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THE $\mathbb{Z}_2$-BETTI NUMBERS OF ORIENTED GRASSMANNIANS

Dedicated to the memory of Professor Tomio Kubota

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Abstract

The purpose of this article is to show a simple method of finding $\mathbb{Z}_2$-Betti numbers of oriented Grassmannians. The actual calculation is performed by computing $\mathbb{Z}_2$-ranks of matrices defined in a combinatorial way. For the reduction of problem from topology to matrix rank, we will analyse Morse homology of certain functions on the oriented Grassmannians.

1. Introduction

Let $Gr_{m,n}$ be the Grassmannian of m-dimensional subspaces in $(n+m)$-dimensional real vector space $\mathbb{R}^{n+m}$, and $\widetilde{Gr}_{m,n}$ the Grassmannian of oriented m-dimensional subspaces in $\mathbb{R}^{n+m}$. We will simply call $\widetilde{Gr}_{m,n}$ an *oriented Grassmannian*. Since there exist canonical diffeomorphisms between $Gr_{m,n}$ and $Gr_{n,m}$, and between $\widetilde{Gr}_{m,n}$ and $\widetilde{Gr}_{n,m}$, we will always consider Grassmannians of (oriented) subspaces of dimension at most the half of the dimension of ambient vector spaces.

Grassmannian has been a central theme in geometry and topology. Their cohomological structure is well understood by means of both Schubert cell decompositions and Stiefel-Whitney characteristic classes of the tautological vector bundles. The Grassmannians can be naturally embedded in linear spaces of matrices, where almost all linear functions restricted to the image are *perfect Morse functions* on $Gr_{m,n}$, namely all Morse inequalities are equalities for certain coefficient field in homology, and the unstable manifold decomposition coincides with the Schubert cell decomposition.

In contrast, the cohomological structure of oriented Grassmannians $\widetilde{Gr}_{m,n}$ is still unclear except for $m = 1, 2$, and even computing $\mathbb{Z}_2$-Betti numbers of $\widetilde{Gr}_{m,n}$ with $m \geq 3$ has not been completed. For recent researches on cohomological structure of oriented Grassmannians $\widetilde{Gr}_{m,n}$ for $m = 3, 4$, see for example [2], [5] and [7], and references therein.

In order to calculate $\mathbb{Z}_2$-Betti numbers of $\widetilde{Gr}_{m,n}$, we consider the following matrices: let $L_{m,n}$ be the set of Young diagrams fitting in an $m \times n$ rectangle, which has a natural ordering and turns into a graded poset. Let $L_{m,n}(k)$ be the set of elements of rank k in $L_{m,n}$. Denote by $\delta_{m,n}(k)$ the adjacency matrix of the sub-bipartite graph with two vertex sets $L_{m,n}(k)$ and $L_{m,n}(k+1)$ extracted from the Hasse diagram of the poset $L_{m,n}$. We will denote by $r_{m,n}(k)$ the $\mathbb{Z}_2$-rank of $\delta_{m,n}(k)$ (see Section 2). The rank generating function of the poset $L_{m,n}$ is the so-called *q-binomial coefficient* (see 2.1), namely the cardinality $s_{m,n}(k)$ of $L_{m,n}(k)$ is the coefficient of q^k in the q-binomial coefficient. Then the matrix $\delta_{m,n}(k)$ is an $s_{m,n}(k) \times$

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$s_{m,n}(k+1)$ matrix whose entries are either 1 or 0. The $\mathbb{Z}_2$-rank of this matrix is at most $\min\{s_{m,n}(k), s_{m,n}(k+1)\}$. These matrices are also called the *incidence matrices* of the poset $L_{m,n}$ (see [3, Definition 1.48]). The usual ranks (not the $\mathbb{Z}_2$-rank) of them are studied in relation with problems in combinatorics such as the Sperner and Lefschetz properties. Here our concern is the ranks $r_{m,n}(k)$ of these matrices as $\mathbb{Z}_2$-module homomorphisms. The author could not find any results on the $\mathbb{Z}_2$-ranks $r_{m,n}(k)$ in the literature.

