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ADDENDUM TO: MAXIMAL TORI OF EXTRINSIC SYMMETRIC SPACES AND MERIDIANS

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Abstract

Improving a theorem in [1] we observe that a maximal torus of an extrinsic symmetric space in a euclidean space V is itself extrinsic symmetric in some affine subspace of V.

A compact *extrinsic symmetric space* is a submanifold $X \subset \mathbb{S}^{p-1} \subset \mathbb{R}^p = V$ such that for any point $x \in X$ the reflection s_x along the normal space $N = N_x X$ keeps X invariant.

Every compact symmetric space X contains a maximal torus T which is unique up to congruence. If $X = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, the maximal torus is a great circle $C = X \cap \mathbb{R}^2$ which is reflective, hence extrinsic symmetric, see [1, Theorem 4]. But for most extrinsic symmetric spaces, the maximal torus is not reflective. However, as we will show, it is an "iterated" reflective subspace, and in particular:

Theorem 1. *A maximal torus T of a compact extrinsic symmetric space $X \subset V$ is itself extrinsic symmetric in some linear subspace $W \subset V$. In fact, there is an affine subspace $W' \subset W$ such that $T \subset W'$ is extrinsically isometric to a Clifford torus $(\mathbb{S}^1)^r \subset \mathbb{C}^r$.*

Example. Consider the Veronese embedding of the real projective plane $\mathbb{RP}^2 \subset S(\mathbb{R}^3)$ (= space of symmetric 3×3-matrices) given by $[x] \mapsto xx^T$ for any $x \in \mathbb{S}^2$. Then $T = \mathbb{RP}^1 = \{xx^T : x \in \mathbb{S}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3\} = \{\begin{pmatrix} A & \\ & 0 \end{pmatrix} : A = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$ with $c^2 + s^2 = 1\}$. This is a fixed component of the reflection $s_3 = \mathrm{diag}(1, 1, -1)$ acting on $S(\mathbb{R}^3)$ by conjugation; its fixed space under this linear action is $W = \{\begin{pmatrix} A & \\ & a \end{pmatrix} : A \in S(\mathbb{R}^2),\ a \in \mathbb{R}\}$. Thus $\mathbb{RP}^1$ is contained in the affine plane $W' \subset W$ which consists of the symmetric matrices $\begin{pmatrix} A & \\ & a \end{pmatrix}$ with $A_{11} + A_{22} = 1$ and $a = 0$.

Proof of Theorem 1. Using a chain of certain reflective subspaces, called meridians, we have shown in [1] that the maximal torus of X is contained in a submanifold which is extrinsic symmetric in a subspace of V and which is intrinsically the Riemannian product of some round spheres $S_1, \ldots, S_r$ with dimensions ≥ 1. Thus we may assume that

$$X = S_1 \times \ldots \times S_r \subset V. \tag{1}$$

Now the maximal torus T of X is the Riemannian product of great circles $C_i \subset S_i$ for $i = 1, \ldots, r$. We have to show that this splitting is extrinsic, more precisely, that each S_i is a round sphere in a subspace $V_i \subset V$ with $V = V_1 \oplus \cdots \oplus V_r$ (orthogonal direct sum). It is well known from a theorem of Ferus [2] that every extrinsic symmetric space is a certain K-orbit in a Lie triple $\mathfrak{p}$ where K is the connected component of $\mathrm{Aut}(\mathfrak{p})$, and it splits extrinsically if the Lie triple $\mathfrak{p}$ splits. Thus to prove that the splitting (1) is extrinsic we only have to check the list of extrinsic symmetric spaces in simple Lie triples (e.g. cf. [3, p. 311]) for intrinsic

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Riemannian products: there are none (although local products occur). Further, there is (up to reduction of codimension and extrinsic isometries) at most one extrinsic symmetric embedding for each compact symmetric space; in particular, there is no extrinsic symmetric embedding of the sphere $\mathbb{S}^p$ other than the standard sphere $\mathbb{S}^p \subset \mathbb{R}^{p+1}$. (This can be checked also directly from the possible choices for the SO_p-equivariant second fundamental form $\alpha : S(T_x) \to N_x$ at any base point $x \in \mathbb{S}^p$.) Thus a maximal torus of X is $C_1 \times \ldots \times C_r$ for great circles $C_i \subset S_i$, more precisely, $C_i = S_i \cap E_i$ for some plane $E_i \subset V_i$. □

The sphere product (1) is an iterated reflective subspace [1], and the maximal torus of a sphere product is also reflective, hence it is an iterated reflective subspace of the given extrinsic symmetric space.

References

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