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CLASSIFICATION OF PARA-REAL FORMS OF ABSOLUTELY SIMPLE PARA-HERMITIAN SYMMETRIC SPACES

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Abstract
We introduce the notion of para-real forms of para-Hermitian symmetric spaces and classify para-real forms of absolutely simple para-Hermitian symmetric spaces of hyperbolic orbit type.

1. Introduction

Let G/L be a para-Hermitian symmetric space and let I be its para-complex structure. We will introduce the notion of para-real forms of para-Hermitian symmetric spaces. A nonempty set $R \subset G/L$ is called a *para-real form*, if there exists an involutive isometry Ξ of G/L such that Ξ is a para-antiholomorphic and that R coincides with a connected component of $(G/L)^{\Xi} := \mathrm{Fix}(G/L, \Xi)$. In addition, two para-real forms R_1 of G/L_1 and R_2 of G/L_2 are *equivalent*, if there exists a homothety Φ from G/L_1 onto G/L_2 such that Φ is para-holomorphic and that $\Phi(R_1) = R_2$. We assume that the complexification of the Lie algebra $\mathfrak{g}$ of G is simple and G/L can be realized as a hyperbolic orbit under the adjoint representation of G on the Lie algebra $\mathfrak{g}$ of G. The main result of this paper is the following theorem:

Theorem 1.1. *Any para-real form R of an absolutely simple para-Hermitian symmetric space G/L of hyperbolic orbit type (see* Definitions 2.3 *and* 2.4*) is equivalent to one in* Table 1.

Here in the first row of Table 1, the symbols $G/C_G(Z)$ and $H/C_H(Z)$ mean an APHS of hyperbolic orbit type and a para-real form of $G/C_G(Z)$, respectively. In addition, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, $\mathbb{R}^+$ is the set of positive numbers, and $S(GL(p,\mathbb{R}) \times GL(q,\mathbb{R}))$ is the set of matrices

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \in SL(p+q,\mathbb{R}),$$

where $X \in GL(p,\mathbb{R})$, $Y \in GL(q,\mathbb{R})$.

We introduce the notion of para-real forms of para-Hermitian symmetric spaces similarly to the notion of real forms of Hermitian symmetric spaces. On real forms, a number of study have been conducted. For example, real forms of (pseudo-) Hermitian symmetric spaces were classified under various conditions (cf. [2], [10], [19]). S. Kaneyuki and M. Kozai introduced the notion of para-Hermitian symmetric spaces and classified them under the

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Table 1. Para-real forms of absolutely simple para-Hermitian symmetric spaces

$G/C_G(Z)$	$H/C_H(Z)$	Condition
Type AI		
$SL(n,\mathbb{R})/S(GL(i,\mathbb{R})\times GL(n-i,\mathbb{R}))$	$SO(n)/(SO(i)\times SO(n-i))$	$2\le n$ $1\le i<((n/2)+1)$
	$SO_0(i,n-i)/(SO(i)\times SO(n-i))$	$3\le n$ $1\le i<((n/2)+1)$
$SL(n,\mathbb{R})/$ $S(GL(k-i,\mathbb{R})\times GL(n-k+i,\mathbb{R}))$	$SO_0(k,n-k)/$ $(SO(k-i)\times SO_0(i,n-k))$	$4\le n$ $1\le k\le n-1$ $1\le i\le k-1$
$SL(n,\mathbb{R})/$ $S(GL(i+j,\mathbb{R})\times GL(n-i-j,\mathbb{R}))$	$SO_0(k,n-k)/$ $(SO_0(i,j)\times SO_0(k-i,n-k-j))$	$4\le n$ $2\le k\le n-1$ $1\le i\le k-1$ $1\le j\le n-k-1$
$SL(2n,\mathbb{R})/S(GL(n,\mathbb{R})\times GL(n,\mathbb{R}))$	$S(GL(n,\mathbb{R})\times GL(n,\mathbb{R}))/SL(n,\mathbb{R})$	$1\le n$
	$(SL(n,\mathbb{C})\times T)/SL(n,\mathbb{R})$	
$SL(2n,\mathbb{R})/$ $S(GL(2i,\mathbb{R})\times GL(2(n-i),\mathbb{R}))$	$Sp(n,\mathbb{R})/(Sp(i,\mathbb{R})\times Sp(n-i,\mathbb{R}))$	$2\le n$ $1\le i<((n/2)+1)$
Type AII		
$SU^*(2n)/$ $(SU^*(2i)\times SU^*(2(n-i))\times\mathbb{R}^+)$	$Sp(n)/(Sp(i)\times Sp(n-i))$	$3\le n$ $1\le i<((n/2)+1)$
	$Sp(i,n-i)/(Sp(i)\times Sp(n-i))$	
	$SO^*(2n)/(SO^*(2i)\times SO^*(2(n-i)))$	
	$Sp(k,n-k)/(Sp(i)\times Sp(k-i,n-k))$	$4\le n$ $1\le k<n-1$ $1\le i\le k-1$
$SU^*(2n)/$ $(SU^*(2(i+j))\times SU^*(2(n-i-j))\times\mathbb{R}^+)$	$Sp(k,n-k)/$ $(Sp(i,j)\times Sp(k-i,n-k-j))$	$4\le n$ $2\le k\le n-1$ $1\le i\le k-1$ $1\le j\le n-k-1$
$SU^*(4n)/(SU^*(2n)\times SU^*(2n)\times\mathbb{R}^+)$	$(SL(2n,\mathbb{C})\times T)/SU^*(2n)$	$2\le n$
	$(SU^*(2n)\times SU^*(2n)\times\mathbb{R}^+)/SU^*(2n)$	
Type AIII		
$SU(n,n)/(SL(n,\mathbb{C})\times\mathbb{R}^*)$	$(SU(n)\times SU(n)\times T)/SU(n)$	$3\le n$
	$SO_0(n,n)/SO(n,\mathbb{C})$	
	$(SL(n,\mathbb{C})\times\mathbb{R}^+)/SU(n)$	
	$SO^*(2n)/SO(n,\mathbb{C})$	
	$(SU(i,n-i)\times SU(n-i,i)\times T)/$ $SU(i,n-i)$	$3\le n$ $1\le i<((n/2)+1)$
	$(SL(n,\mathbb{C})\times\mathbb{R}^+)/SU(i,n-i)$	
$SU(2n,2n)/(SL(2n,\mathbb{C})\times\mathbb{R}^*)$	$Sp(2n,\mathbb{R})/Sp(n,\mathbb{C})$	$2\le n$
	$Sp(n,n)/Sp(n,\mathbb{C})$	
Type BDI		
$SO_0(p,q)/(SO_0(p-1,q-1)\times\mathbb{R}^*)$	$(SO(p)\times SO(q))/$ $(SO(p-1)\times SO(q-1))$	$1\le p\le q$ $p+q\neq 2$
$SO_0(n,n)/(SL(n,\mathbb{R})\times\mathbb{R}^*)$	$(SO(n)\times SO(n))/SO(n)$	$2\le n$
	$SO(n,\mathbb{C})/SO(n)$	
	$SO(n,\mathbb{C})/SO_0(i,n-i)$	$2\le n$ $1\le i<((n/2)+1)$
$SO_0(p,q)/(SO_0(p-1,q-1)\times\mathbb{R}^*)$	$(SO_0(i,j)\times SO_0(p-i,q-j))/$ $(SO_0(i-1,j)\times SO_0(p-i,q-j-1))$	$1\le p\le q$ $1\le i<p-1$ $1\le j<q-1$
$SO_0(2p,2q)/(SO_0(2p-1,2q-1)\times\mathbb{R}^*)$	$(SO_0(p,q)\times SO_0(p,q))/SO_0(p,q)$	$1\le p\le q$
$SO_0(n,n)/(SO_0(n-1,n-1)\times\mathbb{R}^*)$	$SO(n,\mathbb{C})/SO(n-1,\mathbb{C})$	$2\le n$
$SO_0(2n,2n)/(SL(2n,\mathbb{R})\times\mathbb{R}^*)$	$(SU(n,n)\times T)/Sp(n,\mathbb{R})$	
	$(SL(2n,\mathbb{R})\times\mathbb{R}^+)/Sp(n,\mathbb{R})$	

Table 1. Para-real forms of absolutely simple para-Hermitian symmetric spaces (continued)

$G/C_G(Z)$	$H/C_H(Z)$	Condition
Type DIII		
$SO^*(4n)/(SU^*(2n)\times\mathbb{R}^+)$	$(SU(2n)\times T)/Sp(n)$	$3\leq n$
	$SO(2n,\mathbb{C})/SO^*(2n)$	
	$(SO^*(2n)\times SO^*(2n))/SO^*(2n)$	
	$(SU^*(2n)\times\mathbb{R}^+)/Sp(n)$	
	$(SU^*(2n)\times\mathbb{R}^+)/Sp(i,n-i)$	$3\leq n$ $1\leq i<((n/2)+1)$
	$(SU^*(2i,2n-2i)\times\mathbb{R}^+)/Sp(i,n-i)$	$2\leq n$ $1\leq i<((n/2)+1)$
Type CI		
$Sp(n,\mathbb{R})/(SL(n,\mathbb{R})\times\mathbb{R}^*)$	$(SU(n)\times T)/SO(n)$	$3\leq n$
	$(SL(n,\mathbb{R})\times\mathbb{R}^+)/SO(n)$	
	$(SU(i,n-i)\times T)/SO_0(i,n-i)$	$3\leq n$ $1\leq i<((n/2)+1)$
	$(SL(n,\mathbb{R})\times\mathbb{R}^+)/SO_0(i,n-i)$	
$Sp(2n,\mathbb{R})/(SL(2n,\mathbb{R})\times\mathbb{R}^*)$	$(Sp(n,\mathbb{R})\times Sp(n,\mathbb{R}))/Sp(n,\mathbb{R})$	$2\leq n$
	$Sp(n,\mathbb{C})/Sp(n,\mathbb{R})$	
Type CII		
$Sp(n,n)/(SU^*(2n)\times\mathbb{R}^+)$	$(Sp(n)\times Sp(n))/Sp(n)$	$2\leq n$
	$(SU(n,n)\times T)/SO^*(2n)$	
	$(SU^*(2n)\times\mathbb{R}^+)/SO^*(2n)$	
	$Sp(n,\mathbb{C})/Sp(n,\mathbb{R})$	
	$(Sp(i,n-i)\times Sp(i,n-i))/Sp(i,n-i)$	$2\leq n$ $1\leq i<((n/2)+1)$
Type EI		
$E_{6(6)}/(Spin(5,5)\times\mathbb{R}^*)$	$Sp(4)/(Sp(2)\times Sp(2))$	—
	$F_{4(4)}/SO_0(4,5)$	
	$Sp(4,\mathbb{R})/Sp(2,\mathbb{C})$	
	$Sp(4,\mathbb{R})/(Sp(2,\mathbb{R})\times Sp(2,\mathbb{R}))$	
	$Sp(2,2)/(Sp(2)\times Sp(2))$	
	$Sp(2,2)/(Sp(1,1)\times Sp(1,1))$	
	$Sp(2,2)/Sp(2,\mathbb{C})$	
Type EIV		
$E_{6(-26)}/(Spin(1,9)\times\mathbb{R}^+)$	$F_4/SO(9)$	—
	$Sp(1,3)/(Sp(1,1)\times Sp(2))$	
	$F_{4(-20)}/SO(9)$	
	$F_{4(-20)}/SO_0(1,8)$	
Type EV		
$E_{7(7)}/(E_{6(6)}\times\mathbb{R}^*)$	$SU(8)/Sp(4)$	—
	$SU(4,4)/Sp(4,\mathbb{R})$	
	$SU(4,4)/Sp(2,2)$	
	$SL(8,\mathbb{R})/Sp(4,\mathbb{R})$	
	$SU^*(8)/Sp(4)$	
	$SU^*(8)/Sp(2,2)$	
	$(E_{6(2)}\times T)/F_{4(4)}$	
	$(E_{6(6)}\times\mathbb{R}^+)/F_{4(4)}$	
Type EIV		
$E_{7(-25)}/(E_{6(-26)}\times\mathbb{R}^*)$	$(E_6\times T)/F_4$	—
	$(E_{6(-26)}\times\mathbb{R}^+)/F_4$	
	$(E_{6(-26)}\times\mathbb{R}^+)/F_{4(-20)}$	
	$SU^*(8)/Sp(1,3)$	
	$SU(2,6)/Sp(1,3)$	
	$(E_{6(-14)}\times T)/F_{4(-20)}$	

certain condition in [6]. In addition, Kaneyuki and Kozai showed the relation between the symmetric R-spaces and para-Hermitian symmetric spaces. On para-Hermitian symmetric spaces, it seems that there has been no study similar to real forms of Hermitian symmetric spaces. For this reason, we try to introduce the notion of para-real forms of para-Hermitian symmetric spaces and classify them under certain conditions.

Let G be a semisimple connected Lie group, let $\mathfrak{g}$ be the Lie algebra of G, and let Z be a semisimple element of $\mathfrak{g}$ which satisfies all the eigenvalues of $\mathrm{ad}\,Z$ are real. Then the adjoin orbit $\mathrm{Ad}\,G(Z)$ of G through Z is called *hyperbolic orbit*. It is known that the adjoin orbit of G through an element $X \in \mathfrak{g}$ is hyperbolic orbit if and only if $\mathrm{Ad}\,G(X)$ is a para-Kähler homogeneous space (see Remark 2.2). Kaneyuki and Kozai showed a one-to-one correspondence between effective semisimple graded Lie algebras of first kind and semisimple para-Hermitian symmetric spaces of hyperbolic orbit type in [6]. Thus the study of semisimple para-Hermitian symmetric spaces of hyperbolic orbit type means the study of hyperbolic orbits of semisimple Lie groups which correspond to semisimple graded Lie algebras of first kind.

Let $(G/L, \hat{\sigma}, I, g)$ be a simple para-Hermitian symmetric space, let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{l}$ be the Lie algebra of L. Then the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is one or two dimensions over $\mathbb{R}$ (cf. [9]). If $\mathfrak{z}(\mathfrak{l})$ is one dimension (resp. two dimensions) over $\mathbb{R}$, then we call $(G/L, \hat{\sigma}, I, g)$ *first category* (resp. *second category*) (cf. [7]). It is known that $(G/L, \hat{\sigma}, I, g)$ is first category if and only if $(G/L, \hat{\sigma}, I, g)$ is absolutely simple. In addition $(G/L, \hat{\sigma}, I, g)$ is second category if and only if $\mathfrak{g}$ is complexification of some absolutely simple Lie algebra. Thus the study of absolutely simple para-Hermitian symmetric spaces means the study of simple para-Hermitian symmetric spaces of first category.

A para-real form R of an absolutely simple para-Hermitian symmetric space G/L of hyperbolic orbit type has several features similar to real forms of Hermitian symmetric spaces. For instance, R is a totally geodesic, Lagrangian submanifold of G/L (cf. Section 3). We note that the fixed point set $(G/L)^{\Xi}$ is generally not connected in contrast with the case of (simple irreducible pseudo-) Hermitian symmetric spaces (cf. Example 3.1).

This paper is organized as follows. In Section 2, we provide useful symbols and recollect some definitions and facts being related to fundamental proposition of para-Hermitian symmetric spaces. Proposition 2.1 is an important fact related to absolutely simple para-Hermitian symmetric spaces. On an absolutely simple para-Hermitian symmetric space, a para-Hermitian metric is unique up to constant, hence our result Theorem 1.1 does not depend on the choice of para-Hermitian metrics. In Section 3, we introduce the notion of para-real forms of para-Hermitian symmetric spaces and define an equivalent relation of para-real forms. Theorem 3.1 shows us a relation between para-real forms and Lie algebra automorphisms. In Section 4, we construct a method for classifying para-real forms in Lemma 4.3. In addition, we prepare several useful lemmas related to Lemma 4.3. In Section 5, we determine para-real forms based on the way of Lemma 4.3 in some cases. As a classical type, we consider the example of $\mathfrak{g} = \mathfrak{su}(n, n)$. Further, as an exceptional type, we consider the example of $\mathfrak{g} = \mathfrak{e}_{7(7)}$. In similar ways of these, we determine every para-real form of every absolutely simple para-Hermitian symmetric space of hyperbolic orbit type. Consequently, we obtain Theorem 1.1.

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2. Preliminaries.

2.1. Notation. We use the following notation, where M is a manifold, G is a Lie group, and $\mathfrak{g}$ is a Lie algebra:

(n1) $\mathfrak{X}(M)$: the set of vector fields on M,
(n2) T_pM: the tangent space of M at $p \in M$,
(n3) $\mathrm{I}(M, g)$: the group of isometries of a pseudo-Riemannian manifold (M, g),
(n4) $\mathrm{I}(M, g, p)$: the isotropy subgroup at a point $p \in M$ of the group of isometries $\mathrm{I}(M, g)$,
(n5) $\mathrm{Aut}(G)$, $\mathrm{Aut}(\mathfrak{g})$: the groups of automorphisms of G and $\mathfrak{g}$, respectively,
(n6) $\mathrm{Lie}(G)$: the Lie algebra of G,
(n7) $\mathrm{Inv}(G)$, $\mathrm{Inv}(\mathfrak{g})$: the sets of involutive automorphisms (involutions, as an abbreviation) of G and $\mathfrak{g}$, respectively,
(n8) Ad, ad: the adjoint representations of G and $\mathfrak{g}$, respectively,
(n9) $f|_A$: the restriction of a map $f : X \to Y$ to a subset $A \subset X$,
(n10) $\mathrm{ad}_{\mathfrak{h}} Z := \mathrm{ad}\, Z|_{\mathfrak{h}}$ for a subspace $\mathfrak{h} \subset \mathfrak{g}$ and $Z \in \mathfrak{g}$ when $\mathrm{ad}\, Z(\mathfrak{h}) \subset \mathfrak{h}$,
(n11) $B_{\mathfrak{g}}$: the Killing form of $\mathfrak{g}$,
(n12) $\mathrm{Aut}(\mathfrak{g}, \phi) := \{\psi \in \mathrm{Aut}(\mathfrak{g}) \mid \phi \circ \psi = \psi \circ \phi\}$ for $\phi \in \mathrm{Aut}(\mathfrak{g})$,
(n13) $C_G(Z) := \{x \in G \mid \mathrm{Ad}(x)Z = Z\}$ for $Z \in \mathrm{Lie}(G)$,
(n14) $\mathfrak{c}_{\mathfrak{g}}(Z) := \{X \in \mathfrak{g} \mid [Z, X] = 0\}$ for $Z \in \mathfrak{g}$,
(n15) $Z(G)$, $\mathfrak{z}(\mathfrak{g})$: the centers of G and $\mathfrak{g}$, respectively,
(n16) G_0: the identity component of G,
(n17) G^{σ}: the closed subgroup of G which consists of the fixed points of an involution σ of G,
(n18) M^{Ξ}, $\mathfrak{g}^{\xi}$: the fixed point sets in M and $\mathfrak{g}$ of maps $\Xi : M \to M$ and $\xi : \mathfrak{g} \to \mathfrak{g}$, respectively,
(n19) A_x: the inner automorphism of G by an element $x \in G$,
(n20) ϕ_*: the differential map of a smooth map $\phi : G \to G$ at the identity element,
(n21) τ: the action of G onto G/L defined by $\tau_x : aL \mapsto xaL$ for $x \in G$ and $aL \in G/L$,
(n22) o: the origin of G/L.

