denoting by overline the main involution of B. Then isomorphism classes of polarizations of E^n are identified with equivalence classes in $\mathrm{Herm}_n^+(O)$. The Hauptnorm of $\mathrm{Herm}_n(B)$ is defined to be a polynomial map such that $\mathrm{Hm}(H)^2 = N(H)$ for each $H \in \mathrm{Herm}_n(B)$ with reduced norm $N(H)$ and that $\mathrm{Hm}(1_n) = 1$ for the unit matrix 1_n([1, Chapter 2.2, denoted as *det* there]). We define a subset $\mathcal{H}^{(n)}_{\mathrm{prin}}$ of $\mathrm{Herm}_n^+(O)$ by

$$\mathcal{H}^{(n)}_{\mathrm{prin}} = \{H \in \mathrm{Herm}_n^+(O); \mathrm{Hm}(H) = 1\}.$$

We easily see that $\mathcal{H}^{(n)}_{\mathrm{prin}} = GL_n(O) \cap \mathrm{Herm}_n^+(O)$. When $n \geq 2$, we define

$$\mathcal{H}^{(n)}_{\mathrm{np}} = \{H \in \mathrm{Herm}_n^+(O) \cap \pi M_n(O); \mathrm{Hm}(H) = p^{[(n+1)/2]}\},$$

where $[*]$ denotes the maximal integer which does not exceed $*$. A polarization λ of E^n is a principal polarization if and only it $\phi_X^{-1}\lambda$ belongs to $\mathcal{H}^{(n)}_{\mathrm{prin}}$ ([4, Theorem 2.10]). For the moment, for the sake of simplicity, we call the set $\mathcal{H}^{(n)}_{\mathrm{prin}}$ the principal genus and $\mathcal{H}^{(n)}_{\mathrm{np}}$

the non-principal genus. More rigorous definition of notion related to quaternion hermitian lattices will be explained in the next section. The number of equivalence classes in a genus is finite and called the class number of the genus.

Assume that $\dim A = n$ and A is supersingular. We define the finite group scheme $\alpha_p = \mathrm{Spec}(k[x]/(x^p))$ as usual. We call a sequence

$$E^n = Y_{n-1} \overset{\phi_{n-1}}{\longrightarrow} Y_{n-2} \overset{\phi_{n-2}}{\longrightarrow} \cdots \longrightarrow Y_1 \overset{\phi_1}{\longrightarrow} Y_0 = Y$$

a flag type quotient (ftq for short) if $\mathrm{Ker}(\phi_i) \cong (\alpha_p)^i$. We say that a ftp sequence with polarizations μ_i

$$(Y_{n-1}, \mu_{n-1}) \overset{\phi_{n-1}}{\longrightarrow} (Y_{n-2}, \mu_{n-2}) \overset{\phi_{n-2}}{\longrightarrow} \cdots \overset{\phi_1}{\longrightarrow} (Y_0, \mu_0)$$

is a polarized flag type quotient (pftq for short) if μ_0 is a principal polarization, $Y_{n-1} = E^n$, $\phi_i^*(\mu_{i-1}) = \mu_i$ and

$$\ker \mu_i \subset \bigcap_{j=0}^{[i/2]} Y_i[V^j \circ F^{i-j}].$$

where F is the absolute Frobenius and V the Verschiebung ([14], [13, Section 3]). We put $a = \dim \mathrm{Hom}(\alpha_p, A)$. This is called the *a-number* of A.

Proposition 2.1. *We assume that $n \geq 3$ is odd.*
(i) *For any principal polarization λ of A, there exists a pftq starting from $(E^n, p^{(n-1)/2}\lambda_0)$ for a principal polarization λ_0 and ending at (A, λ).*
(ii) *If $a(A) = 1$, then pftq in* (i) *is unique (up to natural isomorphisms of the sequence).*
(iii) *The irreducible components of the supersingular locus $\mathcal{S}_{n,1}$ in $\mathcal{A}_{n,1}$ correspond bijectively to the isomorphism classes of all principal polarizations λ_0 of E^n, or equivalently to the isomorphism classes of $\phi_X^{-1}\lambda_0 \in \mathcal{H}^{(n)}_{\mathrm{prin}}$.*

For the proof of this proposition, see [14, Theorem 2.2], [11, Lemma 4.4, Theorem 6,6, 6.7]. and [13, Theorem 4.9].

We denote by $V(\lambda_0)$ the irreducible component of $\mathcal{S}_{n,1}$ corresponding to a principal polarization λ_0 of E^n in the sense of the above (iii).

Proposition 2.2. *Assume that $n = \dim A = 3$.*
(i) *Assume that $a(A) = 2$ and λ is a principal polarization of A. Then there exists a polarization μ_1 of E^3 and an isogeny ϕ_1*

$$(E^3, \mu_1) \overset{\phi_1}{\longrightarrow} (A, \lambda) \tag{1}$$

such that $\mu_1 = \phi_1^*(\lambda)$, $\mathrm{Ker}(\mu_1) \cong \alpha_p^2 \subset E^3[F]$, and $\mathrm{Ker}(\phi_1) \cong \alpha_p$.
(ii) *For any polarization μ_1 of E^3 such that $\mathrm{Ker}(\mu_1) \cong \alpha_p^2$, there exists a principally polarized abelian threefold (A, λ) with $a(A) = 2$ having a sequence as in* (1) *for some ϕ_1.*
(iii) *Notation and assumption being as in* (i)*, if we regard (A, λ) with $a(A) = 2$ as a point in $S_{3,1}$, then we have $(A, \lambda) \in V(\lambda_0)$ for a principal polarization λ_0 of E^3 if and only if there exists an isogeny ϕ_2*

$$(E^3, p\lambda_0) \overset{\phi_2}{\longrightarrow} (E^3, \mu_1) \tag{2}$$

such that $p\lambda_0 = \phi_2^*(\mu_1)$ *and* $\mathrm{Ker}(\phi_2) \cong \alpha_p^2$.

For a proof, see Li and Oort [13, section 9.4], Katsura and Oort [11, Section 5], and Karemaker, Yobuko and Chia-Fu Yu [10, Proposition 3.12]. See also [19].

We give here a simple remark. For a fixed (A, λ), if there are two sequences

$$(E^3, \mu_1) \xrightarrow{\phi_1} (A, \lambda), \qquad (E^3, \lambda_1) \xrightarrow{\psi_1} (A, \lambda)$$

such that $\mu_1 = \phi_1^*(\lambda)$, $\lambda_1 = \psi_1^*(\lambda)$, $\mathrm{Ker}(\psi_1) \cong \mathrm{Ker}(\phi_1) \cong \alpha_p$ and $\mathrm{Ker}(\mu_1) \cong \mathrm{Ker}(\lambda_1) \cong \alpha_p^2 \subset E^3[F]$, then there exists an isomorphism of $\phi : (E^3, \mu_1) \cong (E^3, \lambda_1)$ such that this is an isomorphism between sequences. In other words, the above claim is summarized by

Lemma 2.3. *The sequence in* Proposition 2.2(i) (1) *is unique up to isomorphism.*

Proof. Since E^3 and A are not isomorphic, ϕ_1 and ψ_1 are minimal isogenies. Since $\mathrm{End}(E^3) = M_3(O)$ is of class number one, the module $\mathrm{Hom}(E^3, A)$ is free as right $\mathrm{End}(E^3)$ module and we have $\phi_0 \in \mathrm{Hom}(E^3, A)$ such that $\mathrm{Hom}(E^3, A) = \phi_0 \mathrm{End}(E^3)$. So $\phi_1 = \phi_0\epsilon_1$, $\psi_1 = \phi_0\epsilon_2$. Since $\deg(\phi_1) = \deg(\psi_1) = \deg(\phi_0) = p$, we see that ϵ_i are isomorphisms of E^3. So $\phi = \epsilon_2^{-1}\epsilon_1$ is an isomorphism of the claim. □

We add here one more remark whose proof will be obtained later as an easy corollary of Theorem 5.2.

Lemma 2.4. *Let* (E^3, λ) *be a principally polarized superspecial abelian threefold. Then there exists a polarization* μ_1 *of* E^3 *and an isogeny* ϕ_1

$$(E^3, \mu_1) \xrightarrow{\phi_1} (E^3, \lambda) \tag{3}$$

such that $\mathrm{Ker}(\mu_1) \cong \alpha_p^2$ *and* $\phi^*(\lambda) = \mu_1$.

For a polarization μ_1 of E^3 with $\mathrm{Ker}(\mu_1) \cong \alpha_p^2$, consider principally polarized abelian threefolds (A, λ) with $a(A) = 2$ such that there are sequences as in Proposition 2.2 (1) starting from (E^3, μ_1). We denote by $W(\mu_1)$ the Zariski closure of all such (A, λ) in $S_{3,1}$. In this setting, we have

Proposition 2.5. *A principally polarized superspecial abelian threefold* (E^3, λ) *belongs to* $W(\mu_1)$ *if and only if there exists the above sequence* (3) *in* Lemma 2.4 *for* λ *and* μ_1.

For the proof, see [11] and [10, Proposition 3.16].

Remark. For a polarized threefold (E^3, μ_1) in Proposition 2.2, of course principal polarizations λ_0 in the sequence (2) of Proposition 2.2 are not unique. This means that (A, λ) belongs to various irreducible components $V(\lambda_0)$ in $S_{3,1}$. Also, polarizations μ_1 in the sequence (3) are not unique. This means that (E^3, λ) belongs to various $W(\mu_1)$.

Next we fix two principal polarizations λ_0 and μ_0 of E^3. Then $(E^3, \mu_0) \in V(\lambda_0)$ if and only if there exists a polarized superspecial abelian threefold (E^3, λ_1) which gives a pftq by

$$(E^3, p\lambda_0) \xrightarrow{\phi_1} (E^3, \mu_1) \xrightarrow{\phi_0} (E^3, \mu_0). \tag{4}$$

Later we will see that this is equivalent to saying that there exists an isogeny $(E^3, p\lambda_0) \to (E^3, \mu_0)$. with $p\lambda_0 = \phi^*(\mu_0)$ (See Theorem 5.3 in section 5).

Next we consider the case $n = 2$. For the following proposition, see [12, Theorem 5.7]

and [13, Theorem 4.9].

