JOIN THEOREM FOR REAL ANALYTIC SINGULARITIES

KAZUMASA INABA

(Received November 6, 2020, revised February 9, 2021)

Abstract

Let $f_1 : (\mathbb{R}^n, 0_n) \to (\mathbb{R}^p, 0_p)$ and $f_2 : (\mathbb{R}^m, 0_m) \to (\mathbb{R}^p, 0_p)$ be analytic germs of independent variables, where $n, m \geq p \geq 2$. In this paper, we assume that $f_1, f_2$ and $f = f_1 + f_2$ satisfy $\alpha_j$-condition. Then we show that the tubular Milnor fiber of $f$ is homotopy equivalent to the join of tubular Milnor fibers of $f_1$ and $f_2$. If $p = 2$, the monodromy of the tubular Milnor fibration of $f$ is equal to the join of the monodromies of the tubular Milnor fibrations of $f_1$ and $f_2$ up to homotopy.

1. Introduction

Let $g : (\mathbb{R}^N, 0_N) \to (\mathbb{R}^p, 0_p)$ be an analytic germ, where $N \geq p \geq 2$, $0_N$ and $0_p$ are the origins of $\mathbb{R}^N$ and $\mathbb{R}^p$ respectively. Take a positive real number $\varepsilon_0$ sufficiently small if necessary. Assume that for any $0 < \varepsilon \leq \varepsilon_0$, there exists a positive real number $\delta$ such that $\delta \ll \varepsilon$ and $g : B^N_\varepsilon \cap g^{-1}(D^p_\delta \setminus \{0_p\}) \to D^p_\delta \setminus \{0_p\}$ is a locally trivial fibration, where $B^N_\varepsilon = \{x \in \mathbb{R}^N | ||x|| \leq \varepsilon\}$ and $D^p_\delta = \{w \in \mathbb{R}^p | ||w|| \leq \delta\}$. In this paper, $B^N_\varepsilon$ is used for the disk in the defining euclidean space. The isomorphism class of the above fibration does not depend on the choice of $\varepsilon$ and $\delta$. This map is called the tubular Milnor fibration of $g$. If $f_1$ and $f_2$ are holomorphic functions of independent variables, the following theorem is known.

**Theorem 1** (Join theorem). Let $f_1 : (\mathbb{C}^n, 0_{2n}) \to (\mathbb{C}, 0_2)$ and $f_2 : (\mathbb{C}^m, 0_{2m}) \to (\mathbb{C}, 0_2)$ be holomorphic functions of independent variables $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_m)$. Set $f(z, w) = f_1(z) + f_2(w)$. Then the Milnor fiber of $f$ is homotopy equivalent to the join of the Milnor fibers of $f_1$ and $f_2$ and the monodromy of $f$ is equal to the join of the monodromies of $f_1$ and $f_2$ up to homotopy.

Join theorem is algebraically proved by M. Sebastiani and R. Thom for isolated singularities [31]. M. Oka showed this for weighted homogeneous singularities [23]. For general complex singularities, this is proved by K. Sakamoto [30]. In [14], L. H. Kauffman and W. D. Neumann studied fiber structures and Seifert forms of links defined by tame isolated singularities of real analytic germs of independent variables. In this paper, we study Join theorem for more general real analytic singularities.

To show the existence of Milnor fibrations for real analytic singularities, we consider stratifications of analytic sets. Let $\varepsilon$ be a small positive real number. Let $g : (\mathbb{R}^N, 0_N) \to $
be a smooth map and $S$ be a stratification of $B^N_e \cap g^{-1}(0_p)$. The map $g$ satisfies the $a_f$-condition if $B^N_e \setminus g^{-1}(0_p)$ has no critical point and satisfies the following condition: For any sequence $p_v \in B^N_e \setminus g^{-1}(0_p)$ such that

$$T_{p_v}g^{-1}(g(p_v)) \to \tau, \ p_v \to p_\infty \in M,$$

where $M \in S$, we have $T_{p_\infty}M \subset \tau$. A stratification $S$ is called Whitney (a)-regular if for any pair of strata $(S_1, S_2)$ of $S$ and any point $p \in S_1 \cap \overline{S_2}$, $(S_1, S_2)$ satisfies the following condition: For any sequence $q_v \in S_2$ satisfying

$$q_v \to p, \ T_{q_v}S_2 \to T,$$

we have $T_pS_1 \subset T$. We say $\varepsilon$ is an $a_f$-stable radius for $g$ with respect to $S$ if it satisfies the following: Each sphere $S^{N-1}_e, 0 < \varepsilon' \leq \varepsilon$ intersects transversely with any stratum of $S$ and $0_p$ is the only critical value of $g : B^N_e \to \mathbb{R}^p$.

Let $f_1 : (\mathbb{R}^n, 0_p) \to (\mathbb{R}^p, 0_p)$ and $f_2 : (\mathbb{R}^m, 0_p) \to (\mathbb{R}^p, 0_p)$ be analytic germs, where $n, m \geq p \geq 2$. Set $V(f_1) = f_1^{-1}(0_p) \cap B^N_e$ and $V(f_2) = f_2^{-1}(0_p) \cap B^m_e$ for $0 < \varepsilon \ll 1$. We denote a stratification of $V(f_1)$ (resp. $V(f_2)$) by $S_1$ (resp. $S_2$). Assume that $f_1$ and $f_2$ satisfy the following conditions:

(a-i) $V(f_j)$ has codimension $p$ at the origin, $f_j$ has an isolated value at the origin and $f_j$ is locally surjective on $V(f_j)$ near the origin for $j = 1, 2$,

(a-ii) $f_j$ satisfies the $a_f$-condition with respect to $S_j$ for $j = 1, 2$.

Here $g : (\mathbb{R}^N, 0_N) \to (\mathbb{R}^p, 0_p)$ is locally surjective near the origin if there exists a positive real number $\varepsilon$ so that for any $x \in V(g) \cap B^N_e$, there exists an open neighborhood $W$ of $x$ so that $0_p$ is an interior point of the image $g(W)$. Since $V(f_j)$ and $V(f_2)$ are real analytic sets, we may assume that $S_1$ and $S_2$ are Whitney stratifications. See [10] for further information.

We take $\varepsilon$ sufficiently small if necessary. Then the sphere $\partial B^N_e$ (resp. $\partial B^m_e$) intersects $M_1$ (resp. $M_2$) transversely for any $M_1 \in S_1$ and $M_2 \in S_2$. See [19, Corollary 2.9] and the proof of [3, Lemma 3.2].

Take a common $a_f$-stable radius $\varepsilon$ for $f_1$ and $f_2$ and take a sufficiently small $\delta$, $0 < \delta \ll \varepsilon$ so that $f_j^{-1}(\eta)$ intersects transversely with the sphere of radius $\varepsilon$ for $j = 1, 2$. See Lemma 1 below for the existence of such a $\delta$. Hereafter we use $0 < \delta \ll \varepsilon$ in this sense.