In addition, we use the notation of Lie groups and Lie algebras in [3].

2.2. Para-Hermitian symmetric spaces. We review basics of para-Hermitian symmetric spaces.

DEFINITION 2.1 (CF. [15, PP. 52–54], [11, P. 64]). (1) Let G be a connected Lie group and let L be a closed subgroup of G. The pair $(G/L, \hat{\sigma})$ of a homogeneous space G/L and an involution $\hat{\sigma}$ of G is called a *symmetric space*, if the following inclusion relation holds:

$$(G^{\hat{\sigma}})_0 \subset L \subset G^{\hat{\sigma}}.$$

(2) Let $(G/L, \hat{\sigma})$ be a symmetric space and let $\Sigma : G/L \to G/L$ be a map defined by $\Sigma(xL) := \hat{\sigma}(x)L$ for $xL \in G/L$. For any point $p := xL \in G/L$, we define an involutive diffeomorphism $S_p : G/L \to G/L$ by $S_p := \tau_x \circ \Sigma \circ \tau_{x^{-1}}$, which is independent of the choice of $x \in G$ satisfying $p = xL$. Then we call S_p the *symmetry at a point* p of

$(G/L, \hat{\sigma})$.

(3) Let $(G/L, \hat{\sigma})$ be a symmetric space and let S_p be the symmetry at a point p of $(G/L, \hat{\sigma})$. Then a diffeomorphism $\Phi : G/L \to G/L$ is called an *automorphism* of a symmetric space $(G/L, \hat{\sigma})$, if the equality $\Phi \circ S_p = S_{\Phi(p)} \circ \Phi$ holds.

(4) A symmetric space $(G/L, \hat{\sigma})$ is uniquely equipped with a G-invariant affine connection ∇^1 which makes a map $\Sigma : G/L \ni xL \mapsto \hat{\sigma}(x)L \in G/L$ an affine transformation. We call the connection ∇^1 the *canonical affine connection* on $(G/L, \hat{\sigma})$.

(5) A symmetric space $(G/L, \hat{\sigma})$ is called *effective*, if G is effective on G/L as a transformation group.

(6) A symmetric space $(G/L, \hat{\sigma})$ is called *semisimple*, if Lie(G) is semisimple.

REMARK 2.1. If a symmetric space $(G/L, \hat{\sigma})$ admits a G-invariant pseudo-Riemannian metric g, then the Levi-Civita connection induced by g coincides with the canonical affine connection ∇^1 (cf. [15], p. 55).

DEFINITION 2.2 (CF. [6, PP. 82–84, PP. 86–87]). (1) Let M be a $2n$-dimensional manifold. A tensor field I of type $(1, 1)$ on M is called a *para-complex structure*, if the following conditions are satisfied:

(i) I^2 is the identity map of $\mathfrak{X}(M)$,

(ii) for each $p \in M$, the $(+1)$ (resp. (-1)) -eigenspace T_p^+M (resp. T_p^-M) of I_p is an n-dimensional subspace of T_pM,

(iii) for each $X, Y \in \mathfrak{X}(M)$, the equality $[IX, IY] - I[IX, Y] - I[X, IY] + [X, Y] = 0$ holds.

We call the pair (M, I) a *para-complex manifold*.

(2) Let (M, I) be a para-complex manifold and let g be a pseudo-Riemannian metric on M. We call g a *para-Hermitian metric* on M, if the equality

$$g(IX, Y) + g(X, IY) = 0$$

holds for each $X, Y \in \mathfrak{X}(M)$. We call the triplet (M, I, g) a *para-Hermitian manifold*.

(3) Let (M, I) and (M', I') be para-complex manifolds. A smooth map $\Phi : M \to M'$ is called *para-holomorphic* (resp. *para-antiholomorphic*), if the equality

$$(\Phi_*)_p \circ I_p = I_{\Phi(p)} \circ (\Phi_*)_p \quad (\text{resp. } (\Phi_*)_p \circ I_p = -I_{\Phi(p)} \circ (\Phi_*)_p)$$

holds for each $p \in M$.

(4) Let (M, I, g) be a para-Hermitian manifold. If a 2-form ω defined by $\omega(X, Y) := g(X, IY)$ for $X, Y \in \mathfrak{X}(M)$ is closed, g is called a *para-Kähler metric*.

(5) A *para-Hermitian symmetric space* is a quadruplet $(G/L, \hat{\sigma}, I, g)$ for a symmetric space $(G/L, \hat{\sigma})$ equipped with a G-invariant para-complex structure I and a G-invariant para-Hermitian metric g.

REMARK 2.2. Let $(G/L, \hat{\sigma}, I, g)$ be a para-Hermitian symmetric space.

(1) A 2-form ω defined by $\omega(X, Y) := g(X, IY)$ for $X, Y \in \mathfrak{X}(G/L)$ is a symplectic form. In other words, g is a para-Kähler metric (cf. [6, p. 86]).

(2) For an arbitrary $x \in G$, τ_x is a para-holomorphic isometry of $(G/L, I, g)$.

DEFINITION 2.3. A real Lie algebra $\mathfrak{g}$ is called *absolutely simple*, if its complexification $\mathfrak{g}_{\mathbb{C}}$ is simple. A Lie group G and a symmetric space $(G/L, \hat{\sigma})$ are called *absolutely simple*, if

Lie(G) is absolutely simple.

REMARK 2.3. We abbreviate "absolutely simple para-Hermitian symmetric space" to "APHS".

Proposition 2.1 (cf. [4, p. 478], [6, pp. 89–92], [9, p. 306])**.** *Let* $(G/L, \hat{\sigma}, I, g)$ *be an APHS and let* $\mathfrak{g} := \mathrm{Lie}(G)$. *In addition, set* $\mathfrak{l} := \mathfrak{g}^{\hat{\sigma}_*}$ *and* $\mathfrak{u} := \mathfrak{g}^{-\hat{\sigma}_*}$. *Then*

(1) *there exists a unique element* $Z \in \mathfrak{z}(\mathfrak{l})$ *such that*
  (i) $\mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(Z)$,
  (ii) $I_o = \mathrm{ad}_{\mathfrak{u}} Z$.

(2) *For this* $Z \in \mathfrak{z}(\mathfrak{l})$, *the followings are satisfied:*
  (i) $C_G(Z)_0 \subset L \subset C_G(Z)$.
  (ii) $\mathfrak{l} = \mathfrak{g}_0$ *and* $\mathfrak{u} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, *where* $\mathfrak{g}_\lambda$ *denotes the* λ*-eigenspace in* $\mathfrak{g}$ *of* $\mathrm{ad}\, Z$ $(\lambda = 0, \pm 1)$.
  (iii) $\hat{\sigma}_* = \exp \sqrt{-1}\pi\, \mathrm{ad}\, Z$.
  (iv) $\mathfrak{z}(\mathfrak{l}) = \mathbb{R}Z$.
  (v) *There exists a Cartan involution* θ *of* $\mathfrak{g}$ *such that* $\theta \circ \hat{\sigma}_* = \hat{\sigma}_* \circ \theta$ *and that* $\theta(Z) = -Z$.
  (vi) *If we take an open subgroup* $\bar{L}$ *of* $C_G(Z)$, *let* $\bar{I}$ *(resp.* $\bar{g}$*) be the* G*-invariant extension of* $\mathrm{ad}_{\mathfrak{u}} Z$ *or* $-\mathrm{ad}_{\mathfrak{u}} Z$ *(resp.* $\lambda B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}}$ *for* $\lambda \in \mathbb{R} \setminus \{0\}$*), then* $(G/\bar{L}, \hat{\sigma}, \bar{I}, \bar{g})$ *becomes an APHS. In addition,* Z *(resp.* $-Z$*) is the element which satisfies the condition* (1) *for the APHS* $(G/\bar{L}, \hat{\sigma}, \bar{I}, \bar{g})$, *if* $\bar{I}_o = \mathrm{ad}_{\mathfrak{u}} Z$ *(resp.* $\bar{I}_o = -\mathrm{ad}_{\mathfrak{u}} Z$*).*

REMARK 2.4. (1) We call the element Z in Proposition 2.1 the *characteristic element* of an APHS $(G/L, \hat{\sigma}, I, g)$. This is a nonzero semisimple element of $\mathfrak{g}$ such that the set of eigenvalues of $\mathrm{ad}\, Z$ on $\mathfrak{g}$ is $\{\pm 1, 0\}$.
(2) On an APHS $(G/L, \hat{\sigma}, I, g)$, for any G-invariant para-Hermitian metric g' with respect to I, there exists the nonzero real number λ such that g' is the G-invariant extension of $\lambda B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}}$ (cf. [18, p. 24]).

DEFINITION 2.4. We call an APHS $(G/L, \hat{\sigma}, I, g)$ *hyperbolic orbit type*, if L coincides with $C_G(Z)$ for the characteristic element Z of G/L. In other words, G/L is the adjoint orbit through Z.

2.3. Para-holomorphic isometries.

Lemma 2.1 (cf. [18, p. 29])**.** *Let* $(G/L, \hat{\sigma}, I, g)$ *be an APHS, let* $\mathfrak{g} := \mathrm{Lie}(G)$, *and let* Z *be the characteristic element of* $(G/L, \hat{\sigma}, I, g)$. *Put* $\mathrm{Aut}(\mathfrak{g}, Z)^+ := \{\phi \in \mathrm{Aut}(\mathfrak{g}) \mid \phi(Z) = Z\}$ *and* $\mathrm{Aut}(\mathfrak{g}, Z)^- := \{\phi \in \mathrm{Aut}(\mathfrak{g}) \mid \phi(Z) = -Z\}$. *Then as a disjoint union we have*

$$\mathrm{Aut}(\mathfrak{g}, \hat{\sigma}_*) = \mathrm{Aut}(\mathfrak{g}, Z)^+ \cup \mathrm{Aut}(\mathfrak{g}, Z)^-.$$

Proposition 2.2 (cf. [18, pp. 29–30])**.** *Let* $(G/L, \hat{\sigma}, I, g)$ *be an APHS of hyperbolic orbit type such that* $Z(G)$ *is trivial and let* $\mathfrak{g} := \mathrm{Lie}(G)$. *For an arbitrary* $\phi \in \mathrm{Aut}(\mathfrak{g}, \hat{\sigma}_*)$, *there exists the unique* $\hat{\phi} \in \mathrm{Aut}(G)$ *such that* (1) $\hat{\phi}(L) = L$ *and* (2) $\hat{\phi}_* = \phi$. *Let* $\Phi : G/L \to G/L$ *be a map defined by* $\Phi(xL) = \hat{\phi}(x)L$ *for* $xL \in G/L$. *Then* $\Phi \in \mathrm{I}(G/L, g, o)$. *In addition the map*

$$f_{\mathrm{isom}} : \mathrm{Aut}(\mathfrak{g}, \hat{\sigma}_*) \longrightarrow \mathrm{I}(G/L, g, o),\ \phi \longmapsto \Phi$$

is a group isomorphism.

Lemma 2.2. *Let $(G/L, \hat{\sigma}, I, g)$ be an APHS of hyperbolic orbit type such that $Z(G)$ is trivial, let $\mathfrak{g} := \mathrm{Lie}(G)$, and let Z be the characteristic element of $(G/L, \hat{\sigma}, I, g)$. For $\phi \in \mathrm{Aut}(\mathfrak{g}, \hat{\sigma}_*)$, $\Phi := f_{\mathrm{isom}}(\phi)$ is para-holomorphic (resp. para-antiholomorphic) if and only if $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^+$ (resp. $\mathrm{Aut}(\mathfrak{g}, Z)^-$).*

Proof. We assume $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^+$. By Propositions 2.1 and 2.2, we have

$$((\Phi_*)_o \circ I_o)(X) = (\phi \circ \mathrm{ad}\, Z)(X) = \phi[Z, X] = \mathrm{ad}\, Z(\phi(X)) = (I_o \circ (\Phi_*)_o)(X)$$

for $X \in \mathfrak{g}^{-\hat{\sigma}_*} \cong T_o(G/L)$. By the G-invariance of I and Remark 2.2 (2), Φ is para-holomorphic. In the case of $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^-$, we obtain Φ is para-antiholomorphic by the similar way to the case of $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^+$.

Conversely, We assume Φ is para-holomorphic (resp. para-antiholomorphic). By Lemma 2.1, $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^+$ or $\mathrm{Aut}(\mathfrak{g}, Z)^-$. Thus $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^+$ (resp. $\phi \in \mathrm{Aut}(\mathfrak{g}, Z)^-$). □

Corollary 2.1. *For $i = 1, 2$, let $(G/C_G(Z_i), \hat{\sigma}_i, I_i, g_i)$ be an APHS such that $Z(G)$ is trivial, where Z_i is the characteristic element of $(G/C_G(Z_i), \hat{\sigma}_i, I_i, g_i)$. Put $\mathfrak{g} := \mathrm{Lie}(G)$. For an arbitrary $\phi \in \mathrm{Aut}(\mathfrak{g})$ which satisfies $\phi(Z_1) = Z_2$ (resp. $\phi(Z_1) = -Z_2$), there exists the unique $\hat{\phi} \in \mathrm{Aut}(G)$ such that (1) $\hat{\phi}(C_G(Z_1)) = C_G(Z_2)$ and (2) $\hat{\phi}_* = \phi$. Let $\Phi : G/C_G(Z_1) \to G/C_G(Z_2)$ be a map defined by $\Phi(xC_G(Z_1)) = \hat{\phi}(x)C_G(Z_2)$ for $xC_G(Z_1) \in G/C_G(Z_1)$. Then the map Φ is a para-holomorphic isometry (resp. para-antiholomorphic isometry).*

Proof. We can prove it in the similar ways to the proofs of Lemma 2 in [18, p. 29] and Lemma 2.2. □

2.4. Affine transformations and connected components. We need the following lemmas to prove Theorem 3.1. We note that Lemma 2.4 is a generalization of Lemma 2.3.1 in [2, p. 42], where we do not assume the irreducibility of symmetric spaces $(G_i/L_i, \hat{\sigma}_i)$ $(i = 1, 2)$ in Lemma 2.4.

Lemma 2.3 (cf. [2, pp. 44–45])**.** *Let $(G/L, \hat{\sigma})$ be a symmetric space and let $\hat{\xi} \in \mathrm{Inv}(G)$ such that $\hat{\sigma} \circ \hat{\xi} = \hat{\xi} \circ \hat{\sigma}$ and that $\hat{\xi}(L) = L$. Let Ξ be a diffeomorphism of G/L defined by $\Xi(aL) := \hat{\xi}(a)L$ for $aL \in G/L$ and let C_o be the connected component of $(G/L)^{\Xi}$ including the origin o. Then C_o is a closed, connected, complete, totally geodesic submanifold of $(G/L, \nabla^1)$ (with the induced topology from G/L). In addition, $C_o = (G^{\hat{\xi}})_0/((G^{\hat{\xi}})_0 \cap L)$ and C_o is a symmetric space with respect to $\hat{\sigma}|_{(G^{\hat{\xi}})_0}$.*

Remark 2.5. We refer to [8, p. 180] for the definition of totally giodesic submanifolds. We note that this definition differs from one in [3, p. 79].

Lemma 2.4. *For $i = 1, 2$, let $(G_i/L_i, \hat{\sigma}_i)$ be an effective semisimple symmetric spaces and let Φ be an affine diffeomorphism from $(G_1/L_1, \nabla^1_1)$ onto $(G_2/L_2, \nabla^1_2)$ such that $\Phi(o_1) = o_2$. Then there exists the unique isomorphism $\hat{\phi}$ from G_1 onto G_2 such that*

$$(1)\ \hat{\phi} \circ \hat{\sigma}_1 = \hat{\sigma}_2 \circ \hat{\phi}, \quad (2)\ \hat{\phi}(L_1) = L_2, \quad (3)\ \Phi \circ \pi_1 = \pi_2 \circ \hat{\phi}.$$

Here we denote by o_i the origin of G_i/L_i, by ∇^1_i the canonical affine connection on $(G_i/L_i, \hat{\sigma}_i)$ and by π_i the natural projection from G_i onto G_i/L_i for $i = 1, 2$.

Proof. Put $M_i := G_i/L_i$ for $i = 1, 2$. We denote by τ^i the action of G_i onto G_i/L_i defined by $\tau^i_x(aL_i) = xaL_i$ for $x \in G_i$, $aL_i \in G_i/L_i$ for $i = 1, 2$. We denote by $S^i_{p_i}$ the symmetry at $p_i \in G_i/L_i$ for $i = 1, 2$. First, we prove the Lie group G_i is isomorphic to $(\mathrm{Aut}(M_i))_0$, where $\mathrm{Aut}(M_i)$ denotes the group of automorphisms of a symmetric space M_i. Since M_i is connected and semisimple, $\mathrm{Aut}(M_i)$ coincides with the group of affine transformations of (M_i, ∇^1_i) and $(\mathrm{Aut}(M_i))_0$ is isomorphic to $G(M_i)$ as a Lie group, where $G(M_i)$ denotes the group generated by $\{S^i_p \circ S^i_q \mid p, q \in M_i\}$ (cf. [11, p. 64, p. 84]). Moreover, $G(M_i)$ is equal to the image $\tau^i_{G_i}$. Indeed, let $p := aL_i, q := bL_i \in M_i$, and let $\Sigma_i := S^i_{o_i}$. Then we have

$$\begin{aligned} S^i_p \circ S^i_q &= (\tau^i_a \circ \Sigma_i \circ \tau^i_{a^{-1}}) \circ (\tau^i_b \circ \Sigma_i \circ \tau^i_{b^{-1}}) \\ &= \tau^i_a \circ \tau^i_{\hat{\sigma}_i(a^{-1})} \circ \tau^i_{\hat{\sigma}_i(b)} \circ \tau^i_{b^{-1}} \\ &= \tau^i_{(a\hat{\sigma}_i(a^{-1})\hat{\sigma}_i(b)b^{-1})}. \end{aligned}$$

Therefore, $G(M_i) \subset \tau^i_{G_i}$. Consequently, $G(M_i) = \tau^i_{G_i}$ because $G(M_i) = (\mathrm{Aut}(M_i))_0$ and $\tau^i_{G_i} \subset (\mathrm{Aut}(M_i))_0$. Since M_i is effective, $\tau^i_{G_i} \cong G_i$. Thus G_i is isomorphic to $(\mathrm{Aut}(M_i))_0$ by the correspondence

$$\tau^i : G_i \longrightarrow (\mathrm{Aut}(M_i))_0,\ a \longmapsto \tau^i_a. \tag{2.4.1}$$

Secondly, we get the following equation:

$$\Phi \circ \Sigma_1 \circ \Phi^{-1} = \Sigma_2. \tag{2.4.2}$$

Indeed, by the definition of Σ_i, we have $\Sigma_i(o_i) = o_i$. Then the differential map $(\Sigma_i)_*$ at the origin o_i equals $-\mathrm{id}$ on $T_{o_i}M_i$. Therefore, $(\Phi \circ \Sigma_1 \circ \Phi^{-1})_* = (\Sigma_2)_*$ on $T_{o_2}(M_2)$. Moreover, by the assumption, we have $(\Phi \circ \Sigma_1 \circ \Phi^{-1})(o_2) = \Sigma_2(o_2)$. Thus we obtain Equation (2.4.2) by Lemma 6 in [14, p. 820].