Proposition 2.6. (i) *If n is even, then the number of irreducible components of $S_{n,1}$ is equal to the class number of the non-principal genus.*
(ii) *Let λ_1 be a polarization of E^2 such that $\phi_X^{-1}\lambda_1 \in \mathcal{H}_{np}^{(2)}$. Denote by $V(\lambda_1)$ the corresponding irreducible component in* (i). *Then a principally polarized abelian surface (E^2, λ) belongs to $V(\lambda_1)$ as a point of $S_{2,1}$ if and only if there exists an isogeny*

$$(E^2, \lambda_1) \xrightarrow{\phi_1} (E^2, \lambda)$$

such that $\lambda_1 = \phi_1^*(\lambda)$.

In Proposition 2.6, comparing the degrees, we see that $\deg(\phi_1) = p$ and $\mathrm{Ker}(\phi_1) \cong \alpha_p$

3. Review on basic arithmetic

For readers' convenience, in this section we shortly review Hecke double cosets of GL_n, quaternion hermitian lattices, and the quaternion hermitian group (See [18], [16], [17]). As before we consider the definite quaternion algebra B with discriminant $p\infty$. We fix a maximal order O of B such that $\pi \in O$ with $\pi^2 = -p$. Such a maximal order always exists. The choice of O is not essential, but this assumption on O sometimes makes explanation simpler. We denote by B_A the adelization of B. For any prime q, we put $O_q = O \otimes_{\mathbb{Z}} \mathbb{Z}_q$. We put $O_A = B_\infty \prod_q O_q$ where $B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R}$. We write $GL_n(O_A) = GL_n(B_\infty) \prod_q GL_n(O_q)$. We assume from now on that $n \geq 2$. Then by the strong approximation theorem of SL_n and the property of the norm images, we have

$$GL_n(B_A) = GL_n(B)GL_n(O_A). \tag{5}$$

For any $g = (g_v) \in M_n(O_A) \cap GL_n(B_A)$, consider a double coset $GL_n(O_A)gGL_n(O_A)$. By (5), we may assume that $g \in GL_n(B)$ in the above double coset. Then we have

$$GL_n(O_A)\backslash GL_n(O_A)gGL_n(O_A) \cong GL_n(O)\backslash GL_n(O)gGL_n(O).$$

By this fact, the Hecke algebras with respect to the pair $(GL_n(B) \cap M_n(O), GL_n(O))$ and the pair $(GL_n(B_A) \cap M_n(O_A), GL_n(O_A))$ are isomorphic as rings. (Of course this is not true in general for $n = 1$). The isomorphism is given by the restriction to $GL_n(B)$. Now for $a_i \in B^\times$ $(i = 1, \ldots, n)$, we denote by $\mathrm{diag}(a_1, \ldots, a_n)$ the $n \times n$ diagonal matrix whose diagonal components are $a_1, \ldots, a_n$. For $g \in M_n(O) \cap GL_n(B)$, if we assume that $g \in GL_n(O_q)$ for any $q \neq p$, then in the double coset we may replace g by a diagonal matrix $\mathrm{diag}(\pi^{e_1}, \ldots, \pi^{e_n})$ for some integers e_i with $0 \leq e_1 \leq \cdots \leq e_n$ (See [18, Lemma 2]). We write

$$T_A(\pi^{e_1}, \ldots, \pi^{e_n}) = GL_n(O_A)\mathrm{diag}(\pi^{e_1}, \ldots, \pi^{e_n})GL_n(O_A),$$
$$T(\pi^{e_1}, \ldots, \pi^{e_n}) = GL_n(O)\mathrm{diag}(\pi^{e_1}, \ldots, \pi^{e_n})GL_n(O).$$

By the assumption $n \geq 2$, we have a bijection

$$GL_n(O_A)\backslash T_A(\pi^{e_1}, \ldots, \pi^{e_n}) \cong GL_n(O)\backslash T(\pi^{e_1}, \ldots, \pi^{e_n}).$$

Next we explain the arithmetic theory of quaternion hermitian lattices. We denote by B^n the left B vector space of row vectors and consider a quaternion hermitian metric on B^n defined by

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}, \qquad x = (x_i), y = (y_i) \in B^n.$$

This is the unique positive definite quaternion hermitian metric up to base change over B. A quaternion hermitian group G is defined by the set of elements $g \in M_n(B)$ such that $(xg, yg) = n(g)(x, y)$ for some $n(g) \in \mathbb{Q}_+^\times$, that is,

$$G = \{g \in M_n(B); gg^* = n(g)1_n\},$$

where 1_n is the $n \times n$ unit matrix. We denote by G_A the adelization of G and for a place v of $\mathbb{Q}$, we denote by G_v the v-component of G_A.

For a lattice L in B^n (i.e. a free module over $\mathbb{Z}$ such that $L \otimes_{\mathbb{Z}} \mathbb{Q} = B^n$), we say that L is a left O lattice if it is a left O module. We say that L_1 and L_2 are in the same class if $L_2 = L_1 g$ for some $g \in G$. For a left O lattice L and a prime q, we put $L_q = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We say that L_1 and L_2 are in the same genus if $L_{2,q} = L_{1,q} g_q$ for some $g_q \in G_q$ for all primes q.

For a fixed left O lattice L, we denote by $\mathcal{G}(L)$ the set of lattices belonging to the same genus as L. For any prime q, we put

$$U_q(L) = \{g \in G_q; L_q g = L_q\},$$

and define an open subgroup $U(L)$ of G_A by

$$U(L) = G_\infty \prod_{q:\text{prime}} U_q(L).$$

For an element $g = (g_v)_{v \le \infty} \in G_A$ and a left O lattice L, we define a left O lattice Lg by

$$Lg = \bigcap_{q:\text{prime}} (L_q g_q \cap B^n).$$

Then we also have $U(L) = \{g \in G_A; Lg = L\}$ and $\mathcal{G}(L)$ is the G_A-orbit of L. Classes in $\mathcal{G}(L)$ correspond bijectively with double cosets in

$$G_A = \bigcup_{i=1}^{h} U(L) g_i G$$

by $L \mapsto Lg_i$. This correspondence depends on the choice of L but the number h does not. The number h of classes in $\mathcal{G}(L)$ is finite and called the class number of $\mathcal{G}(L)$.

If $n \ge 2$, then the class number of the order $M_n(O)$ is 1, so any left O lattice L is O-free and there exists an element $h \in GL_n(B) = M_n(B)^\times$ such that $L = O^n h$. (This is not true for $n = 1$.) Changing L in the same class if necessary, we may assume that $hh^* \in \mathrm{Herm}_n^+(O)$. We say that $H_1, H_2 \in \mathrm{Herm}_n^+(O)$ are equivalent in a wide sense if we have $H_2 = m\epsilon_1 H_1 \epsilon_1^*$ for some $\epsilon \in GL_n(O)$ and $m \in \mathbb{Q}_+^\times$, where $\mathbb{Q}_+^\times$ denotes the set of positive rational numbers. When $m = 1$ in the above relation, we say that H_1 and H_2 are equivalent in a narrow sense. Then we have a bijection between classes of left O-lattices and classes of $\mathrm{Herm}_n^+(O)$ in a wide sense by the map $O^n h \to hh^* \in \mathrm{Herm}_n^+(O)$. If $O^n h_2 = O^n h_1 g$ for $g \in G^1 = \{g \in G; n(g) = 1\}$ and $h_i \in GL_n(B)$ with $h_i h_i^* \in \mathrm{Herm}_n^+(O)$, then $h_2 h_2^*$ and $h_1 h_1^*$ are equivalent in a narrow sense and vice versa. If H_1 and H_2 have the same reduced norm and equivalent in the wide sense, then they are also equivalent in the narrow sense. Geometrically, we often only treat cases when the norm of the matrices in question are fixed, so the difference of the definition has

not much problem. Since $N(B^\times) = \mathbb{Q}_+^\times$, we have $\{n(g); g \in G\} = \mathbb{Q}_+^\times$, and the double cosets for G_A and G_A^1 are essentially the same for the lattices we treat here, so hereafter we always use G and not G^1 and by equivalence class we mean in a wide sense.

We define the norm $N(L)$ of a left O lattice L by the two sided O ideal generated by (x, y) for all $x, y \in L$. Even if we fix a class of L, the norm $N(L)$ depends on a choice of representatives, since $N(Lg) = n(g)N(L)$ for $g \in G$. We may often take a lattice which has a simple $N(L)$ as a representative of a class of lattices. We say that L is a maximal lattice if L is maximal among those that have the same norm. When $n \geq 2$, maximal left O lattices in B^n are divided into two genera, the one is the principal genus and the other is the non-principal genus. The principal genus is represented by $L = O^n$ and in this case $N(L) = O$. The non-principal genus is a little more complicated, but the genus contains a representative L with $N(L) = \pi O$ (with some extra condition). The corresponding matrix version for $n \geq 2$ has been already explained in the previous section as $\mathcal{H}_{\text{prin}}^{(n)}$ and $\mathcal{H}_{\text{np}}^{(n)}$, respectively.

4. Parahoric subgroups and descendable directions

In our dictionary between geometry and arithmetic, polarizations on E^n are elements in $\text{Herm}_n^+(O)$ and isogenies of E^n to E^n are elements in $M_n(O) \cap GL_n(B)$. So the problem that a polarization of E^n has a descent to E^n with some given property by a certain isogeny of given type can be described using only by matrices. (A corresponding lattice version will be explained at the end of this section.) We can ask such problems for very general setting, but here we content ourselves to the case appearing in the pftq for $n = 2$ and 3.

