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NEIGHBORHOOD COMPLEXES AND KRONECKER DOUBLE COVERINGS

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Abstract

The neighborhood complex $N(G)$ is a simplicial complex assigned to a graph G whose connectivity gives a lower bound for the chromatic number of G. We show that if the Kronecker double coverings of graphs are isomorphic, then their neighborhood complexes are isomorphic. As an application, for integers m and n greater than 2, we construct connected graphs G and H such that $N(G) \cong N(H)$ but $\chi(G) = m$ and $\chi(H) = n$. We also construct a graph $KG'_{n,k}$ such that $KG'_{n,k}$ and the Kneser graph $KG_{n,k}$ are not isomorphic but their Kronecker double coverings are isomorphic.

1. Introduction

The neighborhood complex was introduced by Lovász in his proof of Kneser's conjecture [7]. He assigned a simplicial complex $N(G)$ to a graph G, and showed that a certain homotopy invariant $\mathrm{conn}(N(G))$, called the connectivity, gives a lower bound for the chromatic number. He used this method to compute the chromatic number of the Kneser graphs $KG_{n,k}$. After that, topological methods in graph coloring problems have been studied by many authors. We refer to [5] for the background of this subject.

In the study of neighborhood complexes, the following question is quite fundamental: Does the isomorphism type (homeomorphism type, or homotopy type) of $N(G)$ determine the chromatic number $\chi(G)$? Actually, this problem was negatively solved. Walker [10] and Matsushita [9] deal with many examples of graphs whose neighborhood complexes are homotopy equivalent but whose chromatic numbers are different. Moreover, Walker [10] gave examples that for every $n \geq 2$, there are graphs G and H such that $\chi(G) = n$ and $\chi(H) = n + 1$, but their neighborhood complexes are isomorphic.

The purpose of this paper is to improve Walker's result:

Theorem 1.1. *Let m and n be integers greater than 2. Then there are connected graphs G and H such that $\chi(G) = m$, $\chi(H) = n$, but their neighborhood complexes are isomorphic.*

The method employed here is different from Walker's. In this paper, we observe that the following close relation between neighborhood complexes $N(G)$ and Kronecker double coverings $K_2 \times G$ (The precise definitions will be found in Section 2).

Theorem 1.2. *Let G and H be graphs. If $K_2 \times G \cong K_2 \times H$, then $N(G) \cong N(H)$. On the other hand, if G and H are stiff and $N(G) \cong N(H)$, then $K_2 \times G \cong K_2 \times H$.*

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This theorem will be proved in Section 2. Thus to prove Theorem 1.1, it suffices to construct graphs $X(m,n)$ and $Y(m,n)$ such that $\chi(X(m,n)) = m$ and $\chi(Y(m,n)) = n$, but $K_2 \times X(m,n) \cong K_2 \times Y(m,n)$, and this will be done in Example 3.3.

Theorem 1.2 asserts that the neighborhood complex is determined by its Kronecker double covering. Thus the Kronecker double covering gives a restriction on the chromatic number. In Section 3, we construct a simple graph $KG'_{n,k}$ for $n > 2k \geq 4$ such that $K_2 \times KG'_{n,k} \cong K_2 \times KG_{n,k}$ but $KG'_{n,k} \not\cong KG_{n,k}$ (Theorem 3.5). By the connectivity of $N(KG'_{n,k}) = N(KG_{n,k})$, we prove $\chi(KG'_{n,k}) = n - 2k + 2$ (Theorem 3.6).

Finally, we make a remark on the box complex [2, 8]. The box complex $B(G)$ is a $\mathbb{Z}/2$-space assigned to a graph, whose underlying space is homotopy equivalent to $N(G)$. Moreover, a certain $\mathbb{Z}/2$-homotopy invariant of $B(G)$, called $\mathbb{Z}/2$-index, is a lower bound for $\chi(G)$ sharper than $\mathrm{conn}(N(G))$ (see [8]).

One can ask if a similar assertion to Theorem 1.1 holds for box complexes. Since $N(G) \simeq B(G)$, it is clear that $K_2 \times G \cong K_2 \times H$ implies $B(G) \simeq B(H)$. However, there are many definitions of box complexes, and these definitions are not isomorphic but only $\mathbb{Z}/2$-homotopy equivalent. Hence the isomorphism problem concerning box complexes is not so reasonable although $K_2 \times G \cong K_2 \times H$ implies $B(G) \cong B(H)$ for every definition of box complexes as far as the author knows.