Thirdly, we give the unique isomorphism $\hat{\phi}$ from G_1 onto G_2 which satisfies (1), (2), and (3). We define the homeomorphism

$$\mathrm{A}_\Phi : \mathrm{Aut}(M_1) \longrightarrow \mathrm{Aut}(M_2),\ \Psi \longmapsto \Phi \circ \Psi \circ \Phi^{-1}$$

with respect to the compact-open topology. Furthermore, its restriction

$$\mathrm{A}_\Phi : (\mathrm{Aut}(M_1))_0 \longrightarrow (\mathrm{Aut}(M_2))_0,\ \tau^1_a \longmapsto \Phi \circ \tau^1_a \circ \Phi^{-1}$$

is a Lie group isomorphism. Since τ^i in (2.4.1) is an isomorphism, we obtain the isomorphism

$$\hat{\phi} : G_1 \longrightarrow G_2,\ a \longmapsto (\tau^2)^{-1}(\mathrm{A}_\Phi(\tau^1_a)).$$

We prove $\hat{\phi}$ satisfies (1), (2), and (3).

For any $a_1 \in G_1$, there exists an element $a_2 \in G_2$ such that $\Phi \circ \tau^1_{a_1} \circ \Phi^{-1} = \tau^2_{a_2}$, i.e., $\hat{\phi}(a_1) = a_2$.

(1) Since $\hat{\sigma}_i$ is an involutive automorphism of G_i, we have

$$\Sigma_i \circ \tau^i_{a_i} \circ \Sigma_i = \tau^i_{\hat{\sigma}_i(a_i)}.$$

Thus we have

$$\Phi \circ \tau^1_{\hat{\sigma}_1(a_1)} \circ \Phi^{-1} = \Phi \circ (\Sigma_1 \circ \tau^1_{a_1} \circ \Sigma_1) \circ \Phi^{-1}$$

$$\overset{(2.4.2)}{=} \Sigma_2 \circ \Phi \circ \tau^1_{a_1} \circ \Phi^{-1} \circ \Sigma_2 = \Sigma_2 \circ \tau^2_{a_2} \circ \Sigma_2 = \tau^2_{\hat{\sigma}_2(a_2)}.$$

Consequently, we have

$$(\hat{\phi} \circ \hat{\sigma}_1)(a_1) = (\tau^2)^{-1}(\mathrm{A}_\Phi(\tau^1_{\hat{\sigma}_1(a_1)})) = (\tau^2)^{-1}(\Phi \circ \tau^1_{\hat{\sigma}_1(a_1)} \circ \Phi^{-1}) = \hat{\sigma}_2(a_2) = (\hat{\sigma}_2 \circ \hat{\phi})(a_1).$$

Thus (1) holds.
(3) It is obtained by

$$\pi_2(\hat{\phi}(a_1)) = \pi_2(a_2) = \tau^2_{a_2}(o_2) = (\Phi \circ \tau^1_{a_1} \circ \Phi^{-1})(o_2) = \Phi(\tau^1_{a_1}(o_1)) = \Phi(\pi_1(a_1)).$$

(2) For all $a_1 \in L_1$, it follows that $\pi_2(\hat{\phi}(a_1)) = \Phi(\pi_1(a_1)) = \Phi(o_1) = o_2$. As a result, $\hat{\phi}(L_1) \subset L_2$. On the other hand, we have $\pi_1(\hat{\phi}^{-1}(a_2)) = \Phi^{-1}(\pi_2(a_2)) = \Phi^{-1}(o_2) = o_1$ for any $a_2 \in L_2$. Thus $\hat{\phi}^{-1}(L_2) = L_1$. Hence (2) holds.

The uniqueness of $\hat{\phi}$ can be proven in a similar way to the proof of Lemma 2.3.1 in [2, pp. 42–43]. □

3. Relation between para-real forms and Lie algebra automorphisms.

3.1. Para-real forms. In this subsection, we define a para-real form of a para-Hermitian symmetric space.

Definition 3.1. For a para-Hermitian symmetric space $(G/L, \hat{\sigma}, I, g)$, a nonempty subset $R \subset G/L$ is called a *para-real form* of G/L, if there exists an involutive isometry Ξ of G/L such that Ξ is para-antiholomorphic and that R coincides with a connected component of $(G/L)^{\Xi}$.

Lemma 3.1. *Let $(G/L, \hat{\sigma}, I, g)$ be an APHS of hyperbolic orbit type and let ω be the closed 2-form defined by $\omega(X, Y) := g(X, IY)$ for $X, Y \in \mathfrak{X}(G/L)$. Let Ξ be an involutive para-antiholomorphic isometry of G/L which satisfies $\Xi(o) = o$ and let R be a para-real form of G/L with respect to Ξ containing the origin o. In addition, we assume that $Z(G)$ is trivial. Then R is a closed, connected, complete, totally geodesic, Lagrangian submanifold of the symplectic, pseudo-Riemmanian manifold $(G/L, g, \omega)$.*

Proof. Put $M := G/L$. By Proposition 2.2, there exists $\hat{\xi} \in \mathrm{Aut}(G)$ such that $\hat{\sigma} \circ \hat{\xi} = \hat{\xi} \circ \hat{\sigma}$ and that $\Xi(aL) = \hat{\xi}(a)L$ for $aL \in G/L$. Thus R is a closed, connected, complete, totally geodesic submanifold of M by Lemma 2.3. By Remark 2.2 (1), ω is a symplectic form of M. We prove the equalities

$$\dim T_pR = \frac{1}{2}\dim T_pM, \quad g_p(T_pR, I_p(T_pR)) = \{0\}$$

for any $p \in R$. Since $\Xi^2 = \mathrm{id}$, we have

$$T_pM = T_p^+M \oplus T_p^-M.$$

Here T_p^+M (resp. T_p^-M) denotes the (+1) (resp. (-1)) -eigenspace of $(\Xi_*)_p$ in T_pM. We obtain the equalities

$$I_p(T_p^+M) = T_p^-M, \quad I_p(T_p^-M) = T_p^+M$$

because Ξ is para-antiholomorphic. Therefore, $T_p^+M = T_pR$ yields $\dim T_pR = (1/2)\dim T_pM$ and $g_p(T_pR, I_p(T_pR)) = \{0\}$. Thus Lemma 3.1 holds. □

EXAMPLE 3.1. Let $G := SL(2,\mathbb{R})$, let $\mathfrak{g}$ be its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, and let

$$I_{1,1} := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}).$$

The map $\hat{\sigma} : G \ni a \mapsto I_{1,1} a I_{1,1} \in G$ is an involution of G. Then we have

$$G^{\hat{\sigma}} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \setminus \{0\} \right\}.$$

Set $Z := (-1/2)I_{1,1} \in \mathfrak{g}$ and $\mathfrak{u} := \mathfrak{g}^{-\hat{\sigma}_*}$. Then we obtain an APHS $(G/G^{\hat{\sigma}}, \hat{\sigma}, I, g)$, where I is the G-invariant extension of $\mathrm{ad}_{\mathfrak{u}} Z$ and g is the G-invariant extension of $B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}}$. We note that this Z is the characteristic element of the APHS $(G/G^{\hat{\sigma}}, \hat{\sigma}, I, g)$ and $G^{\hat{\sigma}}$ coincides with $C_G(Z)$.

We construct a para-real form of the APHS. We define an involution $\hat{\xi}$ of G by $\hat{\xi}(a) := J_{1,1} a J_{1,1}$ for $a \in G$, where

$$J_{1,1} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This $\hat{\xi}$ satisfies $\hat{\xi}(G^{\hat{\sigma}}) = G^{\hat{\sigma}}$ and $\hat{\xi}_*(Z) = -Z$. Thus we define a map $\Xi : G/G^{\hat{\sigma}} \longrightarrow G/G^{\hat{\sigma}}$ by

$$\Xi(aG^{\hat{\sigma}}) = \hat{\xi}(a)G^{\hat{\sigma}}$$

for $aG^{\hat{\sigma}} \in G/G^{\hat{\sigma}}$. Then the map Ξ is a para-antiholomorphic isometry because $(\Xi_*)_o = \hat{\xi}_*|_{\mathfrak{u}}$ and $\hat{\xi}_*(Z) = -Z$. Then $(G/G^{\hat{\sigma}})^{\Xi}$ is given by

$$(G/G^{\hat{\sigma}})^{\Xi} = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} G^{\hat{\sigma}} \,\middle|\, x^2 - y^2 = 1,\ x, y \in \mathbb{R} \right\}.$$

Thus $(G/G^{\hat{\sigma}})^{\Xi}$ has two connected components. Each connected component of $(G/G^{\hat{\sigma}})^{\Xi}$ is a para-real form of the APHS $(G/G^{\hat{\sigma}}, \hat{\sigma}, I, g)$. We note that $G/G^{\hat{\sigma}}$ is a hyperboloid of one sheet and $(G/G^{\hat{\sigma}})^{\Xi}$ equals a hyperbola in $G/G^{\hat{\sigma}}$.

3.2. Equivalence relations on $\mathcal{R}(G)$ and $d\mathcal{R}(\mathfrak{g})$. We assume that G is an absolutely simple connected Lie group. Let $\mathcal{R}(G)$ denote the set of pairs $(G/L, R)$ of an APHS G/L of hyperbolic orbit type and a para-real form R of G/L. We define an equivalence relation on $\mathcal{R}(G)$ as follows:

DEFINITION 3.2. We call two elements $(G/L_1, R_1)$ and $(G/L_2, R_2) \in \mathcal{R}(G)$ are *equivalent*, which is denoted by $(G/L_1, R_1) \simeq (G/L_2, R_2)$ or $(G/L_1, R_1) \overset{\Phi}{\simeq} (G/L_2, R_2)$, if there exists a homothety Φ from G/L_1 onto G/L_2 such that Φ is para-holomorphic, and that $\Phi(R_1) = R_2$.

Let $\mathfrak{g}$ be an absolutely simple Lie algebra. Let $d\mathcal{R}(\mathfrak{g})$ denote the set of pairs (Z, ξ) of a nonzero semisimple element Z of $\mathfrak{g}$ and an involution ξ of $\mathfrak{g}$ with $\xi(Z) = -Z$, where the set of eigenvalues of $\mathrm{ad}\, Z$ on $\mathfrak{g}$ is $\{0, \pm 1\}$. We define an equivalence relation on $d\mathcal{R}(\mathfrak{g})$ as follows:

DEFINITION 3.3. We call two elements (Z_1, ξ_1) and $(Z_2, \xi_2) \in d\mathcal{R}(\mathfrak{g})$ are *equivalent*, which denoted by $(Z_1, \xi_1) \sim (Z_2, \xi_2)$ or $(Z_1, \xi_1) \overset{\phi}{\sim} (Z_2, \xi_2)$, if there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\phi \circ \xi_1 = \xi_2 \circ \phi$ and that $\phi(Z_1) = Z_2$.

3.3. A correspondence between $d\mathcal{R}(\mathfrak{g})/\sim$ and $\mathcal{R}(G)/\simeq$. We fix an absolutely simple connected Lie group G with trivial center. In this subsection, we define a bijection F from $d\mathcal{R}(\mathfrak{g})/\sim$ onto $\mathcal{R}(G)/\simeq$, where $\mathfrak{g}$ is the Lie algebra of G. First, we construct an element $(G/L, R) \in \mathcal{R}(G)$ from $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$. Take an arbitrary $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$, we obtain the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ with respect to $\mathrm{ad}\, Z$, where $\mathfrak{g}_\lambda$ denotes the λ-eigenspace of $\mathrm{ad}\, Z$ of $\mathfrak{g}$ ($\lambda = 0, \pm 1$). Put $\hat{\sigma} := \mathrm{A}_{\exp \sqrt{-1}\pi Z}$ and $\sigma := \hat{\sigma}_*$. Then we get the following equalities

$$\mathfrak{g}^{\sigma} = \mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}(Z), \quad \mathfrak{u} := \mathfrak{g}^{-\sigma} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

Since Z is a semisimple element of $\mathfrak{g}$ such that all the eigenvalues of $\mathrm{ad}\, Z$ are real, we obtain an APHS $(G/C_G(Z), \hat{\sigma}, I, g)$ such that the element Z is the characteristic element of $(G/C_G(Z), \hat{\sigma}, I, g)$ by Theorem 3.7 in [6]. Here I (resp. g) is the G-invariant extension of $\mathrm{ad}_{\mathfrak{u}}\, Z$ (resp. $B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}}$) (cf. Proposition 2.1). By Lemma 2.1 and Proposition 2.2, ξ induces the unique involutive isometry $\Xi := f_{\mathrm{isom}}(\xi)$ of $(G/C_G(Z), \hat{\sigma}, I, g)$ satisfying $\Xi(o) = o$. Moreover, by Lemma 2.2, Ξ is para-antiholomorphic. Set R_o be the connected component of $(G/C_G(Z))^{\Xi}$ containing the origin o. Then this R_o is a para-real form of $G/C_G(Z)$. Thus

$$(G/C_G(Z), R_o) \in \mathcal{R}(G).$$

Consequently, we define a map

$$f : d\mathcal{R}(\mathfrak{g}) \longrightarrow \mathcal{R}(G),\ (Z, \xi) \longmapsto (G/C_G(Z), R_o).$$

REMARK 3.1. Let $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ and let $(G/C_G(Z), R_o) := f(Z, \xi)$. By Lemma 2.3, R_o is a symmetric space.

We denote by $[(Z, \xi)]$ (resp. $[(G/C_G(Z), R)]$) the equivalence class of $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ (resp. $(G/C_G(Z), R) \in \mathcal{R}(G)$).

Lemma 3.2. *Let* $(Z_1, \xi_1), (Z_2, \xi_2) \in d\mathcal{R}(\mathfrak{g})$. *If* $(Z_1, \xi_1) \sim (Z_2, \xi_2)$, *then* $f(Z_1, \xi_1) \simeq f(Z_2, \xi_2)$.

Proof. By Definition 3.3, there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\phi(Z_1) = Z_2$ and that

$$\phi \circ \xi_1 = \xi_2 \circ \phi. \tag{3.3.1}$$

By Corollary 2.1, there exists the unique $\hat{\phi} \in \mathrm{Aut}(G)$ such that $\hat{\phi}(C_G(Z_1)) = C_G(Z_2)$ and that $\hat{\phi}_* = \phi$. In addition, we obtain a para-holomorphic isometry

$$\Phi : G/C_G(Z_1) \longrightarrow G/C_G(Z_2),\ aC_G(Z_1) \longmapsto \hat{\phi}(a)C_G(Z_2)$$

which satisfies $\Phi(o_1) = o_2$, where o_i is the origin of $G/C_G(Z_i)$ ($i = 1, 2$). For $i = 1, 2$, put $\Xi_i := f_{\mathrm{isom}}(\xi_i)$ and $(G/C_G(Z_i), R_i) := f(Z_i, \xi_i)$. By Equation (3.3.1), we have $\Phi \circ \Xi_1 = \Xi_2 \circ \Phi$. Then for an arbitrary $p \in R_1$,

$$\Xi_2(\Phi(p)) = \Phi(\Xi_1(p)) = \Phi(p).$$

Hence $\Phi(R_1) \subset (G/C_G(Z_2))^{\Xi_2}$. Moreover, $\Phi(R_1) = R_2$ because $\Phi(o_1) = o_2$. Therefore

$$f(Z_1, \xi_1) \overset{\Phi}{\simeq} f(Z_2, \xi_2). \qquad \square$$

We define a map $F : d\mathcal{R}(\mathfrak{g})/\!\sim\, \to \mathcal{R}(G)/\!\simeq$ by

$$F : [(Z, \xi)] \longmapsto [f(Z, \xi)].$$

By Lemma 3.2, the map F is well-defined.

