Tanaka, T.
Osaka J. Math.
58 (2021), 685–696

ON SATELLITE KNOTS WITH SYMMETRIC UNION PRESENTATIONS

Toshifumi TANAKA

(Received February 4, 2020, revised May 13, 2020)

Abstract

A symmetric union in the 3-space $\mathbb{R}^3$ is a knot, obtained from a knot in $\mathbb{R}^3$ and its mirror image, which are symmetric with respect to a 2-plane in $\mathbb{R}^3$, by taking the connected sum of them and moreover by connecting them with some vertical twists along the plane, which is a generalized operation of the connected sum of a knot and its mirror image. In this paper, we show that a satellite symmetric union with minimal twisting number one such that the order of the pattern is an odd number ≥ 3 has at least two disjoint non-parallel essential tori in the complement.

1. Introduction

A *symmetric union* was introduced by Kinoshita and Terasaka [8]. Later, Lamm [9] generalized the definition and gave its properties. A symmetric union is known to be a *ribbon knot* which bounds a smooth disk in the 4-ball (see [11] for the definition). Every ribbon knot with minimal crossing number ≤ 10 is a symmetric union [4, 9] and it is known that all two-bridge ribbon knots can be represented as symmetric unions (see [10, 13]). Then we have a question which asks whether every ribbon knot can be represented as a symmetric union (see [9]).

In this paper, we study a *satellite knot* with a symmetric union presentation. Let $\hat{V}$ be a solid torus which is the complement of the unknot J in S^3. Let $\hat{K}$ be a knot in $\hat{V}$ such that $\hat{K}$ is a geometrically essential (i.e. $\hat{K}$ meets every meridian disk of $\hat{V}$). We define the *order* of the pair $(\hat{V}, \hat{K})$ as the geometric intersection number of $\hat{K}$ to any meridian disk of $\hat{V}$. Let V be a regular neighborhood of a non-trivial knot K_c in S^3. We call a knot K is a *satellite knot* if K is the image $\phi(\hat{K}) \subset V \subset S^3$ for a homeomorphism $\phi : \hat{V} \to V$, the order of $(\hat{V}, \hat{K})$ is not zero and $\hat{K}$ is not the core of $\hat{V}$. The knot K_c is called a *companion knot* and the boundary of V is called a *companion torus* for K with respect to K_c. The link $\hat{K} \cup J$ in S^3 is called the *pattern link* for K and the pair $(\hat{V}, \hat{K})$ is called the *pattern* of K. It is known that a companion torus is an essential torus, that is, a torus which is incompressible and not boundary-parallel in the complement of K (see [14, p. 335]).

Let $K = K_0 \sharp K_1$ be the connected sum of two non-trivial knots K_0 and K_1. Then we have two disjoint non-parallel essential tori, which are called *swallow-follow tori*, in the complement of K if we regard K_0 and K_1 as companion knots for K with the patterns of order one (see [1, Proposition 16.7]). In particular, if K_1 is the mirror image of K_0, then we have two disjoint non-parallel essential tori which are symmetric with respect to a plane

2020 Mathematics Subject Classification. Primary 57K10; Secondary 57K31.

as shown in Figure 1(a) and it is easily seen that each pattern link complement contains an essential torus as shown in Figure 1(b). In this paper, we show the following theorem.

Theorem 1.1. *Let K be a satellite symmetric union with minimal twisting number one. If the order of the pattern of K is an odd number ≥ 3, then the complement of K has two disjoint non-parallel essential tori. In particular, the pattern link complement contains an essential torus.*

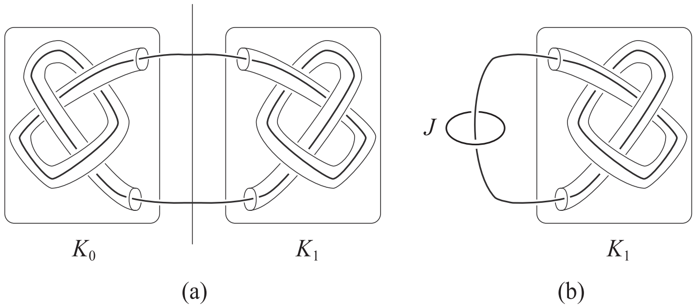


Fig. 1

Throughout this paper, $\sharp\{X\}$ denotes the number of elements of X for a finite set X. We denote the boundary of a surface or a 3-ball M by ∂M. In Section 2, we shall give the definitions of a symmetric union and the minimal twisting number. In Section 3, we introduce a *T-graph* to study an essential torus in the complement of a symmetric union. In Section 4, we shall prove Theorem 1.1. In Section 5, we shall give some examples.

2. The definition of a symmetric union

Let $\mathbb{R}^3$ be the 3-space with x-, y-, and z-axes. Let $\mathbb{R}^3_+ = \{(x, y, z) | x > 0\}$ and $\mathbb{R}^3_- = \{(x, y, z) | x < 0\}$. Throughout this paper, a *tangle* denotes a disjoint union of a finite number of circles and two arcs properly embedded in a 3-ball. A tangle (without circles) in a 3-ball B^3 is *prime* if it is *locally unknotted* (i.e. any 2-sphere in B^3, which meets the tangle transversely in two points, bounds in B^3 a ball meeting the tangle in an unknotted spaning arc) and not *untangled* (i.e. it is not equivalent to the trivial tangle) (see [12] for the definition). We denote the tangle made of $|m|$ half-twists along the z-axis as a diagram by an integer $m \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ with respect to the x-axis as in Figure 2. A symmetric union is defined as follows (see [9] for the original definition).