To make the story more acceptable for readers, first we explain the case $n = 2$, relating them to some known geometry. For $n = 2$, Katsura and Oort defined *good directions* and *very good directions* in their paper [12]. There a direction means the tangent of the line which is the image of an embedding of $\iota : \alpha_p \hookrightarrow \alpha_p^2$. When the natural map $E^2 \to E^2/\iota(\alpha_p)$ is realized by an element g in the double coset $T(1, \pi)$ (and this is equivalent to the claim that $E^2/\iota(\alpha_p) \cong E^2$. See [15, Remark 3]), then the direction is called a good direction. In other words, the set of good directions are identified with $GL_2(O)\backslash T(1, \pi) \cong \mathbb{P}^1(\mathbb{F}_{p^2})$. So there are $p^2 + 1$ good directions. For any $g \in T(1, \pi)$ and a polarization λ of E^2 such that $\phi_X^{-1}\lambda$ belong to $\mathcal{H}_{\text{np}}^{(2)}$, we can easily show that we always have $\phi_X^{-1}\lambda = gHg^*$ for some $H \in \mathcal{H}_{\text{prin}}^{(2)}$. So good directions are characterized in this way, because for every direction defined by $g \in End(E^2)$, the polarization λ does descend to a principal polarization on the quotient. For a principal polarization λ of E^2, if $p\lambda$ has a descent by an isogeny g to a polarization λ_1 on E^2 such that $\text{Ker}(\lambda_1) \cong \alpha_p^2$, (in other words, which belongs to the non-principal genus), then the direction corresponding to $GL_2(O)g$ is called a very good direction in [12, p. 119]. To say this by matrices, for a fixed principal polarization λ of E^2, we write $K_0 = \phi_X^{-1}\lambda \in \text{Herm}_2^+(O)$. Then a very good direction is a coset $GL_2(O)g \in T(1, \pi)$ such that $(g^*)^{-1}(pK_0)g^{-1} \in \mathcal{H}_{\text{np}}^{(2)}$. (By the way, note that $(g^{-1})^* = (g^*)^{-1}$.) It is easy to count such directions (i.e. the number of cosets $GL_2(O)g$) by checking if a representative of each coset satisfies the condition. Anyway, there are $p + 1$ very good directions ([12, (3.4)]).

There is one more interesting thing here. We can interpret these directions by the set of cosets of the minimal parahoric subgroup in two maximal parahoric subgroups of G_p, or by the set for the corresponding adelic subgroups of G_A. For $n = 2$ and $i = 0, 1, 2$, we define subgroups $U_{i,p}$ of G_p as follows. We can choose $\xi \in GL_2(O_p)$ such that

$$\xi\xi^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and fix it. We put $G_p^* = \xi G_p \xi^{-1}$ and define subgroups of G_p^* by

$$U_{2,p}^* = GL_2(O_p) \cap G_p^*,$$
$$U_{1,p}^* = \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix}^\times \cap G_p^*,$$
$$U_{0,p}^* = U_{1,p}^* \cap U_{2,p}^*.$$

We put $U_{i,p} = \xi^{-1} U_{i,p}^* \xi$ for $i = 0, 1, 2$. The group G_p has two maximal compact subgroups up to conjugation by G_p and representatives are given by $U_{1,p}$ and $U_{2,p}$. The group $U_{0,p}$ is the unique minimal parahoric subgroup up to conjugation. For $i = 0, 1, 2$, we define open subgroups of G_A by

$$U_i = G_\infty U_{i,p} \prod_{q \neq p} (GL_2(O_q) \cap G_q).$$

Then it is well-known that $[U_2 : U_0] = [U_{2,p} : U_{0,p}] = p^2 + 1$ and $[U_1 : U_0] = [U_{1,p} : U_{0,p}] = p + 1$ (See [2, III, p, 395]). So it is natural to expect that good directions and very good directions have some natural connection to $U_{0,p}\backslash U_{2,p}$ and $U_{0,p}\backslash U_{1,p}$ respectively (though not categorical sense). The similar thing happens also for $n = 3$. The case $n = 2$ is much simpler than the case $n = 3$. so we mainly explain the case $n = 3$ hereafter.

First we define matrices corresponding to polarizations in the pftq for $n = 3$ and the adelic subgroups corresponding to those. For a polarization of E^3 whose kernel is isomorphic to α_p^2, the corresponding matrix is nothing but an element in

$$\mathcal{H}_1 = \mathrm{Herm}_3^+(O) \cap T(1, \pi, \pi).$$

The local class at p of $\mathcal{H}_1$ up to $GL_3(O_p)$ equivalence is given by

$$\begin{pmatrix} 0 & 0 & \pi \\ 0 & 1 & 0 \\ \overline{\pi} & 0 & 0 \end{pmatrix}.$$

The local class at $q \neq p$ is represented by 1_3, so the set $\mathcal{H}_1$ forms one genus. It is easy to show that there exists an element $\xi \in GL_3(O_p)$ such that

$$\xi\xi^* = J := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We fix such ξ. Then the left O_p lattice corresponding to the above representative of matrices is given by

$$N_p = (\pi O_p, O_p, O_p)\xi = \pi(O_p, \pi^{-1}O_p, \pi^{-1}O_p)\xi.$$

There is a global left O lattice $M \subset B^3$ such that for any prime v, we have

$$M_v = \begin{cases} O_v^3 & \text{for any prime } v \neq p, \\ N_p & \text{for } v = p. \end{cases} \tag{6}$$

This lattice M is not a maximal lattice, since obviously $N(M) = O$ while $M \subsetneq O^3$ and $N(O^3) = O$. For any left O lattice L of B^n, we define the dual lattice L^* of L by

$$L^* = \{x \in B^n; (x, y) \in O \text{ for all } y \in L\}.$$

Then the local dual N_p^* of N_p is given by

$$N_p^* = (O_p, O_p, \pi^{-1}O_p)\xi = \pi^{-1}(\pi O_p, \pi O_p, O_p)\xi$$

and by direct calculation we see that πN_p^* is locally maximal with $N(\pi N_p^*) = \pi O_p$. So πM^* belongs to the non-principal genus. We put

$$G_p^* = \xi G_p \xi^{-1} = \{g \in GL_3(B_p); gJg^* = n(g)J\}.$$

We define three subgroups of G_p^* by

$$U_{2,p}^* = GL_3(O_p) \cap G_p^*,$$
$$U_{1,p}^* = \begin{pmatrix} O_p & O_p & \pi^{-1}O_p \\ \pi O_p & O_p & O_p \\ \pi O_p & \pi O_p & O_p \end{pmatrix}^{\times} \cap G_p^*,$$
$$U_{0,p}^* = U_{1,p}^* \cap U_{2,p}^*.$$

We put $U_{i,p} = \xi^{-1}U_{i,p}^*\xi$ for $i = 0, 1, 2$. The group G_p has two maximal compact subgroups up to conjugation and they are represented by $U_{1,p}$ and $U_{2,p}$. The group $U_{0.p}$ is the unique minimal parahoric subgroup of G_p up to conjugation.

If we write $\mathrm{Aut}(N_p) = \{g_p \in G_p; N_p g_p = N_p\}$, then we have $U_{1,p} = \mathrm{Aut}(N_p)$. Indeed, if we put

$$\Pi_1 = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$N_p = O_p^3 \Pi_1 \xi, \quad \text{and} \quad \mathrm{Aut}(N_p) = (\xi^{-1}\Pi_1^{-1} GL_3(O_p)\Pi_1 \xi) \cap G_p, \tag{7}$$

so we have

$$\xi \mathrm{Aut}(N_p)\xi^{-1} = \begin{pmatrix} O_p & \pi^{-1}O_p & \pi^{-1}O_p \\ \pi O_p & O_p & O_p \\ \pi O_p & O_p & O_p \end{pmatrix}^{\times} \cap G_p^*. \tag{8}$$

Now let $(x, y, z) \in (O_p, \pi^{-1}O_p, \pi^{-1}O_p)$ be the first row of the elements of RHS of (8). Then since RHS is in G_p^*, we have

$$(x, y, z)J \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \mathrm{Tr}(x\overline{z}) + N(y) = 0.$$

It is well known that we have $O_p = O_F + \pi O_F$ for the maximal order O_F of the unramified quadratic extension of $\mathbb{Q}_p$ in O_p, where the multiplication is defined by $\alpha\pi = \pi\alpha^{\sigma}$ for any $\alpha \in F$ and the non-trivial automorphism σ of $Gal(F/\mathbb{Q}_p)$. Since $\mathrm{Tr}(\pi^{-1}O_F) = 0$, we have

$\mathrm{Tr}(\pi^{-1}O_p) = \mathbb{Z}_p$. Since $x \in O_p$ and $z \in \pi^{-1}O_p$, we have $N(y) \in \mathbb{Z}_p$, so $y \in O_p$. In the same way, if we take the third row (a, b, c) of RHS of (8), then we can show that $b \in \pi O_p$. So RHS of (8) is equal to $U^*_{1,p}$. By the same proof, we see $\mathrm{Aut}(\pi N^*_p) = \mathrm{Aut}(N^*_p) = U^*_{1,p}$ (or use the fact that it is the dual of N_p.) Now for $i = 0, 1, 2$, we put

$$U_i = G_\infty U_{i,p} \prod_{q \neq p} (GL_3(O_q) \cap G_q).$$

Then we have

$$U_2 = \{g \in G_A; O^3 g = O^3\},$$
$$U_1 = \{g \in G_A; Mg = M\} = \{g \in G_A; \pi M^* g = \pi M^*\}.$$

So the classes in the principal genus correspond with $U_2 \backslash G_A / G$ and the classes in the genus $\mathcal{G}(M)$ correspond with $U_1 \backslash G_A / G$. The latter also correspond with the classes in the non-principal genus. By standard calculation which we omit here, we can show that

$$[U_2 : U_0] = p^3 + 1, \qquad [U_1 : U_0] = p^2 + 1.$$

For elements H and $K \in \mathrm{Herm}^+_n(O)$, we say that H is *descendable* to K if $H = g^* K g$ for some $g \in M_n(O)$.

By abuse of language, for any $g \in M_3(O) \cap GL_3(B)$, we call the coset $GL_3(O)g$ the direction of g. We consider two kinds of special descents.

The first one is as follows. We fix $H_1 \in \mathcal{H}_1$. For an element $g \in M_3(O) \cap GL_3(B)$, we consider a condition for g that $H_1 = g^* H_0 g$ for some matrix $H_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$. Here note that H_0 is not fixed. If such g exists, then comparing the reduced norms of H_0 and H_1, we see that $g \in T(1, 1, \pi)$. It is clear that the condition of this descent depends only on a direction of g. If there exists $H_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$ as above, then we say that $GL_3(O)g$ is a descendable direction for H_1. The set of $g \in T(1, 1, \pi)$ belonging to some descendable direction is denoted by $T^{H_1}(1, 1, \pi)$.

The second one is as follows. We fix $K_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$ and put $H_2 = pK_0$. We consider a condition for an element $g \in M_3(O)$ that $H_2 = g^* H_1 g$ for some $H_1 \in \mathcal{H}_1$. Again this condition depends only on $GL_3(O)g$, and if the condition is satisfied, we say that $GL_3(O)g$ is a descendable direction for H_2.