Lemma 3.3. *For any $(G/L, Q) \in \mathcal{R}(G)$, there exists (Z, ξ) such that $L = C_G(Z)$ and that $(G/L, Q) \simeq f(Z, \xi)$.*

Proof. Take an arbitrary $(G/L, Q) \in \mathcal{R}(G)$, where Q is a para-real form with respect to an involutive para-antiholomorphic isometry Ξ of G/L. Since G/L is an APHS of hyperbolic orbit type, L can be expressed as $C_G(Z)$ for a nonzero semisimple element Z of $\mathfrak{g}$ which satisfies the eigenvalues of $\operatorname{ad} Z$ on $\mathfrak{g}$ are $\pm 1, 0$. Since Q is a nonempty set, there exists an element $aC_G(Z) \in Q$ $(a \in G)$. Put $\Xi_o := \tau_a^{-1} \circ \Xi \circ \tau_a$, then Ξ_o is also an involutive para-antiholomorphic isometry of $G/C_G(Z)$ which satisfies $\Xi_o(o) = o$ because of Remark 2.2 (2). Furthermore, $\Xi_o(\tau_a^{-1}(p)) = \tau_a^{-1}(\Xi(p)) = \tau_a^{-1}(p)$ for $p \in Q$. Hence $\tau_a^{-1}(Q)$ coincides with the para-real form R_o of $G/C_G(Z)$ with respect to Ξ_o containing the origin o. Therefore, we have

$$(G/C_G(Z), R_o) \overset{\tau_a}{\simeq} (G/C_G(Z), Q)$$

by Remark 2.2 (2). On account of Proposition 2.2 and Lemma 2.2, there exists the unique $\xi \in \operatorname{Aut}^-(\mathfrak{g}, Z)$ such that

$$\Xi_o = f_{\mathrm{isom}}(\xi).$$

Moreover, ξ is an involution because Ξ_o is an involution. Therefore, $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ and

$$f(Z, \xi) = (G/C_G(Z), R_o) \simeq (G/C_G(Z), Q). \qquad \square$$

Theorem 3.1. *The map $F : d\mathcal{R}(\mathfrak{g})/\!\sim\, \to \mathcal{R}(G)/\!\simeq$, $[(Z, \xi)] \mapsto [f(Z, \xi)]$ is a bijection.*

Proof. By Lemma 3.3, F is surjective. We prove that F is injective. For $i = 1, 2$, let $(Z_i, \xi_i) \in d\mathcal{R}(\mathfrak{g})$. We assume that

$$(G/C_G(Z_1), R_1) := f(Z_1, \xi_1) \simeq f(Z_2, R_2) =: (G/C_G(Z_2), R_2).$$

Here R_i is a para-real form of $G/C_G(Z_i)$ with respect to $\Xi_i := f_{\mathrm{isom}}(\xi_i)$ containing the origin o_i of $G/C_G(Z_i)$ for $i = 1, 2$. By Definition 3.2, there exists a para-holomorphic homothety Φ' from $G/C_G(Z_1)$ onto $G/C_G(Z_2)$ such that $\Phi'(R_1) = R_2$. We make a para-holomorphic homothety Φ from $G/C_G(Z_1)$ onto $G/C_G(Z_2)$ which satisfies

$$\Phi(R_1) = R_2, \ \Phi(o_1) = o_2. \tag{3.3.2}$$

By Proposition 2.2, there exists the unique $\hat{\xi}_2 \in \operatorname{Inv}(G)$ such that $(\hat{\xi}_2)_* = \xi_2$ and that $\Xi_2(aC_G(Z_2)) = \hat{\xi}_2(a)C_G(Z_2)$ for $aC_G(Z_2) \in G/C_G(Z_2)$. For $i = 1, 2$, set $\hat{\sigma}_i := \mathrm{A}_{\exp\sqrt{-1}\pi Z_i}$ and $\sigma_i := (\hat{\sigma}_i)_*$. Since $\xi_i(Z_i) = -Z_i$, we have

$$\xi_i \circ \sigma_i = \sigma_i \circ \xi_i \quad (i = 1, 2).$$

Thus we have $\hat{\xi}_2 \circ \hat{\sigma}_2 = \hat{\sigma}_2 \circ \hat{\xi}_2$ because G is connected. On account of Lemma 2.3, we obtain the equality

$$R_2 = (G^{\hat{\xi}_2})_0/((G^{\hat{\xi}_2})_0 \cap C_G(Z_2)).$$

Since $\Phi'(o_1) \in R_2$, there exists an element $a \in (G^{\hat{\xi}_2})_0$ such that $\Phi'(o_1) = a^{-1}C_G(Z_2)$. Then we obtain a para-holomorphic homothety $\Phi : G/C_G(Z_1) \to G/C_G(Z_2)$ which satisfies (3.3.2) by setting $\Phi := \tau_a^2 \circ \Phi'$, where τ^2 is the action of G onto $G/C_G(Z_2)$ defined by $\tau_x^2(bC_G(Z_2)) = xbC_G(Z_2)$ for $x \in G$, $bC_G(Z_2) \in G/C_G(Z_2)$. Moreover, Φ is an affine transformation with respect to the canonical affine connections of $G/C_G(Z_1)$ and $G/C_G(Z_2)$ (cf. Remark 2.1). Then $G/C_G(Z_i)$ is effective because $Z(G)$ is trivial ($i = 1, 2$). Thus by Equation (3.3.2) and Lemma 2.4, there exists the unique $\hat{\phi} \in \mathrm{Aut}(G)$ such that

$$\text{(3.3.3)} \qquad \text{(i) } \hat{\phi} \circ \hat{\sigma}_1 = \hat{\sigma}_2 \circ \hat{\phi}, \quad \text{(ii) } \hat{\phi}(C_G(Z_1)) = C_G(Z_2), \quad \text{(iii) } \Phi \circ \pi_1 = \pi_2 \circ \hat{\phi}.$$

Here π_i is the natural projection from G onto $G/C_G(Z_i)$. Put $\phi := \hat{\phi}_*$. Then we have

$$\phi \circ \sigma_1 = \sigma_2 \circ \phi.$$

Thus we have $\phi(Z_1) \in \mathfrak{z}(\mathfrak{g}^{\sigma_2})$. Since Φ is para-holomorphic, we have $\phi(Z_1) = Z_2$ by Proposition 2.1 (2) (iv). At the end of the proof, we prove the equation

$$\phi \circ \xi_1 = \xi_2 \circ \phi.$$

Set $\mathfrak{u}_i := \mathfrak{g}^{-\sigma_i}$ for $i = 1, 2$. Since $\xi_i \circ \sigma_i = \sigma_i \circ \xi_i$, we obtain the direct sum decomposition

$$\text{(3.3.4)} \qquad \mathfrak{u}_i = \mathfrak{u}_i^+ \oplus \mathfrak{u}_i^-,$$

where $\mathfrak{u}_i^\pm := \{X \in \mathfrak{u}_i \mid \xi_i(X) = \pm X\}$, respectively. Then

$$T_{o_i}(G/C_G(Z_i)) = \mathfrak{u}_i, \quad T_{o_i}(R_i) = \mathfrak{u}_i^+, \quad B_{\mathfrak{g}}(\mathfrak{u}_i^+, \mathfrak{u}_i^-) = \{0\}.$$

Therefore, Equation (3.3.2) and Equation (3.3.3) (iii) imply $\phi(\mathfrak{u}_1^+) = \mathfrak{u}_2^+$. Moreover, we have $\phi(\mathfrak{u}_1^-) = \mathfrak{u}_2^-$ because $\phi(\mathfrak{u}_1) = \mathfrak{u}_2$, Equation (3.3.4), and $B_{\mathfrak{g}}$ is non-degenerate on $\mathfrak{u}_i$ ($i = 1, 2$). Hence we get the equation $\phi \circ \xi_1 = \xi_2 \circ \phi$ on $\mathfrak{u}_1 = \mathfrak{u}_1^+ \oplus \mathfrak{u}_1^-$. Since $\mathfrak{g}$ is absolutely simple, $\mathfrak{g}_1 = [\mathfrak{u}_1, \mathfrak{u}_1] \oplus \mathfrak{u}_1$ (cf. [15], p. 56). Therefore, we get the equation $\phi \circ \xi_1 = \xi_2 \circ \phi$ on $\mathfrak{g}$ and the relation

$$(Z_1, \xi_1) \overset{\phi}{\sim} (Z_2, \xi_2).$$

Hence F is injective. □

Lemma 3.4. *Let $(G/C_G(Z), \hat{\sigma}, I, g)$ be an APHS of hyperbolic orbit type with respect to the characteristic element Z, where it is not necessary to assume that the center of G is trivial. Set $\tilde{G} := G/Z(G)$.*

(1) *$(\tilde{G}/(C_{\tilde{G}}(Z)), \tilde{\sigma}, \tilde{I}, \tilde{g})$ is an APHS of hyperbolic orbit type with respect to the characteristic element Z, where $\tilde{\sigma}$ is an involution of $\tilde{G}$ defined by $aZ(G) \mapsto \hat{\sigma}(a)Z(G)$ for $aZ(G) \in \tilde{G}$, and $\tilde{I}$ (resp. $\tilde{g}$) is the $\tilde{G}$-invariant extension of I_o (resp. g_o).*
(2) *$(\tilde{G}/(C_{\tilde{G}}(Z)), \tilde{\sigma}, \tilde{I}, \tilde{g})$ is para-holomorphic isometric isomorphic to $(G/C_G(Z), \hat{\sigma}, I, g)$.*
(3) *There is a bijection from $\mathcal{R}(G)/\simeq$ onto $\mathcal{R}(\tilde{G})/\simeq$.*

Proof. (1) Set $\mathfrak{g} := \mathrm{Lie}(G)$. Since G is simple, $Z(G)$ is discrete subgroup of G. Thus $\mathrm{Lie}(\tilde{G})$ coincides with $\mathfrak{g}$. In addition, we have $Z(G) \subset C_G(Z)$. Set $L := C_G(Z)$ and $\tilde{L} := L/Z(G)$. We note that $\tilde{L} = C_{\tilde{G}}(Z)$. First, we prove that $(\tilde{G}/\tilde{L}, \tilde{\sigma})$ is a symmetric space. Since $\hat{\sigma}(Z(G)) \subset Z(G)$, we define an involution $\tilde{\sigma}$ of $\tilde{G}$ by $aZ(G) \mapsto \hat{\sigma}(a)Z(G)$. It is clear that $\tilde{L} \subset \tilde{G}^{\tilde{\sigma}}$. We denote by π the natural projection from G onto $\tilde{G}$. Then we have $\tilde{G}^{\tilde{\sigma}} = \pi(G^{\sigma})$ and $(\tilde{G}^{\tilde{\sigma}})_0 = \pi((G^{\hat{\sigma}})_0)$. Thus we have $(\tilde{G}^{\tilde{\sigma}})_0 \subset \tilde{L} \subset \tilde{G}^{\tilde{\sigma}}$. Hence $(\tilde{G}/\tilde{L}, \tilde{\sigma})$ is a symmetric space. Second, we prove that the symmetric space $(\tilde{G}/\tilde{L}, \tilde{\sigma})$ becomes an APHS. By Proposition 2.1 and Remark 2.4 (2), we have $I_o = \mathrm{ad}_{\mathfrak{g}^{-\hat{\sigma}_*}} Z$ and $g_o = \lambda B_{\mathfrak{g}}|_{\mathfrak{g}^{-\hat{\sigma}_*} \times \mathfrak{g}^{-\hat{\sigma}_*}}$ $(\lambda \in \mathbb{R} \setminus \{0\})$. By definition of $\tilde{\sigma}$, we have $\hat{\sigma}_* = \tilde{\sigma}_*$. Thus we have $T_o(G/L) = T_{\tilde{o}}(\tilde{G}/\tilde{L})$, where o (resp. $\tilde{o}$) is the origin of G/L (resp. $\tilde{G}/\tilde{L}$). Therefore, $(\tilde{G}/\tilde{L}, \tilde{\sigma}, \tilde{I}, \tilde{g})$ is an APHS with respect to the characteristic element Z, where $\tilde{I}$ (resp. $\tilde{g}$) is the $\tilde{G}$-invariant extension of $\tilde{I}_{\tilde{o}} := I_o$ (resp. $\tilde{g}_{\tilde{o}} := g_o$). Hence (1) holds.

(2) Since $\pi(L) \subset \tilde{L}$, we define a map

$$\Pi_L : G/L \to \tilde{G}/\tilde{L}, \quad aL \mapsto \pi(a)\tilde{L}.$$

It is clear that the map Π_L is a diffeomorphism satisfying $\Pi_L(o) = \tilde{o}$. By definitions of $\tilde{G}$ and π, the differential map π_* coincides with identity map of $\mathfrak{g}$. Thus we have

$$((\Pi_L)_*)_o \circ I_o = \tilde{I}_{\tilde{o}} \circ ((\Pi_L)_*)_o, \tag{3.3.5}$$

because $I_o = \tilde{I}_{\tilde{o}}$ and $T_o(G/L) = T_{\tilde{o}}(\tilde{G}/\tilde{L})$. We denote by τ (resp. $\tilde{\tau}$) the action of G onto G/L defined by $\tau_a : G/L \ni bL \mapsto abL \in G/L$ for $a \in G$ (resp. $\tilde{G}$ onto $\tilde{G}/\tilde{L}$ defined by $\tilde{\tau}_{\tilde{a}} : \tilde{G}/\tilde{L} \ni \tilde{b}\tilde{L} \mapsto \tilde{a}\tilde{b}\tilde{L} \in \tilde{G}/\tilde{L}$ for $\tilde{a} \in \tilde{G}$). Then we have

$$\Pi_L \circ \tau_a = \tilde{\tau}_{\pi(a)} \circ \Pi_L$$

for $a \in G$ because π is a homomorphism. Thus Equation (3.3.5), G-invariance of I, and $\tilde{G}$-invariance of $\tilde{I}$ imply that the map Π is para-holomorphic. Moreover, $\tilde{g}_{\tilde{o}} = g_o = \lambda B_{\mathfrak{g}}|_{\mathfrak{g}^{-\hat{\sigma}_*} \times \mathfrak{g}^{-\hat{\sigma}_*}}$ $(\lambda \in \mathbb{R} \setminus \{0\})$, G-invariance of g, and $\tilde{G}$-invariance of $\tilde{g}$ imply that the map Π_L is an isometry. Hence (2) holds.

(3) Let $(G/L_0, R) \in \mathcal{R}(G)$. By the Definition 3.1, there exists an involutive para-antiholomorphic isometry Ξ of G/L_0 such that R coincides with a connected component of $(G/L_0)^{\Xi}$. Set $\tilde{L}_0 := L_0/Z(G)$ and $\tilde{\Xi} := \Pi_{L_0} \circ \Xi \circ \Pi_{L_0}^{-1}$. Then $\tilde{\Xi}$ is an involutive para-antiholomorphic isometry of $\tilde{G}/\tilde{L}_0$ because Π_{L_0} is para-holomorphic isometry. For any $p \in R$, we have

$$\tilde{\Xi}(\Pi_{L_0}(p)) = (\Pi_{L_0} \circ \Xi \circ \Pi_{L_0}^{-1})(\Pi(p)) = \Pi_{L_0}(\Xi(p)) = \Pi_{L_0}(p).$$

Thus $\Pi_{L_0}(R) \subset (\tilde{G}/\tilde{L}_0)^{\tilde{\Xi}}$. Moreover, $\Pi_{L_0}(R)$ is a para-real form of $\tilde{G}/\tilde{L}_0$ with respect to $\tilde{\Xi}$ containing $\Pi_{L_0}(p)$ for $p \in R$ because R is para-real form of G/L_0 and Π_{L_0} is a diffeomorphism. Consequently, we define a map

$$f_0 : \mathcal{R}(G) \longrightarrow \mathcal{R}(\tilde{G}) \quad (G/L_0, R) \mapsto (\tilde{G}/\tilde{L}_0, \Pi_{L_0}(R)).$$

Then the map f_0 is a surjection. Indeed, for $(\tilde{G}/\tilde{L}_0, \tilde{R}) \in \mathcal{R}(\tilde{G})$, we have $(G/L_0, \Pi_{L_0}^{-1}(\tilde{R})) \in \mathcal{R}(G)$. Thus we have $f_0(G/L_0, \Pi_{L_0}^{-1}(\tilde{R})) = (\tilde{G}/\tilde{L}_0, \tilde{R})$. For $(G/L_1, R_1), (G/L_2, R_2) \in \mathcal{R}(G)$, we prove that $(G/L_1, R_1) \simeq (G/L_2, R_2)$ if and only if $f_0(G/L_1, R_1) \simeq f_0(G/L_2, R_2)$. We assume that $(G/L_1, R_1) \overset{\Phi}{\simeq} (G/L_2, R_2)$. Then $\Pi_{L_2} \circ \Phi \circ \Pi_{L_1}^{-1}$ is a para-holomorphic homothety from $\Pi_{L_1}(G/L_1)$ onto $\Pi_{L_2}(G/L_2)$ because Π_{L_i} is a para-holomorphic isometry $(i = 1, 2)$. Moreover, we have

$$(\Pi_{L_2} \circ \Phi \circ \Pi_{L_1}^{-1})(\Pi_{L_1}(R_1)) = (\Pi_{L_2} \circ \Phi)(R_1) = \Pi_{L_2}(R_2).$$

Thus we have $f_0(G/L_1, R_1) \simeq f_0(G/L_2, R_2)$. Conversely, if $f_0(G/L_1, R_1) \overset{\Phi'}{\simeq} f_0(G/L_2, R_2)$, then $\Pi_{L_2}^{-1} \circ \Phi' \circ \Pi_{L_1}$ is a para-holomorphic homothety from G/L_1 onto G/L_2 which satisfies $(\Pi_{L_2}^{-1} \circ \Phi' \circ \Pi_{L_1})(R_1) = R_2$. Consequently, we define a map

$$F_0 : \mathcal{R}(G)/\simeq \longrightarrow \mathcal{R}(\tilde{G})/\simeq, \quad [(G/L, R)] \mapsto [f_0(G/L, R)].$$

Here the map F_0 is well-defined and a bijection. Hence (3) holds. □

In order to prove Theorem 1.1, it is enough to determine $d\mathcal{R}(\mathfrak{g})/\sim$ by Theorem 3.1 and Lemma 3.4.

4. A way to the determination of $d\mathcal{R}(\mathfrak{g})/\sim$.

4.1. Noncompact real forms of complex simple Lie algebras. In this subsection, we review the construction of a noncompact real form of a complex simple Lie algebra. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, let $\mathfrak{c}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and let Δ be the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{c}_{\mathbb{C}}$. For each $\alpha \in \Delta$, there exists the unique element $H_\alpha \in \mathfrak{c}_{\mathbb{C}}$ such that $B_{\mathfrak{g}_{\mathbb{C}}}(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{c}_{\mathbb{C}}$. There exists a basis, called a *Weyl basis*, $\{X_\alpha \mid \alpha \in \Delta\}$ of $\mathfrak{g}_{\mathbb{C}}$ mod $\mathfrak{c}_{\mathbb{C}}$ such that

(1) $[H, X_\alpha] = \alpha(H)X_\alpha$ for any $H \in \mathfrak{c}_{\mathbb{C}}$ and $\alpha \in \Delta$,
(2) $[X_\alpha, X_{-\alpha}] = H_\alpha$ for any $\alpha \in \Delta$,
(3) $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \neq 0$ and $\alpha + \beta \notin \Delta$,
(4) $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$.

Here each element of $\{N_{\alpha,\beta} \mid \alpha, \beta, \alpha + \beta \in \Delta\}$ is a nonzero real number and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ holds for any $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$. A Weyl basis gives rise to a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$\mathfrak{g}_u = \sqrt{-1}\mathfrak{c}_{\mathbb{R}} \oplus \mathrm{span}_{\mathbb{R}}\{X_\alpha - X_{-\alpha} \mid \alpha \in \Delta\} \oplus \mathrm{span}_{\mathbb{R}}\{\sqrt{-1}(X_\alpha + X_{-\alpha}) \mid \alpha \in \Delta\}.$$

Here $\mathfrak{c}_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\{H_\alpha \mid \alpha \in \Delta\}$ (cf. [3, p. 182]). Let $\theta \in \mathrm{Inv}(\mathfrak{g}_{\mathbb{C}})$ with $\theta(\mathfrak{g}_u) = \mathfrak{g}_u$. We decompose

$$\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$$

as the direct sum of $\mathfrak{k} := \mathfrak{g}_u^\theta$ and $\sqrt{-1}\mathfrak{p} := \mathfrak{g}_u^{-\theta}$. Then we get a noncompact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

It is known that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition with respect to $\theta|_{\mathfrak{g}}$.