Denote by $R_{m,n}(q)$ the generating function $\sum_{k=0}^{mn} r_{m,n}(k)q^k$ of the series $\{r_{m,n}(k)\}$ of $\mathbb{Z}_2$-ranks. The main theorem of this article (Theorem 2.1) states that *the sum of the $\mathbb{Z}_2$-Poincaré polynomial of oriented Grassmannian $\widetilde{Gr}_{m,n}$ and $(1+q)R_{m,n}(q)$ equals twice the q-binomial coefficient*. For the proof, we use Morse homology of a certain function on the oriented Grassmannian. Even though the existence of perfect Morse functions on $Gr_{m,n}$ is well-known, we do not know whether perfect Morse functions on $\widetilde{Gr}_{m,n}$ exist or not. However if we consider the composition of the above perfect Morse function on $Gr_{m,n}$ with the canonical double covering map $\widetilde{Gr}_{m,n} \to Gr_{m,n}$, we obtain a function on $\widetilde{Gr}_{m,n}$ whose boundary operator of the corresponding Morse complex has a simple form (see (5.1)) which is closely related to the matrix $\delta_{m,n}(k)$.

In Section 6, we will deduce from Theorem 2.1 along with Lemma 6.1 that there exists a positive integer $\kappa_{m,n} \le nm/2$ such that the $\mathbb{Z}_2$-Betti number $b_k(\widetilde{Gr}_{m,n})$ is equal to the difference $s_{m,n}(k) - s_{m,n}(k-1)$ for all positive integers $k \le \kappa_{m,n}$ (Corollary 6.2). Regarding the double covering $\widetilde{Gr}_{m,n} \to Gr_{m,n}$ as a 0-dimensional sphere bundle, we obtain the Gysin exact sequence

$$\cdots \to H^{k-1}(Gr_{m,n}) \overset{\cup w}{\to} H^k(Gr_{m,n}) \to H^k(\widetilde{Gr}_{m,n}) \to H^k(Gr_{m,n}) \to \cdots,$$

where w is the Stiefel-Whitney class of the $\mathbb{R}^1$-bundle associated to the S^0-bundle. Since the Poincaré polynomial of $Gr_{m,n}$ equals the q-binomial coefficient, the existence of the integer $\kappa_{m,n}$ implies that the homomorphisms $H^{k-1}(Gr_{m,n}) \overset{\cup w}{\to} H^k(Gr_{m,n})$ are injective for all $k \le \kappa_{m,n}$. In Example 2.2 we find $\kappa_{4,4} = 3$ and $\kappa_{5,5} = 10$. The existence of $\kappa_{m,n}$ is a consequence of Lemma 6.1 which says that the $\mathbb{Z}_2$-rank $r_{m,n}(k)$ is maximal provided $k < n$. In this way, the maximality of the rank $r_{m,n}(k)$ is closely related to the estimate of *characteristic ranks* of oriented Grassmannians. For details, see for example [2], [5] and [7].

In this article, we will consider Grassmannians defined over the real field $\mathbb{R}$, and homology groups with coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

2. Combinatorics and Main theorem

2.1. Young Lattice. The graded poset $L_{m,n}$ of all Young diagrams fitting in an $m \times n$ rectangle with a usual partial ordering has an alternative description, that will be suitable for our presentation. Let $S_{m,n}$ be the set of m-element subsets $\alpha = \{\alpha(1), \cdots, \alpha(m)\} \subset \{1, \cdots, m+n\}$ with $\alpha(1) < \cdots < \alpha(m)$, and for α and $\beta \in S_{m,n}$ define a partial ordering $\alpha \le \beta$ by the condition that $\alpha(i) \le \beta(i)$ for all $i = 1, \cdots, m$. The relation $\le$ turns $S_{m,n}$ into a graded poset which is isomorphic to the Young lattice $L_{m,n}$. The isomorphism is given by assigning to each $\alpha \in S_{m,n}$ the Young diagram with $\alpha(m-j) - (m-j)$ boxes in the j-th row.

