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ON INFINITESIMAL GENERATORS OF SUBLINEAR MARKOV SEMIGROUPS

Franziska KÜHN

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Abstract

We establish a Dynkin formula and a Courrège-von Waldenfels theorem for sublinear Markov semigroups. In particular, we show that any sublinear operator A on $C_c^\infty(\mathbb{R}^d)$ satisfying the positive maximum principle can be represented as supremum of a family of pseudo-differential operators:

$$Af(x) = \sup_{\alpha \in I}(-q_\alpha(x,D)f)(x).$$

As an immediate consequence, we obtain a representation formula for infinitesimal generators of sublinear Markov semigroups with a sufficiently rich domain. We give applications in the theory of non-linear Hamilton–Jacobi–Bellman equations and Lévy processes for sublinear expectations.

1. Introduction

Let $(T_t)_{t\geq 0}$ be a Markov semigroup of linear operators on the space $\mathcal{B}_b(\mathbb{R}^d)$ of bounded Borel measurable functions, i.e. a family of contractive linear operators $T_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d)$ satisfying the semigroup property and the sub-Markov property ($0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$). Many properties of the semigroup $(T_t)_{t\geq 0}$ can be characterized via the associated infinitesimal generator

$$Af(x) := \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t}, \qquad f \in \mathcal{D}(A),\ x \in \mathbb{R}^d,$$

whose domain $\mathcal{D}(A)$ is defined in such a way that the limit exists in a suitable sense, cf. Section 2. Strongly continuous Markov semigroups are uniquely determined by their generator $(A, \mathcal{D}(A))$, cf. [13, Corollary I.4.1.35]. If the domain $\mathcal{D}(A)$ of the infinitesimal generator is sufficiently rich, in the sense that the compactly supported smooth functions $f \in C_c^\infty(\mathbb{R}^d)$ belong to $\mathcal{D}(A)$, then a result due to Courrège [5] and von Waldenfels [22, 23] states that $A|_{C_c^\infty(\mathbb{R}^d)}$ has a representation of the form

$$\begin{aligned} Af(x) = &-c(x)f(x) + b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\cdot\nabla^2 f(x)) \\ &+ \int_{y\neq 0} (f(x+y) - f(x) - y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu(x,dy), \qquad x \in \mathbb{R}^d. \end{aligned}$$

Equivalently, $A|_{C_c^\infty(\mathbb{R}^d)}$ can be written as a pseudo-differential operator with negative definite symbol, see Section 2 for details. In combination with Dynkin's formula

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$$T_t f - f = \int_0^t T_s A f \, ds, \qquad f \in \mathcal{D}(A),\ t \geq 0,$$

cf. [13, Lemma I.4.1.14], which can be seen as a counterpart of the fundamental theorem of calculus, this representation formula for A has turned out to be a very powerful tool in the study of Markov semigroups and the associated Markov processes, cf. [4, 11, 13].

In this paper, we extend Dynkin's formula and the Courrège-Waldenfels theorem to *sub*linear Markov semigroups $(T_t)_{t\geq 0}$, i.e. the operators T_t are no longer assumed to be linear but only subadditive and positively homogeneous:

$$\forall f, g, \forall \lambda \in [0,\infty) \ : \ T_t(f+g) \leq T_t(f) + T_t(g) \qquad T_t(\lambda f) = \lambda T_t(f).$$

Sublinear Markov semigroups appear naturally in the study of stochastic processes on sublinear expectation spaces. They can be interpreted as stochastic processes under uncertainty, cf. Hollender [12], and in many cases the semigroup has a representation of the form

$$T_t f(x) = \sup_{\mathbb{P} \in \mathfrak{P}^x} \mathbb{E}_{\mathbb{P}} f(X_t) := \sup_{\mathbb{P} \in \mathfrak{P}^x} \int_\Omega f(X_t) \, d\mathbb{P}$$

where the supremum is taken over a family of probability measures $\mathfrak{P}^x$ which depends on the starting point $x \in \mathbb{R}^d$. As in case of classical Markov semigroups, it is possible to associate an evolution equation with sublinear semigroups $(T_t)_{t\geq 0}$,