On the other hand, it is meaningful to ask if $K_2 \times G \cong K_2 \times H$ implies that $B(G)$ and $B(H)$ are $\mathbb{Z}/2$-homotopy equivalent. However, the graphs constructed in Example 3.3 are counter examples to this question (see Remark 3.4).

2. Neighborhood complexes

Here we review definitions and facts concerning neighborhood complexes, and show Theorem 1.2. For a comprehensive introduction to this subject, we refer to [5].

A *graph* is a pair $G = (V(G), E(G))$ consisting of a finite set $V(G)$ together with a symmetric binary relation $E(G)$ of $V(G)$. For a pair v and w of vertices of G, we write $v \sim w$ to mean $(v,w) \in E(G)$. A *graph homomorphism* from a graph G to a graph H is a map $f\colon V(G) \to V(H)$ such that $(f \times f)(E(G)) \subset E(H)$. Let K_n be the graph defined by $V(K_n) = \{1, \cdots, n\}$ and $E(K_n) = \{(i,j) \mid i \neq j\}$. The *chromatic number* $\chi(G)$ *of* G is the number

$$\min\{n \geq 0 \mid \text{There is a graph homomorphism } G \to K_n\}.$$

Let G be a graph and v a vertex of G. Let $N(v)$ be the set of vertices adjacent to v. The *neighborhood complex* $N(G)$ is the simplicial complex

$$N(G) = \{\sigma \subset V(G) \mid \sigma \text{ is finite and } \sigma \subset N(v) \text{ for some } v\}$$

whose underlying set is $V(G)$. Lovász [7] showed that if $N(G)$ is n-connected, then $\chi(G) > n + 2$. He used this method to determine the chromatic numbers of Kneser graphs $KG_{n,k}$ defined as follows: Let n and k be positive integers satisfying $n \geq 2k$. Then the *Kneser graph* $KG_{n,k}$ is the graph defined by

$$V(KG_{n,k}) = \{\sigma \subset \{1, \cdots, n\} \mid |\sigma| = k\},\ E(KG_{n,k}) = \{(\sigma, \tau) \mid \sigma, \tau \in V(KG_{n,k}),\ \sigma \cap \tau = \emptyset\}.$$

It is easy to see $\chi(KG_{n,k}) \leq n-2k+2$. Lovász showed that $N(KG_{n,k})$ is $(n-2k-1)$-connected, and hence $\chi(KG_{n,k}) = n - 2k + 2$.

Next we recall the definition of Kronecker double coverings. The *categorical product* of G and H is the graph $G \times H$ defined by $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{((v, w), (v', w')) \mid (v, v') \in E(G), (w, w') \in E(H)\}$. The *Kronecker double covering of* G is the product $K_2 \times G$. For a more detailed discussion on the Kronecker double covering, see Section 3 or [4]. The projection $K_2 \times G \to G$, $(i, v) \mapsto v$ is a covering. Here a *covering* means a graph homomorphism $f\colon G \to H$ such that $f|_{N(v)}\ N(v) \to N(f(v))$ is bijective for every $v \in V(G)$. It is easy to see that for a connected graph G, $K_2 \times G$ is connected if and only if $\chi(G) > 2$.

Now we start the proof of Theorem 1.2. In fact, this theorem is deduced from an observation of [1] concerning neighborhood hypergraphs. However, we first give a direct short proof for reader's convenience. We start with the following easy observation:

Lemma 2.1. $N(K_2 \times G) \cong N(G) \sqcup N(G)$

Proof. For $i = 1, 2$, define $f_i\colon V(G) \to V(K_2 \times G)$ by $f_i(v) = (i, v)$. Then the sum $f_1 + f_2\colon V(G) \sqcup V(G) \to V(K_2 \times G)$ gives an isomorphism $N(G) \sqcup N(G) \to N(K_2 \times G)$. □

A graph G is *stiff* if for every pair of vertices v and w, $N(v) \subset N(w)$ implies $v = w$. Let $F(N(G))$ denote the set of facets of $N(G)$. Then the stiffness of graphs means the map $V(G) \to F(N(G))$, $v \mapsto N(v)$ is well-defined and bijective.

Before giving the proof of Theorem 1.2, we prove the following lemma:

Lemma 2.2. *Let K and L be finite simplicial complexes. If $K \sqcup K$ and $L \sqcup L$ are isomorphic, then K and L are isomorphic.*

Proof. Let $X_1, \cdots, X_r$ be the connected components of K. We prove this lemma by induction on the number r of connected components of K. The case $r = 0$ is clear.