According to the above procedure, we obtain noncompact real forms of $\mathfrak{g}_{\mathbb{C}}$. Conversely, we obtain every noncompact real form of $\mathfrak{g}_{\mathbb{C}}$ as above (cf. [3, p. 235]).

4.2. A way of the determination of $d\mathcal{R}(\mathfrak{g})/\sim$. In this subsection, we introduce a way to determine $d\mathcal{R}(\mathfrak{g})/\sim$ for every absolutely simple Lie algebra $\mathfrak{g}$.

First, we review symmetric pairs of Lie algebras. Let $\mathfrak{g}$ be a real Lie algebra and let $\xi \in \mathrm{Inv}(\mathfrak{g})$. Then we call $(\mathfrak{g}, \mathfrak{g}^\xi)$ a symmetric pair with respect to ξ. We call two symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\xi})$ and $(\mathfrak{g}', \mathfrak{g}'^{\xi'})$ are *isomorphic*, if there exists a Lie algebra isomorphism ϕ from $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $\phi \circ \xi \circ \phi^{-1} = \xi'$.

REMARK 4.1. (1) The classification of symmetric pairs was given by M. Berger in Tableau II of [1] up to isomorphism when $\mathfrak{g}$ is simple. Thus for any involution ξ of a simple Lie algebra, there exists the unique symmetric pair $(\mathfrak{g}, \mathfrak{h})$ such that $(\mathfrak{g}, \mathfrak{h})$ appears in Tableau II of [1] and that the symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\xi})$ is isomorphic to $(\mathfrak{g}, \mathfrak{h})$.

(2) Let $\mathfrak{g}$ be an absolutely simple Lie algebra and let $(Z, \xi), (Z', \xi') \in d\mathcal{R}(\mathfrak{g})$. We assume that symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\xi})$ and $(\mathfrak{g}, \mathfrak{g}^{\xi'})$ are not isomorphic each other. Then for any $\phi \in \mathrm{Aut}(\mathfrak{g})$, $\phi \circ \xi \circ \phi^{-1} \neq \xi'$. By Definition 3.3, we have $(Z, \xi) \nsim (Z', \xi')$.

Let θ be a Cartan involution of $\mathfrak{g}$ which satisfies $\xi \circ \theta = \theta \circ \xi$. We obtain the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to θ. Set $\mathfrak{h} := \mathfrak{g}^{\xi}$ and $\mathfrak{m} := \mathfrak{g}^{-\xi}$. It is clear that $\theta\xi := \theta \circ \xi$ is also an involution of $\mathfrak{g}$. Thus we obtain another direct sum decomposition $\mathfrak{g} = \mathfrak{h}^a \oplus \mathfrak{m}^a$ with respect to $\theta\xi$. Here

$$\mathfrak{h}^a = \{X \in \mathfrak{g} \mid \theta\xi(X) = X\}, \quad \mathfrak{m}^a = \{X \in \mathfrak{g} \mid \theta\xi(X) = -X\}.$$

We note the following relations:

$$\mathfrak{h}^a = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{p}), \quad \mathfrak{m}^a = (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{m} \cap \mathfrak{k}).$$

The symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ is called the *associated symmetric pair* of $(\mathfrak{g}, \mathfrak{h})$ (cf. [16, p.436]).

We review a restricted root system (cf. [16, 17]). Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{m} \cap \mathfrak{p}$ and let $\Delta(\mathfrak{g}, \mathfrak{a})$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$, we denote by $\mathfrak{g}_\alpha$ the root subspace of $\mathfrak{g}$ with respect to α. Then we have $\theta\xi(\mathfrak{g}_\alpha) \subset \mathfrak{g}_\alpha$ for any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$. Thus we obtain the direct sum decomposition $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha^+ \oplus \mathfrak{g}_\alpha^-$, where $\mathfrak{g}_\alpha^+$ (resp. $\mathfrak{g}_\alpha^-$) is the $(+1)$ (resp. (-1)) -eigenspace of $\theta\xi|_{\mathfrak{g}_\alpha}$. We note that $\mathfrak{g}_\alpha^+ = \mathfrak{h}^a \cap \mathfrak{g}_\alpha$. Set $\Delta(\mathfrak{h}^a, \mathfrak{a}) := \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid \mathfrak{g}_\alpha^+ \neq \{0\}\}$. It is clear that $\Delta(\mathfrak{h}^a, \mathfrak{a})$ is a root system. Then we call $\Delta(\mathfrak{h}^a, \mathfrak{a})$ a *restricted root system* of $\mathfrak{h}^a$ with respect to $\mathfrak{a}$.

Lemma 4.1. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra and let θ be a Cartan involution of $\mathfrak{g}$. For any $(Z_0, \xi_0) \in d\mathcal{R}(\mathfrak{g})$, there exists a pair $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ such that*

(1) $\xi \circ \theta = \theta \circ \xi$,

(2) $(Z_0, \xi_0) \sim (Z, \xi)$,

(3) $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$.

Here $\mathfrak{h} := \mathfrak{g}^{\xi}$, $\mathfrak{m} := \mathfrak{g}^{-\xi}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition with respect to θ, and $\mathfrak{h}^a := (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{p})$. In addition, $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{m} \cap \mathfrak{p}$, $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$ is a fundamental system of the restricted root system $\Delta(\mathfrak{h}^a, \mathfrak{a})$, and $W(\Delta^1(\mathfrak{h}^a, \mathfrak{a})) := \{X \in \mathfrak{a} \mid \alpha(X) \geq 0$ for all $\alpha \in \Delta^1(\mathfrak{h}^a, \mathfrak{a})\}$.

Proof. First, we construct a Cartan involution θ_0 of $\mathfrak{g}$ which satisfies

$$\theta_0 \circ \xi_0 = \xi_0 \circ \theta_0, \quad \theta_0(Z_0) = -Z_0. \tag{4.2.1}$$

Put $\sigma := \exp \sqrt{-1}\pi \,\mathrm{ad}\, Z_0 \in \mathrm{Inv}(\mathfrak{g})$. By definition of $d\mathcal{R}(\mathfrak{g})$, $\xi_0 \in \mathrm{Aut}(\mathfrak{g}, Z_0)^- \cap \mathrm{Inv}(\mathfrak{g})$. Lemma 2.1 implies that $\xi_0 \circ \sigma = \sigma \circ \xi_0$. On account of Lemma 2.7 in [13, p. 71], there exists a Cartan involution θ_0 of $\mathfrak{g}$ such that

$$\theta_0 \circ \xi_0 = \xi_0 \circ \theta_0, \quad \theta_0 \circ \sigma = \sigma \circ \theta_0.$$

Thus $\theta_0 \in \mathrm{Aut}(\mathfrak{g},\sigma)$. Since Z_0 is a semisimple element of $\mathfrak{g}$ such that every eigenvalue of $\mathrm{ad}\, Z$ on $\mathfrak{g}$ is real and by Lemma 2.1, $\theta_0(Z_0) = -Z_0$.

Next, we construct a pair $(Z_1,\xi) \in d\mathcal{R}(\mathfrak{g})$ which is equivalent to (Z_0,ξ_0). Since θ and θ_0 are Cartan involutions of $\mathfrak{g}$, there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\theta = \phi \circ \theta_0 \circ \phi^{-1}. \tag{4.2.2}$$

Put $Z_1 := \phi(Z_0)$ and

$$\xi := \phi \circ \xi_0 \circ \phi^{-1}. \tag{4.2.3}$$

Then $\xi(Z_1) = -Z_1$ and Z_1 is also a nonzero semisimple element of $\mathfrak{g}$ which satisfies the eigenvalues of $\mathrm{ad}(Z_1)$ on $\mathfrak{g}$ are $0, \pm 1$. Thus

$$(Z_1,\xi) \in d\mathcal{R}(\mathfrak{g}), \quad (Z_0,\xi_0) \overset{\phi}{\sim} (Z_1,\xi).$$

Moreover, we have $\xi \circ \theta \circ \xi^{-1} = \theta$ because

$$\begin{aligned} \xi \circ \theta \circ \xi^{-1} &\overset{(4.2.3)}{\underset{(4.2.2)}{=}} (\phi \circ \xi_0 \circ \phi^{-1}) \circ (\phi \circ \theta_0 \circ \phi^{-1}) \circ (\phi \circ \xi_0 \circ \phi^{-1})^{-1} \\ &= \phi \circ \xi_0 \circ \theta_0 \circ \xi_0^{-1} \circ \phi^{-1} \overset{(4.2.1)}{=} \phi \circ \theta_0 \circ \phi^{-1} \\ &\overset{(4.2.2)}{=} \theta. \end{aligned}$$

Hence (1) holds.

At the end of the proof, we construct an element Z which satisfies (2) and (3) with keeping ξ fixed. Let K, H, and H^a be the connected Lie subgroups of G whose Lie algebras are $\mathfrak{k}$, $\mathfrak{h}$, and $\mathfrak{h}^a$, respectively. Here G is the connected Lie group whose Lie algebra is $\mathfrak{g}$. Then the Weyl group of $\Delta(\mathfrak{h}^a,\mathfrak{a})$ coincides with $N_{H\cap K}(\mathfrak{a})/Z_{H\cap K}(\mathfrak{a}) = N_{H^a}(\mathfrak{a})/Z_{H^a}(\mathfrak{a})$, where $N_M(\mathfrak{a})$ (resp. $Z_M(\mathfrak{a})$) denotes the normalizer (resp. the centralizer) of $\mathfrak{a}$ in M (cf [17, p. 170]). Moreover, any maximal abelian subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in $\mathfrak{m} \cap \mathfrak{p}$ are conjugate under the action of $H \cap K$ (cf. [16, pp. 445-446]). In addition, we have $\theta(Z_1) = (\phi \circ \theta_0)(Z_0) = -Z_1$ because of (4.2.1). Thus there exists $k \in H \cap K$ such that $\mathrm{Ad}\, k$ transfers $Z_1 \in \mathfrak{m} \cap \mathfrak{p}$ into $W(\Delta^1(\mathfrak{h}^a,\mathfrak{a}))$ because the Weyl group acts transitively on the set of Weyl chambers. Since H is connected and $\mathfrak{h} = \mathfrak{g}^\xi$, we have $\mathrm{Ad}\, k = \xi \circ \mathrm{Ad}\, k \circ \xi^{-1}$. Therefore, by setting $Z := \mathrm{Ad}\, k(Z_1)$, we obtain the following:

$$(Z,\xi) \in d\mathcal{R}(\mathfrak{g}), \quad (Z_0,\xi_0) \overset{\phi}{\sim} (Z_1,\xi) \overset{\mathrm{Ad}\, k}{\sim} (Z,\xi), \quad Z \in W(\Delta^1(\mathfrak{h}^a,\mathfrak{a})).$$

They imply (2) and (3). Hence Lemma 4.1 holds. □

Lemma 4.2. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra and let $\mathfrak{g}_{\mathbb{C}}$ be the its complexification. Let $(Z,\xi) \in d\mathcal{R}(\mathfrak{g})$, let θ be a Cartan involution of $\mathfrak{g}$ which satisfies $\theta \circ \xi = \xi \circ \theta$ and $\theta(Z) = -Z$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. In addition, set $\mathfrak{h} := \mathfrak{g}^\xi$, $\mathfrak{k}^d := (\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})$, and $(G/C_G(Z), R) := f(Z,\xi)$.*

(1) *The symmetric pair $(\mathfrak{h}, \mathfrak{c}_{\mathfrak{h}}(Z))$ corresponds to the para-real form R.*
(2) *The symmetric pair $(\mathfrak{k}^d, \mathfrak{c}_{\mathfrak{k}^d}(\sqrt{-1}Z))$ corresponds to a real form of a compact Hermitian symmetric space $G_u/C_{G_u}(\sqrt{-1}Z)$ which corresponds to $(\mathfrak{g}_u, \mathfrak{c}_{\mathfrak{g}_u}(\sqrt{-1}Z))$, where $\mathfrak{g}_u := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. (1) By Lemma 2.3, we have $R = (G^{\hat{\xi}})_0/((G^{\hat{\xi}})_0 \cap C_G(Z))$. Here $\hat{\xi}$ is the unique automorphism of G which satisfies $\hat{\xi}(C_G(Z)) = C_G(Z)$ and $\hat{\xi}_* = \xi$ (cf. Proposition 2.2). Thus (1) holds.

(2) Let $\mathfrak{m} := \mathfrak{g}^{-\xi}$, let $\mathfrak{g}^d := (\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{m} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{m} \cap \mathfrak{p})$, and let $\mathfrak{p}^d := (\mathfrak{m} \cap \mathfrak{p}) \oplus \sqrt{-1}(\mathfrak{m} \cap \mathfrak{k})$. Then $\mathfrak{g}^d$ is a real form of $\mathfrak{g}_{\mathbb{C}}$. Let $\xi_{\mathbb{C}}$ be the complex linear extension of ξ to $\mathfrak{g}_{\mathbb{C}}$. Then we have $\xi_{\mathbb{C}}(\mathfrak{g}^d) = \mathfrak{g}^d$ and $\xi^d := \xi_{\mathbb{C}}|_{\mathfrak{g}^d} \in \mathrm{Inv}(\mathfrak{g}^d)$. Moreover, we obtain the Cartan decomposition $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ with respect to ξ^d (cf. [16, p. 435]). Then we have $\mathfrak{g}^d_u := \mathfrak{k}^d \oplus \sqrt{-1}\mathfrak{p}^d = \mathfrak{g}_u$. Since the element $Z \in \mathfrak{m} \cap \mathfrak{p}$ satisfies the set of eigenvalues of $\mathrm{ad}\, Z$ on $\mathfrak{g}$ is $\{0, \pm 1\}$, the element $\sqrt{-1}Z \in \sqrt{-1}(\mathfrak{m} \cap \mathfrak{p}) \subset \sqrt{-1}\mathfrak{p}^d$ satisfies the set of eigenvalues of $\mathrm{ad}(\sqrt{-1}Z)$ on $\mathfrak{g}_u$ is $\{0, \pm\sqrt{-1}\}$. Hence $(\mathfrak{g}_u, \mathfrak{c}_{\mathfrak{g}_u}(\sqrt{-1}Z))$ is a compact Hermitian symmetric pair and the symmetric pair $(\mathfrak{k}^d, \mathfrak{c}_{\mathfrak{k}^d}(\sqrt{-1}Z))$ corresponds to a real form of a compact Hermitian symmetric space $G_u/C_{G_u}(\sqrt{-1}Z)$ (cf. [19, pp. 294–296]). □

By Lemmas 4.1 and 4.2, we obtain the following lemma.

Lemma 4.3. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. For any real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, we obtain $d\mathcal{R}(\mathfrak{g})/\!\sim$ by the following steps.*

(Step 1) *Take a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_{\mathbb{C}}$ and an involution $\theta \in \mathrm{Inv}(\mathfrak{g}_{\mathbb{C}})$ such that $\theta(\mathfrak{g}_u) = \mathfrak{g}_u$. Set $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ (resp. $\sqrt{-1}\mathfrak{p}$) is the $(+1)$ (resp. (-1)) -eigenspace of θ in $\mathfrak{g}_u$. Here $\mathfrak{g}$ is a noncompact real form of $\mathfrak{g}_{\mathbb{C}}$ and θ is a Cartan involution of $\mathfrak{g}$ (cf.* Subsection 4.1*). If $\mathfrak{g}_{\mathbb{C}}$ is classical type, realize $\mathfrak{g}$ as a set of matricies. If $\mathfrak{g}_{\mathbb{C}}$ is exceptional type, realize $\mathfrak{g}$ by the way of* Subsection 4.1.

(Step 2) *Take an involution ξ of $\mathfrak{g}$ which satisfies $\theta \circ \xi = \xi \circ \theta$ (by* Remark 4.1 (1)*, the symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\xi})$ is isomorphic to a symmetric pair $(\mathfrak{g}, \mathfrak{h})$ which appears in Tableau II of [1]).*

(Step 3) *Decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as the direct sum of $\mathfrak{h} := \mathfrak{g}^{\xi}$ and $\mathfrak{m} := \mathfrak{g}^{-\xi}$, and set $\mathfrak{h}^a := (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{p})$. For any semisimple element $S \in \mathfrak{g}_u$ which satisfies that the set of eigenvalues of $\mathrm{ad}\, S$ on $\mathfrak{g}_u$ is $\{0, \pm\sqrt{-1}\}$, if the symmetric pair $(\mathfrak{k}^d, \mathfrak{c}_{\mathfrak{k}^d}(S))$ does not correspond to a real form of a compact Hermitian symmetric space $G_u/C_{G_u}(S)$ which corresponds to $(\mathfrak{g}_u, \mathfrak{c}_{\mathfrak{g}_u}(S))$, then $(Z, \xi) \notin d\mathcal{R}(\mathfrak{g})$ for any $Z \in \mathfrak{g}$ by* Lemma 4.2 (2)*. Here $\mathfrak{k}^d := (\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})$.*

(Step 4) *Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m} \cap \mathfrak{p}$.*

(Step 5) *Take a fundamental system $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$ of the restricted root system $\Delta(\mathfrak{h}^a, \mathfrak{a})$.*

(Step 6) *Choose all nonzero elements $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a})) := \{X \in \mathfrak{a} \mid \alpha(X) \geq 0$ for all $\alpha \in \Delta^1(\mathfrak{h}^a, \mathfrak{a})\}$ which satisfy that the eigenvalues of $\mathrm{ad}_{\mathfrak{h}^a} Z$ are contained in $\{0, \pm 1\}$.*

(Step 7) *Choose all elements Z such that the set of eigenvalues of $\mathrm{ad}\, Z$ on $\mathfrak{g}$ is $\{0, \pm 1\}$, among the elements chosen in* (Step 6).

(Step 8) *Determine all pairs which are equivalent to each other among the pairs $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ for ξ taken in* (Step 2) *and for Z chosen in* (Step 7).

Repeat from (Step 2) *to* (Step 8) *until the symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\xi})$ exhaust the pair $(\mathfrak{g}, \mathfrak{h})$ isomorphic to $(\mathfrak{g}, \mathfrak{g}^{\xi})$ in Tableau II of [1].*

Proof. By Remark 4.1 (2) and Lemmas 4.1, 4.2, Lemma 4.3 holds . □

REMARK 4.2. (1) Related to (Step 1), if $\mathfrak{g}$ is classical type (resp. exceptional type), a Cartan involution θ of $\mathfrak{g}$ is realized explicitly in [3, pp. 451-455] (resp. [12, p. 297, p. 305]).