$$\frac{\partial}{\partial t} u(t,x) - A_x u(t,x) = 0 \qquad u(0,x) = f(x),$$

where A is the (sublinear) infinitesimal generator, cf. [12, Proposition 4.10]. In a recent paper, Denk et al. [8] studied under which conditions a sublinear semigroup is uniquely determined by its infinitesimal generator. The Courrège-von Waldenfels theorem which we derive in this paper, cf. Corollary 4.3, shows that sublinear generators with a sufficiently rich domain have a representation of the form

$$\begin{aligned} Af(x) = \sup_{\alpha \in I} \Big(& - c_\alpha(x) f(x) + b_\alpha(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q_\alpha(x) \cdot \nabla^2 f(x)) \\ & + \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \, \nu_\alpha(x, dy) \Big), \quad f \in C_c^\infty(\mathbb{R}^d), \end{aligned}$$

and therefore sublinear Markov semigroups play an important role in the study of non-linear Hamilton–Jacobi–Bellman (HJB) equations,

$$\begin{aligned} (1) \quad \partial_t u(t,x) - \sup_{\alpha \in I} \Big(& - c_\alpha(x) u(t,x) + b_\alpha(x) \cdot \nabla_x u(t,x) + \frac{1}{2} \operatorname{tr}(Q_\alpha(x) \cdot \nabla_x^2 u(t,x)) \\ & + \int_{y \neq 0} (u(t,x+y) - u(t,x) - y \cdot \nabla_x u(t,x) \mathbb{1}_{(0,1)}(|y|)) \, \nu_\alpha(x, dy) \Big) = 0. \end{aligned}$$

The idea to approach non-linear equations via stochastic processes on non-linear expectation spaces goes back to Peng [19] who introduced the so-called G-Brownian motion to study the G-heat equation

$$\partial_t u(t,x) - \frac{1}{2} \sup_{\alpha \in I} (\operatorname{tr}(Q_\alpha \cdot \nabla_x^2 u(t,x))) = 0.$$

More recently, the connection between non-linear integro-differential equations and nonlinear semigroups has been investigated in [7, 12, 16, 18]. We will establish a general result which shows that for any "nice" sublinear semigroup $(T_t)_{t\geq 0}$ the mapping $u(t,x) := T_t f(x)$ is a viscosity solution to an HJB equation of the form (1), cf. Section 5.

The paper is structured as follows. After introducing basic definitions and notation in Section 2, we present a generalization of Dynkin's formula for sublinear Markov semigroups in Section 3. The Courrège-von Waldenfels theorem for sublinear operators is stated and proved in Section 4. We use the Courrège-von Waldenfels theorem to study the connection between HJB equations (1) and sublinear Markov semigroups, cf. Section 5. Some applications in the theory of Lévy processes for sublinear expectations are presented in Section 6.

2. Definitions and notation

Sublinear semigroups: Let $\mathcal{H}$ be a family of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\alpha f + \beta g \in \mathcal{H}$ for all $\alpha, \beta \in \mathbb{R}$, $f, g \in \mathcal{H}$ and $c\mathbb{1}_{\mathbb{R}^d} \in \mathcal{H}$ for all $c \in \mathbb{R}$. If $(T_t)_{t\geq 0}$ is a family of sublinear operators on $\mathcal{H}$, i.e.

$$\forall f, g \in \mathcal{H},\ \lambda \in [0,\infty)\ :\ T_t(f+g) \leq T_t f + T_t g \qquad T_t(\lambda f) = \lambda T_t f,$$

then we call $(T_t)_{t\geq 0}$ a *sublinear Markov semigroup (on $\mathcal{H}$)* if the following properties are satisfied:

(i) $T_{t+s} = T_t T_s$ for all $s, t \geq 0$, and $T_0 = \mathrm{id}$ (semigroup property),
(ii) $f, g \in \mathcal{H}$, $f \leq g$ implies $T_t f \leq T_t g$ for all $t \geq 0$ (monotonicity),
(iii) $T_t(\mathbb{1}_{\mathbb{R}^d}) \leq \mathbb{1}_{\mathbb{R}^d}$.