Let X_i' be a copy of X_i, and so $K \sqcup K = (X_1 \sqcup X_1') \sqcup \cdots \sqcup (X_r \sqcup X_r')$. Similarly, let $Y_1, \cdots, Y_s$ be the connected components of L and so that $L \sqcup L = (Y_1 \sqcup Y_1') \sqcup \cdots \sqcup (Y_s \sqcup Y_s')$. Let $f\colon K \sqcup K \to L \sqcup L$ be an isomorphism. By changing indices of Y_i and exchanging Y_i and Y_i', we can assume $f(X_1) = Y_1$. Then $f(X_1')$ is a connected component of $L \sqcup L$ other than Y_1. Note that $f(X_1')$ and Y_1' are isomorphic since $f(X_1') \cong X_1' \cong X_1 \cong Y_1 \cong Y_1'$ Let $g\colon L \sqcup L \to L \sqcup L$ be an isomorphism which exchanges $f(X_1)$ and Y_1' and fixes other components. Then we have $gf(X_1) = Y_1$ and $gf(X_1') = Y_1'$.

Set $K' = X_2 \sqcup \cdots \sqcup X_r$ and $L' = Y_2 \sqcup \cdots \sqcup Y_s$. Then gf induces an isomorphism between $K' \sqcup K'$ and $L' \sqcup L'$. By the inductive hypothesis, we have $K' \cong L'$. Since X_1 and Y_1 are isomorphic, we conclude $K = X_1 \sqcup K' \cong Y_1 \sqcup L' = L$. □

Proof of Theorem 1.2. If $K_2 \times G \cong K_2 \times H$, then Lemma 2.1 implies $N(G) \sqcup N(G) \cong N(H) \sqcup N(H)$, and hence Lemma 2.2 implies $N(G) \cong N(H)$.

On the other hand, suppose G and H are stiff, and let $\varphi\colon V(G) \to V(H)$ be an isomorphism from $N(G)$ to $N(H)$. Define the maps $f\colon V(G) \to V(H)$ and $g\colon V(H) \to V(G)$ by $N(f(v)) = \varphi(N(v))$ and $N(g(w)) = \varphi^{-1}(N(w))$ for all $v \in V(G)$ and $w \in V(H)$. Moreover, define the maps $\tilde{f}\colon V(K_2 \times G) \to V(K_2 \times H)$ and $\tilde{g}\colon V(K_2 \times H) \to V(K_2 \times G)$ by

$$\tilde{f}(0, v) = (0, \varphi(v)),\ \tilde{f}(1, v) = (1, f(v)),\ \tilde{g}(0, w) = (0, \varphi^{-1}(w)),\ \tilde{g}(1, w) = (1, g(w))$$

for $v \in V(G)$ and $w \in V(H)$. Then $\tilde{f}$ and $\tilde{g}$ are graph homomorphisms, and $\tilde{g}$ is the inverse of $\tilde{f}$. □

Now we explain that Theorem 1.2 is easily deduced from an observation in [1] concerning neighborhood hypergraphs. To see this, we need some terminology and notation.

Recall that a *(multi-)hypergraph* is a pair $\mathcal{H} = (V(\mathcal{H}), \mathcal{H})$ consisting of a set $V(\mathcal{H})$ together with a multi-set of $V(\mathcal{H})$, i.e. a function $\mathcal{H}\colon 2^{V(\mathcal{H})} \to \mathbb{N}$. The *neighborhood hypergraph $\mathcal{N}(G)$ of a graph* G is the multi-hypergraph on $V(G)$ whose multi-set of hyperedges is $\mathcal{N}(G) = \{N(v) \mid v \in V(G)\}$, in other words, $\mathcal{N}(G)(S) = \#\{S = N(v) \mid v \in V(G)\}$ for $S \in 2^{V(G)}$.

For a hypergraph $\mathcal{H}$, define the bigraph representation $B_{\mathcal{H}}$ (the precise definition of bigraphs will be found in the beginning of Section 3) as follows: the vertex set of $B_{\mathcal{H}}$ is $V(\mathcal{H}) \sqcup \mathcal{H}$, and $v \in V(\mathcal{H})$ and $S \in \mathcal{H}$ are adjacent if and only if $v \in S$. There is no other adjacent relation among vertices of $B_{\mathcal{H}}$. The bigraph $B_{\mathcal{H}}$ determines the original hypergraph $\mathcal{H}$. In fact, they used this method to show that for bipartite graphs G and H, $G \cong H$ if and only if $\mathcal{N}(G) \cong \mathcal{N}(H)$.