DEFINITION 2.1. We take a knot $\tilde{K}$ in $\mathbb{R}^3_-$ and its mirror image $\tilde{K}^*$ in $\mathbb{R}^3_+$ such that $\tilde{K}$ and $\tilde{K}^*$ are symmetric with respect to the yz-plane $\mathbb{R}^2_{yz}$ as in Figure 3(a). Here we consider a diagram of a knot in the xz-plane $\mathbb{R}^2_{xz}$ and we denote the diagrams of $\tilde{K}$ and $\tilde{K}^*$ by $\tilde{D}$ and $\tilde{D}^*$, respectively. Each disk-arc pair of $T_0, T_1, \ldots, T_k$ as in Figure 3(a) denotes a diagram of the tangle 0. Then we replace the tangles $T_0, T_1, \ldots, T_k$ with tangles ∞, $m_1, m_2, \ldots, m_k$ as in Figure 3(b) (see Figure 4 for example). Here we assume that $m_i \neq \infty$ $(1 \leq i \leq k)$. The resultant diagram is called a *symmetric union presentation* and we denote it by $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$.
If a knot K has a diagram $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$, then the knot K is called a *symmetric union*.

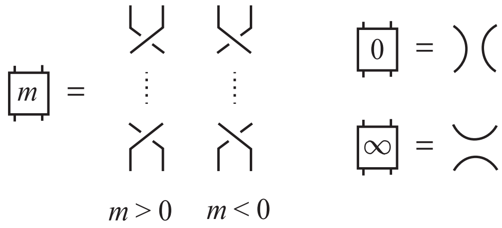


Fig. 2

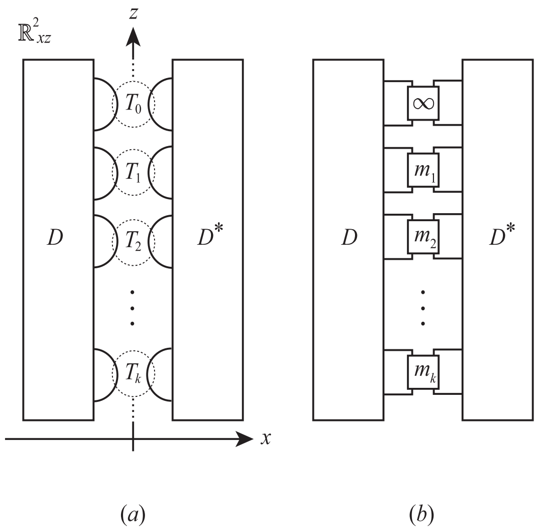


Fig. 3

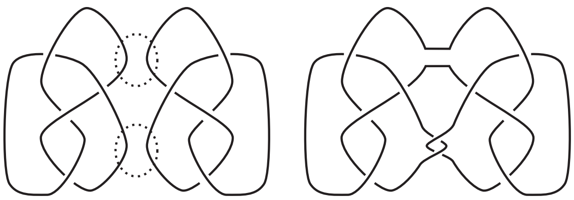


Fig. 4

The sphere $S^2_T = \mathbb{R}^2_{yz} \cup \{\infty\}$ is called the *symmetry sphere* of the symmetric union.

Here we define the minimal twisting number for a symmetric union which was originally introduced in [15] as follows.

Definition 2.2. We call the number of non-zero elements in $\{m_1, \ldots, m_k\}$ the *twisting number* for $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$. The *minimal twisting number* of a symmetric union K is the smallest number of the twisting numbers of all symmetric union presentations to K.

Remark 2.3. The minimal twisting number is an invariant of K. The minimal twisting number of a two-bridge symmetric union is equal to either one or two [10]. An example of a symmetric union with minimal twisting number two was given in [15].

3. A trivalent graph on an essential torus

Let K be a satellite symmetric union with the diagram $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$. Let T be a companion torus in the complement of K and V, a solid torus bounded by T such that $V \supset K$. Recall that $S^2_T = \mathbb{R}^2_{yz} \cup \{\infty\}$ is the symmetry sphere. Let B_i be the 3-ball which corresponds to each tangle m_i ($i = 1, 2, \ldots, k$). The symmetry sphere S^2_T divides $\mathbb{S}^3$ into two 3-balls, denoted by $\mathbb{B}^3_+$ and $\mathbb{B}^3_-$. Let $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$) be S^2_T with each disk of S^2_T inside each B_i replaced by the hemisphere $\partial B_i \cap \mathbb{B}^3_+$ (or $\partial B_i \cap \mathbb{B}^3_-$) and let $\mathbb{W} = \mathbb{S}^2_+ \cup \mathbb{S}^2_-$ and $\overline{\mathbb{W}} = \mathbb{W} \cup S^2_T$. Then the sphere $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$) bounds a 3-ball in $\mathbb{B}^3_+$ (or $\mathbb{B}^3_-$). We denote the 3-ball by $\overline{\mathbb{B}}^3_+$ (or $\overline{\mathbb{B}}^3_-$).