Lemma 4.1. *Notation being as above, if $GL_3(O)g$ is descendable for H_2, then $g \in T(1, \pi, \pi)$.*

Proof. We see easily that the reduced norms of H_2 and H_1 are given by $N(H_2) = p^6$ and $N(H_1) = p^2$, so we have $N(g) = p^2$. So we have $g \in T(1, \pi, \pi)$ or $T(1, 1, p)$. We show that the latter case does not occur. Assume $g \in T(1, 1, p)$ and write $g = \epsilon_1 \mathrm{diag}(1, 1, p)\epsilon_2$ for some $\epsilon_i \in GL_3(O)$. Put $H'_1 = \mathrm{diag}(1, 1, p^{-1})(p\epsilon_2^{-*} K_0 \epsilon_2^{-1})\mathrm{diag}(1, 1, p^{-1})$. Then the condition $(g^*)^{-1} p K_0 g^{-1} \in T(1, \pi, \pi)$ is equivalent to the condition $H'_1 \in T(1, \pi, \pi)$. We write $\epsilon_2^{-*} K_0 \epsilon_2^{-1} = (a_{ij})$. Since $K_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$, we have $(a_{ij}) \in \mathcal{H}^{(3)}_{\mathrm{prin}} \subset GL_3(O)$. Now assume that $H'_1 \in M_3(O)$. Then we have

$$H'_1 = \begin{pmatrix} pa_{11} & pa_{12} & a_{13} \\ p\overline{a_{12}} & pa_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & p^{-1}a_{33} \end{pmatrix} \in M_3(O).$$

So $a_{33} \in pO$. Then since $(a_{ij}) \in GL_3(O)$, we have $a_{13} \in O_p^\times$ or $a_{23} \in O_p^\times$. So by multiplication of elementary matrices, we can show that $H_1' \in T(1,1,p)$. This is a contradiction. □

We note that even if $g \in T(1,\pi,\pi)$, it might happen that $(g^*)^{-1}H_2g^{-1} \in T(1,1,p)$. We denote the set of $g \in T(1,\pi,\pi)$ belonging to a descendable direction for H_2 by

$$T^{H_2}(1,\pi,\pi). \tag{9}$$

Lemma 4.2. *Notation being as above, we have*

$$\#(GL_3(O)\backslash T^{H_1}(1,1,\pi)) = p^2+1 = [U_1 : U_0] \tag{10}$$

$$\#(GL_3(O)\backslash T^{H_2}(1,\pi,\pi)) = p^3+1 = [U_2 : U_0] \tag{11}$$

where #(*set*) *means the cardinality of the set.*

Proof. We have the following coset representatives.

$$GL_3(O)\backslash T(1,1,\pi) = \left\{ \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & \pi \end{pmatrix} \right.$$
$$\left. : a,b,c \text{ are representatives of } O/\pi O \cong \mathbb{F}_{p^2} \right\},$$

$$GL_3(O)\backslash T(1,\pi,\pi) = \left\{ \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & \pi \end{pmatrix}, \begin{pmatrix} 1 & b & c \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} \right.$$
$$\left. : a,b,c \text{ are representatives of } O/\pi O \cong \mathbb{F}_{p^2} \right\}.$$

In particular, we have $\deg T(1,1,\pi) = \deg T(1,\pi,\pi) = 1+p^2+p^4$. (By the way, we have $\deg T(1,1,p) = p^4+p^6+p^8$.) Now we prove Lemma 4.2 for the second case. We have $K_0 = \epsilon_0^*\epsilon_0$ for some $\epsilon_0 \in GL_3(O_A)$. Since $\epsilon_0 g^{-1} = (g\epsilon_0^{-1})^{-1}$, it is enough to consider a representative of the coset $GL(O_A)g\epsilon_0^{-1} \in T_A(1,\pi,\pi)$ for the descent from $p1_3$, so we may assume that $g\epsilon_0^{-1}$ is a representative in the set given above. For $g \in T(1,\pi,\pi)$ in the above representatives, we check if

$$p(g^*)^{-1}g^{-1} \tag{12}$$

is in $T_A(1,\pi,\pi)$ or not. For $g = \mathrm{diag}(\pi,\pi,1)$, (12) is equal to $\mathrm{diag}(1,1,p) \in T_A(1,1,p)$ so this is not good. For $g = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & \pi \end{pmatrix}$, (12) is equal to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & a\pi \\ 0 & -\pi\overline{a} & 1+N(a) \end{pmatrix}.$$

This is in $T(1,\pi,\pi)$ if and only if $1+N(a) \equiv 0 \bmod p$. We have $O_p/\pi O_p \cong \mathbb{F}_{p^2}$ and

$N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\mathbb{F}_{p^2}) = \mathbb{F}_p$. So the number of such a mod π is $p+1$. Next for $g = \begin{pmatrix} 1 & b & c \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$, (12) is equal to

$$\begin{pmatrix} p & b\pi & c\pi \\ -\pi\overline{b} & 1+N(b) & \pi^{-1}\overline{b}c\pi \\ -\pi\overline{c} & \pi\overline{c}b\pi^{-1} & 1+N(c) \end{pmatrix}.$$

This is in $T(1,\pi,\pi)$ if and only if the matrix mod π has rank 1. So this is equivalent to $(N(b)+1)(N(c)+1) - N(b)N(c) = N(b)+N(c)+1 \in O_p^\times$. For each element of $\{(x,y) \in \mathbb{F}_{p^2}^2; x+y+1=0\}$, the number of b mod π and c mod π such that $N(b)=x$, $N(c)=y$ is as follows. If $x=0$, then $b \equiv 0$ mod π and $N(c)$ mod $\pi = y = -1$. The number of such c mod π is $p+1$. If $x \neq 0$ and $y=0$, then again the number of (b,c) mod π is $p+1$. If $x+y+1=0$ and $x \neq 0$, $y \neq 0$, then the number of (x,y) is $p-2$ and for each such (x,y), the number of (b,c) mod π is $(p+1)^2$. So the total of this case is $2(p+1)+(p+1)^2(p-2) = p(p^2-1)$. So the total number of cosets is $(p+1)+p(p^2-1) = 1+p^3$. The proof for the case (10) is obtained similarly. The second equalities of (10) and (11) are proved directly by calculating $[U_i : U_0]$ for $i = 1, 2$ independently. □

Now here the equalities of the numbers of descendable directions and the group indices $[U_i : U_0]$ are given just by a coincidence of independently calculated numbers, but we could ask if there is more direct relation. We answer this question next.

Before going further, we give lemmas which will be used later. In particular, the first one is a key lemma for the proof.

Lemma 4.3. *We assume $n = 2$ or 3. For an element $v \in G_p^*$ with $n(v) \in \mathbb{Z}_p^\times$, assume that every component in the first row of v belongs to $\pi^{-1}O_p$ and that all the other components are in O_p. Then we have*

$$v \in U_{1,p}^* \cdot U_{2,p}^*. \quad \textit{(semi-direct product)}$$

Proof. We prove the case $n=3$. We write $v = (v_{ij}) \in M_3(B_p)$. If $v_{11}, v_{12}, v_{13} \in O_p$, then we have $v \in M_3(O_p) \cap G_p^*$ and since $n(v) \in \mathbb{Z}_p^\times$, we also have $v^{-1} = n(v)^{-1}Jv^*J \in M_3(O_p) \cap G_p^*$, so $v \in U_{2,p}^*$ and we are done. So we assume that $v_{1i} \in \pi^{-1}O_p^\times$ for some $i = 1, 2$, or 3. Comparing the $(1,1)$ component of the relation $vJv^* = n(v)J$, we have

$$\mathrm{Tr}(v_{11}\overline{v_{13}}) + N(v_{12}) = 0. \tag{13}$$

If $v_{12} \in \pi^{-1}O_p^\times$, we have $N(v_{12}) \in p^{-1}\mathbb{Z}_p^\times$. But we have $\mathrm{Tr}(\pi^{-1}O_p) = \mathbb{Z}_p$ so both v_{11} and v_{13} should be in $\pi^{-1}O_p^\times$. If $v_{12} \in O_p$, then we may assume that $v_{11} \in \pi^{-1}O_p^\times$ or $v_{13} \in \pi^{-1}O_p^\times$. Since $J \in U_{2,p}^*$, we may change v by vJ if necessary, and v_{11} and v_{13} can be exchanged by this. So in any case. we can assume that $v_{13} \in \pi^{-1}O_p^\times$. For $x, y \in B_p$, we put

$$[x,y] = \begin{pmatrix} 1 & 0 & 0 \\ -\overline{x} & 1 & 0 \\ y & x & 1 \end{pmatrix}.$$

Then we have $[x,y] \in U_{2,p}^*$ if and only if $x, y \in O_p$ and $N(x) + \mathrm{Tr}(y) = 0$ and we also have

$n([x,y]) = 1$ in this case. We write $v_{1i} = \pi^{-1}a_{1i}$ with $a_{1i} \in O_p$ for $i = 1, 2, 3$. We have $a_{13} \in O_p^\times$ by the assumption. Since $\mathrm{Tr}(O_p) = \mathbb{Z}_p$, there exists an element $y \in O_p$ such that $\mathrm{Tr}(y) = N(a_{13}^{-1}a_{12})$. By taking $v[-a_{13}^{-1}a_{12}, -y]$, we may assume $v_{12} = 0$. Then by (13), we have $\mathrm{Tr}(\pi^{-1}a_{11}\overline{\pi^{-1}a_{13}}) = p^{-1}\mathrm{Tr}(a_{11}\overline{a_{13}}) = 0$. This also means $\mathrm{Tr}(a_{13}^{-1}a_{11}) = \mathrm{Tr}(a_{11}a_{13}^{-1}) = \mathrm{Tr}(a_{11}\overline{a_{13}}/N(a_{13})) = 0$. So we have $[0, -a_{13}^{-1}a_{11}] \in U_{2,p}^*$ and taking $v[0, -a_{13}^{-1}a_{11}]$ instead of v, we may assume $v_{11} = v_{12} = 0$. Comparing the $(1,2)$ and $(1,3)$ components of $vJv^* = n(v)J$, we have $v_{21} = 0$ and $v_{13}\overline{v_{31}} = n(v)$. So $v_{31} = n(v)\overline{\pi}(\overline{a_{13}})^{-1}$. Comparing the $(2,2)$ components, we have $N(v_{22}) = n(v) \in \mathbb{Z}_p^\times$. So multiplying

$$\begin{pmatrix} n(v)^{-1}\overline{a_{13}} & 0 & 0 \\ 0 & v_{22}^{-1} & 0 \\ 0 & 0 & a_{13}^{-1} \end{pmatrix} \in U_{2,p}^*$$

to v from the right, we may assume

$$v = \begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & v_{23} \\ \overline{\pi} & v_{32} & v_{33} \end{pmatrix}$$

with $v_{13}, v_{23}, v_{33} \in O_p$. Since this is in G_p^*, we have $v_{32} + \overline{v_{23}\pi} = 0$, and this means $v_{32} \in \pi O$. Also we have

$$v^{-1} = Jv^*J = \begin{pmatrix} \overline{v_{33}} & \overline{v_{23}} & \overline{\pi}^{-1} \\ \overline{v_{32}} & 1 & 0 \\ \pi & 0 & 0 \end{pmatrix}.$$

So we have

$$v, v^{-1} \in \begin{pmatrix} O_p & O_p & \pi^{-1}O_p \\ \pi O_p & O_p & O_p \\ \pi O_p & \pi O_p & O_p \end{pmatrix}$$

so $v \in U_{1,p}^*$. The proof for $n = 2$ is similar but easier and we omit it here. □

We give one more well-known lemma.