(2) Related to (Step 2), for any involution $\xi_0 \in \mathrm{Inv}(\mathfrak{g})$, there exist an involution $\xi \in \mathrm{Inv}(\mathfrak{g})$ and $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\xi = \phi \circ \xi_0 \circ \phi^{-1}$ and that $\xi \circ \theta = \theta \circ \xi$, where θ is the Cartan involution which fixed in (Step 1) by Lemma 4.1. Here symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\xi_0})$ and $(\mathfrak{g}, \mathfrak{g}^{\xi})$ are isomorphic each other.

(3) Related to (Step 4) and (Step 5), we can take an arbitrary maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m} \cap \mathfrak{p}$ and an arbitrary fundamental system $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$ of $\Delta(\mathfrak{h}^a, \mathfrak{a})$ by the proof of Lemma 4.1.

4.3. Lemmas related to Lemma 4.3. The following lemmas enable us to carry out (Step 6) and (Step 7) in Lemma 4.3 in systematic ways.

Lemma 4.4. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra. We assume that* (Step 1)-(Step 5) *in* Lemma 4.3 *have been achieved. If an element $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$ satisfies that the eigenvalues of* $\mathrm{ad}_{\mathfrak{h}^a} Z$ *are* 0 *or* ± 1, *then one of the following cases holds:*

(1) *Any eigenvalue of* $\mathrm{ad}_{\mathfrak{h}^a} Z$ *is* 0.

(2) *The eigenvalues of* $\mathrm{ad}_{\mathfrak{h}^a} Z$ *are* $0, \pm 1$.

In addition, the eigenvalue of $\mathrm{ad}_{\mathfrak{h}^a} Z$ *is only* 0 *if and only if* $Z \in \mathfrak{z}(\mathfrak{h}^a) \cap \mathfrak{a}$.

Proof. Related to the eigenvalues of $\mathrm{ad}_{\mathfrak{h}^a} Z$, the possible cases are

(i) Only 0, (ii) ± 1, (iii) $0, 1$, (iv) $0, -1$, (v) $0, \pm 1$.

Case (i): It is clear that (i) holds if and only if $Z \in \mathfrak{z}(\mathfrak{h}^a) \cap \mathfrak{a}$.

Case (ii): Since $\mathrm{ad}_{\mathfrak{h}^a} Z(Z) = 0$ and $Z \neq 0$, the case does not occur.

Case (iii): For an arbitrary $\alpha \in \Delta(\mathfrak{h}^a, \mathfrak{a})$, $\alpha(Z)$ is one of the eigenvalues of $\mathrm{ad}_{\mathfrak{h}^a} Z$. Thus $\alpha(Z) = 0$ because $-\alpha \in \Delta(\mathfrak{h}^a, \mathfrak{a})$ and by the assumption of (iii). In particular, for every $\beta \in \Delta^1(\mathfrak{h}^a, \mathfrak{a})$, $\beta(Z) = 0$. Hence, the case does not occur.

Case (iv): We can prove that the case does not occur by the similar way in Case (iii). □

Lemma 4.5. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra. We assume that* (Step 1)-(Step 5) *in* Lemma 4.3 *have been achieved and $\Delta(\mathfrak{h}^a, \mathfrak{a})$ is irreducible. Let γ be the highest root of $\Delta(\mathfrak{h}^a, \mathfrak{a})$ with respect to $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$. If an element $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$ satisfies that the set of eigenvalues of* $\mathrm{ad}_{\mathfrak{h}^a} Z$ *is* $\{0, \pm 1\}$, *then*

$$\gamma(Z) = 1.$$

Proof. By the assumption, for an arbitrary $\alpha \in \Delta(\mathfrak{h}^a, \mathfrak{a})$, $\alpha(Z) = 0$ or ± 1. Thus $\gamma(Z) = 1$ because γ is the highest root and $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$. □

Corollary 4.1. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra. We assume that* (Step 1)-(Step 5) *in* Lemma 4.3 *have been achieved and $\Delta(\mathfrak{h}^a, \mathfrak{a})$ is irreducible. Set $\Delta^1(\mathfrak{h}^a, \mathfrak{a}) = \{\alpha_1, \ldots, \alpha_l\}$. Let $\{Z_1, \ldots, Z_l\}$ be the dual basis of $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$ and let γ be the highest root of $\Delta(\mathfrak{h}^a, \mathfrak{a})$. If an element $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$ satisfies that the set of eigenvalues of* $\mathrm{ad}_{\mathfrak{h}^a} Z$ *is* $\{0, \pm 1\}$, *then there exist $Z_i \in \{Z_1, \ldots, Z_l\}$ satisfying $\gamma(Z_i) = 1$ and $C \in \mathfrak{z}(\mathfrak{h}^a) \cap \mathfrak{a}$ such that*

$$Z = Z_i + C.$$

Proof. Since $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$ is a fundamental system of $\Delta(\mathfrak{h}^a, \mathfrak{a})$,

$$\gamma = \sum_{i=1}^{l} \mu_i \alpha_i \quad (\mu_i \in \mathbb{Z},\ \mu_i \geq 1,\ 1 \leq i \leq l).$$

We have

$$Z = \sum_{i=1}^{l} \lambda_i Z_i + C \quad (C \in \mathfrak{z}(\mathfrak{h}^a) \cap \mathfrak{a},\ \lambda_i \in \mathbb{R},\ \lambda_i \geq 0,\ 1 \leq i \leq l).$$

Since $\{Z_1, \dots, Z_l\}$ is the dual basis of $\Delta^1(\mathfrak{h}^a, \mathfrak{a})$, we have

$$1 = \gamma(Z) = \sum_{i=1}^{l} \mu_i \lambda_i$$

by Lemma 4.5. Then $\lambda_i = \alpha_i(Z)$ is the eigenvalue of $\mathrm{ad}_{\mathfrak{h}^a} Z$ $(1 \leq i \leq l)$. Thus $\lambda_i = 0$ or 1 because $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$. Thus there exists $1 \leq i \leq l$ such that $\lambda_i = 1$ and that $\lambda_k = 0$, if $k \neq i$. Hence Corollary 4.1 holds. □

We consider the case $\Delta(\mathfrak{h}^a, \mathfrak{a})$ is not irreducible. As a result, it turns out that it is enough to consider the case where $\Delta(\mathfrak{h}^a, \mathfrak{a})$ is the disjoint union of two irreducible root systems by Section 5. We assume that $\Delta(\mathfrak{h}^a, \mathfrak{a})$ is decomposed as the disjoint union of two irreducible root systems Δ_1 and Δ_2. For $i = 1, 2$, let $\Delta_i^1 := \{\alpha_1^i, \dots, \alpha_{k_i}^i\}$ be a fundamental system of Δ_i, let $\{Z_1^i, \dots, Z_{k_i}^i\}$ be the dual basis of Δ_i^1, and let γ_i be the highest root of Δ_i with respect to Δ_i^1. Under these notations, we obtain the following.

Corollary 4.2. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra. If an element $Z \in W(\Delta^1(\mathfrak{h}^a, \mathfrak{a}))$ satisfies that the set of eigenvalues of* $\mathrm{ad}_{\mathfrak{h}^a} Z$ *is* $\{0, \pm 1\}$, *then*

(1) $\gamma_i(Z) = 0$ *or* 1 *and* $\gamma_1(Z) + \gamma_2(Z) \neq 0$,
(2) *there exist* $Z_n^1 \in \{Z_1^1, \dots, Z_{k_1}^1\}$ *satisfying* $\gamma_1(Z_n^1) = 1$, $Z_m^2 \in \{Z_1^2, \dots, Z_{k_2}^2\}$ *satisfying* $\gamma_2(Z_m^2) = 1$, *and* $C \in \mathfrak{z}(\mathfrak{h}^a) \cap \mathfrak{a}$ *such that*

$$Z = \lambda Z_n^1 + \mu Z_m^2 + C,$$

where $\lambda, \mu = 0$ *or* 1 *and* $\lambda + \mu \neq 0$.

Proof. We can prove it by the similar ways to the proofs of Lemma 4.5 and Corollary 4.1. □

Lemma 4.6. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra. We assume that* (Step 1)-(Step 6) *in* Lemma 4.3 *have been achieved. The following* (1), (2) *and* (3) *are equivalent:*

(1) *The set of eigenvalues of* $\mathrm{ad}\, Z$ *on* $\mathfrak{g}$ *is* $\{0, \pm 1\}$.
(2) $(\mathrm{ad}_{[Z,\mathfrak{g}]} Z)^2 = \mathrm{id}$.
(3) $(\mathrm{ad}\, Z)^3 = \mathrm{ad}\, Z$.

Proof. Since $Z \in \mathfrak{a} \subset \mathfrak{p}$, Z is a semisimple element of $\mathfrak{g}$. Thus we can decompose $\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(Z) \oplus [Z, \mathfrak{g}]$ as a direct sum. Hence (1) holds if and only if (2) holds. It is clear that (1) holds if and only if (3) holds. □

In order to carry out (Step 8) in Lemma 4.3, Lemma 4.2 and the following lemma are useful.

DEFINITION 4.1 (CF. [6, P. 88]). Let $(\mathfrak{g}, \mathfrak{l})$ be a symmetric pair with respect to σ and let $\mathfrak{u} := \mathfrak{g}^{-\sigma}$. Then $(\mathfrak{g}, \mathfrak{l})$ is called *para-Hermitian*, if there exist a linear endomorphism I of $\mathfrak{u}$ and a non-degenerate symmetric bilinear form $\langle , \rangle$ on $\mathfrak{u}$ such that

(1) $I^2 = \mathrm{id}$, $I \neq \mathrm{id}$,
(2) $[I, \mathrm{ad}_{\mathfrak{u}}\, \mathfrak{l}] = 0$,
(3) $\langle IX, Y\rangle + \langle X, IY\rangle = 0$ for any $X, Y \in \mathfrak{u}$,
(4) $\langle \mathrm{ad}\, X(Y_1), Y_2\rangle + \langle Y_1, \mathrm{ad}\, X(Y_2)\rangle = 0$ for any $X \in \mathfrak{l}$, $Y_1, Y_2 \in \mathfrak{u}$.

Lemma 4.7. *Let* $\mathfrak{g}$ *be an absolutely simple Lie algebra. We assume that* (Step 1)-(Step 7) *in* Lemma 4.3 *have been achieved.*

(1) *The symmetric pair* $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z))$ *is para-Hermitian.*
(2) *The symmetric pair* $(\mathfrak{h}^a, \mathfrak{c}_{\mathfrak{h}^a}(Z))$ *is also para-Hermitian, if the element* Z *is not contained in* $\mathfrak{z}(\mathfrak{h}^a)$.
(3) *The pair* $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}^a}(Z))$ *is the associated symmetric pair of* $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}}(Z))$.

Proof. (1) By the assumption and Lemma 2.1 in [4, p. 477], $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z))$ is para-Hermitian.

(2) Put $\sigma := \exp \mathrm{ad}_{\mathfrak{h}^a} \sqrt{-1}\pi Z$. Then $(\mathfrak{h}^a, \mathfrak{c}_{\mathfrak{h}^a}(Z))$ is a symmetric pair with respect to σ, if Z is not contained $\mathfrak{z}(\mathfrak{h}^a)$. Set $\mathfrak{l}^a := (\mathfrak{h}^a)^{\sigma} = \mathfrak{c}_{\mathfrak{h}^a}(Z)$ and $\mathfrak{u}^a := (\mathfrak{h}^a)^{-\sigma}$. By settings $I := \mathrm{ad}_{\mathfrak{u}^a} Z$ and $\langle , \rangle := B_{\mathfrak{g}}|_{\mathfrak{u}^a \times \mathfrak{u}^a}$, $(\mathfrak{h}^a, \mathfrak{c}_{\mathfrak{h}^a}(Z))$ becomes a para-Hermitian symmetric pair with respect to I and $\langle , \rangle$.

(3) We obtain the equalities $\mathfrak{c}_{\mathfrak{g}}(Z) = \mathfrak{g}^{\sigma}$, $\mathfrak{c}_{\mathfrak{h}}(Z) = \mathfrak{g}^{\xi} \cap \mathfrak{g}^{\sigma}$, and $\mathfrak{c}_{\mathfrak{h}^a}(Z) = \mathfrak{g}^{\theta\xi} \cap \mathfrak{g}^{\sigma}$. Thus $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}}(Z))$ and $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}^a}(Z))$ are symmetric pairs with respect to $\theta\xi$ and ξ, respectively. It is clear that $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}^a}(Z))$ is the associated symmetric pair of $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}}(Z))$. □

5. The determination of $d\mathcal{R}(\mathfrak{g})/\sim$.

In this section, we determine para-real forms of ASPH's of hyperbolic orbit type. Let $\mathfrak{g}$ be an absolutely simple Lie algebra. According to the classification of simple para-Hermitian symmetric pairs in [6, p. 97], we consider the case where $\mathfrak{g}$ is one of the following:

List 1

Type	$\mathfrak{g}$	Condition
AI	$\mathfrak{sl}(n, \mathbb{R})$	$2 \leq n$
AII	$\mathfrak{su}^*(2n)$	$3 \leq n$
AIII	$\mathfrak{su}(n, n)$	$3 \leq n$
BDI	$\mathfrak{so}(p, q)$	$1 \leq p \leq q,\ p + q \neq 2$
DIII	$\mathfrak{so}^*(4n)$	$3 \leq n$
CI	$\mathfrak{sp}(n, \mathbb{R})$	$3 \leq n$
CII	$\mathfrak{sp}(n, n)$	$2 \leq n$
EI	$\mathfrak{e}_{6(6)}$	—
EIV	$\mathfrak{e}_{6(-26)}$	—
EV	$\mathfrak{e}_{7(7)}$	—
EVII	$\mathfrak{e}_{7(-25)}$	—

We setup the following notations:

$M_{(p,q)}(\mathbb{K})$: the set of all matrices of order $p \times q$ over $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$),

E_l : the identity matrix of order l,

$$A \times B := \begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad A \nearrow B := \begin{pmatrix} O & B \\ A & O \end{pmatrix} \text{for matrices } A, B,$$

$$I_{p,q} := -E_p \times E_q, \quad J_l := -E_l \nearrow E_l.$$

5.1. An example of classical type. We consider the case of Type AIII: $\mathfrak{su}(n,n)$.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra $\mathfrak{sl}(2n,\mathbb{C})$ and let $\mathfrak{g}_u := \mathfrak{su}(2n)$, a compact real form of $\mathfrak{g}_{\mathbb{C}}$. Put $\theta := \operatorname{Ad} I_{n,n}$. Then we obtain the noncompact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ {}^t\bar{X}_2 & X_3 \end{pmatrix} \middle| \begin{array}{c} X_l \in \mathfrak{gl}(n,\mathbb{C}) \text{ for } l = 1,2,3, \operatorname{Tr}(X_1 + X_3) = 0, \\ X_1, X_3 : \text{skew Hermitian} \end{array} \right\} = \mathfrak{su}(n,n).$$

Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} := \mathfrak{g}_u^{\theta}$ is the set of matrices $X_1 \times X_3$ in $\mathfrak{g}$, which is isomorphic to $\mathfrak{su}(n) \oplus \mathfrak{su}(n) \oplus \sqrt{-1}\mathbb{R}$, and $\mathfrak{p} := \sqrt{-1}\mathfrak{g}_u^{-\theta}$ is the set of matrices ${}^t\bar{X}_2 \nearrow X_2$ in $\mathfrak{g}$.

We take an involution ξ of $\mathfrak{g}$ which satisfies $\theta \circ \xi = \xi \circ \theta$. According to Tableau II of [1], ξ is conjugate to one of the following:

$\xi_0 := \theta, \quad \xi_1 : X \longmapsto -{}^tX \ (X \in \mathfrak{g}), \quad \xi_2 := \operatorname{Ad} J_n, \quad \xi_3 := \xi_1 \circ \xi_2, \quad \xi_4 := \theta \circ \xi_3,$

$\xi_5 := \operatorname{Ad}(I_{p,n-p} \times I_{q,n-q}) \ (1 \leq p,q \leq n-1),$

$\xi_6 := \xi_1 \circ \operatorname{Ad}(J_k \times J_k) \circ \xi_1$ (when $n = 2k$, $2 \leq k$).