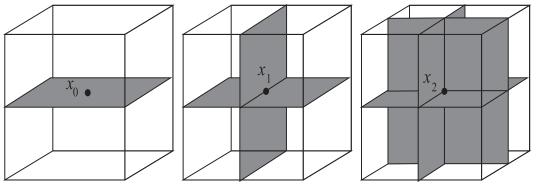


Fig. 5

We assume that T, S^2_T and ∂B_i ($i = 1, 2, \ldots, m$) meet transversely, that is, at each point x of $T \cup \overline{\mathbb{W}}$ $(= T \cup S^2_T \cup (\bigcup_{i=1}^k \partial B_i))$, we have a neighborhood N_x of x in $\mathbb{S}^3$ such that $N_x \cap (T \cup \overline{\mathbb{W}})$ is described as in Figure 5(a), (b) or (c), where $x = x_0$, x_1 or x_2. Then we regard $T \cap \overline{\mathbb{W}}$ as a 4-valent graph on T by considering each point of $T \cap S^2_T \cap \partial B_i$ ($1 \leq i \leq k$), corresponding to x_2 in Figure 5, as a vertex as in Figure 6. We consider (the closure of) each component of $T \cap \overline{\mathbb{W}} - T \cap S^2_T \cap (\bigcup_{i=1}^k \partial B_i)$ as an edge of the graph on T. (Each point of the interior of an edge corresponds to x_1 of Figure 5 as in Figure 6.)

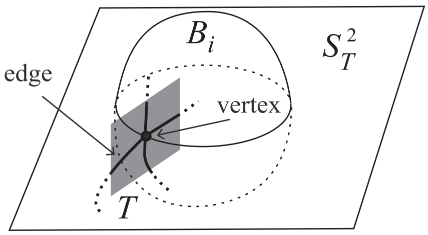
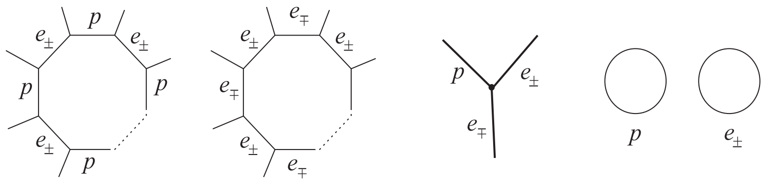


Fig. 6

Similarly, we regard $T \cap \mathbb{W} \subset T \cap \overline{\mathbb{W}}$ as a trivalent graph on T, denoted by G_T. We call G_T a *T-graph* on T. We call each edge of G_T on S^2_T, *p-edge* and denoted by p. We call each edge of G_T on $\mathbb{S}^2_+ \cap \partial B_i$ (or $\mathbb{S}^2_- \cap \partial B_i$), *$t$-edge* and denoted by e_+ (or e_-).

Note that each innermost inessential cycle of G_T on T bounds a disk of T in one of $\overline{\mathbb{B}}^3_+$, $\overline{\mathbb{B}}^3_-$ and B_i for some i. Each of cycles (with some vertices) of G_T on $\mathbb{S}^2_\pm$ and ∂B_i ($i = 1, 2, \ldots, k$) are one of the cycles as in Figure 7(a) and (b). The adjascent edges of a vertex are configurated as in Figure 7(c). We call the cycle of (a) (or (b)) a *p-cycle* (or a *t-cycle*). We assume that a p-cycle has at least one p edge. We note that a p-cycle on $S^2_T - (B_1 \cup B_2 \cup \cdots \cup B_k)$ and a t-cycle on $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$) have no vertices and we describe them as shown in Figure 7(d).



(a) (b) (c) (d)

Fig. 7

4. Proof of Theorem 1.1

Proposition 4.1. *Let K be a satellite symmetric union with the diagram $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$ with twisting number k. If the companion torus does not meet any of $B_1, B_2, \ldots, B_k$ and the order of the pattern is an odd number $\geq k+2$, then the complement of K has two disjoint non-parallel essential tori. In particular, the pattern link complement contains an essential torus.*

Proof. Let T be the companion torus in the complement of K. By assumption, we know that G_T consists of disjoint unions of cycles with no vertices on $S^2_T - (B_1 \cup B_2 \cup \cdots \cup B_k)$. Let ℓ be an innermost inessential cycle on T and d, an innermost disk, bounded by ℓ. Let d_c be the disk which is symmetric to d with respect to S^2_T in S^3. We have $K \cap d_c = \emptyset$ since $K \cap d = \emptyset$. Then $d \cup d_c$ bounds a 3-ball, $\hat{B}$, which does not meet K. Thus we remove ℓ by an isotopy along $\hat{B}$ as shown in Figure 8. (The isotopy removes any circles on $\hat{B} \cap S^2_T$.) By repeating this process, we may assume that there are no inessential cycles on $\mathbb{S}^2_\pm$. □