The *rank* $\rho(\alpha)$ of $\alpha \in S_{m,n}$ is defined by

$$\rho(\alpha) = \sum_{i=1}^{m} (\alpha(i) - i),$$

that satisfies $0 \leq \rho(\alpha) \leq mn$ for all $\alpha \in S_{m,n}$. Let $S_{m,n}(k)$ be the subset of $S_{m,n}$ consisting of elements α of rank $\rho(\alpha) = k$, and denote the cardinality $s_{m,n}(k) = |S_{m,n}(k)|$, whose generating function with respect to k is equal to the q-binomial coefficient;

$$\sum_{k=0}^{mn} s_{m,n}(k) q^k = \begin{bmatrix} n+m \\ m \end{bmatrix}_q = \prod_{i=1}^{m} \frac{1-q^{n+i}}{1-q^i}. \tag{2.1}$$

The sequence $\{s_{m,n}(k)\}$ is symmetric and unimodal. See [8].

2.2. Bipartite graph and biadjacency matrix. We linearize the set $S_{m,n}(k)$ by the reverse lexicographic ordering $<_{rl}$ for each $k = 0, \cdots, mn$; let α and $\beta \in S_{m,n}(k)$, and set $\alpha <_{rl} \beta$ if and only if there exists an index j such that $\alpha(i) = \beta(i)$ for all $i > j$, and $\alpha(j) < \beta(j)$.

Define a matrix $\delta_{m,n}(k)$ whose entries are indexed by $(\alpha, \beta) \in S_{m,n}(k) \times S_{m,n}(k+1)$, and satisfies

$$(\delta_{m,n}(k))_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The rank of $\delta_{m,n}(k)$ as a $\mathbb{Z}_2$-module homomorphism is denoted by $r_{m,n}(k)$, and the generating function of the series $r_{m,n}(k)$ with respect to k is denoted by $R_{m,n}(q)$;

$$r_{m,n}(k) = \mathrm{rank}_{\mathbb{Z}_2}(\delta_{m,n}(k)), \quad \text{and} \quad R_{m,n}(q) = \sum_{k=0}^{mn-1} r_{m,n}(k) q^k. \tag{2.3}$$

In Figure 1, elements α_i of $S_{4,4}(4)$ are on the left, and elements β_j of $S_{4,4}(5)$ on the right, where $\alpha_i <_{rl} \alpha_j$ and $\beta_i <_{rl} \beta_j$ for $i > j$, and the first row shows $\{1, 2, 3, 8\} = \alpha_1$ and $\beta_1 = \{1, 2, 4, 8\}$. The line segments in the middle join pairs (α_i, β_j) satisfying $\alpha_i \leq \beta_j$, that is, (α_i, β_j) with $(\delta_{m,n}(4))_{\alpha_i,\beta_j} = 1$. For example, α_2 is joined to β_3, because $\beta_3(i) = \alpha_2(i)$ except for $i = 2$, and $\beta_3(2) = 3 = \alpha_2(2) + 1$. The $\mathbb{Z}_2$-rank of $\delta_{4,4}(4)$ is

$$r_{4,4}(4) = \mathrm{rank}_{\mathbb{Z}_2} \delta_{4,4}(4) = \mathrm{rank}_{\mathbb{Z}_2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = 4. \tag{2.4}$$

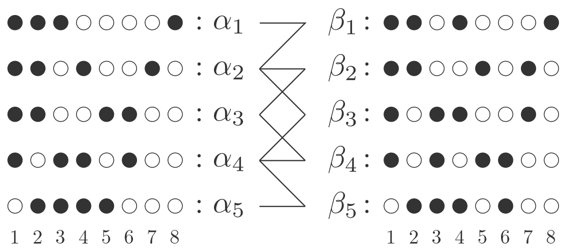

Fig. 1. Example $\delta_{4,4}(4)$. The bullets mark the positions of integers belonging to α_i's on the left, and to β_i's on the right.