We decompose $\mathfrak{g} = \mathfrak{h}_l \oplus \mathfrak{m}_l$ where $\mathfrak{h}_l := \mathfrak{g}^{\xi_l}$ and $\mathfrak{m}_l := \mathfrak{g}^{-\xi_l}$ for ξ_l ($0 \leq l \leq 6$), and then

$$\mathfrak{h}_0 = \mathfrak{k} \cong \mathfrak{su}(n) \oplus \mathfrak{su}(n) \oplus \sqrt{-1}\mathbb{R},$$

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} X_1 & \sqrt{-1}X_2 \\ \sqrt{-1}\,{}^tX_2 & X_3 \end{pmatrix} \middle| \begin{array}{c} X_l \in \mathfrak{gl}(n,\mathbb{R}) \text{ for } l = 1,2,3, \\ X_1, X_3 : \text{skew symmetric} \end{array} \right\} \cong \mathfrak{so}(n,n),$$

$$\mathfrak{h}_2 = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n,\mathbb{C}),\ \operatorname{Tr}(X_1) = 0, \\ X_1, X_2 : \text{skew Hermitian} \end{array} \right\} \cong \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R},$$

$$\mathfrak{h}_3 = \left\{ \begin{pmatrix} X_1 & X_2 \\ \bar{X}_2 & \bar{X}_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n,\mathbb{C}),\ \operatorname{Tr}(X_1) = 0, \\ X_1 : \text{skew Hermitian},\ X_2 : \text{symmetric} \end{array} \right\} \cong \mathfrak{sp}(n,\mathbb{R}),$$

$$\mathfrak{h}_4 = \left\{ \begin{pmatrix} X_1 & X_2 \\ -\bar{X}_2 & \bar{X}_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n,\mathbb{C}), \operatorname{Tr}(X_1) = 0, \\ X_1 : \text{skew Hermitian},\ X_2 : \text{skew symmetric} \end{array} \right\} \cong \mathfrak{so}^*(2n),$$

$$\mathfrak{h}_5 = \left\{ \begin{pmatrix} X_{11} & O & X_{13} & O \\ O & X_{22} & O & X_{24} \\ {}^t\bar{X}_{13} & O & X_{33} & O \\ O & {}^t\bar{X}_{24} & O & X_{44} \end{pmatrix} \middle| \begin{array}{c} X_{11} \in \mathfrak{gl}(p,\mathbb{C}),\ X_{22} \in \mathfrak{gl}(n-p,\mathbb{C}), \\ X_{33} \in \mathfrak{gl}(q,\mathbb{C}),\ X_{44} \in \mathfrak{gl}(n-q,\mathbb{C}), \\ X_{13} \in M_{(p,q)}(\mathbb{C}),\ X_{24} \in M_{(n-p,n-q)}(\mathbb{C}), \\ \operatorname{Tr}(X_{11}) + \operatorname{Tr}(X_{22}) + \operatorname{Tr}(X_{33}) + \operatorname{Tr}(X_{44}) = 0, \\ X_{ll} : \text{skew Hermitian for } 1 \leq l \leq 4 \end{array} \right\}$$

$$\cong \mathfrak{su}(p,q) \oplus \mathfrak{su}(n-p,n-q) \oplus \sqrt{-1}\mathbb{R},$$

$$\mathfrak{h}_6 = \left\{ \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ -\bar{X}_{12} & -{}^t X_{11} & \bar{X}_{14} & -\bar{X}_{13} \\ {}^t\bar{X}_{13} & {}^t X_{14} & X_{33} & X_{34} \\ {}^t\bar{X}_{14} & -{}^t X_{13} & -\bar{X}_{34} & -{}^t X_{33} \end{pmatrix} \middle| \begin{array}{c} X_{lm} \in \mathfrak{gl}(k, \mathbb{C}) \text{ for } 1 \leq l, m \leq 4, \\ X_{11}, X_{33} : \text{skew Hermitian} \\ X_{12}, X_{34} : \text{symmetric} \end{array} \right\} \cong \mathfrak{sp}(k, k).$$

The case of $\xi = \xi_1$: $\mathfrak{h}_1^a := (\mathfrak{h}_1 \cap \mathfrak{k}) \oplus (\mathfrak{m}_1 \cap \mathfrak{p})$ is given by

$$\mathfrak{h}_1^a = \left\{ \begin{pmatrix} X_1 & X_2 \\ {}^t X_2 & X_3 \end{pmatrix} \middle| \begin{array}{c} X_l \in \mathfrak{gl}(n, \mathbb{R}) \text{ for } l = 1, 2, 3, \\ X_1, X_3 : \text{skew symmetric} \end{array} \right\} \cong \mathfrak{so}(n, n).$$

We take a maximal abelian subspace $\mathfrak{a}_1$ in

$$\mathfrak{m}_1 \cap \mathfrak{p} = \left\{ \begin{pmatrix} O & X_2 \\ {}^t X_2 & O \end{pmatrix} \middle| \; X_2 \in \mathfrak{gl}(n, \mathbb{R}) \right\}$$

as

$$\mathfrak{a}_1 = \left\{ A = \begin{pmatrix} O & \mathrm{diag}(a_1, \ldots, a_n) \\ \mathrm{diag}(a_1, \ldots, a_n) & O \end{pmatrix} \middle| \; a_1, \ldots a_n \in \mathbb{R} \right\},$$

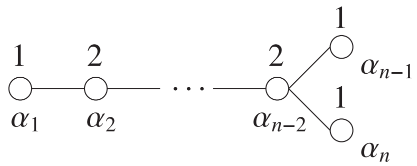
where $\mathrm{diag}(a_1, \ldots, a_n)$ denotes the diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Here we note that $\mathfrak{z}(\mathfrak{h}_1^a) \cap \mathfrak{a}_1 = \{0\}$. Then the restricted root system $\Delta(\mathfrak{h}_1^a, \mathfrak{a}_1)$ is

$$\Delta(\mathfrak{h}_1^a, \mathfrak{a}_1) = \{\pm(e_l - e_m), \; \pm(e_l + e_m) \mid 1 \leq l < m \leq n\}.$$

Here $e_l : \mathfrak{a}_1 \longrightarrow \mathbb{R}$ $(1 \leq l \leq n)$ is a linear map defined by

$$e_l(A) = a_l, \; A = \begin{pmatrix} O & \mathrm{diag}(a_1, \ldots, a_n) \\ \mathrm{diag}(a_1, \ldots, a_n) & O \end{pmatrix} \in \mathfrak{a}_1.$$

Set $\alpha_l := e_l - e_{l+1}$ for $1 \leq l \leq n-1$ and $\alpha_n := e_{n-1} + e_n$. Then $\Delta^1(\mathfrak{h}_1^a, \mathfrak{a}_1) := \{\alpha_1, \ldots, \alpha_n\}$ is a fundamental system of $\Delta(\mathfrak{h}_1^a, \mathfrak{a}_1)$. The Dynkin diagram of $\Delta^1(\mathfrak{h}_1^a, \mathfrak{a}_1)$ with the coefficients of the highest root is:



$: D_n.$

Let $\{Z_1, \ldots, Z_n\}$ be the dual basis of $\Delta^1(\mathfrak{h}_1^a, \mathfrak{a}_1)$. By Corollary 4.1, the elements satisfying the property in (Step 6) in Lemma 4.3 are only Z_1, Z_{n-1}, and Z_n. Here

$$Z_1 = \begin{pmatrix} O & \mathrm{diag}(1, 0, \ldots, 0) \\ \mathrm{diag}(1, 0, \ldots, 0) & O \end{pmatrix}, \quad Z_{n-1} = \frac{1}{2}(-I_{n-1,1} \nearrow -I_{n-1,1}),$$

$$Z_n = \frac{1}{2}(E_n \nearrow E_n).$$

Moreover, by Lemma 4.6, the set of eigenvalues of $\mathrm{ad}\, Z_l$ on $\mathfrak{g}$ is $\{0, \pm 1\}$ for $l = n-1, n$. However, the set of eigenvalues of $\mathrm{ad}\, Z_1$ on $\mathfrak{g}$ is not $\{0, \pm 1\}$. Indeed, they include 2. In fact, the restricted root system $\Delta(\mathfrak{g}, \mathfrak{a}_1)$ is Type C_n and we obtain a fundamental system $\Delta^1(\mathfrak{g}, \mathfrak{a}_1) = \{\alpha_1, \ldots \alpha_{n-1}, \alpha_n'\}$, where $\alpha_n' := 2e_n$. Then the Dynkin diagram of $\Delta^1(\mathfrak{g}, \mathfrak{a}_1)$ with the coefficients of the highest root is:

$: C_n.$

Let γ be the highest root of $\Delta(\mathfrak{g}, \mathfrak{a}_1)$ and $X_\gamma \in \mathfrak{g}_\gamma := \{X \in \mathfrak{g} \mid \operatorname{ad} A(X) = \gamma(A)X \text{ for } {}^\forall A \in \mathfrak{a}_1\}$. Then $\operatorname{ad} Z_1(X_\gamma) = \gamma(Z_1)X_\gamma = 2X_\gamma$.

We define a map $\phi \in \operatorname{Aut}(\mathfrak{g})$ by $\phi := \operatorname{Ad} I_{2n-1,1}$. It is clear that $\xi_1 \circ \phi = \phi \circ \xi_1$ and $\phi(Z_{n-1}) = Z_n$. Hence $(Z_{n-1}, \xi_1) \sim (Z_n, \xi_1)$. In addition, we have

$$\mathfrak{c}_{\mathfrak{g}}(Z_n) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & X_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n, \mathbb{C}),\ \operatorname{Tr}(X_1) = 0, \\ X_1 : \text{skew Hermitian},\ X_2 : \text{Hermitian} \end{array} \right\} \cong \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R},$$

$$\mathfrak{c}_{\mathfrak{h}_1}(Z_n) = \left\{ \begin{pmatrix} X_1 & \sqrt{-1}X_2 \\ \sqrt{-1}X_2 & X_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n, \mathbb{R}), \\ X_1, X_2 : \text{skew symmetric} \end{array} \right\} \cong \mathfrak{so}(n, \mathbb{C}).$$

Hence we obtain the following proposition:

Proposition 5.1. *In the case of $\xi = \xi_1$, each elements $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ are equivalent to (Z_n, ξ). Here $Z_n = (1/2)(E_n \nearrow E_n)$. Moreover, we get the following equalities:*

$$(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z_n)) = (\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}),$$
$$(\mathfrak{h}_1, \mathfrak{c}_{\mathfrak{h}_1}(Z_n)) = (\mathfrak{so}(n, n), \mathfrak{so}(n, \mathbb{C})).$$

The case of $\xi = \xi_2$: We obtain $\mathfrak{h}_2^a := (\mathfrak{h}_2 \cap \mathfrak{k}) \oplus (\mathfrak{m}_2 \cap \mathfrak{p})$ as

$$\mathfrak{h}_2^a = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & X_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n, \mathbb{C}),\ \operatorname{Tr}(X_1) = 0, \\ X_1 : \text{skew Hermitian},\ X_2 : \text{Hermitian} \end{array} \right\} \cong \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}.$$

We take a maximal abelian subspace $\mathfrak{a}_2$ in

$$\mathfrak{m}_2 \cap \mathfrak{p} = \left\{ \begin{pmatrix} O & X_2 \\ X_2 & O \end{pmatrix} \middle| \begin{array}{c} X_2 \in \mathfrak{gl}(n, \mathbb{C}), \\ X_2 : \text{Hermitian} \end{array} \right\}$$

as

$$\mathfrak{a}_2 = \left\{ A = \begin{pmatrix} O & \operatorname{diag}(a_1, \ldots, a_n) \\ \operatorname{diag}(a_1, \ldots, a_n) & O \end{pmatrix} \middle| a_1, \ldots a_n \in \mathbb{R} \right\}.$$

Here, we note that $\mathfrak{z}(\mathfrak{h}_2^a) \cap \mathfrak{a}_2 = \mathbb{R}(E_n \nearrow E_n)$. Then the restricted root system $\Delta(\mathfrak{h}_2^a, \mathfrak{a}_2)$ is given by

$$\Delta(\mathfrak{h}_2^a, \mathfrak{a}_2) = \{\pm(e_l - e_m) \mid 1 \leq l < m \leq n\}.$$

Set $\alpha_l := e_l - e_{l+1}$ for $1 \leq l \leq n-1$. Then we obtain a fundamental system $\Delta^1(\mathfrak{h}_2^a, \mathfrak{a}_2) = \{\alpha_1, \ldots, \alpha_{n-1}\}$. The Dynkin diagram of $\Delta^1(\mathfrak{h}_2^a, \mathfrak{a}_2)$ with the coefficients of the highest root is:

$: A_{n-1}.$

Then the dual basis $\{Z_1, \ldots, Z_{n-1}\}$ of $\Delta^1(\mathfrak{h}_2^a, \mathfrak{a}_2)$ is given by

$$Z_l = \frac{1}{n}(((n-l)E_l \times -lE_{n-l}) \nearrow ((n-l)E_l \times -lE_{n-l})),$$

for $1 \leq l \leq n-1$. By Corollary 4.1, the elements satisfying the property in (Step 6) in Lemma 4.3 are only $\lambda_0 C$ and $Z_l + \lambda_l C$ $(1 \leq l \leq n-1)$. Here $C := E_n \nearrow E_n \in \mathfrak{z}(\mathfrak{h}_2^a) \cap \mathfrak{a}_2$ and $\lambda_m \in \mathbb{R}$ $(0 \leq m \leq n-1)$ are arbitrary. By the similar way to the case of $\xi = \xi_1$, it turns out that $\lambda_0 C$ and $Z_l + \lambda_l C$ are satisfy the property in (Step 7) in Lemma 4.3 when $\lambda_0 = \pm 1/2$ and $\lambda_l = (-n+2l)/2n$ for $1 \leq l \leq n-1$. Then we get the following equalities

$$\begin{aligned}
\mathfrak{c}_{\mathfrak{g}}(\lambda_0 C) &= \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & X_1 \end{pmatrix} \middle| \begin{array}{c} X_1, X_2 \in \mathfrak{gl}(n,\mathbb{C}), \\ X_1 : \text{skew Hermitian},\ X_2 : \text{Hermitian} \end{array} \right\} \cong \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}, \\
&\quad \lambda_0 = \pm\frac{1}{2}, \\
\mathfrak{c}_{\mathfrak{g}}(Z_l + \lambda_l C) &= \left\{ \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ -{}^t\bar{X}_{12} & X_{22} & -{}^t\bar{X}_{14} & X_{24} \\ X_{13} & -X_{14} & X_{11} & -X_{12} \\ {}^t\bar{X}_{14} & X_{24} & {}^t\bar{X}_{12} & X_{22} \end{pmatrix} \middle| \begin{array}{c} X_{11}, X_{13} \in \mathfrak{gl}(l,\mathbb{C}), \\ X_{22}, X_{24} \in \mathfrak{gl}(n-l,\mathbb{C}), \\ X_{12}, X_{14} \in M_{(l,n-l)}(\mathbb{C}), \\ X_{11}, X_{22} : \text{skew Hermitian}, \\ X_{13}, X_{24} : \text{Hermitian} \end{array} \right\} \\
&\cong \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}, \\
\mathfrak{c}_{\mathfrak{h}_2}(\lambda_0 C) &= \left\{ \begin{pmatrix} X_1 & O \\ O & X_1 \end{pmatrix} \middle| \begin{array}{c} X_1 \in \mathfrak{gl}(n,\mathbb{C}), \\ X_1 : \text{skew Hermitian} \end{array} \right\} \cong \mathfrak{su}(n),\ \lambda_0 = \pm\frac{1}{2}, \\
\mathfrak{c}_{\mathfrak{h}_2}(Z_l + \lambda_l C) &= \left\{ \begin{pmatrix} X_{11} & O & O & X_{14} \\ O & X_{22} & -{}^t\bar{X}_{14} & O \\ O & -X_{14} & X_{11} & O \\ {}^t\bar{X}_{14} & O & O & X_{22} \end{pmatrix} \middle| \begin{array}{c} X_{11} \in \mathfrak{gl}(l,\mathbb{C}),\ X_{22} \in \mathfrak{gl}(n-l,\mathbb{C}), \\ X_{14} \in M_{(l,n-l)}(\mathbb{C}), \\ X_{11}, X_{22} : \text{skew Hermitian} \end{array} \right\} \\
&\cong \mathfrak{su}(l, n-l).
\end{aligned}$$

By Definition 3.3, if $(\mathfrak{h}_2, \mathfrak{c}_{\mathfrak{h}_2}(Z_l + \lambda_l C))$ and $(\mathfrak{h}_2, \mathfrak{c}_{\mathfrak{h}_2}(Z_k + \lambda_k C))$ are not isomorphic, then $(Z_l + \lambda_l C, \xi_2)$ and $(Z_k + \lambda_k C, \xi_2)$ are not equivalent to $(1 \leq l, k \leq n-1)$. Hence we obtain the following proposition:

Proposition 5.2. *In the case of $\xi = \xi_2$, each element $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ is equivalent to one of the following:*

$$(\lambda_0 C, \xi),\ (Z_1 + \lambda_1 C, \xi), \ldots, (Z_k + \lambda_k C, \xi).$$

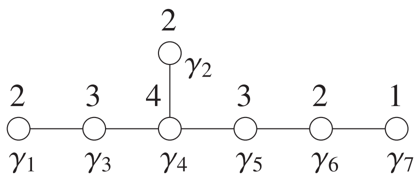
Here k is the largest integer which dose not exceed $n/2+1$, $Z_l = (1/n)(((n-l)E_l \times -lE_{n-l}) \nearrow ((n-l)E_l \times -lE_{n-l}))$, $C = E_n \nearrow E_n$, $\lambda_0 = \pm 1/2$, and $\lambda_l = (-n+2l)/2n$ for $1 \leq l \leq k$. Therefore, $Z_l + \lambda_l C = (1/2)(-I_{l,n-l} \nearrow -I_{l,n-l})$ $(1 \leq l \leq k)$. Moreover, we get the following equalities:

$$(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(\lambda_0 C)) = (\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}),$$
$$(\mathfrak{h}_2, \mathfrak{c}_{\mathfrak{h}_2}(\lambda_0 C)) = (\mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}, \mathfrak{su}(n)),$$
$$(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z_l + \lambda_l C)) = (\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}) \quad (1 \le l \le k),$$
$$(\mathfrak{h}_2, \mathfrak{c}_{\mathfrak{h}_2}(Z_l + \lambda_l C)) = (\mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}, \mathfrak{su}(l, n-l)) \quad (1 \le l \le k).$$

For the other cases, (Z, ξ) are determined in a similar way.