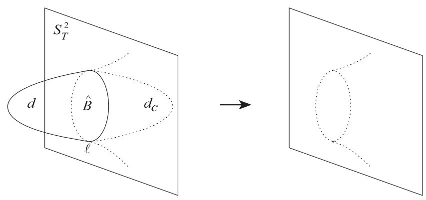


Fig. 8

Let $\bar{\ell}$ be an essential cycle on T, innermost on $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$), and $\bar{d}$, an innermost disk bounded by $\bar{\ell}$ in $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$). Note that $\bar{d}$ is a meridian disk for T since $\bar{\ell}$ is essential. By assumption, we know that $\sharp\{\bar{d} \cap K\}$ is an odd number and $\geq k+2$. Since $\bar{\ell}$ is essential, there exists another innermost essential cycle $\bar{\ell}'$ in $\mathbb{S}^2_+ - \bar{d}$ (or $\mathbb{S}^2_- - \bar{d}$). (In fact, there exists an annulus S of $\overline{\mathbb{B}}^3_+ \cap T$ (or $\overline{\mathbb{B}}^3_- \cap T$) which has $\bar{\ell}$ as a boundary component. Since $\bar{\ell}$ is essential, the other boundary component of S is an essential cycle. If the cycle is not innermost on $\mathbb{S}^2_+ - \bar{d}$ (or $\mathbb{S}^2_- - \bar{d}$), then we can take an innermost one, other than $\bar{\ell}$.) Let $\bar{d}'$ be the innermost disk for $\bar{\ell}'$ on $\mathbb{S}^2_+ - \bar{d}$ (or $\mathbb{S}^2_- - \bar{d}$). Then we know that $\bar{d}'$ is a meridian disk and $\sharp\{\bar{d}' \cap K\} \leq k$ since $\sharp\{\bar{d} \cap K\} + \sharp\{\bar{d}' \cap K\} \leq 2k+2$. This is contrary to the assumption. Thus we know that there are no essential circles on $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$. Therefore we may assume that $\mathbb{S}^2_\pm \cap T = \emptyset$ and we know that T is embedded in either $\overline{\mathbb{B}}^3_+$ or $\overline{\mathbb{B}}^3_-$.

Without loss of generality, we may assume that $T \subset \overline{\mathbb{B}}^3_+$. Note that $\overline{\mathbb{B}}^3_-$ is in the interior of V and then $\overline{\mathbb{B}}^3_+$ contains $S^3 \setminus V$, denoted by E. Now we take a 3-manifold, denoted by E', which is symmetric to E with respect to S^2_T. Then $\partial E'$ is a torus, denoted by T', and T' is symmetric to T with respect to S^2_T. Let V' be the solid torus bounded by T' in S^3. Note that $V' \neq E'$ since E is not a solid torus. Then $\overline{\mathbb{B}}^3_+$ is in the interior of V' since $\overline{\mathbb{B}}^3_-$ contains E'.

Now we show that T' is incompressible. Suppose that T' is compressible in $S^3 \setminus K$, denoted by E_0, and d_0 is a compressing disk for T' such that $d_0 \cap K = \emptyset$. Then we know that T' is incompressible in E' since T is incompressible in E. Thus d_0 must lie in V'. Let d_m be a meridian disk for V. Note that $\partial d_m \subset T \subset \overline{\mathbb{B}}^3_+$. By an innermost argument on d_m to the loops of $d_m \cap S^2_+$, we isotope d_m in V so that $d_m \cap S^2_+ = \emptyset$. We may assume that the resultant disk, also denoted by d_m, meets K transversely. By assumption, we know that d_m meets K in an odd number of points $\geq k+2$ since it is a meridian disk. Let $\overline{d}_m$ be a disk which is symmetric to d_m with respect to S^2_T. Then $\overline{d}_m$ meets K in an odd number of points, is a compressing disk for T' and is a meridian disk for V'. We may assume that $\overline{d}_m \cap d_0$ consists of a finite number of disjoint loops by an isotopy. Then by repeating surgery on $\overline{d}_m$ along an innermost disk in d_0, we remove all circles of the intersection so that we have $\overline{d}_m \cap d_0 = \emptyset$. Note that by the surgery, we remove an even number of points of the intersection of $\overline{d}_m$ and K. Thus the resultant disk, also denoted by $\overline{d}_m$, meets K in an odd number of points ($\geq k + 2$). Now by surgery for T' along d_0, we obtain a sphere S_0 embedded in S^3. One 3-ball bounded by S_0 must contain d_m. Again by surgery for S_0 along d_m, we obtain a sphere which meets K in an odd number of points. This is impossible. Thus we know that T' is incompressible in E_0.

Now we show that T' is not boundary-parallel. Suppose that T' is boundary-parallel. Then E' is isotopic to E_0. Since E is symmetric to E' with respect to S^2_T, the companion knot is equivalent to K by [5]. On the one hand, since the order of the pattern is larger than two, the bridge number of K is strictly larger than that of the companion knot by [1, Theorem 16.28]. This is a contradiction.

Now we show that T' is not parellel to T. Suppose that T and T' are parallel. Then T' bounds a solid torus which is isotopic to V in S^3. Since T' is knotted, the solid torus is exactly equal to V'. Then we have either $V \subset V'$ or $V' \subset V$. In particular, we have either $T \subset V'$ or $T' \subset V$. (They do not happen simultaneously.) However, we have $T = \partial E \subset V'$ and $T' = \partial E' \subset V$ as in the argument above. This is a contradiction.