We assume that $\varepsilon$ is a common $a_f$-stable radius for $f_j$ with respect to $S_j$ for $j = 1, 2$ and take $U_1 = U_1(\varepsilon, \delta)$ and $U_2 = U_2(\varepsilon, \delta)$ for $0 < \delta \ll \varepsilon$. Here $U_j(\varepsilon, \delta) = \{x \in B^N_e \mid \|f_j(x)\| \leq \delta\}$ with $n_1 = n$ and $n_2 = m$. By the above conditions and the Ehresmann fibration theorem [33], we may assume that

$$f_j : U_j \setminus V(f_j) \to D^p_\delta \setminus \{0_p\}$$

is a locally trivial fibration for $j = 1, 2$. We call these fibrations stable tubular Milnor fibrations of $f_j$ for $j = 1, 2$.

Let $f : (\mathbb{R}^n \times \mathbb{R}^m, 0_{n+m}) \to (\mathbb{R}^p, 0_p)$ be the analytic germ defined by $f = f_1 + f_2$. Put $V(f) = f^{-1}(0) \cap (U_1 \times U_2)$. By [1, Proposition 5.2], $f$ also satisfies the conditions (a-i) and (a-ii) with respect to the stratification $S$ for $f$ which will be defined in Section 2. See Section 2.1. The main theorem of this paper is the following.

**Theorem 2.** Let $f_1 : (\mathbb{R}^n, 0_n) \to (\mathbb{R}^p, 0_p)$ and $f_2 : (\mathbb{R}^m, 0_m) \to (\mathbb{R}^p, 0_p)$ be analytic germs of independent variables, where $n, m \geq p \geq 2$. Assume that $f_1$ and $f_2$ satisfy the conditions
(a-i) and (a-ii). Set $f = f_1 + f_2$. Then the fiber of the tubular Milnor fibration of $f$ is homotopy equivalent to the join of the fibers of the tubular Milnor fibrations of $f_1$ and $f_2$.

Moreover, if $p = 2$, the monodromy of the tubular Milnor fibration of $f$ is equal to the join of the monodromies of $f_1$ and $f_2$ up to homotopy.

Moreover, we assume that $f_1, f_2$ and $f$ satisfy the following condition:

(a-iii) there exists a positive real number $r'$ such that

$$P/|P| : \partial B_r^N \setminus K_p \to S^{p-1}$$

is a locally trivial fibration and this fibration is isomorphic to the tubular Milnor fibration of $P$, where $K_p = \partial B_r^N \cap P^{-1}(0)$ and $0 < r \leq r'$ for $(P, N) = (f_1, n), (f_2, m), (f, n+m)$.

The fibration in (a-iii) is called the spherical Milnor fibration of $P$. By using Theorem 2 and the condition (a-iii), we have

**Corollary 1.** Let $f_1 : (\mathbb{R}^p, 0_p) \to (\mathbb{R}^p, 0)$ and $f_2 : (\mathbb{R}^m, 0_m) \to (\mathbb{R}^p, 0)$ be analytic germs in Theorem 2. Assume that $f_1, f_2$ and $f = f_1 + f_2$ satisfy the condition (a-iii). Then the fiber of the spherical Milnor fibration of $f$ is homotopy equivalent to the join of the fibers of the spherical Milnor fibrations of $f_1$ and $f_2$.

If $p$ is equal to 2, analytic germs which satisfy the above conditions were studied by Oka [25, 26]. Let $(\rho_1, \rho_2) : (\mathbb{R}^{2n}, 0_{2n}) \to (\mathbb{R}^2, 0_2)$ be an analytic map germ with real 2n-variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. Then $(\rho_1, \rho_2)$ is represented by a complex-valued function of variables $z = (z_1, \ldots, z_n)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ as

$$P(z, \bar{z}) := \rho_1 \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\sqrt{-1}} \right) + \sqrt{-1} \rho_2 \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\sqrt{-1}} \right).$$

Here any complex variable $z_j$ of $\mathbb{C}^n$ is represented by $x_j + \sqrt{-1}y_j$ and $\bar{z}_j$ is the complex conjugate of $z_j$ for $j = 1, \ldots, n$. Then a map $P : (\mathbb{C}^n, 0_{2n}) \to (\mathbb{C}, 0_2)$ is called a mixed function map. For mixed weighted homogeneous singularities, Join theorem is proved by J. L. Cisneros-Molina [4]. Oka introduced the notion of Newton boundaries of mixed functions and the concept of strong non-degeneracy. If $P$ is a convenient strongly non-degenerate mixed function or a strongly non-degenerate mixed function which is locally tame along vanishing coordinate subspaces, then $P$ satisfies the conditions (a-i), (a-ii) and (a-iii). See [25, 26, 8].

We study the topology of Milnor fibrations of join type. If a mixed function $P$ satisfies the condition (a-iii) and the origin is an isolated singularity of $P$, the Seifert form is determined by the spherical Milnor fibration of $P$. Note that the Seifert form is a topological invariant of fibrations. Then we calculate Seifert forms defined by joins of Milnor fibrations of 1-variable mixed functions in Corollary 5. This is a generalization of [29, Corollary 3].

We also study homotopy types of fibered links defined by isolated singularities of join type. In [20, 21, 22], W. Neumann and L. Rudolph defined the enhanced Milnor number and the enhancement to the Milnor number of a fibered link. These are invariants of homotopy types of fibered links in $S^{2k+1}$. If $k = 1$, it is shown that for any $d \in \mathbb{Z}$, there exists a mixed polynomial $P$ such that the enhancement to the Milnor number of $K_p$ is equal to $d$ [11]. If $k$ is greater than 1, the enhanced Milnor number is represented by $((-1)^{k+1} \ell, r)$, where $\ell \in \mathbb{N}$.
and \( r \in \{0, 1\} \). Note that there exists a complex polynomial \( Q \) such that the enhanced Milnor number determined by the Milnor fibration of \( Q \) is equal to \((-1)^{k+1} \ell, 0\) for \( \ell \in \mathbb{N} \) and \( k \geq 2 \). We show that there exists a mixed polynomial of join type such that the enhanced Milnor number of a link defined by a mixed polynomial is equal to \((-1)^{k+1} \ell, 1\) for \( \ell \in \mathbb{N} \) and \( k \geq 2 \).

This paper is organized as follows. In Section 2 we give some Join type statements, the definition of zeta functions of monodromies and strongly non-degenerate mixed functions. In Section 3 we prove Theorem 2 and Corollary 1. In Section 4 we consider Join theorem of Seifert forms of links defined by 1-variable mixed polynomials. In Section 5 we study homotopy types of Milnor fibrations defined by mixed polynomial of Join type.

2. Preliminaries

2.1. Join type statements. Let \( g : (\mathbb{R}^N, 0_N) \rightarrow (\mathbb{R}^p, 0_p) \) be an analytic germ which satisfies the conditions (a-i) and (a-ii) with respect to a Whitney regular stratification \( S \). By using the same argument in [26, Proposition 11], we can show the following lemma.

**Lemma 1.** Assume that an analytic germ \( g : (\mathbb{R}^N, 0_N) \rightarrow (\mathbb{R}^p, 0_p) \) satisfies the conditions (a-i) and (a-ii). Take an \( \alpha f \)-stable radius \( r_0 \) for \( g \). For any positive real number \( r_1 \) which satisfies \( r_1 \leq r_0 \), there exists a positive real number \( \tilde{\delta} \) such that \( g^{-1}(\eta) \) intersects transversely with the sphere \( S_p^{k-1} \) for \( r_1 \leq r \leq r_0 \) and \( 0 < ||\eta|| \leq \tilde{\delta} \).