Lemma 4.4. *Let $\mathfrak{G}$ be a group, $\mathfrak{H}_1$ and $\mathfrak{K}$ subgroups of $\mathfrak{G}$, and $\mathfrak{H}_0$ a subgroup of $\mathfrak{H}_1$. Fix $g \in \mathfrak{G}$ and put $\Gamma = \mathfrak{H}_1 \cap g\mathfrak{K}g^{-1}$. We make Γ act on $\mathfrak{H}_0\backslash\mathfrak{H}_1$ by right multiplication. Then the set of double cosets in $\mathfrak{H}_0\backslash\mathfrak{H}_1 g\mathfrak{K}/\mathfrak{K}$ is bijective to the set of right Γ-orbits in $\mathfrak{H}_0\backslash\mathfrak{H}_1$,*

Proof. Take two double cosets $\mathfrak{H}_0 hg\mathfrak{K}$ and $\mathfrak{H}_0 h'g\mathfrak{K}$ in $\mathfrak{H}_0\backslash\mathfrak{H}_1 g\mathfrak{K}/\mathfrak{K}$ where $h, h' \in \mathfrak{H}_1$. If they are the same, then we have $h'g = h_0hgk$ for some $h_0 \in \mathfrak{H}_0$ and $k \in \mathfrak{K}$. So $h_0^{-1}h' = h(gkg^{-1})$. Here $gkg^{-1} \in g\mathfrak{K}g^{-1} \cap \mathfrak{H}_1 = \Gamma$. So $\mathfrak{H}_0 h' \in \mathfrak{H}_0 h\Gamma$. The converse is proved similarly. □

Now we come back to our main theme. First we consider the condition on descent for a fixed $H_2 = pK_0$ for $K_0 \in \mathcal{H}_{\mathrm{prin}}^{(3)}$ by $g \in T(1,\pi,\pi)$. We take $k_0 \in GL_3(B)$ such that $K_0 = k_0k_0^*$ and put

$$h_2 = k_0\pi. \tag{14}$$

Then $H_2 = h_2h_2^*$. Of course there are several choices of h_2 or k_0 but we fix them for sim-

plicity. The standard lattice corresponding to the genus containing H_2 is $O^3\pi$. So we choose $g_2 \in G_A$ such that

$$O^3 h_2 = O^3 \pi g_2 \tag{15}$$

and fix it. By our choice, comparing the norm of lattices, we have $n(g_2) \in \mathbb{Z}_A^\times := \mathbb{R}_+^\times \prod_q \mathbb{Z}_q^\times$. (By the way, the lattices $O^3\pi$ and O^3 are in the same class by G and the automorphism groups of these are both U_2, but the double coset $U_2\pi g_2 G$ and $U_2 g_2 G$ are different in general. This is natural since the correspondence with double cosets depend on the choice of a lattice representative of the genus. The lattices $O^3\pi g_2$ and $O^3 g_2$ are not in the same class but belong to the same G type in the sense of [8, p. 372].) For any $H \in \mathrm{Herm}_n^+(O)$, we define

$$\mathrm{Aut}(H) = \{\alpha \in M_n(O); \alpha^* H \alpha = H\}.$$

By comparing the reduced norms of the defining equality, we see that $\mathrm{Aut}(H) \subset GL_n(O)$. Since H is positive definite, it is also obvious that $\mathrm{Aut}(H)$ is a finite group. Going back to the above setting, we have $\mathrm{Aut}(H_2) = \mathrm{Aut}(K_0)$. The group $\mathrm{Aut}(H_2)$ acts on $T^{H_2}(1,\pi,\pi)$ by the right multiplication since if $g \in T^{H_2}(1,\pi,\pi)$, then there exists $H_1 \in \mathcal{H}_1$ such that $g^* H_1 g = H_2$ and this means that $\alpha^* g^* H_1 g \alpha = \alpha^* H_2 \alpha = H_2$. Now we put

$$\Gamma_2 = g_2^{-1} U_2 g_2 \cap G,$$
$$\widehat{\Gamma}_2 = U_2 \cap g_2 G g_2^{-1} = g_2 \Gamma_2 g_2^{-1}.$$

Then the group $\widehat{\Gamma}_2$ acts on $U_0\backslash U_2$ by the right multiplication. If $\alpha \in \mathrm{Aut}(H_2)$, then obviously $h_2^{-1}\alpha^* h_2 \in G^1$ and we have $O^3 h_2 (h_2^{-1}\alpha^* h_2) = O^3\alpha^* h_2 = O^3 h_2$. The stabilizer of $O^3 h_2 = O^3\pi g_2 = \pi O^3 g_2$ in G_A is $g_2^{-1} U_2 g_2$, so we have

$$h_2^{-1}\mathrm{Aut}(H_2)^* h_2 = \Gamma_2 = g_2^{-1}\widehat{\Gamma}_2 g_2.$$

where we write $\mathrm{Aut}(H_2)^* = \{\alpha^*; \alpha \in \mathrm{Aut}(H_2)\}$. Here we define an isomorphism R_2 of $\mathrm{Aut}(H_2)$ to $\widehat{\Gamma}_2$ by

$$\mathrm{Aut}(H_2) \ni \alpha \longrightarrow R_2(\alpha) = g_2 h_2^{-1} (\alpha^*)^{-1} h_2 g_2^{-1} \in \widehat{\Gamma}_2.$$

(Here we take $(\alpha^*)^{-1}$ instead of α^* to make the map an isomorphism and not an anti-isomorphism.)

The group $\mathrm{Aut}(H_2)$ acts on $GL_3(O)\backslash T^{H_2}(1,\pi,\pi)$ and $\widehat{\Gamma}_2$ on $U_0\backslash U_2$, both by right multiplication.

Proposition 4.5. *Notation and assumption being as above, we can define a map ρ from $T^{H_2}(1,\pi,\pi)$ to $U_0\backslash U_2$ which has the following properties.*
(i) *The map ρ induces a bijection*

$$\rho : GL_3(O)\backslash T^{H_2}(1,\pi,\pi) \cong U_0\backslash U_2.$$

(ii) *The action of* $\mathrm{Aut}(H_2)$ *and the action of* $\widehat{\Gamma}_2$ *defined above are compatible with ρ. That is, we have*

$$\rho(g\alpha) = \rho(g) R_2(\alpha), \qquad \text{for } \alpha \in \mathrm{Aut}(H), g \in GL_3(O)\backslash T^{H_2}(1,1,\pi).$$

This induces a bijection between the following set of orbits.

$$GL_3(O)\backslash T^{H_2}(1,\pi,\pi)/\mathrm{Aut}(H_2) \cong U_0\backslash U_2/\widehat{\Gamma}_2.$$

(iii) *The map ρ induces a bijection between the set of* $\mathrm{Aut}(H_2)$ *orbits in* $GL_3(O)\backslash T^{H_2}(1,\pi,\pi)$ *and the set of all double cosets in* $U_0\backslash U_2 g_2 G/G$, *where the map is given by* $g \to U_0\rho(g)g_2G$.
(iv) *We fix* $H_2 = pK_0$ *for* $K_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$ *and* $H_1 \in \mathcal{H}_1$. *We choose* h_2, h_1 *so that* $h_2h_2^* = H_2$ *and* $h_1h_1^* = H_1$. *We choose* g_2 *and* $g_1 \in G_A$ *so that* $O^3h_2 = O^3\pi g_2$ *and* $O^3h_1 = Mg_1$, *where* M *is the standard lattice defined by* (6) *in the genus corresponding to matrices in* $\mathcal{H}_1$. *Then* H_2 *is descendable to* H_1 *if and only if* $U_2g_2G \cap U_1g_1G \neq \emptyset$.

We postpone the proof of this proposition after stating a claim for the other case which is quite similar.

Now we explain the other case. We fix $H_1 \in \mathcal{H}_1$ and also fix $h_1 \in GL_3(B)$ such that $H_1 = h_1h_1^*$. We fix $g_1 \in G_A$ such that $O^3h_1 = Mg_1$. where M is the standard lattice in $\mathcal{H}_1$ defined in (6). More precisely we use following notation. We write $\Pi_1 = \mathrm{diag}(\pi, 1, 1)$. By Π_1 and ξ (being defined by $\xi\xi^* = J$), we also mean elements in $GL_3(B_A)$ such that the p-components are Π_1 and ξ respectively and the other components are 1_3, Then there exists $\epsilon_1 \in GL_3(O_A)$ such that

$$\epsilon_1 h_1 = \Pi_1 \xi g_1. \tag{16}$$

Such elements ϵ_1 and g_1 are not unique, but we fix them. We write

$$\mathrm{Aut}(H_1) = \{\alpha \in M_3(O); \alpha^* H_1 \alpha = H_1\}$$
$$\Gamma_1 = g_1^{-1}U_1g_1 \cap G$$
$$\widehat{\Gamma}_1 = U_1 \cap g_1Gg_1^{-1} = g_1\Gamma_1 g_1^{-1}.$$

We have $h_1^{-1}\mathrm{Aut}(H_1)^*h_1 = \Gamma_1 = g_1^{-1}\widehat{\Gamma}_1 g_1$. We define an isomorphism R_1 from $\mathrm{Aut}(H_1)$ to $\widehat{\Gamma}_1$ by

$$R_1(\alpha) = g_1h_1^{-1}(\alpha^*)^{-1}h_1g_1^{-1}.$$

Here $\mathrm{Aut}(H_1)$ acts on $T^{H_1}(1,1,\pi)$ and $\widehat{\Gamma}_1$ acts on $U_0\backslash U_1$ both by right multiplication.