2.3. Main theorem. For a pair (m, n) of integers with $0 < m \leq n$, let $\widetilde{Gr}_{m,n}$ be the Grassmannian of oriented m-dimensional subspaces in $\mathbb{R}^{m+n}$, and $P_{\widetilde{Gr}_{m,n}}(q)$ the $\mathbb{Z}_2$-Poincaré polynomial of $\widetilde{Gr}_{m,n}$;

$$P_{\widetilde{Gr}_{m,n}}(q) = \sum_{k=0}^{mn} (\dim_{\mathbb{Z}_2} H_k(\widetilde{Gr}_{m,n}))q^k.$$

We will prove the following in Section 5:

Theorem 2.1. *The $\mathbb{Z}_2$-Poincaré polynomial $P_{\widetilde{Gr}_{m,n}}(q)$ and the polynomial $R_{m,n}(q)$ defined in* (2.3) *satisfy*

$$P_{\widetilde{Gr}_{m,n}}(q) + (1+q)R_{m,n}(q) = 2\prod_{i=1}^{m} \frac{1-q^{n+i}}{1-q^i}.$$

EXAMPLE 2.2. The $\mathbb{Z}_2$-Poincaré polynomials of $\widetilde{Gr}_{4,4}$ and $\widetilde{Gr}_{5,5}$. There are $|S_{4,4}| = \binom{8}{4} = 70$ subsets α of $\{1, \cdots, 8\}$ of cardinality 4. The coefficients $s_{4,4}(k)$ of $\begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$ are shown in the second row in Table 1 below. We may calculate the $\mathbb{Z}_2$-ranks $r_{4,4}(k)$ of $\delta_{4,4}(k)$ one by one as in Figure 1 and (2.4), which are shown in the third row. By Theorem 2.1, we obtain the $\mathbb{Z}_2$-Betti numbers $b_k = \dim_{\mathbb{Z}_2} H_k(\widetilde{Gr}_{4,4})$, as shown in the last row. Since we have Poincaré duality, we have shown the first half of the $\mathbb{Z}_2$-Betti numbers b_k; $k \leq \dim \widetilde{Gr}_{4,4}/2 = 8$.

Table 1. The $\mathbb{Z}_2$-Betti numbers $b_k = \dim_{\mathbb{Z}_2} H_k(\widetilde{Gr}_{4,4})$.

k	0	1	2	3	4	5	6	7	8
$s_{4,4}(k)$	1	1	2	3	5	5	7	7	8
$r_{4,4}(k)$	1	1	2	3	**4**	5	**6**	**6**	**6**
b_k	1	0	1	1	3	1	3	2	4

(cf. [7, Theorem 3.3].) We also show the Betti numbers of $\widetilde{Gr}_{5,5}$ in Table 2.

Table 2. The $\mathbb{Z}_2$-Betti numbers $b_k = \dim_{\mathbb{Z}_2} H_k(\widetilde{Gr}_{5,5})$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_{5,5}(k)$	1	1	2	3	5	7	9	11	14	16	18	19	20
$r_{5,5}(k)$	1	1	2	3	5	7	9	11	14	16	18	**18**	20
b_k	1	0	1	1	2	2	2	2	3	2	2	2	2

In Section 6.1, we will show that the $\mathbb{Z}_2$-ranks of $\delta_{m,n}(k)$ are maximal for small k's. In Tables 1 and 2, the $\mathbb{Z}_2$-ranks of $\delta_{4,4}(k)$ with $k = 4, 6, 7, 8$ and $\delta_{5,5}(11)$ are not maximal.

3. Morse functions on $Gr_{m,n}$

The main reference of this section is [4], where complex Grassmannians are treated in detail, but the same idea can be applied for real Grassmannians.

We use notations in Section 2. Denote by $\mathrm{Sym}(N; \mathbb{R})$ the vector space of real symmetric matrices of size $N = m + n$, endowed with inner product defined by $\langle X, Y\rangle = \mathrm{trace}(XY)$. Given a diagonal matrix $D = \mathrm{diag}(a_1, \cdots, a_N)$ satisfying

$$a_1 < \cdots < a_N, \tag{3.1}$$

let $h_D : \mathrm{Sym}(N; \mathbb{R}) \to \mathbb{R}$ be a linear function defined by $h_D(X) = \langle X, D\rangle$. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^N$.