**Corollary 2.** Take an \( \alpha f \)-stable radius \( r_0 \) for \( g \) and take any \( \tilde{\delta}_1 \leq \tilde{\delta} \). Then for any \( r_1 \leq r \leq r_0 \), the isomorphism class of the tubular Milnor fibration \( g : U(r, \tilde{\delta}_1) \setminus g^{-1}(0_p) \rightarrow D_P^{\tilde{\delta}_1} \setminus \{0_p\} \) is independent of the choice of \( r \) and \( \tilde{\delta}_1 \).

Let \( f_1 : (\mathbb{R}^n, 0_n) \rightarrow (\mathbb{R}^p, 0_p) \) and \( f_2 : (\mathbb{R}^m, 0_m) \rightarrow (\mathbb{R}^p, 0_p) \) be analytic germs which satisfy the conditions (i) and (ii), where \( n, m \geq p \geq 2 \). Let \( f : (\mathbb{R}^n \times \mathbb{R}^m, 0_{n+m}) \rightarrow (\mathbb{R}^p, 0_p) \) be the analytic germ defined by \( f = f_1 + f_2 \). Put \( V(f) = f^{-1}(0) \cap (U_1(\varepsilon_0, \delta) \times U_2(\varepsilon_0, \delta)) \). We take the stratification \( S \) of \( V(f) \) as follows:

\[
S : (S_1 \times S_2) \cup (V(f) \setminus (V(f_1) \times V(f_2))),
\]

where \( S_j \) is a stratification of \( V(f_j) \) in Section 1 for \( j = 1, 2 \). By [9, p.12], we may assume that \( S_1 \times S_2 \) is Whitney (a)-regular. By using the stratification \( S \) of \( V(f) \), R. N. Araújo dos Santos, Y. Chen and M. Tibăr showed the following lemma.

**Lemma 2** ([1, Proposition 5.2]). Let \( f_1 : (\mathbb{R}^n, 0_n) \rightarrow (\mathbb{R}^p, 0_p) \) and \( f_2 : (\mathbb{R}^m, 0_m) \rightarrow (\mathbb{R}^p, 0_p) \) be analytic germs which satisfy the conditions (a-i) and (a-ii), where \( n, m \geq p \geq 2 \). Then the analytic germ \( f = f_1 + f_2 : (\mathbb{R}^n \times \mathbb{R}^m, 0_{n+m}) \rightarrow (\mathbb{R}^p, 0_p) \) also satisfies the conditions (a-i) and (a-ii).

By Lemma 1 and Lemma 2, we have

**Corollary 3.** Take \( \varepsilon_0 \) sufficiently small so that we assume that \( \varepsilon_0 \) is also an \( \alpha f \)-stable radius for \( f \) with respect to the above \( S \). Then \( f \) satisfies the conditions (a-i) and (a-ii) on \( B^{n+m}_\varepsilon \) for \( 0 < \varepsilon \leq \varepsilon_0 \). Moreover, there exists a positive real number \( \delta \) such that

\[
f : B^{n+m}_\varepsilon \cap f^{-1}(D_p^\delta \setminus \{0_p\}) \rightarrow D_p^\delta \setminus \{0_p\}
\]

is a locally trivial fibration.
2.2. Divisors and Zeta functions of monodromies. Take 1-variable polynomials \( q_1(t) \) and \( q_2(t) \) with \( q_1(0) = q_2(0) = 0 \). Set \( q_1(t) = a \prod_{j=1}^{k}(t - \alpha_j) \) and \( q_2(t) = b \prod_{j=1}^{\ell}(t - \beta_j) \), where \( a, b, \alpha_j, \beta_j \in \mathbb{C}^{\ast} := \mathbb{C} \setminus \{0\} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, \ell \). Then we define the divisor of \( q_1(t)/q_2(t) \) by

\[
\left( \frac{q_1(0)}{q_2(0)} \right) = \sum_{i=1}^{k} (\alpha_i) - \sum_{j=1}^{\ell} (\beta_j) \in \mathbb{Z}(\mathbb{C}^{\ast}),
\]

where \( \mathbb{Z}(\mathbb{C}^{\ast}) \) is the group ring of \( \mathbb{C}^{\ast} \).

Let \( F \) be the fiber of the spherical Milnor fibration of \( g : (\mathbb{R}^{2n}, \mathbf{0}_{2n}) \to (\mathbb{R}^{2}, \mathbf{0}_2) \) and \( h : F \to F \) be the monodromy of this fibration. Set \( P_j(t) = \det(\text{Id} - th_{\nu, j}) \), where \( h_{\nu, j} : H_j(F, \mathbb{Q}) \to H_j(F, \mathbb{Q}) \) is an isomorphism induced by \( h \). Then the zeta function \( \zeta(t) \) of the monodromy is defined by

\[
\zeta(t) = \prod_{j=0}^{2n-2} P_j(t)^{(-1)^{j+1}}.
\]

See [19, Section 9] and [24, Chapter I]. Assume that \( g \) satisfies the following properties:

(a) \( \mathbf{0}_{2n} \) is an isolated singularity of \( g \).

(b) \( F \) has a homotopy type of a finite CW-complex of dimension \( \leq n - 1 \).

(c) \( F \) is \((n - 2)\)-connected.

Then the zeta function \( \zeta(t) \) is equal to \( P_{n-1}(t)^{(-1)^{n}}/(t - 1) \) and the reduced zeta function is defined by \( \tilde{\zeta}(t) = (t - 1)^{-1}\zeta(t) \).

2.3. Strongly non-degenerate mixed functions. In this subsection, we introduce a class of mixed functions which admit tubular Milnor fibrations and spherical Milnor fibrations given by Oka in [25]. Let \( P(z, \bar{z}) \) be a mixed function, i.e., \( P(z, \bar{z}) \) is a function expanded in a convergent power series of variables \( z = (z_1, \ldots, z_n) \) and \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \)

\[
P(z, \bar{z}) := \sum_{\nu, \mu} c_{\nu, \mu} z^{\nu} \bar{z}^{\mu},
\]

where \( z^{\nu} = z_1^{\nu_1} \cdots z_n^{\nu_n} \) for \( \nu = (\nu_1, \ldots, \nu_n) \) (respectively \( \bar{z}^{\mu} = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n} \) for \( \mu = (\mu_1, \ldots, \mu_n) \)). The Newton polygon \( \Gamma_+(P; z, \bar{z}) \) is defined by the convex hull of

\[
\bigcup_{(\nu, \mu)} \{(\nu + \mu) + \mathbb{R}_+^m | c_{\nu, \mu} \neq 0\},
\]