Proposition 4.6. *Notation and assumption being as above, we can define a map ρ from $T^{H_1}(1,1,\pi)$ to $U_0\backslash U_1$ which has the following properties.*
(i) *The map ρ induces a bijection*

$$\rho : GL_3(O)\backslash T^{H_1}(1,1,\pi) \to U_0\backslash U_1.$$

(ii) *The map ρ is compatible with the action of* $\mathrm{Aut}(H_1)$ *and* $\widehat{\Gamma}_1$. *In other words, for any* $g \in T^{H_1}(1,1,\pi)$. *we have*

$$\rho(g\alpha) = \rho(g)R_1(\alpha).$$

This induces a bijection between the following set of orbits.

$$GL_3(O)\backslash T^{H_1}(1,1,\pi)/\mathrm{Aut}(H_1) \cong U_0\backslash U_1/\widehat{\Gamma}_1.$$

(iii) *The map ρ induces a bijection between the above set of orbits and the set of all double cosets in $U_0\backslash U_1 g_1 G/G$, where the map is given by $g \to U_0\rho(g)g_1G$.*
(iv) *We fix $H_1 \in \mathcal{H}_1$ and $H_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$. We fix $h_i \in GL_3(O)$ for $i = 0, 1$ such that $h_ih_i^* = H_i$. We*

choose g_1 and $g_0 \in G_A$ such that $O^3 h_1 = M g_1$ and $O^3 h_0 = O^3 g_0$. Then H_1 is descendable to H_0 if and only if $U_1 g_1 G \cap U_0 g_0 G \neq \emptyset$.

Proof of Proposition 4.5. First, before defining the map ρ in the proposition, we prepare several notation and explain necessary properties. We fix $h_2 \in GL_3(B)$ and $g_2 \in G_A$ as in (14) and (15). For $g \in T^{H_2}(1, \pi, \pi)$, we put

$$h_1 = (g^*)^{-1} h_2.$$

Then by definition of $T^{H_2}(1, \pi, \pi)$ at (9), we have $H_1 = h_1 h_1^* \in \mathcal{H}_1$. By our choice as in (15) and (16), we have

$$\begin{aligned} h_1 &= \epsilon_1^{-1} \Pi_1 \xi g_1, \qquad (\Pi_1 = \mathrm{diag}(\pi, 1, 1)), \\ h_2 &= \epsilon_2^{-1} \pi g_2 \end{aligned}$$

for some $\epsilon_1, \epsilon_2 \in GL_3(O_A)$, $g_1 \in G_A$. In order to see a relation between H_1 and H_2 by adelic elements, we put

$$\eta = g_2 g_1^{-1} \in G_A. \tag{17}$$

Then we have

$$\eta = \pi^{-1} \epsilon_2 h_2 h_1^{-1} \epsilon_1^{-1} \Pi_1 \xi = \pi^{-1} \epsilon_2 g^* \epsilon_1^{-1} \Pi_1 \xi,$$

and

$$\eta^{-1} = \xi^{-1} (\Pi_1^{-1} \epsilon_1 (g^*)^{-1} \epsilon_2^{-1} \pi \xi^{-1}) \xi.$$

Here we have $GL_3(O_A)\pi = \pi GL_3(O_A)$ and $(g^*)^{-1} \in T(1, \pi^{-1}, \pi^{-1})$, so we have

$$\epsilon_1 (g^*)^{-1} \epsilon_2^{-1} \pi \in T(1, 1, \pi).$$

By this fact, the first row of the p-component of $\Pi_1^{-1} \epsilon_1 (g^*)^{-1} \epsilon_2^{-1} \pi \xi^{-1} \in G_p^*$ is in $\pi^{-1}(O_p)^3$ and the other components are in O_p. So this is in $U_{1,p}^* \cdot U_{2,p}^*$ by Lemma 4.3. All the other local components of η^{-1} at $q \neq p$ are in $GL_3(O_q)$, so we have $\eta^{-1} \in U_1 \cdot U_2$ and $\eta \in U_2 \cdot U_1$. So there exists $w \in U_2$ such that $w\eta \in U_1$.

Now under these preparations, we define the map ρ and then prove (i) of the proposition. The coset $U_0 w$ does not depend on the choice of w, since if $w\eta$, $w'\eta \in U_1$ and w, $w' \in U_2$, then we have $w' w^{-1} = (w'\eta)(w\eta)^{-1} \in U_2 \cap U_1 = U_0$. So we would like to define $\rho(g)$ by $\rho(g) = U_0 w$. But we must show that this is well-defined, since this might depend on the choice of h_1 and g_1. Also we want to show that $\rho(g)$ is the same for any element in $GL_3(O)g$. So to prove both at the same time, we take another $g' \in T^{H_2}(1, \pi, \pi)$, $h_1' = (g'^*)^{-1} h_2$. We also write

$$h_1' = \epsilon_1'^{-1} \Pi_1 \xi g_1', \quad (\epsilon_1' \in GL_3(O_A), g_1' \in G_A), \tag{18}$$

$$\eta' = g_2 g_1'^{-1}. \tag{19}$$

Then we have

$$\eta' = \pi^{-1} \epsilon_2 g'^* \epsilon_1'^{-1} \Pi_1 \xi = \eta (\xi^{-1} \Pi_1^{-1} \epsilon_1 (g^*)^{-1} g'^* \epsilon_1'^{-1} \Pi_1 \xi).$$

If $g' \in GL_3(O)g$, then we have $g' = \epsilon g$ for some $\epsilon \in GL_3(O)$ and $(g^*)^{-1} g'^* = \epsilon^*$. So the p

component of $\Pi_1^{-1}\epsilon_1\epsilon^*\epsilon_1'^{-1}\Pi_1$ is in

$$\begin{pmatrix} O_p & \pi^{-1}O_p & \pi^{-1}O_p \\ \pi O_p & O_p & O_p \\ \pi O_p & O_p & O_p \end{pmatrix}^{\times} \cap G_p^*.$$

This is equal to $U_{1,p}^*$, so we have

$$\eta' \in \eta U_1.$$

This means that if $w \in U_2$ and $w\eta \in U_1$, then $w\eta' \in U_1$. So we have $\rho(g) = \rho(g') = U_0 w$, and ρ is a well defined map from $GL_3(O)\backslash T^{H_2}(1,\pi,\pi)$.

Next we show that ρ is injective from $GL_3(O)\backslash T^{H_2}(1,\pi,\pi)$. We take g, $g' \in T^{H_2}(1,\pi,\pi)$ and define η, η' as (17), (19). If $w\eta$, $w\eta' \in U_1$ for $w \in U_2$, then we have $\eta^{-1}\eta' \in U_1$ and

$$\epsilon_1(g^*)^{-1}(g')^*\epsilon_1'^{-1} = \Pi_1\xi(\eta^{-1}\eta')\xi^{-1}\Pi_1^{-1} \in M_3(O_A).$$

So we have $g'^* \in g^*M_3(O)$ and $g' \in M_3(O)g$. Since $N(g) = N(g') = p^2$. we have $g' \in GL_3(O)g$, so ρ is injective.

Next we show that ρ is surjective to $U_0\backslash U_2$. In order to find g for any $w \in U_2$, we put $g_1 = wg_2$ and take $h_1 \in GL_3(B)$ such that $Mg_1 = O^3h_1$. Then we have

$$h_1 = \epsilon_1^{-1}\Pi_1\xi g_1$$

for some $\epsilon_1 \in GL_3(O_A)$. We define g by $g^* = h_2h_1^{-1}$. Then we have

$$g^* = \epsilon_2^{-1}\pi g_2 g_1^{-1}\xi^{-1}\Pi_1^{-1}\epsilon_1 = \epsilon_2^{-1}\pi w^{-1}\xi^{-1}\Pi_1^{-1}\epsilon_1.$$

Since $w \in U_2$, obviously we have $g^* \in GL_3(O_A) \cap GL_3(B)$ and by Lemma 4.1, we have $g \in T(1,\pi,\pi)$. For g, we have $\eta = \pi^{-1}\epsilon_2 g^*\epsilon_1^{-1}\Pi_1\xi = w^{-1}$. So $\rho(g) = U_0 w$. Hence ρ is surjective and we finished the proof of (i).

Now we see the correspondence of the orbits and prove (ii) and (iii) of the proposition. We put $g' = g\alpha$ for $\alpha \in \mathrm{Aut}(H_2)$. Here we will show that $\rho(g') \in U_0\rho(g)\widehat{\Gamma}_2$. We define g_1, g_1', h_1, h_1', ϵ_1, ϵ_1', η, η' as in the proof of (i). Then we have

$$\begin{aligned}
\eta' &= \eta(\xi^{-1}\Pi_1^{-1}\epsilon_1(g^*)^{-1}g'^*)\epsilon_1'^{-1}\Pi_1\xi \\
&= \eta(\xi^{-1}\Pi_1^{-1}\epsilon_1(g^*)^{-1}\alpha^*g^*\epsilon_1'^{-1}\Pi_1\xi \\
&= \eta(\xi^{-1}\Pi_1^{-1}\epsilon_1 h_1)(h_2^{-1}\alpha^*h_2)(h_1^{-1}\epsilon_1'^{-1}\Pi_1\xi) \\
&= \eta g_1(h_2^{-1}\alpha^*h_2)g_1^{-1}(\xi^{-1}\Pi_1^{-1}\epsilon_1\epsilon_1'^{-1}\Pi_1\xi) \\
&= (g_2h_2^{-1}\alpha^*h_2g_2^{-1})(g_2g_1^{-1})(\xi^{-1}\Pi_1^{-1}\epsilon_1\epsilon_1'^{-1}\Pi_1\xi) \\
&= (g_2h_2^{-1}\alpha^*h_2g_2^{-1})\eta(\xi^{-1}\Pi_1^{-1}\epsilon_1\epsilon_1'^{-1}\Pi_1\xi).
\end{aligned}$$

Since η, η', g_2 and $h_2^{-1}\alpha^*h_2 \in G_A$, we have $\xi^{-1}\Pi_1^{-1}\epsilon_1\epsilon_1'^{-1}\Pi_1\xi \in G_A$ and this also belongs to U_1. So if $w\eta \in U_1$ for $w \in U_2$, then we have

$$w(g_2h_2^{-1}(\alpha^*)^{-1}h_2g_2^{-1})\eta' \in U_1.$$

If $\alpha \in \mathrm{Aut}(H_2)$, then $g_2h_2^{-1}(\alpha^*)^{-1}h_2g_2^{-1} = R_2(\alpha) \in U_2$, so for $w \in U_2$ such that $\rho(g) = U_0 w$, we have $\rho(g') = \rho(g\alpha) = U_0\rho(g)R_2(\alpha)$. On the contrary, if $U_0\rho(g') = U_0\rho(g)R_2(\alpha) = U_0\rho(g\alpha)$ for some $\alpha \in \mathrm{Aut}(H_2)$, then by injectivity we have $GL_3(O)g' = GL_3(O)g\alpha$. So we

prove (ii). The claim (iii) is obvious by Lemma 4.4.