From the above observation of [1], one can easily show Theorem 1.2 as follows: Clearly, the bigraph representation $B_{\mathcal{N}(G)}$ of the neighborhood hypergraph $\mathcal{N}(G)$ coincides with the Kronecker double covering $K_2 \times G$. This means that $K_2 \times G \cong B_{\mathcal{N}(G)}$ determines $\mathcal{N}(G)$. Since the neighborhood complex $N(G)$ is determined by $\mathcal{N}(G)$, we have that $K_2 \times G$ determines $N(G)$.

On the other hand, if a graph G is stiff, then the neighborhood complex $N(G)$ determines the neighborhood hypergraph $\mathcal{N}(G)$. In fact, the multi-set of hyperedges of $\mathcal{N}(G)$ is the set of facets of $N(G)$ in this case. Thus if G and H are stiff and $N(G) \cong N(H)$, then we have $\mathcal{N}(G) \cong \mathcal{N}(H)$ and hence $K_2 \times G \cong K_2 \times H$. This completes the proof of Theorem 1.2.

We close this section with a few remarks.

Remark 2.3. There are graphs whose neighborhood complexes are isomorphic but whose Kronecker double coverings are different. In fact, consider the 4-cycle graph C_4 and the path graph P_4 with 4 vertices. Then the neighborhood complexes of these graphs are two 1-simplices, but $K_2 \times C_4 = C_4 \sqcup C_4$ and $K_2 \times P_4 = P_4 \sqcup P_4$.

Remark 2.4. Theorem 1.2 asserts that the neighborhood complex $N(G)$ is determined by the Kronecker double covering $K_2 \times G$. Thus if $N(G)$ is n-connected and $K_2 \times G \cong K_2 \times H$, then $N(H)$ is also n-connected, and hence we have $\chi(H) > n + 2$. This means that the Kronecker double covering restricts the chromatic number.

We construct graphs $KG'_{n,k}$ in Section 3 such that $K_2 \times KG'_{n,k} \cong K_2 \times KG_{n,k}$ but $KG'_{n,k} \not\cong KG_{n,k}$ for $n > 2k \geq 4$. Since $N(KG_{n,k})$ is $(n-2k-1)$-connected (see Section 2), this means $\chi(KG'_{n,k}) \geq n - 2k + 2$.

3. Kronecker double coverings

In this section, we review the theory of Kronecker double coverings, and construct graphs $X(m,n)$ and $Y(m,n)$ such that $\chi(X(m,n)) = m$ and $\chi(Y(m,n)) = n$ but $K_2 \times X(m,n) \cong K_2 \times Y(m,n)$ in Example 3.3. This shows Theorem 1.1. Moreover, we construct a family of graphs $KG'_{n,k}$ such that $K_2 \times KG_{n,k} \cong K_2 \times KG'_{n,k}$ but $KG_{n,k} \not\cong KG'_{n,k}$.

We review the Kronecker double coverings from a viewpoint of bigraphs, that is, graphs with 2-colorings. For the sake of this treatment, one can obtain a simple description of the categorical equivalence given in Theorem 3.1.

A *bigraph*[1] is a graph X equipped with a 2-coloring $\varepsilon_X\colon X \to K_2$. A *bigraph homomorphism* is a graph homomorphism $f\colon X \to Y$ such that $\varepsilon_Y \circ f = \varepsilon_X$. Let $\mathcal{G}$ be the category of graphs whose morphisms are graph homomorphisms, and $\mathcal{G}_{/K_2}$ the category of bigraphs whose morphisms are bigraph homomorphisms. For a graph G, the Kronecker double covering $K_2 \times G$ is a bigraph whose 2-coloring is the 1st projection $K_2 \times G \to K_2$.

An *odd involution of a bigraph X* is a graph homomorphism (not necessarirly a bigraph homomorphism) $\tau\colon X \to X$ satisfying $\tau^2 = \mathrm{id}_X$ and $\varepsilon_X(\tau(v)) \neq \varepsilon_X(v)$ for every $v \in V(X)$. A typical example of odd involutions is the involution $(1, v) \leftrightarrow (2, v)$ of the Kronecker double covering $K_2 \times G$. In fact, the following theorem (Theorem 3.1) asserts that every odd involution is obtained in this way.