If $T_t(c\mathbb{1}_{\mathbb{R}^d}) = c\mathbb{1}_{\mathbb{R}^d}$ for all $c \in \mathbb{R}$, then $(T_t)_{t\geq 0}$ is *conservative*. The *(strong) infinitesimal generator* $(A, \mathcal{D}(A))$ of a sublinear Markov semigroup $(T_t)_{t\geq 0}$ is defined by

$$\mathcal{D}(A) := \left\{ f \in \mathcal{H};\ \exists g \in \mathcal{H}\ :\ \lim_{t\to 0} \left\| \frac{T_t f - f}{t} - g \right\|_\infty = 0 \right\},$$

$$Af := \lim_{t\downarrow 0} \frac{T_t f - f}{t}, \qquad f \in \mathcal{D}(A).$$

If the limit $g(x) := \lim_{t\to 0} t^{-1}(T_t f(x) - f(x))$ exists for all $x \in \mathbb{R}^d$ and defines a function in $\mathcal{H}$, then f is in the domain $\mathcal{D}(A^{(p)})$ of the *pointwise infinitesimal generator* $A^{(p)}$, and we set

$$A^{(p)} f(x) := \lim_{t\downarrow 0} \frac{T_t f(x) - f(x)}{t}, \qquad x \in \mathbb{R}^d,\ f \in \mathcal{D}(A^{(p)}).$$

By definition, the pointwise infinitesimal generator $(A^{(p)}, \mathcal{D}(A^{(p)})$ is an extension of the (strong) infinitesimal generator $(A, \mathcal{D}(A))$. It is immediate that $(A, \mathcal{D}(A))$ and $(A^{(p)}, \mathcal{D}(A^{(p)}))$ are sublinear operators. The next lemma is simple to prove but will play an important role lateron when we investigate the structure of sublinear generators.

Lemma 2.1. *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup. The associated pointwise infinitesimal generator $(A^{(p)}, \mathcal{D}(A^{(p)})$ satisfies the* positive maximum principle*, i.e.*

$$f \in \mathcal{H},\ f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geq 0 \implies A^{(p)} f(x_0) \leq 0.$$

Proof. Fix $f \in \mathcal{H}$ and $x_0 \in \mathbb{R}^d$ with $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x)$. Since T_t is monotone, positively homogeneous and $T_t(1) \leq 1$, we have

$$T_t f(x_0) \leq T_t(\|f\|_\infty)(x_0) \leq \|f\|_\infty = f(x_0).$$

Subtracting $f(x_0)$ on both sides, dividing by $t > 0$ and letting $t \downarrow 0$ yields $A^{(p)} f(x_0) \leq 0$. □

The positive maximum principle clearly also holds for the (strong) generator $(A, \mathcal{D}(A))$. Our standard reference for non-linear semigroups is the monograph by Miyadera [17].

Pseudo-differential operators: Let $q(x, \cdot)$, $x \in \mathbb{R}^d$ be a family of *continuous negative definite* functions with representation

$$(2)\quad q(x,\xi) = c(x) - ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{y\neq 0} (1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|))\, \nu(x, dy), \quad x, \xi \in \mathbb{R}^d,$$

where $c(x) \geq 0$, $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ is positive semidefinite and $\nu(x, dy)$ is a measure such that $\int_{y\neq 0} \min\{1, |y|^2\}\, \nu(x, dy) < \infty$ for each fixed $x \in \mathbb{R}^d$. We will sometimes call $(c(x), b(x), Q(x), \nu(x, dy))$ *characteristics* of $q(x, \cdot)$. The associated *pseudo-differential operator* is defined on the smooth compactly supported functions $C_c^\infty(\mathbb{R}^d)$ by

$$(3)\qquad q(x, D)f(x) := -\int_{\mathbb{R}^d} q(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where $\hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx$ is the Fourier transform of f, and q is called *symbol* of the operator. Equivalently,

$$(4)\qquad \begin{aligned} q(x, D)f(x) &= -c(x)f(x) + b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\cdot\nabla^2 f(x)) \\ &\quad + \int_{y\neq 0} (f(x+y) - f(x) - y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\, \nu(x, dy). \end{aligned}$$

An application of Taylor's formula shows that the pseudo-differential operator extends via (4) to an operator on $C_b^2(\mathbb{R}^d)$ satisfying

(5)

$$|q(x, D)f(x)| \leq M\|f\|_{C_b^2(\mathbb{R}^d)} \left(|c(x)| + |b(x)| + |Q(x)| + \int_{y\neq 0} \min\{1, |y|^2\}\, \nu(x, dy) \right),\ f \in C_b^2(\mathbb{R}^d)$$

for some absolute constant $M > 0$. Pseudo-differential operators appear naturally in the study of stochastic processes, e.g. Feller processes, stochastic differential equations and martingale problems, cf. [4, 11, 13, 15].