We prove (iv). Assume that $H_2 = g^* H_1 g$ for some $g \in M_3(O)$. Since $U_0\rho(g) = U_0 w$ for $w \in U_2$ such that $w\eta = wg_2 g_1^{-1} \in U_1$, we have $U_0\rho(g)g_2G \subset U_2g_2G \cap U_1g_1G$. So this is non empty. Conversely, if $U_2g_2G \cap U_1g_1G \neq \emptyset$, then we have $U_0wg_2G \subset U_2g_2G \cap U_1g_1G$ for some $w \in U_2$. So we have $wg_2 = u_1g_1\delta$ for some $\delta \in G$ and $u_1 \in U_1$. Since $U_1u_1g_1G = U_1g_1G$, we may write u_1g_1 by g_1. Since we may assume $n(g_2), n(g_1) \in \mathbb{Z}_A^\times$ and since we have $n(w) \in \mathbb{Z}_A^\times$, we have $n(\delta) = 1$. So we have $H_1 = h_1h_1^* = (h_1\delta)(h_1\delta)^*$. If we put $g^* = h_2(h_1\delta)^{-1}$, then we have $g^*H_1g = H_2$ and as in the proof of the surjectivity of ρ, we can show that $g \in M_3(O) \cap GL_3(B)$. (We can also show that $\rho(g) = U_0w$.) □

Proof of Proposition 4.6. The proof is almost the same as the proof of Proposition 4.5, so we explain shortly. We fixed $h_1 \in GL_3(B)$ such that $H_1 = h_1h_1^*$. We also fix $g_1 \in G_A$ such that

$$O^3h_1 = Mg_1.$$

We have

$$h_1 = \epsilon_1^{-1}\Pi_1\xi g_1$$

for a certain $\epsilon_1 \in GL_3(O_A)$. For $g \in T^{H_1}(1,1,\pi)$, we put $h_0 = (g^*)^{-1}h_1$ and define $\epsilon_0 \in GL_3(O_A)$ and $g_0 \in G_A$ by

$$h_0 = \epsilon_0^{-1}g_0.$$

We put

$$\eta = g_1g_0^{-1}.$$

Then as in the proof of Proposition 4.5, we can show that there exists $w \in U_1$ such that $w\eta \in U_2$. We define

$$\rho(g) = U_0w \subset U_1.$$

For this ρ, the proof of the claims is almost the same as that of Proposition 4.5, so we omit it here. □

Proposition 4.7. *Fix $K_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$ and put $H_2 = pK_0$. We also fix $H_0 \in \mathcal{H}^{(3)}_{\mathrm{prin}}$. If H_2 is descendable to H_0 by $g \in M_3(O)$, i.e. $H_2 = g^*H_0g$, then there exist some $H_1 \in \mathcal{H}_1$ and $\alpha_1 \in T(1,1,\pi)$, $\alpha_2 \in T(1,\pi,\pi)$ such that $g = \alpha_1\alpha_2$, $H_2 = \alpha_2^*H_1\alpha_2$, and $\alpha_1^*H_0\alpha_1 = H_1$.*

Proof. Comparing the norm of $H_2 = g^*H_0g$, we have $N(g) = p^3$, so we have $g \in T(\pi,\pi,\pi)$, $T(1,\pi,\pi^2)$, or $T(1,1,\pi^3)$. We see that the last case does not happen. Indeed, if $g = \epsilon_1\mathrm{diag}(1,1,\pi^3)\epsilon_2$ for $\epsilon_i \in GL_3(O)$, then

$$\mathrm{diag}(1,1,\overline{\pi}^3)\epsilon_1^*H_0\epsilon_1\mathrm{diag}(1,1,\pi^3) \in pGL_3(O).$$

We write

$$\epsilon_1^*H_0\epsilon_1 = (a_{ij}).$$

Then we see that $a_{11}, a_{12}, a_{21}, a_{22} \in pO$. Then obviously the rank of the matrix $((a_{ij}) \bmod \pi)$ is at most 2. This contradicts to the fact $(a_{ij}) \in GL_3(O)$. So this case does not happen. Next,

we assume that $g \in T(1,\pi,\pi^2)$. We may write

$$g = \epsilon_1 \mathrm{diag}(1,\pi,\pi^2)\epsilon_2, \qquad \epsilon_1, \epsilon_2 \in GL_3(O).$$

By definition we have $g^* H_0 g = H_2 = pK_0$, If we write $H_0' = \epsilon_1^* H_0 \epsilon_1 = (b_{ij})$, then we have

$$\mathrm{diag}(1,\overline{\pi},\overline{\pi}^2)(b_{ij})\mathrm{diag}(1,\pi,\pi^2) \in pGL_3(O).$$

So we have $b_{11} \in pO$, $b_{12} = \overline{b_{21}} \in \pi O$. Since $(b_{ij}) \in GL_3(O)$, we have b_{22}, $\overline{b_{31}} = b_{13} \in O_p^\times$. Now put

$$\alpha_1 = \epsilon_1 \mathrm{diag}(1,1,\pi),$$
$$\alpha_2 = \mathrm{diag}(1,\pi,\pi)\epsilon_2.$$

Then we have

$$\alpha_1^* H_0 \alpha_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13}\pi \\ \overline{b_{12}} & b_{22} & b_{23}\pi \\ \overline{b_{13}\pi} & \overline{b_{23}\pi} & \overline{\pi} b_{33}\pi. \end{pmatrix}.$$

Since $b_{11}, b_{12} \equiv 0 \bmod \pi$, we have

$$\alpha_1^* H_0 \alpha_1 \bmod \pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{22} \bmod \pi & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $N(\alpha_1^* H_0 \alpha_1) = p^2$ and $b_{22} \in O_p^\times$, we see that $\alpha_1^* H_0 \alpha_1 \in T(1,\pi,\pi) \cap \mathrm{Herm}_3^+(O) = \mathcal{H}_1$. So if we put $H_1 = \alpha_1^* H_0 \alpha_1$, then $\alpha_2^* H_1 \alpha_2 = g^* H_0 g = H_2$. So this case is proved.

Next we assume that $g \in T(\pi,\pi,\pi)$. We have $g = \pi\epsilon$ for some $\epsilon \in GL_3(O)$. Put $\alpha_1 = \epsilon_1 \mathrm{diag}(1,1,\pi)$ for arbitrary $\epsilon_1 \in GL_3(O)$. Then there exists $\epsilon_2 \in GL_3(O)$ such that we have $g = \alpha_1\alpha_2$ for $\alpha_2 = \mathrm{diag}(\pi,\pi,1)\epsilon_1$. Indeed since $GL_3(O)\pi = \pi GL_3(O)$, writing α_i as above, we have

$$\alpha_1\alpha_2 = \epsilon_1(\pi 1_3)\epsilon_2 = \pi\epsilon_1'\epsilon_2$$

for some ϵ_1'. So we may put $\epsilon_2 = \epsilon_1'^{-1}\epsilon$. So all we should do is to find $\epsilon_1 \in GL_3(O)$ such that $\alpha_1^* H_0 \alpha_1 \in \mathcal{H}_1$. For any $\epsilon_1 \in GL_3(O)$, if we define α_1 as above, we have $\alpha_1^* H_0 \alpha_1 \in T(1,\pi,\pi)$ or $T(1,1,p)$ and the condition that this belongs to $T(1,\pi,\pi)$ is equivalent to the condition that the matrix rank of $\alpha_1^* H_0 \alpha_1 \bmod \pi$ is 1. This means that the first 2×2 diagonal block modulo π of the matrix $\epsilon_1^* H_0 \epsilon_1$ is of rank 1. Since H_0 belongs to the principal genus, we have $\epsilon_0 \in GL_3(O_A)$ such that $H_0 = \epsilon_0\epsilon_0^*$. So we have

$$\epsilon_1^* H_0 \epsilon_1 = (\epsilon_1^*\epsilon_0)(\epsilon_1^*\epsilon_0)^*.$$

First of all, we show that there exists an element $\delta \in SL_3(O_A)$ such that the first 2×2 block of $\delta_0\delta_0^*$ is of rank 1 modulo π. There exists $y \in O_p \subset O_A$ such that $N(y) + 1 = 0$. Identifying y with an element of O_A such that the p-component is y and the other components are 0, put

$$\delta_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in SL_3(O_A).$$

Then we have

$$\delta_0\delta_0^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & y \\ 0 & \overline{y} & 1 \end{pmatrix}.$$