where \( \nu + \mu \) is the sum of the multi-indices of \( z^{\nu} \bar{z}^{\mu} \), i.e., \( \nu + \mu = (\nu_1 + \mu_1, \ldots, \nu_n + \mu_n) \). The Newton boundary \( \Gamma(P; z, \bar{z}) \) is the union of compact faces of \( \Gamma_+(P; z, \bar{z}) \). The strongly non-degeneracy is defined from the Newton boundary as follows: let \( \Delta_1, \ldots, \Delta_m \) be the faces of \( \Gamma(P; z, \bar{z}) \). For each face \( \Delta_k \), the face function \( P_{\Delta_k}(z, \bar{z}) \) is defined by \( P_{\Delta_k}(z, \bar{z}) := \sum_{(\nu + \mu) \in \Delta_k} c_{\nu, \mu} z^{\nu} \bar{z}^{\mu} \). If \( P_{\Delta_k}(z, \bar{z}) : \mathbb{C}^{2n} \to \mathbb{C} \) has no critical point, and \( P_{\Delta_k} \) is surjective if \( \dim \Delta_k \geq 1 \), we say that \( P(z, \bar{z}) \) is strongly non-degenerate for \( \Delta_k \), where \( \mathbb{C}^{2n} = \{z = (z_1, \ldots, z_n) \mid z_j \neq 0, j = 1, \ldots, n\} \). If \( P(z, \bar{z}) \) is strongly non-degenerate for any \( \Delta_k \) for \( k = 1, \ldots, m \), we say that \( P(z, \bar{z}) \) is strongly non-degenerate. If \( P((0, \ldots, 0, z_j, 0, \ldots, 0), (0, \ldots, 0, \bar{z}_j, 0, \ldots, 0)) \neq 0 \) for each \( j = 1, \ldots, n \), then we say that \( P(z, \bar{z}) \) is convenient. Oka showed that a convenient strongly non-degenerate mixed function \( P(z, \bar{z}) \) has both tubular and spherical Milnor fibrations and also two fibrations are isomorphic [25].
Theorem 3 ([25, 26, 8]). Let \( P(\mathbf{z}, \overline{\mathbf{z}}) : (\mathbb{C}^n, 0_{2n}) \to (\mathbb{C}, 0_2) \) be a convenient strongly non-degenerate mixed function. Then \( 0_{2n} \) is an isolated singularity of \( P \) and \( P \) satisfies the conditions (a-i), (a-ii) and (a-iii).

Let \( f_t \) be an analytic family of convenient strongly non-degenerate mixed polynomials such that the Newton boundary of \( f_t \) is constant for \( 0 \leq t \leq 1 \). C. Eyral and M. Oka showed that the topological type of \((V(f_t), 0_{2n})\) is constant for any \( t \) and their tubular Milnor fibrations are equivalent [8].

3. Proof of Theorem 2

Assume that \( f_1 \) and \( f_2 \) satisfy the conditions (a-i) and (a-ii) in Section 1. Then the proof of Theorem 2 is analogous to the holomorphic case [30]. We assume that \( \varepsilon_0 \) is a common \( a_f \)-stable radius of \( f_1, f_2 \) and \( f_1 + f_2 \). Taking \( \delta_0 \ll \varepsilon_0 \) sufficiently small and put

\[
U_1 := U_1(\varepsilon_0, \delta_0), \quad U_2 := U_2(\varepsilon_0, \delta_0).
\]

Set \( X_t = f_1^{-1}(t) \cap U_1, Y_t = f_2^{-1}(t) \cap U_2 \) and \( Z_t = f_t^{-1}(t) \cap (U_1 \times U_2) \). We fix a point \( t \in \mathbb{R}^p \) with \( 0 < ||t|| \ll \delta_0 \) and define the map

\[
F_1 : Z_t \to A_t \text{ as } (x,y) \mapsto f_1(x),
\]

where \( A_t = \{ w \in \mathbb{R}^p \mid ||w|| \leq \delta_0, ||t-w|| \leq \delta_0 \} \).

Lemma 3. The restriction map \( F_1 : Z_t \setminus F_1^{-1}(\{0_p, t\}) \to A_t \setminus \{0_p, t\} \) is a locally trivial fibration.

Proof. From the tubular Milnor fibrations of \( f_1 \) and \( f_2 \), for each \( w \in A_t \setminus \{0_p, t\} \), we may find a neighborhood \( V_w \subset A_t \setminus \{0_p, t\} \) of \( w \) such that there exist local trivializations

\[
\phi_1 : V_w \times X_w \xrightarrow{\cong} f_1^{-1}(V_w) \cap U_1, \quad \phi_2 : V_{t-w} \times Y_{t-w} \xrightarrow{\cong} f_2^{-1}(V_{t-w}) \cap U_2,
\]

where \( V_{t-w} = \{ t-w \mid w \in V_w \} \subset A_t \setminus \{0_p, t\} \). We define the map on \( V_w \times F_1^{-1}(w) = V_w \times (X_w \times Y_{t-w}) \) as follows:

\[
\psi : V_w \times (X_w \times Y_{t-w}) \to F_1^{-1}(V_w), \quad (w', x, y) \mapsto (\phi_1(w', x), \phi_2(w', y)).
\]

Since \( \phi_1 \) and \( \phi_2 \) are local trivializations, \( \psi \) is a continuous map. For any \( (x', y') \in F_1^{-1}(V_w) \), we put \( (w', x) = \phi_1^{-1}(x') \) and \( (t-w', y) = \phi_2^{-1}(y') \). Then \( \psi^{-1}(x', y') = (w', x, y) \) and thus we see that \( \psi^{-1} \) is a continuous map. Thus \( \psi \) is a homeomorphism. This shows the local triviality of \( F_1 \).

Lemma 4. Let \( J \) be the line segment with endpoints \( 0_p \) and \( t \). The inclusion \( F_1^{-1}(J) \hookrightarrow Z_t \) is a homotopy equivalence.

Proof. Since \( Z_t \) is semi-analytic, there is a triangulation of \( Z_t \) such that \( F_1^{-1}(J) \) is a subcomplex [17]. Since \( Z_t \) is compact, by using the local triviality of \( F_1 \) and the partition of unity, \( Z_t \) is deformed into a regular neighborhood of \( F_1^{-1}(J) \). Thus \( F_1^{-1}(J) \) and \( Z_t \) are homotopy equivalent. See [28, Chapter 3].

Let \( \pi : U_1 \times U_2 \to (U_1/V(f_1)) \times (U_2/V(f_2)) \) be the identification map where we use the quotient topology for \( U_1/V(f_1) \) and \( U_2/V(f_2) \).
Lemma 5. The identification map $\pi : F_1^{-1}(J) \to \pi(F_1^{-1}(J))$ is a homotopy equivalence.

Proof. The semi-analytic set $V(f_j)$ has a conic structure for $j = 1, 2$ by [19, Theorem 2.10] and [3], i.e.,

$$V(f_1) \cong \Cone(V(f_1) \cap S_{\varepsilon_0}^{n-1}) = ([0, 1] \times (V(f_1) \cap S_{\varepsilon_0}^{n-1})) / ([0] \times (V(f_1) \cap S_{\varepsilon_0}^{n-1})),$$

$$V(f_2) \cong \Cone(V(f_2) \cap S_{\varepsilon_0}^{m-1}) = ([0, 1] \times (V(f_2) \cap S_{\varepsilon_0}^{m-1})) / ([0] \times (V(f_2) \cap S_{\varepsilon_0}^{m-1})).$$

So $V(f_1)$ and $V(f_2)$ retract to the origins of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively by strong deformation retracts. We can construct deformation retractions from $F_1^{-1}(0_p^p) = V(f_1) \times Y_1$ to $\{0_n\} \times Y_1$ and from $F_1^{-1}(t) = X_1 \times V(f_2)$ to $X_1 \times \{0_m\}$. By applying a triangulation of $F_1^{-1}(J)$ and using the homotopy extension property of a polyhedral pair [32, p.118], the above homotopies can extend to a homotopy $H_s : F_1^{-1}(J) \to F_1^{-1}(J)$, where $0 \leq s \leq 1$ so that

$$H_0 = \text{id}_{F_1^{-1}(J)}, \quad H_1(F_1^{-1}(0_p) \cup F_1^{-1}(t)) = \{0_n\} \times Y_1 \cup X_1 \times \{0_m\}.$$