3.1. A Morse function. Let $Gr_{m,n}$ be the Grassmannian of m-dimensional subspaces of $\mathbb{R}^N$. The *standard embedding* $\varphi : Gr_{m,n} \to \mathrm{Sym}(N, \mathbb{R})$ is the mapping that assigns to each $V \in Gr_{m,n}$ a matrix $\varpi_V \in \mathrm{Sym}(N, \mathbb{R})$ satisfying $\varpi_V^2 = \varpi_V$ and $\varpi_V(\mathbb{R}^N) = V$, that is, the orthogonal projection of $\mathbb{R}^N$ onto V. We identify $Gr_{m,n}$ and the image $\varphi(Gr_{m,n})$, and regard it as a Riemannian manifold with the metric induced by φ.

Consider the composition $F = h_D \circ \varphi$:

$$Gr_{m,n} \xrightarrow{\varphi} \mathrm{Sym}(N, \mathbb{R}) \xrightarrow{h_D} \mathbb{R}.$$

Since the diagonal entries $a_1, \cdots, a_N$ are all distinct, $F = h_D \circ \varphi$ is a Morse function. The critical point set is given by

$$\mathrm{Crt}(F) = \{V_\alpha; \alpha \in S_{m,n}\}, \tag{3.2}$$

where V_α is the coordinate subspace of $\mathbb{R}^N$ corresponding to $\alpha \in S_{m,n}$;

$$V_\alpha = \mathbb{R}e_{\alpha(1)} \oplus \cdots \oplus \mathbb{R}e_{\alpha(m)}.$$

3.2. The gradient flow. The equation of the gradient flow of the Morse function $F : \varphi(Gr_{m,n}) \to \mathbb{R}$ is a restriction to the image $\varphi(Gr_{m,n})$ of the following ordinary differential equation

$$\frac{d\gamma}{dt}(t) = -\gamma(t)D(I - \gamma(t)) - (I - \gamma(t))D\gamma(t)$$

in $\mathrm{Sym}(N, \mathbb{R})$. *The solutions to this equation are the curves*

$$\mathbb{R} \ni t \mapsto e^{-tD}V \in Gr_{m,n}.$$

This result was carefully explained for complex Grassmannians in [4, Theorem 3.1.1], and the method can be applied to real Grassmannians, where we need only minor and natural modifications such as replacing unitary group by orthogonal group. From this, we obtain the stable and unstable manifolds of each critical point V_α in the same way as complex case (cf. [ibid., Section3.1]).

The condition (3.1) implies that the Morse index of a critical point $V_\alpha \in \mathrm{Crt}(F)$ equals the rank of α

$$\rho(\alpha) = \sum_{i=1}^{m}(\alpha(i) - i). \tag{3.3}$$

The stable and unstable manifolds, $W^s(V_\alpha)$ and $W^u(V_\alpha)$ respectively, of those critical points V_α are exactly the Schubert cells of $Gr_{m,n}$ with respect to the canonical filtration and the orthogonal filtration of $\mathbb{R}^N$. The stable manifolds $W^s(V_\alpha)$ and the unstable manifolds $W^u(V_\beta)$ intersect transversally, and hence F is a Morse-Smale function.

For a pair $(\alpha,\beta) \in S_{m,n}(k) \times S_{m,n}(k+1)$ with $\alpha \leq \beta$, the intersection $W^s(V_\alpha) \cap W^u(V_\beta)$ consists of two gradient flow lines, which are obtained as follows: the condition $\alpha \leq \beta$ implies that there is a unique integer i such that $\alpha(i) + 1 = \beta(i)$ (cf. (2.2)). Those curves are obtained by rotating $e_{\alpha(i)}$ in the plane $\mathbb{R}e_{\alpha(i)} \oplus \mathbb{R}e_{\beta(i)}$ both positively and negatively and leaving other basis vectors $e_{\alpha(j)}$ fixed. Those curves are exactly the gradient flow lines from V_β to V_α in the intersection $W^s(V_\alpha) \cap W^u(V_\beta)$, up to parametrization.