We consider the category $\mathcal{G}^{odd}_{/K_2}$ defined as follows. An object of $\mathcal{G}^{odd}_{/K_2}$ is a pair (X, τ) consisting of a bigraph X together with an odd involution τ of it. A morphism from (X, τ) to (X', τ') is a bigraph homomorphism $f\colon X \to X'$ which is equivariant, i.e. $\tau' \circ f = f \circ \tau$. Clearly, the Kronecker double covering gives a functor $\mathcal{K}\colon \mathcal{G} \to \mathcal{G}^{odd}_{/K_2}$, $G \mapsto K_2 \times G$. Moreover, we have the following theorem (see [6] for the terminology of category theory):

Theorem 3.1. *The functor* $\mathcal{K}\colon K_2 \times (-)\colon \mathcal{G} \to \mathcal{G}^{odd}_{/K_2}$ *is a categorical equivalence.*

Proof. We construct a quasi-inverse $\mathcal{Q}\colon \mathcal{G}^{odd}_{/K_2} \to \mathcal{G}$ of $\mathcal{K}$ as follows. For an object (X, τ) of $\mathcal{G}^{odd}_{/K_2}$, define the graph X/τ by $V(X/\tau) = \{\{x, \tau(x)\} \mid x \in V(X)\}$ and

$$E(X/\tau) = \{(\alpha, \beta) \mid \alpha, \beta \in V(X/\alpha), (\alpha \times \beta) \cap E(X) \neq \emptyset\}.$$

Roughly speaking, the graph $\mathcal{Q}(X) = X/\tau$ is the quotient of the graph X by the $\mathbb{Z}/2$-action τ. Then a morphism $f\colon (X, \tau) \to (X', \tau')$ in $\mathcal{G}^{odd}_{/K_2}$ induces a graph homomorphism $\mathcal{Q}(f)\colon X/\tau \to X'/\tau'$, and hence we have a functor $\mathcal{Q}\colon \mathcal{G}^{odd}_{/K_2} \to \mathcal{G}$.

This functor $\mathcal{Q}$ is a quasi-inverse of $\mathcal{K}$. In fact, it is clear that $\mathcal{Q} \circ \mathcal{K}$ and $1_{\mathcal{G}}$ are naturally isomorphic. The natural isomorphism $1_{\mathcal{G}^{odd}_{/K_2}} \to \mathcal{K} \circ \mathcal{Q}$ is given by the map $f\colon X \to K_2 \times (X/\tau)$ defined by $f(x) = (\varepsilon(x), q(x))$, where $q\colon X \to X/\tau$ is the quotient map. It is clear that f is a graph isomorphism. □

Now we turn to the case of bipartite graphs. For a bipartite graph X, an involution $\tau\colon X \to X$ is *odd* if for every $x \in X$, there is no path with even length joining x to $\tau(x)$. If (X, τ) is a bigraph with an odd involution, then τ is odd in the sense of bipartite graphs.

Let X be a bipartite graph with an odd involution τ. In this case, one can construct the quotient graph X/τ in the same way as the proof of Theorem 3.1. Moreover, there is a 2-coloring $\varepsilon\colon X \to K_2$ such that $(X, \tau) \in \mathcal{G}^{odd}_{/K_2}$. Therefore by Theorem 3.1, we have $K_2 \times (X/\tau) \cong X$ as graphs.

REMARK 3.2. Define the category $\mathcal{G}'$ as follows. An object of $\mathcal{G}'$ is a bipartite graph X together with its odd involution τ. A morphism from (X, τ) to (X', τ') is a graph homomorphism $f\colon X \to X'$ satisfying $\tau' \circ f = f \circ \tau$. Then the Kronecker double covering gives a functor $\mathcal{K}'\colon \mathcal{G} \to \mathcal{G}'$. However, this functor is not a categorical equivalence. In fact, there is

[1]This terminology is due to [1].

no map $f\colon G \to G$ such that $K_2 \times f = \tau$, where τ is the canonical odd involution of $K_2 \times G$.

Now we are ready to prove Theorem 1.1.

Example 3.3. We construct graphs $X(m,n)$ and $Y(m,n)$ such that $K_2 \times X(m,n) \cong K_2 \times Y(m,n)$ but $\chi(X(m,n)) = m$ and $\chi(Y(m,n)) = n$. By Theorem 1.2, this completes the proof of Theorem 1.1.

First, set $X_1 = X_2 = K_2 \times K_n$ and $Y_1 = Y_2 = K_2 \times K_m$. Define the graph $Z(m,n)$ by identifying the following vertices of $X_1 \sqcup X_2 \sqcup Y_1 \sqcup Y_2$:

- $(1,1) \in V(X_1)$ and $(1,1) \in V(Y_1)$.
- $(2,1) \in V(X_1)$ and $(1,1) \in V(Y_2)$.
- $(1,1) \in V(X_2)$ and $(2,1) \in V(Y_1)$.
- $(2,1) \in V(X_2)$ and $(2,1) \in V(Y_2)$.

It is clear that $Z(m,n)$ is bipartite and connected. Figure 1 depicts the graph $Z(m,n)$ in the case $m = 4$ and $n = 3$.