Now put put $\delta = (\epsilon_0^*)^{-1}\delta_0^*$, then $\delta^* H_0 \delta = \delta_0\delta_0^*$ and its $(2,2)$ block modulo π is of rank 1. If $N(\delta) = e \in \mathbb{Z}_A^\times$, then we replace δ by $\delta\mathrm{diag}(1,1,a)$ for some $a \in O_A^\times$ such that $N(a) = e^{-1}$. This belongs to $SL_3(O_A)$ and the $(2,2)$ block does not change, so we assume that $\delta \in SL_3(O_A)$ from the first. We write $\Gamma(1) = (1_3 + \pi M_3(O_A)) \cap SL_3(O_A)$. Since this is a open subgroup of $SL_3(B_A)$, by the strong approximation theorem, we have

$$\delta = \epsilon_3\epsilon_4$$

for some $\epsilon_3 \in GL_3(B)$ and $\epsilon_4 \in \Gamma(1)$. Since $\delta \in SL_3(O_A)$, we have $\epsilon_3 \in GL_3(O)$. Since $\epsilon_4 \equiv 1_3 \bmod \pi$, the $(2,2)$ block of $\epsilon_3^* H_0 \epsilon_3 \bmod \pi$ is of rank 1 and if we put $\alpha_1 = \epsilon_3\mathrm{diag}(1,1,\pi)$, then $H_1 := \alpha_1^* H_0 \alpha_1 \in \mathcal{H}_1$. Since $\pi GL_3(O) = GL_3(O)\pi$, we can choose $\epsilon_5 \in GL_3(O)$ such that

$$g = \epsilon_3\mathrm{diag}(1,1,\pi)\mathrm{diag}(\pi,\pi,1)\epsilon_5.$$

So we put $\alpha_2 = \mathrm{diag}(\pi,\pi,1)\epsilon_5$. Then we have

$$H_2 = g^* H_0 g = \alpha_2^*(\alpha_1^* H_0 \alpha_1)\alpha_2 = \alpha_2^* H_1 \alpha_2.$$

□

The similar statement for very good directions for the case $n = 2$ is now much easier. Assume that $K_0, H_0 \in \mathcal{H}^{(2)}_{\mathrm{prin}}$ and $H_1 \in \mathcal{H}^{(2)}_{\mathrm{np}}$. Put $H_2 = pK_0$. We fix $h_i \in GL_2(B)$ such that $h_i h_i^* = H_i$. Define $g_i \in G_A$ such that

$$O^2 h_2 = O^2\pi g_2, \quad O^2 h_0 = O^2 g_0 \quad O^2 h_1 = L g_1$$

where L is a standard global lattice in the non-principal genus such that $L_q = O_q^2$ and $L_p = O_p^2\mathrm{diag}(\pi,1)\xi$.

Proposition 4.8. *Assume that $n = 2$. Notation and assumption being the same as above, we have*
(i) *H_2 is descendable to H_1 if and only if $U_1 g_1 G \cap U_2 g_2 G \neq \emptyset$.*
(ii) *H_1 is descendable to H_0 if and only if $U_1 g_1 G \cap U_2 g_0 G \neq \emptyset$.*

We have also bijective correspondences between the set of orbits of directions and the orbits in $U_0\backslash U_2$ by $U_2 \cap g_2 G g_2^{-1}$ and those in $U_0\backslash U_1$ by $U_1 \cap g_1 G g_1^{-1}$. This is also bijective to the set of $U_0 - G$ double cosets in $U_2 g_2 G$ or $U_1 g_1 G$. Since the statement is similar to the case when $n = 3$ and obvious, we omit that part here. The proof of Proposition 4.8 uses Lemma 4.3 as in the case of $n = 3$. This is an easy exercise after completing the proofs for $n = 3$, so we omit the details.

In this section, we treated a lot about existence of $g \in M_n(O)$ such that $K_2 = g^* K_1 g$ for some positive quaternion hermitian matrices K_1 and K_2. We add here an easy lemma to interpret this into lattice terminology.

Lemma 4.9. *Let K_1, K_2 be $n \times n$ positive definite quaternion hermitian matrices. For $i = 1, 2$, we write $K_i = k_i k_i^*$ for some $k_i \in GL_n(B)$ and define left O lattices L_i by $L_i = O^n k_i$.*

Then there exists $g \in M_n(O)$ such that $K_2 = g^* K_1 g$ if and only if $L_2\gamma \subset L_1$ for some $\gamma \in G^1$.

Proof. If $L_2\gamma \subset L_1$ for $\gamma \in G^1$, then we have $O^n k_2 \gamma k_1^{-1} \subset O^n$, so if we put

$$g^* = k_2 \gamma k_1^{-1}$$

then we have $g \in M_n(O)$. Also we have $g^* K_1 g = k_2 \gamma\gamma^* k_2^* = k_2 k_2^* = K_2$. Conversely, if $K_2 = g^* K_1 g$ for some $g \in M_n(O)$, then put $\gamma = k_2^{-1} g^* k_1$. Then obviously we have $\gamma\gamma^* = 1_n$ and $L_2\gamma = O^n g^* k_1 \subset O^n k_1 = L_1$. □

The lattice terminology looks conceptually simpler but it does not mean that the proofs in this section become simpler by that.

5. Geometric Theorems

The results in the last section are actually all very geometric. Most interpretations are quite obvious and maybe there is no need to be repeated here. But some geometric theorems would be worth to be mentioned. In particular, we see how adelic double cosets describe existence of pftq explained in section 2 for $n = 2$ and 3. These are directly deduced from the results in the last section and the proofs will be mostly omitted here.

First we state our results for $n = 2$. We fix a polarization μ_1 of E^2 which belongs to the non-principal genus. We denote by $V(\mu_1)$ the component of $\mathcal{S}_{2,1}$ corresponding to μ_1. We denote by $U_1 g_1 G$ the double coset corresponding to μ_1. We also fix a principal polarization μ_0 of E^2 and denote by $U_2 g_0 G$ the double coset corresponding to μ_0. Here g_i are chosen as in the last section for $\phi_X^{-1}\mu_i \in \mathcal{H}^{(2)}_{\mathrm{prin}}$ or $\mathcal{H}^{(2)}_{\mathrm{np}}$.

Theorem 5.1. *The principally polarized superspecial abelian surface (E^2, μ_0) is on $V(\mu_1)$ if and only if $U_2 g_0 G \cap U_1 g_1 G \neq \emptyset$.*

The proof is a direct interpretation of Proposition 4.8 and omitted here.

For a fixed $U_2 g_0 G$, of course the double cosets $U_1 g_1 G$ with $U_2 g_0 G \cap U_1 g_1 G \neq \emptyset$ are not unique in general. In the same way, for a fixed $U_1 g_1 G$, the double cosets $U_2 g_0 G$ with $U_2 g_0 G \cap U_1 g_1 G \neq \emptyset$ are not unique in general.

Hereafter, we assume that $n = \dim A = 3$. A polarization μ_1 of E^3 satisfies $\mathrm{Ker}(\mu_1) \cong (\alpha_p)^2$ if and only if $\phi_X^{-1}\mu_1 \in \mathcal{H}_1 \subset T(1, \pi, \pi)$. For $\phi_X^{-1}\mu_1 = H_1$, we write $H_1 = h_1 h_1^*$ for $h_1 \in GL_3(B)$ and choose $g_1 \in G_A$ such that $O^3 h_1 = M g_1$, where M is the standard global lattice defined in the last section. We fix principal polarizations λ_0 and μ_0 of E^3 and denote by K_0 and H_0 the corresponding matrices in $\mathcal{H}^{(3)}_{\mathrm{prin}}$ respectively. We put $H_2 = pK_0$, We choose elements h_2 and $h_0 \in GL_3(B)$ such that $H_i = h_i h_i^*$ for $i = 0, 2$. Choose g_2, $g_0 \in G_A$ such that $O^3 h_2 = O^3 \pi g_2$ and $O^3 h_0 = O^3 g_0$. We have $n(g_i) \in \mathbb{Z}_A^\times = \mathbb{R}_+^\times \prod_{q:prime} \mathbb{Z}_q^\times$ for $i = 0$, 1, 2.

We denote by $V(\lambda_0)$ the irreducible component of $\mathcal{S}_{3,1}$ corresponding to $p\lambda_0$ by Proposition 2.1.

Theorem 5.2. *Notation and assumption being as above, we have the following results.*
(i) *Assume that for a principally polarized supersingular abelian three fold (A, λ) with $a(A) = 2$, there exists an isogeny $\phi_1 : (E^3, \mu_1) \longrightarrow (A, \lambda)$ such that $\mu_1 = \phi^*(\lambda)$. Then $V(\lambda_0)$ contains (A, λ) if and only if $U_2 g_2 G \cap U_1 g_1 G \neq \emptyset$.*

(ii) *The principally polarized superspecial abelian variety* (E^3, μ_0) *is in* $W(\mu_1)$ *if and only if* $U_1 g_1 G \cap U_2 g_0 G \neq \emptyset$.

The proof is a direct consequence of Propositions 4.5 and 4.6 and omitted here.

We add a few remarks on the intersection of adelic double cosets. Since $U_0 \subset U_1$ and U_2, any fixed $U_i - G$ double coset for each $i = 1, 2$ is a union of several $U_0 - G$ double cosets, and this inclusion relations describe a configuration of the algebraic subsets of fixed a-numbers defined above. For example, for a fixed g_2, we have

$$U_2 g_2 G = \bigcup_{j=1}^{m} U_0 g_{2,j} G \quad \text{(finite disjoint union)}$$

for some $g_{2,j} \in G_A$. If we fix $U_1 g_1 G$ ($g_1 \in G_A$), then the number of j such that $U_0 g_{2,j} G \subset U_1 g_1 G$, which might be 0, counts the number of $Aut(H_2)$-orbits of descendable directions from H_2 to (isomorphism classes of) H_1 as shown in Proposition 4.5 and in Theorem 5.2. But it seems we cannot expect a concrete formula for that. A similar thing is said for descendable directions from H_1 to H_0 based on claims in Propposition 4.6. When $n = 2$, see [12, section 8] for some explicit descriptions of such orbits in an irreducible component for small primes p. Also for $n = 2$, a list of the automorphism groups of irreducible components and the numbers of different components having the same automorphism group have been given in [5, Theorem 7.1].

Now, Lemma 2.4 is an easy corollary of Theorem 5.2. Indeed for $U_2 g_0 G$ corresponding to λ in the lemma, we have

$$U_2 g_0 G \subset G_A = \bigcup_{i=1}^{h} U_1 g_i' G$$

for some set of $g_i' \in G_A$, so obviously we have $U_2 g_0 G \cap U_1 g_i' G \neq \emptyset$ for some i. This means that there exists μ_1 in the lemma.

Next, we fix principal polarizations λ_0 and μ_0 of E^3 and see the condition that $(E^3, \mu_0) \in V(\lambda_0)$. The following result is a direct consequence of Proposition 4.7.

Theorem 5.3. *Notation and assumption being as above, we see that* (E^3, μ_0) *belongs to the component* $V(\lambda_0) \subset \mathcal{S}_{3,1}$ *if and only if there exists* $\phi : E^3 \to E^3$ *such that* $p\lambda_0 = \phi^*(\mu_0)$.

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