4. Morse homology

We recall the definitions of Morse complex and Morse homology. See [1] for details.

4.1. Morse complex and Morse homology. Suppose $f : M \to \mathbb{R}$ is a Morse-Smale function on a closed Riemannian manifold M. Let $\mathrm{Crt}_k(f)$ be the set of all critical points of f of Morse index k, and $C_k(f)$ the $\mathbb{Z}_2$-module freely generated by $\mathrm{Crt}_k(f)$. A homomorphism $\partial_k(f) : C_k(f) \to C_{k-1}(f)$, called *boundary operator*, is defined as follows: the matrix element $n(x,y)$ of $\partial_k(f)$ corresponding to a pair $(x,y) \in \mathrm{Crt}_k(f) \times \mathrm{Crt}_{k-1}(f)$ is the number modulo 2 of the gradient flow lines in $W^u(x) \cap W^s(y)$;

$$\partial_k(f)\langle x\rangle = \sum_{y\in \mathrm{Crt}_{k-1}(f)} n(x,y)\langle y\rangle.$$

Theorem 4.1. *The boundary operators satisfy* $\partial_{k-1}(f) \circ \partial_k(f) = 0$*, and the resulting homology group* $H_*(C_*(f), \partial_*(f))$ *is isomorphic to the singular homology group* $H_*(M)$.

(For a proof, see [ibid, Section 7].) The chain complex $(C_*(f), \partial_*(f))$ is called the *Morse complex of* (M, f), and the homology of Morse complexes is called the *Morse homology*.

4.2. Morse homology on Grassmannians. The function $F : Gr_{m,n} \to \mathbb{R}$ in the previous section is Morse-Smale, and for all pairs $(\alpha,\beta) \in S_{m,n}(k) \times S_{m,n}(k+1)$ with $\alpha \leq \beta$ the intersection $W^s(V_\alpha) \cap W^u(V_\beta)$ consists of two gradient flow lines. Since we are working in $\mathbb{Z}_2$, all boundary operators $\partial_k(F)$ are *null homomorphisms*. Therefore, from (3.2) and (3.3), it follows that the $\mathbb{Z}_2$-Poincaré polynomial of $Gr_{m,n}$ is equal to

$$P_{Gr_{m,n}}(q) = \sum_{k=0}^{mn} (\dim_{\mathbb{Z}_2} H_k(Gr_{m,n}))q^k = \prod_{i=1}^{m} \frac{1-q^{n+i}}{1-q^i}.$$

(cf. Equation (2.1).)

We remark that the Poincaré polynomial of the *complex* Grassmannian $Gr_{m,n}(\mathbb{C})$ equals $P_{Gr_{m,n}}(q^2)$, since $Gr_{m,n}(\mathbb{C})$ also decomposes into the Schubert cells labeled by the set $S_{m,n}$, and the real dimension of the corresponding cell is equal to $2\rho(\alpha)$ for each $\alpha \in S_{m,n}$.

5. Morse homology on $\widetilde{Gr}_{m,n}$

Let $\pi : \widetilde{Gr}_{m,n} \to Gr_{m,n}$ be the canonical double covering map. We investigate the $\mathbb{Z}_2$-Morse complex of the function $\tilde{F} := F \circ \pi : \widetilde{Gr}_{m,n} \to \mathbb{R}$, where $F : Gr_{m,n} \to \mathbb{R}$ is defined in Section 3 by using a diagonal matrix $D = \mathrm{diag}(a_1, \cdots, a_{m+n})$ satisfying (3.1), and continue to use notations in Section 2.

5.1. Proof of Theorem 2.1. Let $\{V_\alpha^\pm\} \subset \widetilde{Gr}_{m,n}$ be the pre-image $\pi^{-1}(V_\alpha)$. Then the $\mathbb{Z}_2$-module of chains in the Morse complex $(C_k(\tilde{F}), \tilde{\partial}_k)$ is

$$C_k(\tilde{F}) = \bigoplus_{\alpha \in S_{m,n}(k)} \mathbb{Z}_2\langle V_\alpha^+ \rangle \oplus \mathbb{Z}_2\langle V_\alpha^- \rangle,$$

whose dimension equals $2s_{m,n}(k)$.