Let $\hat{K} \cup J$ be the pattern link for K such that $(\hat{K}, \hat{V})$ is the pattern of K and $\phi(\hat{K}) = K \subset V \subset S^3$ where ϕ: $\hat{V} \to V$ is a homeomorphism. Note that T' lies in a 3-ball $\subset \overline{\mathbb{B}}^3_- \subset V$. Let $\tilde{T} = \phi^{-1}(T')$. Suppose that $\tilde{T}$ is compressible in the complement of $\hat{K} \cup J$. Then we have a compressing disk, d_1, to $\tilde{T}$ in the complement of $\hat{K} \cup J$. Then $\phi(d_1)$ is a compressing disk to T' and this is contrary to the conclusion that T' is incompressible. Suppose that $\tilde{T}$ is boundary-parallel in $S^3 - \hat{K} \cup J$. If $\tilde{T}$ is parallel to the boundary of a regular neighborhood of $\hat{K}$, then we know that T' is boundary-parallel. This is contrary to the conclusion that T' is not boundary-parallel. If $\tilde{T}$ is parallel to $\partial\hat{V}$, then T' is parallel to T since $\phi(\partial\hat{V}) = T$. This is contrary to the conclusion that T and T' are not parallel. This completes the proof.

REMARK 4.2. The knotted torus T' can be also regarded as a companion torus for K since it is essential.

Proof of Theorem 1.1. By assumption, the satellite knot K has a diagram given by $\tilde{D} \cup \tilde{D}^*(m)$. Let G_T be the T-graph for the companion torus T in the complement of K such that $K \subset V$ where V is a solid torus with $\partial V = T$. Let B be the 3-ball which corresponds to the tangle m. □

Suppose that there exists an essential cycle, ℓ, of G_T on T, which is innermost on $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$). Let d_ℓ be an innermost disk bounded by ℓ. By performing surgeries on d_ℓ along an innermost disk in T, we can remove all inessential cycles the intersection of the interior of d_ℓ and T. The resultant disk is a meridian disk for T. By assumption, the number of points of the intersection of the meridian disk and K is odd ≥ 3. Each sugery on d_ℓ may remove an even number of points of the intersection of d_ℓ and K. Thus we know that $\sharp\{d_\ell \cap K\} = 3$ since $\sharp\{d_\ell \cap K\} \leq 4$.

Note that there are at least two essential cycles of G_T, innermost on $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$) since ℓ is essential. (see Figure 9 for example.) Let $\bar{\ell}$ be an essential cycle, innermost on $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$), other than ℓ and $d_{\bar{\ell}}$, an innermost disk for $\bar{\ell}$. We may assume that $d_\ell \cap d_{\bar{\ell}} = \emptyset$ since d_ℓ is innermost. By the same argument above, we know that $\sharp\{d_{\bar{\ell}} \cap K\} = 3$. However, we must have $\sharp\{d_\ell \cap K\} + \sharp\{d_{\bar{\ell}} \cap K\} \leq 4$ since $d_\ell \cap d_{\bar{\ell}} = \emptyset$. This is a contradiction. Thus we may assume that there are no essential cycles of G_T on T.

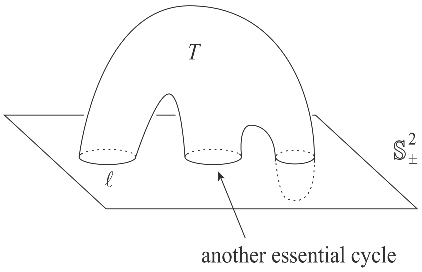


Fig. 9

Suppose that ℓ_p is an inessential p-cycle of G_T, innermost as a p-cycle on T. Let d be an innermost disk on T, bounded by ℓ_p and d_p, a disk bounded by ℓ_p on $\mathbb{S}^2_+$ (or $\mathbb{S}^2_-$). We may assume that $\sharp\{d_p \cap K\} \leq 2$.

Now we remove any innermost (t-)cycles on d. Suppose that there exists a t-cycle, denoted by ℓ_0, on d. If ℓ_0 has a vertex on the interior of d, then we have a p-edge on the interior of d and we have a p-cycle on d, other than ℓ (see Figure 10(a)). This contradicts the assumption that d is an innermost disk. Thus we may assume that all vertices of ℓ_0 are on ℓ. If ℓ_0 has a vertex, then at least one edge of ℓ_0 is adjacent to a p-edge of ℓ and we have a p-cycle, other than ℓ on d (see Figure 10(b)). This is also a contradiction. Thus ℓ_0 has no vertices.

Now we take an innermost t-cycle, ℓ_t, (with no vertices) on d. Then we know that ℓ_t bounds a disk, denoted by d_t, of T in either B or $\overline{\mathbb{B}}^3_\pm$. Let $\bar{d}_t$ be a disk bounded by ℓ_t on $\partial B \cap \overline{\mathbb{B}}^3_+$ (or $\partial B \cap \overline{\mathbb{B}}^3_-$). Suppose that ℓ_t bounds a disk d_t in $\overline{\mathbb{B}}^3_+$ (or $\overline{\mathbb{B}}^3_-$). Then $d_t \cup \bar{d}_t$ bounds a 3-ball, $\tilde{B}$, in $\overline{\mathbb{B}}^3_+$ (or $\overline{\mathbb{B}}^3_-$). Note that $\bar{d}_t \cap K = \emptyset$. In fact, if $\sharp\{\bar{d}_t \cap K\} \geq 1$, then we have $d_t \cap K \neq \emptyset$ by the configuration of the symmetric union in $\overline{\mathbb{B}}^3_\pm$. Thus we know that $\tilde{B} \cap K = \emptyset$ and we can remove the t-cycle by an isotopy along $\tilde{B}$.