5.2. An example of exceptional type. We consider the case of Type EV : $\mathfrak{e}_{7(7)}$.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra $(\mathfrak{e}_7)_{\mathbb{C}}$ and let $\mathfrak{c}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then we have a fundamental system $\Delta^1(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}}) := \{\gamma_1, \dots, \gamma_7\}$ of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ and the Dynkin diagram of $\Delta^1(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ with the coefficients of the highest root is:



$: E_7.$

Let $\{X_\alpha\}_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})}$ be a Weyl basis of $\mathfrak{g}_{\mathbb{C}}$ mod $\mathfrak{c}_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{H_{\gamma_1}, \dots, H_{\gamma_7}\}$, let $\mathfrak{g}_u$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$, and let $\{T_1, \dots, T_7\}$ be the dual basis of $\Delta^1(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$. In addition, we define linear maps ${}^t\tilde{\theta}_1 : \mathfrak{c}_{\mathbb{C}}^* \to \mathfrak{c}_{\mathbb{C}}^*$ and $\tilde{\theta}_2 : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ by

$${}^t\tilde{\theta}_1(\gamma_l) := -\gamma_l \quad (1 \le l \le 7), \quad \tilde{\theta}_2 := \exp \sqrt{-1}\pi \,\mathrm{ad}\, T_2.$$

Then $\tilde{\theta}_2 \in \mathrm{Inv}(\mathfrak{g}_{\mathbb{C}})$. Let $\tilde{\theta}_1$ be an involution of $\mathfrak{g}_{\mathbb{C}}$ induced by ${}^t\tilde{\theta}_1$. Then $\tilde{\theta}_1(\mathfrak{g}_u) = \mathfrak{g}_u$ and $\tilde{\theta}_2(\mathfrak{g}_u) = \mathfrak{g}_u$. Thus we obtain noncompact real forms $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ of $\mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{k}_i := \mathfrak{g}_u^{\tilde{\theta}_i}$ and $\mathfrak{p}_i := \sqrt{-1}\mathfrak{g}_u^{-\tilde{\theta}_i}$ for $i = 1, 2$. Put $\theta_i := \tilde{\theta}_i|_{\mathfrak{g}_i}$ for $i = 1, 2$. It turns out that $\mathfrak{k}_i$ is isomorphic to $\mathfrak{su}(8)$ and $\mathfrak{g}_i$ is isomorphic to $\mathfrak{e}_{7(7)}$ for $i = 1, 2$. Thus we describe these real forms $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}_{\mathbb{C}}$ as the same symbol $\mathfrak{g}$. Therefore, it is possible to choose a Cartan involution θ of $\mathfrak{g}$ from $\{\theta_1, \theta_2\}$. Let γ be the highest root of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$. We take an involution ξ of $\mathfrak{g}$ which satisfies $\theta \circ \xi = \xi \circ \theta$ ($\theta = \theta_1$ or θ_2). According to Tableau II of [1], this ξ is conjugate to one of ξ_l's in the followings:

$$(\theta, \xi_0, \mathfrak{h}_0) = (\theta_1, \theta_1, \mathfrak{su}(8)),$$
$$(\theta, \xi_1, \mathfrak{h}_1) = (\theta_1, \exp \sqrt{-1}\pi \,\mathrm{ad}\, T_1, \mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})),$$
$$(\theta, \xi_2, \mathfrak{h}_2) = (\theta_1, \theta_1 \circ \xi_1, \mathfrak{su}(4,4)),$$
$$(\theta, \xi_3, \mathfrak{h}_3) = (\theta_1, \theta_1 \circ \exp \sqrt{-1}\pi \,\mathrm{ad}\, T_2, \mathfrak{sl}(8,\mathbb{R})),$$
$$(\theta, \xi_4, \mathfrak{h}_4) = (\theta_1, \xi_4, \mathfrak{e}_{6(6)} \oplus \mathbb{R}),$$
$$(\theta, \xi_5, \mathfrak{h}_5) = (\theta_1, \theta_1 \circ \xi_4, \mathfrak{su}^*(8)),$$
$$(\theta, \xi_6, \mathfrak{h}_6) = (\theta_2, \exp \sqrt{-1}\pi \,\mathrm{ad}\, T_7, \mathfrak{e}_{6(2)} \oplus \sqrt{-1}\mathbb{R}),$$
$$(\theta, \xi_7, \mathfrak{h}_7) = (\theta_2, \theta_2 \circ \xi_6, \mathfrak{so}^*(12) \oplus \mathfrak{su}(2)).$$

Here $\mathfrak{h}_l := \mathfrak{g}^{\xi_l}$ for $0 \le l \le 7$, and ξ_4 means an involution of $\mathfrak{g}$ induced by

$$\gamma_1 \mapsto \gamma_6, \quad \gamma_2 \mapsto \gamma_2, \quad \gamma_3 \mapsto \gamma_5, \quad \gamma_4 \mapsto \gamma_4,$$
$$\gamma_5 \mapsto \gamma_3, \quad \gamma_6 \mapsto \gamma_1, \quad \gamma_7 \mapsto -\gamma.$$

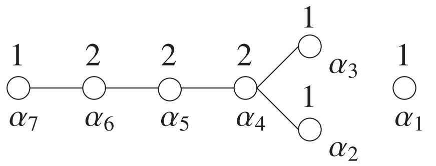
Then we cannot construct a para-real form in the case of $\xi = \xi_1$ or ξ_7 by Lemma 4.2 (2). In fact, by the classification of real forms of compact Hermitian symmetric spaces in [10, 19], a symmetric pair which corresponds to a real form of compact Hermitian symmetric space $E_7/(E_6 \times T)$ is isomorphic to one of the following:

$$(\mathfrak{su}(8), \mathfrak{sp}(4)), \quad (\mathfrak{e}_6 \oplus \sqrt{-1}\mathbb{R}, \mathfrak{f}_4).$$

Put $\mathfrak{k}_l^d := (\mathfrak{h}_l \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h}_l \cap \mathfrak{p})$ for $0 \leq l \leq 7$. Then

$$\mathfrak{k}_l^d = \mathfrak{su}(8) \quad (l = 0, 2, 3, 5),$$
$$\mathfrak{k}_l^d = \mathfrak{e}_6 \oplus \sqrt{-1}\mathbb{R} \quad (l = 4, 6),$$
$$\mathfrak{k}_l^d = \mathfrak{so}(12) \oplus \mathfrak{su}(2) \quad (l = 1, 7).$$

The case of $\xi = \xi_2$: We take a Cartan involution θ of $\mathfrak{g}$ as θ_1. Let $\mathfrak{m}_2 := \mathfrak{g}^{-\xi}$ and let $\mathfrak{h}_2^a := (\mathfrak{h}_2 \cap \mathfrak{k}_1) \oplus (\mathfrak{m}_2 \cap \mathfrak{p}_1)$. Then $\mathfrak{g}$ coincides with $\mathfrak{c}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})} \mathbb{R}X_\alpha$, where $\mathfrak{c}_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\{H_{\gamma_1}, \ldots, H_{\gamma_7}\}$. Then $\mathfrak{c}_{\mathbb{R}}$ is a maximal abelian subspace in $\mathfrak{m}_2 \cap \mathfrak{p}_1$. Moreover, we have $\mathfrak{h}_2^a \cong \mathfrak{so}(6, 6) \oplus \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{h}_1$ because the symmetric pair $(\mathfrak{g}, \mathfrak{h}_1)$ is the associated symmetric pair of $(\mathfrak{g}, \mathfrak{h}_2)$. Here we note that $\mathfrak{z}(\mathfrak{h}_2^a) \cap \mathfrak{a}_2 = \{0\}$. Set $\mathfrak{a}_2 := \mathfrak{c}_{\mathbb{R}}$, $\alpha_1 := \gamma|_{\mathfrak{a}_2}$, and $\alpha_l := \gamma_l|_{\mathfrak{a}_2}$ for $2 \leq l \leq 7$. Then $\Delta^1(\mathfrak{h}_2^a, \mathfrak{a}_1) := \{\alpha_1, \ldots, \alpha_7\}$ is a fundamental system of the restricted root system $\Delta(\mathfrak{h}_2^a, \mathfrak{a}_2)$. The Dynkin diagram of $\Delta^1(\mathfrak{h}_2^a, \mathfrak{a}_2)$ with the coefficients of the highest root is:



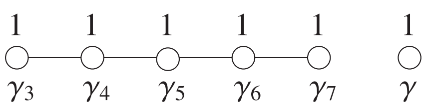
: $D_6 \oplus A_1$.

Let $\{Z_1, \ldots, Z_6\}$ be the dual basis of the D_6 part $\{\alpha_7, \alpha_6, \ldots, \alpha_2\}$ and let $\{W_1\}$ be the dual basis of the A_1 part $\{\alpha_1\}$:

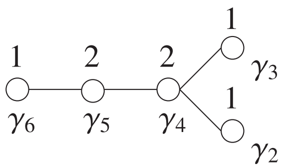
$$W_1 = (1/2)T_1, \quad Z_1 = (-1/2)T_1 + T_7, \quad Z_2 = -T_1 + T_6, \quad Z_3 = (-3/2)T_1 + T_5,$$
$$Z_4 = -2T_1 + T_4, \quad Z_5 = (-3/2)T_1 + T_3, \quad Z_6 = -T_1 + T_2.$$

By Corollary 4.2, the elements satisfying the property in (Step 6) in Lemma 4.3 are only Z_l, W_1, or $Z_l + W_1$ $(l = 1, 5, 6)$. Moreover, only Z_6 and $Z_1 + W_1$ satisfy the property in (Step 7) in Lemma 4.3 among them by Lemma 4.6.

We consider $\mathfrak{c}_{\mathfrak{h}_2}(Z)$ $(Z = Z_6$ or $Z_1 + W_1)$. Set $\sigma_1 := \exp \sqrt{-1}\pi \,\mathrm{ad}\, Z_6$, $\sigma_2 := \exp \sqrt{-1}\pi \,\mathrm{ad}(Z_1 + W_1)$. Then $\tilde{\mathfrak{c}}_i := ((\mathfrak{c}_{\mathbb{C}})^{\sigma_i})^{\xi_1}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}_i := ((\mathfrak{g}_{\mathbb{C}})^{\sigma_i})^{\xi_1}$ for $i = 1, 2$. We note that $(\mathfrak{c}_{\mathfrak{h}_2^a}(Z_6))_{\mathbb{C}} = \tilde{\mathfrak{g}}_1$ and $(\mathfrak{c}_{\mathfrak{h}_2^a}(Z_1 + W_1))_{\mathbb{C}} = \tilde{\mathfrak{g}}_2$. Then $\Delta^1(\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{c}}_1) := \{\gamma_3, \gamma_4, \ldots, \gamma_7, \gamma\}$ is a fundamental system of the root system $\Delta(\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{c}}_1)$ and $\Delta^1(\tilde{\mathfrak{g}}_2, \tilde{\mathfrak{c}}_2) := \{\gamma_2, \gamma_3, \ldots, \gamma_6\}$ is a fundamental system of the root system $\Delta(\tilde{\mathfrak{g}}_2, \tilde{\mathfrak{c}}_2)$. For $i = 1, 2$, the Dynkin diagrams of $\Delta^1(\tilde{\mathfrak{g}}_i, \tilde{\mathfrak{c}}_i)$ with the coefficients of the highest root are:



: $A_5 \oplus A_1$ $(i = 1)$,



$: D_5 \quad (i = 2).$

By Lemma 4.7 and the classification of simple para-Hermitian symmetric pairs in [6, p. 97], we have $\mathfrak{c}_{\mathfrak{g}}(Z) = \mathfrak{e}_{6(6)} \oplus \mathbb{R}$ for $Z = Z_6$ or $Z_1 + W_1$. Since $(\mathfrak{c}_{\mathfrak{g}}(Z), \mathfrak{c}_{\mathfrak{h}_2^a}(Z))$ is a symmetric pair for $Z = Z_6$ or $Z_1 + W_1$, $\mathfrak{c}_{\mathfrak{h}_2^a}(Z_6)$ is isomorphic to $\mathfrak{sl}(6, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{c}_{\mathfrak{h}_2^a}(Z_1 + W_1)$ is isomorphic to $\mathfrak{so}(5, 5) \oplus \mathbb{R} \oplus \mathbb{R}$ by Tableau II of [1]. By Lemma 4.7 (3) and [16, p. 441], we have $\mathfrak{c}_{\mathfrak{h}_2}(Z_6) \cong \mathfrak{sp}(4, \mathbb{R})$ and $\mathfrak{c}_{\mathfrak{h}_2}(Z_1 + W_1) \cong \mathfrak{sp}(2, 2)$. Hence we obtain the following proposition:

Proposition 5.3. *In the case of* $\xi = \xi_2$, *each elements* $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ *are equivalent to one of the following:*

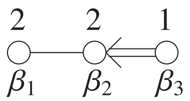
$$(Z_6, \xi), \ (Z_1 + W_1, \xi).$$

Here $Z_6 = -T_1 + T_2$ *and* $Z_1 + W_1 = T_7$. *Moreover, we get the following equalities:*

$$\begin{aligned}
&(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z_6)) = (\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}),\\
&(\mathfrak{h}_2, \mathfrak{c}_{\mathfrak{h}_2}(Z_6)) = (\mathfrak{su}(4, 4), \mathfrak{sp}(4, \mathbb{R})),\\
&(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z_1 + W_1)) = (\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}),\\
&(\mathfrak{h}_2, \mathfrak{c}_{\mathfrak{h}_2}(Z_1 + W_1)) = (\mathfrak{su}(4, 4), \mathfrak{sp}(2, 2)).
\end{aligned}$$

The case of $\xi = \xi_6$: We take a Cartan involution θ of $\mathfrak{g}$ as θ_2. Let $\mathfrak{m}_6 := \mathfrak{g}^{-\xi_6}$ and $\mathfrak{h}_6^a := (\mathfrak{h}_6 \cap \mathfrak{k}) \oplus (\mathfrak{m}_6 \cap \mathfrak{p})$. Since the symmetric pair $(\mathfrak{g}, \mathfrak{h}_7)$ is the associated symmetric pair of $(\mathfrak{g}, \mathfrak{h}_6)$, $\mathfrak{h}_6^a$ is isomorphic to $\mathfrak{h}_7 = \mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$. Here we note that $\mathfrak{z}(\mathfrak{h}_6^a) = \{0\}$.

Set $\delta_1 := \gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + \gamma_7$, $\delta_2 := \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 + \gamma_6 + \gamma_7$, and $\delta_3 := \gamma_1 + \gamma_2 + 2\gamma_3 + 2\gamma_4 + \gamma_5 + \gamma_6 + \gamma_7$. Then we obtain a maximal abelian subspace $\mathfrak{a}_6$ in $\mathfrak{m}_6 \cap \mathfrak{p}_2$ as $\mathfrak{a}_6 := \mathrm{span}_{\mathbb{R}}\{\sqrt{-1}(X_{\delta_t} - X_{-\delta_t}) \mid t = 1, 2, 3\}$. Set $\beta_1 := (1/2)(\gamma_3 - \gamma_5)$, $\beta_2 := (1/2)(\gamma_1 - \gamma_6)$, and $\beta_3 := \gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + \gamma_7$. Then we obtain a fundamental system $\Delta^1(\mathfrak{h}_6^a, \mathfrak{a}_6) := \{\beta_1, \beta_2, \beta_3\}$ of the restricted root system $\Delta(\mathfrak{h}_6^a, \mathfrak{a}_6)$. The Dynkin diagram of $\Delta^1(\mathfrak{h}_6^a, \mathfrak{a}_6)$ with the coefficients of the highest root is:



$: C_3.$

Let $\{Z_1, Z_2, Z_3\}$ be the dual basis of $\{\beta_1, \beta_2, \beta_3\}$. By Lemma 4.5 and Corollary 4.1, only Z_3 satisfies the property in (Step 6) in Lemma 4.3. In addition, Z_3 satisfies the property in (Step 7) in Lemma 4.3 by Lemma 4.7. By the classification of simple para-Hermitian symmetric pairs in [6, p. 97], $\mathfrak{c}_{\mathfrak{g}}(Z_3) = \mathfrak{e}_{6(6)} \oplus \mathbb{R}$. Then $\mathfrak{c}_{\mathfrak{h}_5}(Z_3)$ is isomorphic to $\mathfrak{f}_{4(4)}$ because of Lemma 4.2 (2) and the classification of real forms of compact Hermitian symmetric spaces in [10, 19]. Hence we obtain the following proposition:

Proposition 5.4. *In the case of* $\xi := \xi_6$, *each elements* $(Z, \xi) \in d\mathcal{R}(\mathfrak{g})$ *are equivalent to* (Z_3, ξ). *Moreover, we get the following equalities:*

$$(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(Z_3)) = (\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}),$$

$$(\mathfrak{h}_5, \mathfrak{c}_{\mathfrak{h}_5}(Z_3)) = (\mathfrak{e}_{6(2)} \oplus \sqrt{-1}\mathbb{R}, \mathfrak{f}_{4(4)}).$$

For the other cases, (Z, ξ) are determined by the similar methods.

5.3. APHS's of hyperbolic orbit type. In this subsection, we give the classification of APHS's of hyperbolic orbit type.

Lemma 5.1. *Let $(G/L, \hat{\sigma}, I, g)$ be an APHS of hyperbolic orbit type.*
(1) *If $G = E_{6(6)}$, then $L = Spin(5,5) \times \mathbb{R}^*$.*
(2) *If $G = E_{6(-26)}$, then $L = Spin(1,9) \times \mathbb{R}^+$.*
(3) *If $G = E_{7(7)}$, then $L = E_{6(6)} \times \mathbb{R}^*$.*
(4) *If $G = E_{7(-25)}$, then $L = E_{6(-26)} \times \mathbb{R}^*$.*

Proof. By the proof of Lemma 6 in [18, p. 39], (1) holds.

Since $(G/L, \hat{\sigma}, I, g)$ is an APHS of hyperbolic orbit type, there exists the characteristic element $Z \in \mathrm{Lie}(G)$ such that $L = C_G(Z)$ and that $\hat{\sigma} = \exp \sqrt{-1}\pi \,\mathrm{ad}\, Z$ by Proposition 2.1. We consider (2). By Theorem 3.6.8 in [20, p. 219], we have $G^{\hat{\sigma}} = Spin(1,9) \times \mathbb{R}^+ = (G^{\hat{\sigma}})_0$. Hence (2) holds because $(G^{\hat{\sigma}})_0 \subset L \subset G^{\hat{\sigma}}$.

Next, we consider (3). By Theorem 4.4.6 in [21, p. 387], we have $G^{\hat{\sigma}} = E_{6(6)} \times \mathbb{R}^*$ and $(G^{\hat{\sigma}})_0 = E_{6(6)} \times \mathbb{R}^+ \subset C_G(Z)$. In addition, we have $\{e\} \times \mathbb{R}^* \subset Z(G^{\hat{\sigma}}) \subset C_G(Z)$. Thus we obtain

$$G^{\hat{\sigma}} = E_{6(6)} \times \mathbb{R}^* \subset (\{e\} \times \mathbb{R}^*)(E_{6(6)} \times \{1\}) \subset C_G(Z)C_G(Z) \subset C_G(Z).$$

Hence (3) holds. We can prove (4) in the similar ways to the proofs of (3) by Theorem 4.4.6 in [21, p. 387]. □

S. Kaneyuki gives the classification of classical APHS's of hyperbolic orbit type in [5, p. 368]. Therefore, we obtain the classification of APHS's of hyperbolic orbit type by Kaneyuki [5, p. 368], Kaneyuki-Kozai [6, p. 97], and Lemma 5.1:

List 2

Type	$G/C_G(Z)$	Condition
AI	$SL(n,\mathbb{R})/S(GL(i,\mathbb{R}) \times GL(n-i,\mathbb{R}))$	$2 \le n$
AII	$SU^*(2n)/(SU^*(2i) \times SU^*(2(n-i)) \times \mathbb{R}^+)$	$3 \le n$
AIII	$SU(n,n)/(SL(n,\mathbb{C}) \times \mathbb{R}^*)$	$3 \le n$
BDI	$SO_0(p,q)/(SO_0(p-1,q-1) \times \mathbb{R}^*)$	$1 \le p \le q,\ p+q \ne 2$
	$SO_0(n,n)/(SL(n,\mathbb{R}) \times \mathbb{R}^*)$	$2 \le n$
DIII	$SO^*(4n)/(SU^*(2n) \times \mathbb{R}^+)$	$3 \le n$
CI	$Sp(n,\mathbb{R})/(SL(n,\mathbb{R}) \times \mathbb{R}^*)$	$3 \le n$
CII	$Sp(n,n)/(SU^*(2n) \times \mathbb{R}^+)$	$2 \le n$
EI	$E_{6(6)}/(Spin(5,5) \times \mathbb{R}^*)$	—
EIV	$E_{6(-26)}/(Spin(1,9) \times \mathbb{R}^+)$	—
EV	$E_{7(7)}/(E_{6(6)} \times \mathbb{R}^*)$	—
EVII	$E_{7(-25)}/(E_{6(-26)} \times \mathbb{R}^*)$	—

By the procedure in Lemma 4.3, we determine $d\mathcal{R}(\mathfrak{g})/\sim$ for each absolutely simple Lie algebra $\mathfrak{g}$ in List 1. Thus we obtain Theorem 1.1.

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