Let $\tilde{H}_s : \pi(F_1^{-1}(J)) \to \pi(F_1^{-1}(J))$ be the homotopy which satisfies $\pi(H_s(x, y)) = \tilde{H}_s(\pi(x, y))$, where $(x, y) \in F_1^{-1}(J)$ and $0 \leq s \leq 1$. Note that $\pi(F_1^{-1}(J)) \setminus (\{0_n\} \times Y_1 \cup X_1 \times \{0_m\}) = F_1^{-1}(J) \setminus (F_1^{-1}(0_p) \cup F_1^{-1}(t))$. The map $\varphi : \pi(F_1^{-1}(J)) \to F_1^{-1}(J)$ is defined by

$$\varphi|_{\pi(F_1^{-1}(J))\setminus(\{0_n\}\times Y_1\cup X_1\times \{0_m\})} = H_1|_{F_1^{-1}(J)\setminus(F_1^{-1}(0_p)\cup F_1^{-1}(t))},$$

$$\varphi(\{0_n\} \times Y_1) = \{0_n\} \times Y_1, \quad \varphi(X_1 \times \{0_m\}) = X_1 \times \{0_m\}.$$

Then $\varphi$ is continuous and $H_1 = \varphi \circ \pi$. By the definition of $\tilde{H}_s$, $\pi \circ \varphi = \tilde{H}_1$. Thus the identification map $\pi$ is a homotopy equivalence. \hfill $\Box$

Lemma 6. Let $X_t \times Y_t$ be the join of $X_t$ and $Y_t$. Then $X_t \times Y_t$ is homeomorphic to $\pi(F_1^{-1}(J))$.

Proof. Put $I = [0, 1]$. By the local trivialities of the tubular Milnor fibrations of $f_1$ and $f_2$, there exist homeomorphisms

$$\tilde{\phi}_1 : (I \setminus \{0\}) \times X_t \to f_1^{-1}(J \setminus \{0_p\}) \cup U_1, \quad \tilde{\phi}_2 : (I \setminus \{0\}) \times Y_t \to f_2^{-1}(J \setminus \{0_p\}) \cup U_2$$

such that $f_1(\tilde{\phi}_1(s, x)) = f_2(\tilde{\phi}_2(s, y))$ for $0 < s \leq 1$. We define the map

$$\Phi : X_t \times I \times Y_t \to \pi(F_1^{-1}(J))$$

as $(x, s, y) \mapsto \pi(\tilde{\phi}_1(s, x), \tilde{\phi}_2(1 - s, y))$, where $\tilde{\phi}_1(0, x) = 0_n$ and $\tilde{\phi}_2(0, y) = 0_m$. Since $V(f_1)$ and $V(f_2)$ have conic structures, $\Phi$ is a continuous map. Let $\Psi : X_t \times Y_t \to \pi(F_1^{-1}(J))$ be the map defined by $\Psi([x, s, y]) = \Phi(x, s, y)$, where $[x, s, y]$ is the equivalence class of $(x, s, y)$. By the definition of $\Phi$ and conic structures of $V(f_1)$ and $V(f_2)$, $\Psi$ is a continuous and bijective map. Thus $\Psi$ is a homeomorphism. \hfill $\Box$

Lemma 7. The fiber $Z_t$ is homotopy equivalent to $f^{-1}(t) \cap B_{\varepsilon^+}^{n+m}$, where $0 < \varepsilon' < 1$.

Proof. Let $\varepsilon_0$ be a common $a_j$-stable radius of $f_1, f_2$ and $f = f_1 + f_2$. Put

$$U(\varepsilon_0, \delta_0) := \{(x, y) \in \mathbb{R}^{n+m} \mid \|(x, y)\| \leq \varepsilon_0, \|f_1(x) + f_2(y)\| \leq \delta_0\},$$

where $0 < \delta_0 \ll \varepsilon_0$. As the families $\{U_1(\varepsilon_0, \delta_0) \times U_2(\varepsilon_0, \delta_0)\}_{0 < \delta_0 \ll \varepsilon_0}$ and $\{U(\varepsilon_0, \delta_0)\}_{0 < \delta_0 \ll \varepsilon_0}$ are cofinal systems of neighborhoods of the origin $0_{n+m}$ respectively for $\varepsilon_0$ which is an $a_j$-stable radius of $f_1, f_2, f$ and $0 < \delta_0 \ll \varepsilon_0$. Thus we can choose positive real numbers $\varepsilon_3 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$ and sufficiently small $\delta_3 < \delta_2 < \delta_1 < \delta_0$ with $\delta_j \ll \varepsilon_j$ for $j = 0, \ldots, 3$ so that
By Corollary 2, the inclusions
\[
\iota_j : U_1(\varepsilon_j, \delta_j) \times U_2(\varepsilon_j, \delta_j) \to U_1(\varepsilon_j, \delta_j) \times U_2(\varepsilon_j, \delta_j),
\]
are isomorphisms for \( j = 1, \ldots, 3 \). The homotopies of the above sequence can be defined by \( \iota_j \) and \( \iota_j' \) for \( j = 1, \ldots, 3 \). Thus by a standard homotopy argument, we can see that the inclusion maps in the above sequence are homotopy equivalences. Take \( t \in \mathbb{R}^p \) which satisfies \( \|t\| \leq \delta_3 \). Then we see also the restriction of the homotopies of the above sequence to \( f^{-1}(t) \) is also homotopy equivalences.

Proof of Theorem 2. By using Lemma 4, Lemma 5 and Lemma 6, we can show that \( X_t \ast Y_t \) is homotopy equivalent to \( Z_t \). By Lemma 7, the fiber of the tubular Milnor fibration of \( f \) is homotopy equivalent to \( X_t \ast Y_t \).