Let (α, β) be a pair in $S_{m,n}(k) \times S_{m,n}(k+1)$, and assume $\alpha \leq \beta$. The pre-images in $\widetilde{Gr}_{m,n}$ of the intersection $W^s(V_\alpha) \cap W^u(V_\beta)$ consist of four gradient flow lines, two of which issue from V_β^+ and converge to different critical points $V_\alpha^\pm$, and the other two issue from V_β^- and converge to different ones $V_\alpha^\pm$ as well (cf. Figure 2).

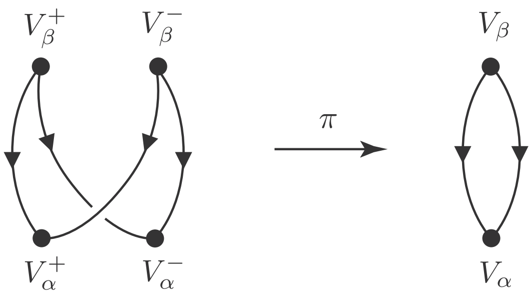

Fig. 2. Double cover of gradient flow lines.

We conclude that the k-th boundary operator is given by

$$\tilde{\partial}_k = \delta_{m,n}(k) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.1}$$

This implies that the $\mathbb{Z}_2$-ranks of $\tilde{\partial}_k$ and $\delta_{m,n}(k)$ *coincide*, and that

$$\begin{aligned} \dim_{\mathbb{Z}_2} H_k(\widetilde{Gr}_{m,n}) &= \dim_{\mathbb{Z}_2} \ker \tilde{\partial}_k - \dim_{\mathbb{Z}_2} \operatorname{image} \tilde{\partial}_{k-1} \\ &= (2s_{m,n}(k) - r_{m,n}(k)) - r_{m,n}(k-1). \end{aligned}$$

This completes the proof of Theorem 2.1. □

6. Corollary

In general the $\mathbb{Z}_2$-rank $r_{m,n}(k)$ of $\delta_{m,n}(k)$ in (2.2) is not necessarily maximal;

$$r_{m,n}(k) \leq \min\{s_{m,n}(k), s_{m,n}(k+1)\}.$$

Recall that the sequence $\{s_{m,n}(k)\}_k$ is unimodal and symmetric in k. For a given positive integer κ, if they are maximal for all $0 \leq k \leq \kappa$, then the first $\kappa + 1$ $\mathbb{Z}_2$-Betti numbers are given by

$$b_k(\widetilde{Gr}_{m,n}) = s_{m,n}(k) - s_{m,n}(k-1).$$

Table 1 shows $r_{4,4}(k) = s_{4,4}(k)$ holds for $k \leq 3$, but not for $k = 4$, and Table 2 shows $r_{5,5}(k) = s_{5,5}(k)$ holds for $k \leq 10$, but not for $k = 11$.

6.1. Low dimensional Betti numbers. We only confirm the rank $r_{m,n}(k)$ is maximal in a restricted range.

Lemma 6.1. *For any $k < \max\{m, n\}$, it holds that $r_{m,n}(k) = s_{m,n}(k)$.*

Proof. The notation $<_{rl}$ stands for the reverse lexicographic ordering in $S_{m,n}(k)$ as used in Section 2.2. Suppose $k < \max\{m, n\}$. Consider, for each $\alpha \in S_{m,n}(k)$, the maximal element $\max_{rl}\{\beta \in S_{m,n}(k+1) | \alpha \leq \beta\}$ with respect to the ordering $<_{rl}$. For α_1 and $\alpha_2 \in S_{m,n}(k)$, we have, if $\alpha_1 <_{rl} \alpha_2$,

$$\max_{rl}\{\beta \in S_{m,n}(k+1) | \alpha_2 \leq \beta\} <_{rl} \max_{rl}\{\beta \in S_{m,n}(k+1) | \alpha_1 \leq \beta\}.$$