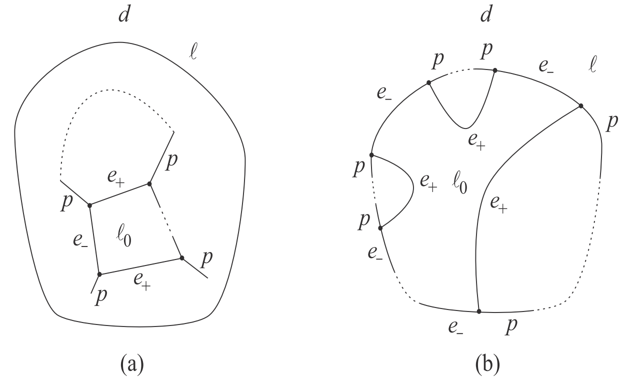


Fig. 10

Next we assume that ℓ_t bounds a disk in B. Note that $\sharp\{\bar{d}_t \cap K\} \leq 2$. If $\sharp\{\bar{d}_t \cap K\} = 0$, then we remove ℓ_t by an isotopy along the 3-ball bounded by $d_t \cup \bar{d}_t$ in B. If $\sharp\{\bar{d}_t \cap K\} = 1$, then K must meet d_t and this is a contradiction. If $\sharp\{\bar{d}_t \cap K\} = 2$, then we have $m = 0$ and this is contrary to the assumption. By repeating this argument, we may assume that d has no cycles, other than ℓ_p, that is, ℓ_p is actually an innermost inessential cycle on T.

We may assume that $\sharp\{d_p \cap K\} \leq 2$. In the case when $d_p \cap K = \emptyset$, we remove the p-cycle by an isotopy along a 3-ball bounded by $d \cup d_p$ since $d \cap K = \emptyset$. (We remove all extra cycles on d_p by the isotopy.) In the case when $\sharp\{d_p \cap K\} = 1$, we have $d \cap K \neq \emptyset$ since $d \cup d_p$ meets K in an even number of points and this is a contradiction. In the case when $\sharp\{d_p \cap K\} = 2$, if we have d_p so that $d \cup d_p$ bounds a 3-ball which meets K in knotted arc, then we take the innermost inessential cycle, ℓ_s, which bounds the innermost disk, d_s, on d_p. We may assume that d_s meets K in two points by removing any inessential cycle that bounds a disk, in d_p, that does not meet K, by an isotopy. Then there exists a disk, $\tilde{d}$, on T such that $\tilde{d} \cup d_s$ bounds a 3-ball in V, which meets K in a knotted arc α since ℓ_p is inessential. Here we take a swallow-follow torus, $\hat{T}$, for α as in Figure 11. Let $\hat{V}$ be the solid torus bounded by $\hat{T}$. Note that $\hat{T}$ is an essential torus in E_0 ($= S^3 \setminus K$) since $\hat{T}$ is a swallow-follow torus [1, p.338] in E_0.

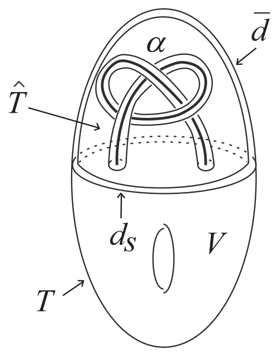


Fig. 11

Now we show that $\hat{T}$ and T are not parallel in $S^3 \setminus K$. Suppose that $\hat{T}$ and T are parallel. Then $\hat{T}$ bounds a solid torus which is isotopic to V in S^3. Since $\hat{T}$ is knotted, the solid torus is exactly equal to $\hat{V}$. Then we have either $V \subset \hat{V}$ or $\hat{V} \subset V$. In particular, we have either $T \subset \hat{V}$ or $\hat{T} \subset V$. (They do not happen simultaneously.) However we have $T \subset \hat{V}$ and $\hat{T} \subset V$ by the construction of $\hat{T}$. This is a contradiction.

As in the proof of Proposition 4.1, let $\hat{K} \cup J$ be the pattern link for K such that $(\hat{K}, \hat{V})$ is the pattern of K and $\phi(\hat{K}) = K \subset V \subset S^3$ where $\phi : \hat{V} \to V$ is a homeomorphism. Let $\tilde{T} = \phi^{-1}(\hat{T})$. Then by using the same argument as in the proof of Proposition 4.1, we know that $\tilde{T}$ is essential in $S^3 \setminus (\hat{K} \cup J)$.

If we cannot choose d_p so that $d \cup d_p$ bounds a 3-ball which meets K in knotted arc, then we can easily show that K is a 1-*fusion ribbon knot*, which is a *banding* of a 2-component trivial link (see [3] for the definition of a banding). In fact, we can take the band β in B along the tangle m as shown in Figure 12. By performing a surgery β along a proper arc on the band, we have a 2-component trivial link. On the one hand, we know that the bridge number is less than or equal to three by isotoping the tangle m as in Figure 13.