If \( p = 2 \), set
\[
E = \{(x, y) \in U_1 \times U_2 \mid 0 < \|f(x, y)\| \leq \rho\},
\]
where \( 0 < \rho \ll \varepsilon \). Then the map \( \tilde{f} : \pi(E) \to D^2_{\rho} \setminus \{0_2\} \) is defined by \( \tilde{f}(\pi(x, y)) = f(x, y) \). By the local trivialities of \( f_1 \) and \( f_2 \), there are continuous one-parameter families of homeomorphisms
\[
\alpha_\theta : U_1 \setminus V(f_1) \to U_1 \setminus V(f_1), \quad \beta_\theta : U_2 \setminus V(f_2) \to U_2 \setminus V(f_2)
\]
such that \( f_1(\alpha_\theta(x)) = e^{i\theta}f_1(x) \) and \( f_2(\beta_\theta(y)) = e^{i\theta}f_2(y) \), where \( \theta \in [0, 2\pi] \). Then we define the map \( \gamma_\theta : \pi(E) \to \pi(E) \) as follows:
\[
\gamma_\theta(\pi(x, y)) = \begin{cases} 
\pi(\alpha_\theta(x), \beta_\theta(y)) & x \in U_1 \setminus V(f_1), y \in U_2 \setminus V(f_2) \\
\pi(0_n, \beta_\theta(y)) & x \in V(f_1), y \in U_2 \setminus V(f_2) \\
\pi(\alpha_\theta(x), 0_m) & x \in U_1 \setminus V(f_1), y \in V(f_2). 
\end{cases}
\]
Note that \( \{\gamma_\theta\} \) is well-defined and a continuous one-parameter family of homeomorphisms such that \( \tilde{f}(\gamma_\theta(z)) = e^{i\theta}\tilde{f}(z) \), where \( z \in \pi(E) \) and \( \theta \in [0, 2\pi] \). Hence \( \{\gamma_\theta\} \) gives the local triviality of \( \tilde{f} \). Then the monodromy of \( \tilde{f} \) can be identified with \( \alpha_{2\pi} \ast \beta_{2\pi} \) up to homotopy. Here the map \( \alpha_{2\pi} \ast \beta_{2\pi} \) is defined by
\[
\alpha_{2\pi} \ast \beta_{2\pi}([x, s, y]) = [\alpha_{2\pi}(x), s, \beta_{2\pi}(y)],
\]
where \( [x, s, y] \in X_t \ast Y_t \).

By Lemma 7, the fiber of \( \tilde{f} \) is homotopy equivalent to the fiber of \( f \). Since \( D^2_{\rho} \setminus \{0_2\} \) is a CW-complex and \( f^{-1}(t) \) is homotopy equivalent to \( f^{-1}(t) \) for any \( t \in D^2_{\rho} \setminus \{0_2\} \), \( \tilde{f} \) is fiber homotopy equivalent to \( f \) by [5]. Then the monodromy of the tubular Milnor fibration of \( f \) is equal to \( \alpha_{2\pi} \ast \beta_{2\pi} \).

Proof of Corollary 1. By Theorem 2 and the condition (a-iii), the fiber of the spherical Milnor fibration of \( f \) is homotopy equivalent to \( X_t \ast Y_t \), where \( 0 < \|t\| \ll 1 \). By the condition (a-iii), \( X_t \) and \( Y_t \) are diffeomorphic to the fibers of the spherical Milnor fibrations of \( f_1 \) and \( f_2 \) respectively. This completes the proof.
Let $F_j$ be the fiber of the spherical Milnor fibration of $f_j$ which satisfies the assumptions in Section 2.2 for $j = 1, 2$. By [18], the reduced homology $\hat{H}_{n+m-1}(F_1 \ast F_2)$ satisfies
\[
\hat{H}_{n+m-1}(F_1 \ast F_2) = \sum_{i+j=n+m-2} \hat{H}_i(F_1, \mathbb{Z}) \otimes \hat{H}_j(F_2, \mathbb{Z}) + \sum_{i'+j'=n+m-3} \text{Tor}(\hat{H}_i(F_1, \mathbb{Z}), \hat{H}_{j'}(F_2, \mathbb{Z})).
\]
Let $F$ be the fiber of the spherical Milnor fibration of $f = f_1 + f_2$ and $\tau : F \to F_1 \ast F_2$ be the homotopy equivalence in Theorem 2. Then $f$ also satisfies the assumptions in Section 2.2 and we have the following commutative diagram:
\[
\begin{array}{ccc}
\hat{H}_{n+m-1}(F, \mathbb{Z}) & \xrightarrow{\gamma_*} & \hat{H}_{n+m-1}(F, \mathbb{Z}) \\
\downarrow\tau & & \downarrow\tau \\
\hat{H}_{n-1}(F_1, \mathbb{Z}) \otimes \hat{H}_{n-1}(F_2, \mathbb{Z}) & \xrightarrow{\alpha_* \otimes \beta_*} & \hat{H}_{n-1}(F_1, \mathbb{Z}) \otimes \hat{H}_{n-1}(F_2, \mathbb{Z})
\end{array}
\]
where $\alpha_*, \beta_*$ and $\gamma_*$ are the linear transformations induced by the monodromy of the spherical Milnor fibrations of $f_1, f_2$ and $f$ respectively. Since the eigenvalues of the linear transformation $\alpha_* \otimes \beta_* : \hat{H}_{n-1}(F_1, \mathbb{Z}) \otimes \hat{H}_{n-1}(F_2, \mathbb{Z}) \to \hat{H}_{n-1}(F_1, \mathbb{Z}) \otimes \hat{H}_{n-1}(F_2, \mathbb{Z})$ are given by the product of the eigenvalues of $\alpha_*$ and $\beta_*$, we obtain the following corollary.

**Corollary 4.** Assume that $f_1$ and $f_2$ satisfy the assumptions in Section 2.2. Let $\tilde{\xi}_1(t), \tilde{\xi}_2(t)$ and $\tilde{\xi}(t)$ of the reduced zeta functions defined by $\alpha_*, \beta_*$ and $\gamma_*$ respectively. Then the divisors of the reduced zeta functions are related by
\[
\left(\tilde{\xi}(t)\right) = \left(\tilde{\xi}_1(t)\right) \cdot \left(\tilde{\xi}_2(t)\right).
\]

4. **Seifert forms of simple links defined by mixed functions**

Let $K$ be a link in the $(2k + 1)$-sphere $S^{2k+1}$, i.e., $K$ is an oriented codimension-two closed smooth submanifold in $S^{2k+1}$. A link $K$ is said to be fibered if there exists a trivialization $K \times D^2 \to N(K)$ of a tubular neighborhood $N(K)$ of $K$ in $S^{2k+1}$ and a fibration of the link exterior $E(K) = S^{2k+1} \setminus \text{Int}(N(K)), \xi_1 : E(K) \to S^1$ such that $\xi_0|\partial N(K) = \xi_1|\partial N(K)$, where $\xi_0 : N(K) \to D^2$ is a trivialization $K \times D^2 \to N(K)$ composed with the second factor. This fibration is also called an open book decomposition of $S^{2k+1}$. A fiber of $\xi_1$ is called a fiber surface of the fibration of $K$. If $f(z, \bar{z})$ is convenient strongly non-degenerate, $K_f$ is a fibered link by [25].

We assume that a fibered link $K$ in $S^{2k+1}$ is $(k-2)$-connected and its fiber surface $F$ is $(k-1)$-connected. Then $K$ is called a simple fibered link. Let $\alpha, \beta \in \hat{H}_k(F; \mathbb{Z})$ and $a$ and $b$ be cycles on $F$ representing $\alpha$ and $\beta$ respectively. Set
\[
L_K(\alpha, \beta) := \text{link}(a^+, b),
\]
where $a^+$ is a pushed off of $a$ to the positive side of $F$ by a transverse vector field and $\text{link}(a^+, b)$ is the linking number of $a^+$ and $b$. The Seifert form $L_K$ of $K$ is the non-singular bilinear form
\[
L_K : \hat{H}_k(F; \mathbb{Z}) \times \hat{H}_k(F; \mathbb{Z}) \to \mathbb{Z}
\]
on the $k$-th homology group $\hat{H}_k(F; \mathbb{Z})$ with respect to a choice of basis of $\hat{H}_k(F; \mathbb{Z})$. By [14], we can show the following proposition.
Proposition 1 ([29, 14]). Let $f_1 : (\mathbb{C}^n, 0_{2n}) \to (\mathbb{C}, 0_2)$ and $f_2 : (\mathbb{C}^m, 0_{2m}) \to (\mathbb{C}, 0_2)$ be mixed function germs of independent variables which satisfy the conditions (a-i), (a-ii) and (a-iii). Suppose that the origin is an isolated singularity of $f_j$ and $K_{f_j}$ is a simple fibered link for $j = 1, 2$. Then $L_{K_j}$ is congruent to $(-1)^{nm}L_{K_{f_1}} \otimes L_{K_{f_2}}$.