Next we define the odd involutions τ_1, τ_2 of $Z(m,n)$. First τ_1 maps X_i to X_i for each i and $\tau_1|_{X_i}$ is the natural involution of $X_1 = X_2 = K_2 \times K_n$, flipping K_2. On $Y_1 \sqcup Y_2$, the involution τ_1 exchanges Y_1 and Y_2, and is given by $V(Y_1) \ni (\varepsilon, x) \leftrightarrow (\varepsilon, x) \in V(Y_2)$. Similarly, τ_2 maps Y_i to Y_i for each i and $\tau_2|_{Y_i}$ is the natural involution of $K_2 \times K_m$, flipping K_2. On $X_1 \sqcup X_2$, the involution τ_2 is given by $V(Y_1) \ni (\varepsilon, x) \leftrightarrow (\varepsilon, x) \in V(X_2)$.

Set $X(m,n) = Z(m,n)/\tau_1$ and $Y(m,n) = Z(m,n)/\tau_2$. To complete the proof, we need to check $\chi(X(m,n)) = m$ and $\chi(Y(m,n)) = n$. We only prove $\chi(X(m,n)) = n$ since the other is similarly shown. However, this clearly follows from the following description of $X(m,n)$: $X(m,n)$ is obtained by identifying the following vertices of $X'_1 \sqcup X'_2 \sqcup (K_2 \times K_m)$, where $X'_1 = X'_2 = K_m$:

- $1 \in V(X'_1) = V(K_m)$ and $(1,1) \in V(K_2 \times K_n)$.
- $1 \in V(X'_2) = V(K_m)$ and $(2,1) \in V(K_2 \times K_n)$.

Figure 1 depicts the graphs $X(m,n)$ and $Y(m,n)$ in the case $m = 4$ and $n = 3$. In this

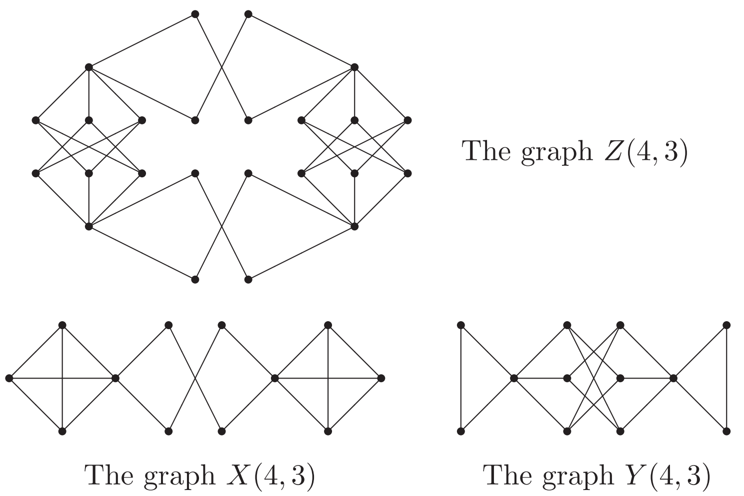

Fig. 1

figure, the involution τ_1 is the reflection in the horizontal line, and the involution τ_2 is the reflection in the vertical line.

REMARK 3.4. The box complexes of $X(m,n)$ and $Y(m,n)$ are not $\mathbb{Z}/2$-homotopy equivalent if $m \neq n$. To see this, we need the following fact: The box complex is a functor from the category of graphs to the category of $\mathbb{Z}/2$-spaces, and $B(K_n)$ and S^{n-2} are $\mathbb{Z}/2$-homotopy equivalent (Proposition 5 of [8]).

One can suppose $m < n$. Then K_n is a subgraph of $Y(m,n)$ and hence there is a $\mathbb{Z}/2$-map from $B(K_n) \simeq_{\mathbb{Z}/2} S^{n-2}$ to $B(Y(m,n))$. If $B(X(m,n)) \simeq_{\mathbb{Z}/2} B(Y(m,n))$, then there is a $\mathbb{Z}/2$-map from S^{n-2} to $B(X(m,n))$. However, since $\chi(X(m,n)) = m$, there is a $\mathbb{Z}/2$-map from $B(X(m,n))$ to $B(K_m) \simeq_{\mathbb{Z}/2} S^{m-2}$. Thus we have a $\mathbb{Z}/2$-map from S^{n-2} to S^{m-2}, but this contradicts the Borsuk-Ulam theorem.