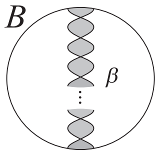


Fig. 12

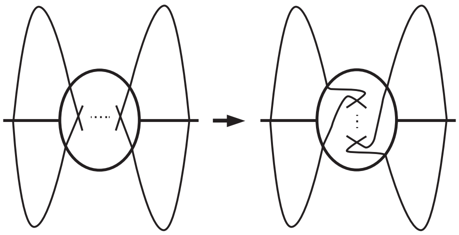


Fig. 13

However, since the order of the pattern is larger than two by assumption, the bridge number of K is larger than three. This is a contradiction (see [1], Theorem 16.28).

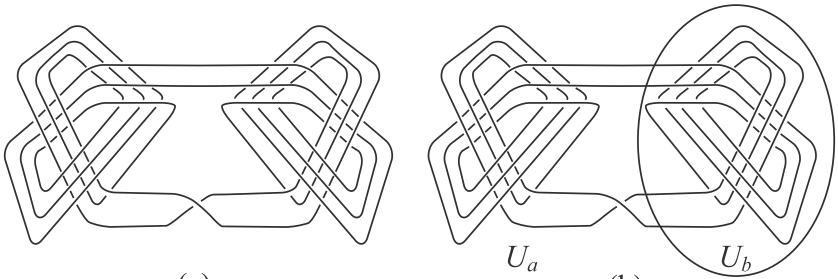
By repeating the argument above, we may assume that we have only inessential t-cycles with no vertices on T. Let $\tilde{\ell}_t$ be an inessential t-cycle in T. Then $\tilde{\ell}_t$ can be removed by the same argument above. By repeating the argument, we remove all inessential t-cycles.

Now we may assume that there are no cycles on S^2_+ (or S^2_-) . Therefore we have $T \cap S^2_\pm = \emptyset$. If $T \subset \overline{\mathbb{B}}^3_\pm$, then we have the conclusion by Proposition 4.1. If $T \subset B$, then we push out T to the outside of B along the twists of the tangle so that we have $T \cap B = \emptyset$. Then by Proposition 4.1, we also have the conclusion.

5. Examples

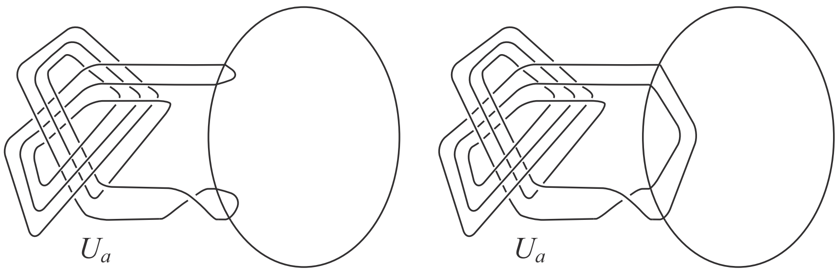
In this section, we give some examples of satellite ribbon knots.

EXAMPLE 5.1. First we consider a satellite symmetric union with minimal twisting number one as shown in Figure 14(a), where the pattern and the pattern link are described as in Figure 16(a) and (b). The order of the pattern is three since the linking number of the link is either 3 or -3 according to the orientation. The minimal twisting number is clearly ≤ 1. Since the symmetric union can be decomposed into two prime tangles, U_a and U_b, as in Figure 14(b), we know that the symmetric union is a prime knot by a result of [12]. In fact, the primeness of U_a can be shown as follows. Clearly, U_a is not untangled since each strand is knotted. Suppose that U_a is not locally unknotted. Then there exists a 2-sphere, which meets U_a transversely in two points, bounds a 3-ball meeting U_a in a knotted spaning arc. Then the knotted spaning arc corresponds to the trefoil knot since the knot in Figure 15(a) is the trefoil knot (which is a prime knot). However, we know that the knot in Figure 15(b) cannot have the trefoil knot as a connected summand by calculating the Jones polynomial. We can also show that U_b is a prime tangle by the same argument. Thus we know that the symmetric union has the minimal twisting number one. The patten link complement contains an essential torus which is indicated as the dotted line as in Figure 16(b).



(a) (b)

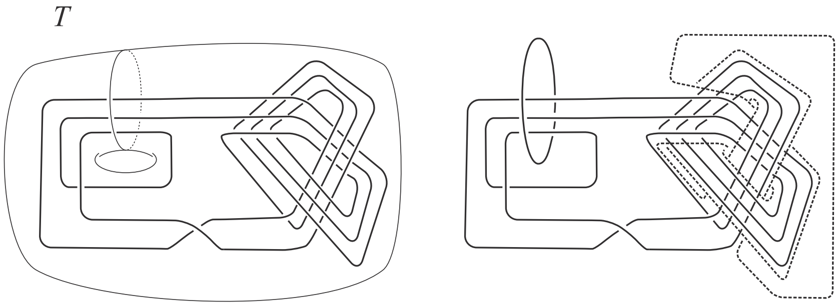
Fig. 14



(a) (b)

Fig. 15

Next we give a satellite ribbon knot for which the pattern link complement does not contain an essential torus. We consider the pattern $(\hat{V}, \hat{K})$ as in Figure 17(a), where $1 \leq m \leq 3$. We know that the order of $(\hat{V}, \hat{K})$ is three. In fact, we can show this as follows. The pattern link is described in Figure 17(b). Since the linking number is either 1 or -1 according to the orientation, we know that the order of the pattern is either 1 or 3. If the order is 1, then the pattern link is equivalent to the link as in Figure 17(c). However the links in Figures 17(b) and 17(c) are not equivalent since they have different Jones polynomials.