Kauffman and Neumann studied Seifert forms of non-simple fibered links. See [14].

Let $A = (a_{i,j})$ and $A'$ be integral unimodular matrices. We say that $A'$ is an extension of $A$ if $A'$ is congruent to

$$
\begin{pmatrix}
  a_{1,1} & \ldots & a_{1,n} & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{n,1} & \ldots & a_{n,n} & 0 \\
  b_1 & \ldots & b_n & \varepsilon
\end{pmatrix},
$$

where $n$ is the rank of $A$, $b_i \in \mathbb{Z}, i = 1, \ldots, n$ and $\varepsilon = \pm 1$. Let $K$ and $K'$ be simple fibered links in $S^{2k+1}$. Set $F$ and $F'$ to be the fiber surfaces of $K$ and $K'$ respectively. If a fiber surface $F$ is obtained from $F'$ by a plumbing of a Hopf band, the Seifert form of $F$ is an extension of the Seifert form of $F'$ (cf. [16]). If a fiber surface is obtained from a disk by successive plumbings of Hopf bands then its Seifert form becomes a unimodular lower triangular matrix for a suitable choice of the basis. D. Lines studied high dimensional fibered knots by using plumbings [15, 16].

Proposition 2 ([15]). Let $F$ be the fiber surface of a simple fibered link $K$ in $S^{2k+1}$, where $k \geq 3$. Then $F$ is obtained from a disk by successive plumbings of Hopf bands if and only if it admits a unimodular lower triangular Seifert form. For $k = 1$, the above condition is necessary.

Example 1. Let $f_1$ be a 2-variable complex polynomial which has an isolated singularity at the origin and $f_2(w) = \sum_{j=1}^{m} w_j^2$, where $m \geq 2$. In [13, Corollary 1.2], the Milnor fiber of $f_1$ is obtained from a disk by successive plumbings of Hopf bands. By Proposition 1 and Proposition 2, the Milnor fiber of $f_1 + f_2$ is also obtained from a disk by successive plumbings of Hopf bands.

For mixed singularities, we can find a 2-variable mixed polynomial $f_1'$ such that the Milnor fiber of $f_1' + f_2$ cannot be obtained from a disk by successive plumbings of Hopf bands. We will give an explicit example below. Suppose that $\alpha_j \neq \alpha_j'$ ($j \neq j'$). Then we define a mixed polynomial as follows:

$$
f_1'(z) := (z_1 + \alpha_1 z_2)(z_1 + \alpha_2 z_2)(z_1 + \alpha_3 z_2).
$$

Note that $f_1'$ is a convenient strongly non-degenerate mixed polynomial. Thus $f_1'$ satisfies the conditions (a-i), (a-ii) and (a-iii). See also [27]. By [12, Lemma 1], the Seifert form of $K_{f_1'}$ is equal to

$$
A_1 := \begin{pmatrix}
  0 & -1 \\
  -1 & 2
\end{pmatrix}.
$$

Since the Seifert form of $K_{f_1}$ is equal to $(-1)^{m(m-1)/2}$ [6, Proposition 2.2], by Proposition 1, the Seifert form of $K_{f_1'}$ is equal to
\[ \Gamma_1 := \epsilon A_1 = (-1)^\frac{m(m-1)}{2} \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \]

where \( f' = f_1' + f_2 \) and \( \epsilon = (-1)^\frac{m(m-1)}{2} \). We show the following assertion.

**Assertion 1.** The Milnor fibers of \( f_1' \) and \( f' \) for \( m \geq 2 \) cannot be obtained by series of plumbings from a disk.

Proof. By the assertion of Proposition 2.4 of [15], it is enough to show that the corresponding Seifert form of \( K_f \) cannot be a lower triangle unimodular matrix. Let \( e_1, e_2 \) be the basis of the homology group of the Milnor fiber of \( f' \) which gives \( \Gamma_1 \) and take a unimodular matrix \( \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \), \( a_1a_4 - a_2a_3 = \pm 1 \).

We show that the assertion by contradiction. Consider the base change \( e_1' = a_1 e_1 + a_2 e_2, e_2' = a_3 e_1 + a_4 e_2 \) and consider the corresponding Seifert form \( \Gamma'_1 \) which should be a lower triangle unimodular matrix. Let \( \tilde{e}_j \) and \( \tilde{e}_j' \) be cycles of the Milnor fiber representing \( e_j \) and \( e_j' \) for \( j = 1, 2 \). By the definition of \( e_1, e_2, e_1' \) and \( e_2' \), we have

\[
\text{link} (\tilde{e}_1', \tilde{e}_2') = \epsilon (a_1 a_3 \text{link}(\tilde{e}_1', \tilde{e}_1) + a_1 a_4 \text{link}(\tilde{e}_1', \tilde{e}_2) + a_2 a_3 \text{link}(\tilde{e}_2', \tilde{e}_1) + a_2 a_4 \text{link}(\tilde{e}_2', \tilde{e}_2)) = 
\epsilon (2a_2 a_4 - (a_1 a_4 + a_2 a_3)) = 
\epsilon (2a_2 a_4 - (2a_2 a_3 \pm 1)).
\]

If \( \Gamma'_1 \) is a lower triangle unimodular matrix, we have the equality \( 2(a_2 a_4 - a_2 a_3) \pm 1 = 0 \) which has no integer solution. Thus the Seifert form of \( K_f \) cannot be a lower triangle unimodular matrix.

By Proposition 1, Proposition 2 and Assertion 1, the Milnor fiber of \( f' \) cannot be obtained from a disk by successive plumbings of Hopf bands.

By using the notion of strongly non-degenerate mixed functions and Proposition 1, we show a generalization of [29, Corollary 3].