In the rest of this paper, we discuss a family of simple graphs $KG'_{n,k}$ which satisfies the following interesting property: The Kronecker double covering of $KG'_{n,k}$ is isomorphic to the Kronecker double covering of $KG_{n,k}$, but $KG'_{n,k} \not\cong KG_{n,k}$ for $n > 2k \geq 4$. In the case of $n = 5$ and $k = 2$, Imrich and Pisanski [4] shows that there is a graph G such that $K_2 \times G \cong K_2 \times KG_{5,2}$ but $G \not\cong KG_{5,2}$.

Let n and k be integers satisfying $n > 2k \geq 4$. First, let α be the automorphism of the n-point set $\{1, \cdots, n\}$ which exchanges n and $n-1$ and fixes the remaining points. Define the odd involution τ of $K_2 \times KG_{n,k}$ by

$$(1, \sigma) \leftrightarrow (2, \alpha(\sigma))$$

for $\sigma \in V(KG_{n,k})$. Then we set $KG'_{n,k} = (K_2 \times KG_{n,k})/\tau$.

Theorem 3.5. *$KG'_{n,k}$ is simple and $K_2 \times KG'_{n,k} \cong K_2 \times KG_{n,k}$ but $KG'_{n,k} \not\cong KG_{n,k}$.*

Proof. It clearly follows from Theorem 3.1 that $K_2 \times KG'_{n,k} \cong K_2 \times KG_{n,k}$. We show that $KG_{n,k} \not\cong KG'_{n,k}$. Since there is no vertex x of $K_2 \times KG_{n,k}$ such that $x \sim \tau(x)$, $KG'_{n,k}$ is a simple graph.

First we introduce the following notation which indicates a vertex of $KG'_{n,k}$. Let $\{i_1, \cdots, i_k\}$ be a k-subset of $\{1, \cdots, n\}$ with $i_1 < \cdots < i_k$. If $n, n-1 \notin \{i_1, \cdots, i_k\}$ or $\{n-1, n\} \subset \{i_1, \cdots, i_k\}$, we write $(i_1, \cdots, i_k)$ to indicate the vertex $\{(1, \{i_1, \cdots, i_k\}), (2, \{i_1, \cdots, i_k\})\}$ of $KG'_{n,k}$. If $i_k = n-1$, then we denote by $(i_1, \cdots, i_{k-1}, \alpha)$ the vertex $\{(1, \{i_1, \cdots, i_k\}), (2, \alpha\{i_1, \cdots, i_k\})\}$ of $KG'_{n,k}$, and by $(i_1, \cdots, i_{k-1}, \beta)$ the vertex $\{(1, \alpha\{i_1, \cdots, i_k\}), (2, \{i_1, \cdots, i_k\})\}$ of $KG'_{n,k}$. In this notation, we have the following adjacent relation:

- If $i_k, j_k < n-1$, then $(i_1, \cdots, i_k) \sim (j_1, \cdots, j_k)$ iff $\{i_1, \cdots, i_k\} \cap \{j_1, \cdots, j_k\} = \emptyset$.
- $(i_1, \cdots, i_{k-1}, \alpha) \not\sim (j_1, \cdots, j_{k-1}, \beta)$
- $(i_1, \cdots, i_{k-1}, \alpha) \sim (j_1, \cdots, j_{k-1}, \alpha)$ and $(i_1, \cdots, i_{k-1}, \beta) \sim (j_1, \cdots, j_{k-1}, \beta)$ iff $\{i_1, \cdots, i_{k-1}\} \cap \{j_1, \cdots, j_{k-1}\} = \emptyset$.

Next we recall the following property of the maximum independent sets of the Kneser graphs. For $i = 1, \cdots, n$, let A_i be the set of vertices of $KG_{n,k}$ which contains i. Recall that the Erdős-Ko-Rado theorem [3] states that $A_1, \cdots, A_n$ are the maximum independent sets of $KG_{n,k}$. This family of maximum independent sets of $KG_{n,k}$ clearly satisfies the following property: For a pair of k-subsets $\{i_1, \cdots, i_k\}$ and $\{j_1, \cdots, j_k\}$ of $\{1, \cdots, n\}$, the intersection $A_{i_1} \cap \cdots \cap A_{i_k}$ is a one point set, and if $A_{i_1} \cap \cdots \cap A_{i_k} = A_{j_1} \cap \cdots \cap A_{j_k}$, then we have

$\{i_1, \cdots, i_k\} = \{j_1, \cdots, j_k\}$.

Now we are ready to prove $KG'_{n,k} \not\cong KG_{n,k}$. Suppose $KG_{n,k} \cong KG'_{n,k}$. For $i = 1, \cdots, n-2$, let B_i be the set of vertices of $KG'_{n,k}$ containing i. Then each B_i is a maximum independent set of $KG'_{n,k}$ since $KG_{n,k} \cong KG'_{n,k}$ and $|B_i| = \binom{n-1}{k-1}$. There are two maximum independent sets C_1 and C_2 of $KG'_{n,k}$ different from $B_1, \cdots, B_{n-2}$.