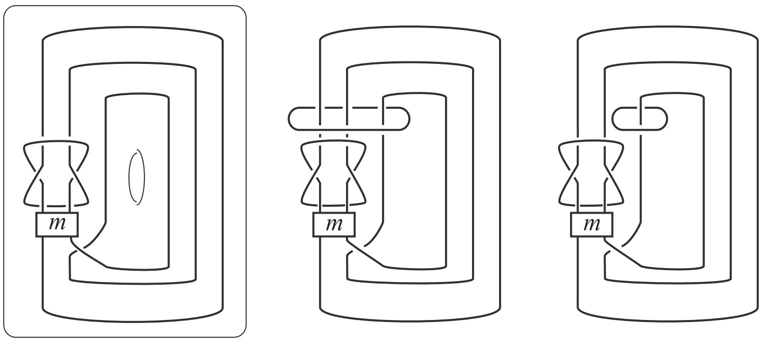
(a) (b)

Fig. 16

Thus we know that the order is three.

Let K be a non-trivial ribbon knot. Then we consider a satellite knot K^* as the image of $\hat{K}$ by a homeomorphism $\hat{V} \to V$, where V is a regular neighborhood of K. We know that K^* is also a ribbon knot.

On the other hand, we can show that the pattern link in Figure 17(b) is hyperbolic if $1 \leq m \leq 3$ by the computer program HIKMOT [6] which is integrated into SnapPy [2]. (If HIKMOT shows that a link is hyperbolic, then the link is truly hyperbolic.) In particular, the complement of link does not contain an essential torus (for example, see [7]). Thus by Theorem 1.1, we know that the satellite knot K^* is not a symmetric union with minimal twisting number one.



(a) (b) (c)

Fig. 17

Remark 5.2. By considering a tangle decomposition of the pattern link, L_m, as $L_m = U_m \cup V$ where U_m and V as in Figure 18, it might be possible to give an alternative proof that the complement of L_m does not contain an essential torus.

Acknowledgements. The author is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research(C), 2019-2021 (19K03465).

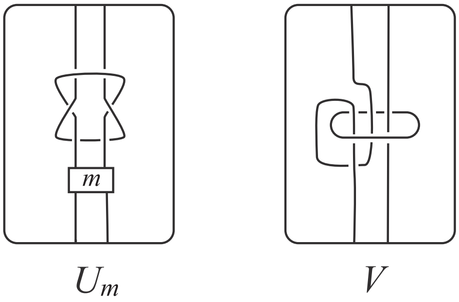


Fig. 18

References

[1] G. Burde, H. Zieschang and M. Heusener: Knots, Third, fully revised and extended edition, De Gruyter Studies in Mathematics, 5, De Gruyter, Berlin, 2014.
[2] M. Culler, N. Dunfield and J. Weeks: SnapPy, a computer program for studying the geometry and topology of 3-manifolds, available at http://snappy.computop.org/.
[3] M. Eudave Muñoz: *Band sums of links which yield composite links, The cabling conjecture for strongly invertible knots*, Trans. Amer. Math. Soc. **330** (1992), 463–501.
[4] M. Eisermann and C. Lamm: *Equivalence of symmetric union diagrams*, J. Knot Theory Ramifications **16** (2007), 879–898.
[5] C. McA Gordon and J. Luecke: *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.
[6] N. Hoffman, K. Ichihara, M. Kashiwagi, H. Masai, S. Oishi and A. Takayasu: *Verified computations for hyperbolic 3-manifolds*, Exp. Math. **25** (2016), 66–78.
[7] A. Kawauchi: A survey of knot theory, Translated and revised from the 1990 Japanese original by the author, Birkhäuser Verlag, Basel, 1996.
[8] S. Kinoshita and H.Terasaka: *On unions of knots*, Osaka Math. J. **9** (1957), 131–153.
[9] C. Lamm: *Symmetric unions and ribbon knots*, Osaka J. Math., **37** (2000), 537–550.
[10] C. Lamm: *Symmetric union presentations for 2-bridge ribbon knots*, arXiv:math.GT/0602395.
[11] W.B.R. Lickorish: An introduction to knot theory, Graduate Texts in Mathematics **175**, Springer-Verlag, New York, 1997.
[12] W.B.R. Lickorish: *Prime knots and tangles*, Trans. Amer. Math. Soc. **267** (1981), 321–332.
[13] P. Lisca: *Lens spaces, rational balls and the ribbon conjecture*, Geom. Topol. **11** (2007), 429–472.
[14] J. Schultens: Introduction to 3-manifolds, Graduate Studies in Mathematics **151**, American Mathematical Society, Providence, RI, 2014.
[15] T. Tanaka: *The Jones polynomial of knots with symmetric union presentations*, J. Korean Math. Soc. **52** (2015), 389–402.

Department of Mathematics, Faculty of Education
Gifu University
Yanagido 1–1, Gifu, 501–1193
Japan
e-mail: tanakat@gifu-u.ac.jp