**Corollary 5.** Let \( f_j(\zeta_j) \) be a strongly non-degenerate mixed polynomial of 1-variable \( \zeta_j \) for \( j = 1, \ldots, n \). Set \( m_j \) to be the mapping degree of \( f_j | f_j : S^1_{\zeta_j} = \{ \zeta_j \in \mathbb{C} \mid |\zeta_j| = \epsilon_j \} \to S^1 \) and

\[
g_j(\zeta_j) = \begin{cases} \zeta_j^{m_j + \ell_j} & m_j > 0 \\ \zeta_j^{-m_j + \ell_j} & m_j < 0 \end{cases},
\]

where \( 0 < \epsilon_j \ll 1 \) for \( j = 1, \ldots, n \). Suppose that the Newton boundary of \( f_j \) is equal to that of \( g_j \) for \( j = 1, \ldots, n \). For any \( j \in \{ 1, \ldots, n \} \), assume that there exists an analytic family \( f_{j,t} \) of strongly non-degenerate mixed polynomials such that \( f_{j,0} = f_j, f_{j,1} = g_j \) and the Newton boundaries of \( f_{j,t} \) is constant for \( 0 \leq t \leq 1 \). Then the Milnor fibration of \( f(\zeta) = f_1(\zeta_1) + \cdots + f_n(\zeta_n) \) is equivalent to that of \( g(\zeta) = g_1(\zeta_1) + \cdots + g_n(\zeta_n) \). Set the \( (m-1) \times (m-1) \) matrix \( \Lambda'_m \) as follows:

\[
\Lambda'_m = \begin{cases} \Lambda_m & m > 0 \\ \Lambda_m^t & m < 0 \end{cases},
\]

where \( \Lambda_m \) is the \( (m-1) \times (m-1) \) matrix given by
The homotopy class of $\tilde{\Xi}$ is equivalent to that of $g$. By Proposition 1 and [29, Corollary 3], the Seifert form $\Gamma_K$ is congruent to

$$(-1)^{n(n+1)/2} \Lambda_{m_1} \otimes \cdots \otimes \Lambda_{m_n}'.$$

This completes the proof. \qed

5. Enhanced Milnor numbers of simple links defined by mixed functions

Let $K$ be a fibered link in $S^{2k+1}$. By gluing $\xi_0$ and $\xi_1$, we give a piecewise smooth map $\xi : S^{2k+1} \to D^2$. By [14], $\xi$ can be extended to a continuous map $\Xi : B^{2k+2} \to D^2$ which is a smooth submersion except at $0_2$ and a corner along $\partial N(K)$. Then we consider the following map:

$$B^{2k+2} \setminus \{0_{2k+2}\} \to G(2k, 2k + 2), \ x \mapsto \ker D(\Xi(x)),$$

where $D(\Xi(x))$ is the differential of $\Xi$ at $x$ and $G(2k, 2k + 2)$ is the Grassman manifold of oriented $2k$-planes in $\mathbb{R}^{2k+2}$. This map defines an element of $\pi_{2k+1}(G(2k, 2k + 2))$. Note that $\pi_{2k+1}(G(2k, 2k + 2))$ is isomorphic to

$$\begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & k = 1 \\
\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k > 1
\end{cases}.$$

The homotopy class of $\Xi$ has the form $((-1)^{k+1}\mu(K), \lambda(K))$. This pair $((-1)^{k+1}\mu(K), \lambda(K))$ is called the enhanced Milnor number of $K$ and $\lambda(K)$ is called the enhancement to the Milnor number. See [20, 21, 22]. Note that if $K$ is a fibered link coming from an isolated singularity of a complex hypersurface, $\lambda(K)$ always vanishes. By [21], we have

**Theorem 4** ([21]). Let $f_1 : (\mathbb{C}^n, 0_{2n}) \to (\mathbb{C}, 0_2)$ and $f_2 : (\mathbb{C}^m, 0_{2m}) \to (\mathbb{C}, 0_2)$ be mixed function germs of independent variables. Assume that $f_1, f_2$ and $f$ satisfy the conditions (a-i), (a-ii) and (a-iii). Suppose that $0_{2n}$ and $0_{2m}$ are isolated singularities of $f_1$ and $f_2$. Then $\mu(K_f) = \mu(K_{f_1})\mu(K_{f_2})$ and $\lambda(K_f) \equiv \lambda(K_{f_1})\mu(K_{f_2}) + \mu(K_{f_1})\lambda(K_{f_2}) \mod 2$.

For any $\ell \in \mathbb{N}$ and $k \geq 2$, there exists a $(k+1)$-variables Brieskorn polynomial $P$ such that $((-1)^{k+1}\mu(K_P), \lambda(K_P)) = ((-1)^{k+1}\ell, 0)$. See [2]. By Theorem 4 and [11], we calculate

$$\Lambda_m = \begin{pmatrix} 
1 & 0 & \ldots & \ldots & 0 \\
-1 & 1 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -1 & 1
\end{pmatrix}.$$
the enhanced Milnor numbers of simple fibered links defined by mixed polynomials of join type as follows.

**Theorem 5.** For any $\ell \in \mathbb{N}$, there exists a $(k+1)$-variables mixed polynomial $P = P_1 + P_2$ of join type such that $P_1, P_2$ and $P$ satisfies the conditions (a-i), (a-ii) and (a-iii) and $K_\ell$ is a simple fibered link which satisfies $((-1)^{k+1}\mu(K_\ell), \lambda(K_\ell)) = ((-1)^{k+1} \ell, 1)$, where $k \geq 2$.

**Proof.** We define a mixed polynomial and a complex polynomial as follows:

$$f_1(z) = (z_1^p + \alpha_1 z_2)(z_1^p + \alpha_2 z_2)(z_1^p + \alpha_3 z_2), \quad f_2(z) = z_1^2 + z_2^2, \quad f_3(w) = \sum_{i=1}^{m} w_i^{a_i},$$

where $\alpha_j \neq \alpha_{j'} (j \neq j'), a_j \geq 2$ and $m \geq 1$. Then $f_j$ is a convenient strongly non-degenerate mixed polynomial and $K_{f_j}$ is a simple fibered link for $j = 1, 2, 3$. By [7, 11, 19], we have

$$\mu(K_{f_j}), \lambda(K_{f_j})) = (2p, 1), \quad (\mu(K_{f_j}), \lambda(K_{f_j})) = (1, 1),$$

$$\mu(K_{f_j}), \lambda(K_{f_j})) = ((a_1 - 1) \cdots (a_m - 1), 0).$$

If $\ell$ is a positive even integer, we set $p = \frac{\ell}{2}$, $P_1 = f_1$ and $P_2 = f_3$. By Corollary 1, $K_\ell$ is also a simple fibered link. By Theorem 4, we have

$$\mu(K_\ell), \lambda(K_\ell)) = ((a_1 - 1) \cdots (a_m - 1), (a_1 - 1) \cdots (a_m - 1) \mod 2).$$

We set $a_i = 2$ for $i = 1, \ldots, m$. Then $((-1)^{k+1}\mu(K_\ell), \lambda(K_\ell))$ is equal to $((-1)^{k+1} \ell, 1)$.

If $\ell$ is a positive odd integer, put $P_1 = f_2$ and $P_2 = f_3$. Then we have

$$\mu(K_\ell), \lambda(K_\ell)) = ((a_1 - 1) \cdots (a_m - 1), (a_1 - 1) \cdots (a_m - 1) \mod 2).$$

We set $a_1 = \ell + 1$ and $a_i = 2$ for $i = 2, \ldots, m$. Then $((-1)^{k+1}\mu(K_\ell), \lambda(K_\ell))$ is equal to $((-1)^{k+1} \ell, 1)$.

\[\square\]

**Acknowledgements.** The author would like to thank Masaharu Ishikawa, Mutsuo Oka and Mihai Tibâr for precious comments and fruitful suggestions. He also thanks to the referee for careful reading of the manuscript and several accurate comments.

References


Faculty of Education, Iwate University
18–33 Ueda 3–chome Morioka
Iwate 020–8550
Japan
email: inaba@iwate-u.ac.jp

[Current Address:]
Faculty of Science, Okayama University of Science
1–1 Ridai-cho, Kita-ku, Okayama
Okayama 700–0005
Japan
e-mail: inaba@das.ous.ac.jp