Consider the intersection $B_1 \cap \cdots \cap B_{k-1} \cap C_1$. By the above property of Kneser graphs, this determines a vertex. If $B_1 \cap \cdots \cap B_{k-1} \cap C_1 = \{(1, \cdots, k-1, m)\}$ with $m < n-1$, then we have $B_1 \cap \cdots \cap B_{k-1} \cap B_m = B_1 \cap \cdots \cap B_{k-1} \cap C_1$, and this contradicts the above property of Kneser graphs. Hence we have $B_1 \cap \cdots \cap B_{k-1} \cap C_1 = \{(1, \cdots, k-1, \alpha)\}$ or $\{(1, \cdots, k-1, \beta)\}$. We assume that $B_1 \cap \cdots \cap B_{k-1} \cap C_1 = \{(1, \cdots, k-1, \alpha)\}$ since the other is similarly proved. In particular, we have $(1, \cdots, k-1, \alpha) \in C_1$.

By indcution, we show $(m, \cdots, m+k-2, \alpha) \in C_1$ for $m = 1, 2, \cdots, k$. Suppose that $(m, \cdots, m+k-2, \alpha) \in C_1$. Let $\{i_1, \cdots, i_{k-1}\}$ be a $(k-1)$-subset of $\{1, \cdots, n-2\}$ such that $\{m, \cdots, m+k-1\} \cap \{i_1, \cdots, i_{k-1}\} = \emptyset$. Considering the intersection $B_{i_1} \cap \cdots \cap B_{i_{k-1}} \cap C_1$, we deduce that $(i_1, \cdots, i_k, \alpha) \in C_1$ or $(i_1, \cdots, i_k, \beta) \in C_1$ in a similar way. Since C_1 is independent and $(m, \cdots, m+k-2, \alpha) \sim (i_1, \cdots, i_{k-1}, \alpha)$, we have that $(i_1, \cdots, i_{k-1}, \beta) \in C_1$. Next by considering the intersection $B_{m+1} \cap \cdots \cap B_{m+k-1} \cap C_1$, we have that $(m+1, \cdots, m+k-1, \alpha) \in C_1$ or $(m+1, \cdots, m+k-1, \beta) \in C_1$. Since C_1 is independent and the $(i_1, \cdots, i_{k-1}, \beta) \sim (m+1, \cdots, m+k-1, \beta)$, we have $(m+1, \cdots, m+k-1, \alpha) \in C_1$. Thus the induction follows.

Hence we have $(1, \cdots, k-1, \alpha), (k, \cdots, 2k-2, \alpha) \in C_1$. However, C_1 is independent and $(1, \cdots, k-1, \alpha) \sim (k, \cdots, 2k-2, \alpha)$. This is a contradiction. □

We close this paper with determining the chromatic number of $KG'_{n,k}$.

Theorem 3.6. $\chi(KG'_{n,k}) = n - 2k + 2$

Proof. Since $K_2 \times KG'_{n,k} \cong K_2 \times KG_{n,k}$, it follows from Theorem 1.2 that $N(KG'_{n,k}) = N(KG_{n,k})$. Since $N(KG_{n,k})$ is $(n-2k-1)$-connected, we have that $\chi(KG'_{n,k}) \geq n - 2k + 2$. So it suffices to construct an $(n-2k+2)$-coloring on $KG'_{n,k}$.

This is proved by induction on n. First, note that $KG_{2k,k}$ is copies of K_2, and hence $K_2 \times KG_{2k,k}$ is also copies of K_2. Since $KG'_{2k,k} = (K_2 \times KG_{2k,k})/\tau$ is simple, we have that $KG'_{2k,k}$ is copies of K_2.

By the notation introduced in the proof of Theorem 3.5, it is clear that $KG'_{n,k}$ is an induced subgraph of $KG'_{n+1,k}$. The set of vertices of $KG'_{n+1,k}$ not contained in $KG'_{n,k}$ is B_{n-1} in the proof of Theorem 3.5. Since B_{n-1} is an independent set, we can construct an $(n-2k+3)$-coloring c of $KG'_{n+1,k}$ as follows:

$$c(x) = \begin{cases} c'(x) & (x \in V(KG'_{n,k})) \\ n - 2k + 3 & (x \in B_{n-1}). \end{cases}$$

Here c' is an $(n-2k+2)$-coloring of $KG'_{n,k}$. □

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