(This does not necessarily hold for $k \geq \max\{m, n\}$.) This implies the biadjacency matrix $\delta_{m,n}(k)$ is in echelon form provided the matrix is formed by using reverse lexicographic ordering for both rows and columns, and has no zero rows. Therefore the rank is maximal. By unimodality and symmetricity of $s_{m,n}(k)$, the inequality $s_{m,n}(k) \leq s_{m,n}(k+1)$ holds for any $k < \max\{m, n\}$. (Recall we are assuming that $2 \leq m \leq n$.) Therefore we have $r_{m,n}(k) = s_{m,n}(k)$. □

Corollary 6.2. *If $2 \leq m \leq n$, then the $\mathbb{Z}_2$-Poincaré polynomial $P_{\widetilde{Gr}_{m,n}}(q)$ and the formal power series $\prod_{\ell=2}^{m}(1 - q^\ell)^{-1}$ coincide up to degree* $\max\{m, n\} - 1$.

Proof. Theorem 2.1 states that the k-th Betti number $b_k(\widetilde{Gr}_{m,n})$ equals $2s_{m,n}(k) - r_{m,n}(k) - r_{m,n}(k-1)$ for any k. Therefore Lemma 6.1 implies

$$b_k(\widetilde{Gr}_{m,n}) = s_{m,n}(k) - s_{m,n}(k-1)$$

for $k < \max\{m, n\}$, where we set $s_{m,n}(-1) = 0$. The q-binomial coefficient times the factor $1 - q$ coincides with $\prod_{\ell=2}^{m}(1 - q^\ell)^{-1}$ up to degree $\max\{m, n\} - 1$. □

Remark that $\widetilde{Gr}_{1,n}$, where $m = 1$, is a sphere of dimension n, and the Poincaré polynomial $P_{\widetilde{Gr}_{1,n}}(q)$ is $1 + q^n$. For reference, we record the formal power series $\prod_{\ell=2}^{m}(1 - q^\ell)^{-1}$ for $m = 3, 4, 5$;

$$\prod_{\ell=2}^{3}(1 - q^\ell)^{-1} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + \cdots,$$

$$\prod_{\ell=2}^{4}(1 - q^\ell)^{-1} = 1 + q^2 + q^3 + 2q^4 + q^5 + 3q^6 + 2q^7 + 4q^8 + 3q^9 + 5q^{10} + \cdots,$$

$$\prod_{\ell=2}^{5}(1 - q^\ell)^{-1} = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 5q^9 + 7q^{10} + \cdots.$$

6.2. Infinite dimensional oriented Grassmannians. Immediately from Corollary 6.2, we deduce the following for infinite dimensional oriented Grassmannians $\widetilde{Gr}_{m,\infty}$ (cf. [6, Theorem 12.4]) :

Corollary 6.3. *If $2 \leq m$, then the $\mathbb{Z}_2$-Poincaré polynomial $P_{\widetilde{Gr}_{m,\infty}}(q)$ of the oriented Grassmannian $\widetilde{Gr}_{m,\infty}$ equals the formal power series*

$$P_{\widetilde{Gr}_{m,\infty}}(q) = \prod_{\ell=2}^{m}(1 - q^\ell)^{-1}.$$

Proof. The infinite dimensional oriented Grassmannian $\widetilde{Gr}_{m,\infty}$ is the direct limit of the sequence of natural embeddings

$$\cdots \hookrightarrow \widetilde{Gr}_{m,n} \hookrightarrow \widetilde{Gr}_{m,n+1} \hookrightarrow \cdots ,$$

which are compatible with CW-complex structures defined by Schubert cells of the natural filtration of $\mathbb{R}^{m+n}$'s. If n is sufficiently large, then all cells of dimension less than or equal to k of $\widetilde{Gr}_{m,\infty}$ is contained in $\widetilde{Gr}_{m,n}$. Hence the homology groups of dimension less than k of $\widetilde{Gr}_{m,\infty}$ and $\widetilde{Gr}_{m,n}$ coinside. Therefore Corollary 6.2 gives the stable Betti numbers of $\widetilde{Gr}_{m,\infty}$. □

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