Sublinear expectation spaces: A *sublinear expectation space* $(\Omega, \mathcal{H}, \mathcal{E})$ consists of a set $\Omega \neq \emptyset$, a linear space $\mathcal{H}$ of functions $f : \Omega \to \mathbb{R}$ and a functional $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ with the following properties:

(i) $\mathcal{E}$ is subadditive, i.e. $\mathcal{E}(X + Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$ for all $X, Y \in \mathcal{H}$,
(ii) $\mathcal{E}$ is positively homogeneous, i.e. $\mathcal{E}(\lambda X) = \lambda\mathcal{E}(X)$ for all $\lambda \geq 0$ and $X \in \mathcal{H}$,
(iii) $\mathcal{E}$ preserves constants, i.e. $\mathcal{E}(c) = c$ for all $c \in \mathbb{R}$,
(iv) $\mathcal{E}$ is monotone, i.e. $\mathcal{E}(X) \leq \mathcal{E}(Y)$ for all $X, Y \in \mathcal{H}$ with $X \leq Y$.

To introduce classical notions, such as random variables and independence, one needs to fix a class of test functions $\mathcal{T}$. In this paper, we take $\mathcal{T} := C_b^{\mathrm{uc}}(\mathbb{R}^d)$, the space of bounded uniformly continuous functions. A ($\mathbb{R}^d$-valued) *random variable* on a sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$ is a mapping $X : \Omega \to \mathbb{R}^d$ such that $\varphi(X) \in \mathcal{H}$ for all $\varphi \in C_b^{\mathrm{uc}}(\mathbb{R}^d)$. We also say that X is *adapted*. Two adapted $\mathbb{R}^d$-valued random variables X and Y are *equal in* *distribution*, $X \overset{d}{=} Y$, if

$$\forall \varphi \in C_b^{\mathrm{uc}}(\mathbb{R}^d) \ : \ \mathcal{E}(\varphi(X)) = \mathcal{E}(\varphi(Y)).$$

The random variables X and Y are called *independent* if

$$\forall \varphi \in C_b^{\mathrm{uc}}(\mathbb{R}^d \times \mathbb{R}^d) \ : \ \mathcal{E}(\varphi(X,Y)) = \mathcal{E}(\mathcal{E}(\varphi(x,Y))\big|_{x=X});$$

it is implicitly assumed that all terms are well-defined. For an introduction to sublinear expectation spaces and their connection to stochastic processes on sublinear expectation spaces we refer to [12] and the references therein.

Function spaces: The space of bounded Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ is denoted by $\mathcal{B}_b(\mathbb{R}^d)$. We write $C_b(\mathbb{R}^d)$ (resp. $C_b^{\mathrm{uc}}(\mathbb{R}^d)$) for the bounded continuous (resp. uniformly continuous) functions $f : \mathbb{R}^d \to \mathbb{R}$. The compactly supported smooth functions $f : \mathbb{R}^d \to \mathbb{R}$ are denoted by $C_c^\infty(\mathbb{R}^d)$.

3. Dynkin's formula for sublinear Markov semigroups

Let $(T_t)_{t\geq 0}$ be a Markov semigroup of linear operators on $\mathcal{H} := \mathcal{B}_b(\mathbb{R}^d)$ with infinitesimal generator $(A, \mathcal{D}(A))$. Dynkin's formula states that

$$T_t f(x) - f(x) = \int_0^t T_s A f(x)\, ds, \qquad t \geq 0,\ x \in \mathbb{R}^d, \tag{6}$$

for all $f \in \mathcal{D}(A)$. If $T_t f(x) = \mathbb{E}^x f(X_t)$ is the semigroup associated with a Markov process $(X_t)_{t\geq 0}$, then (6) can be written equivalently in a probabilistic way:

$$\mathbb{E}^x f(X_t) - f(x) = \int_0^t \mathbb{E}^x A f(X_s)\, ds, \qquad t \geq 0,\ x \in \mathbb{R}^d.$$

Dynkin's formula (6) holds more generally for functions in the domain of the weak generator, cf. Dynkin [9], and for functions in the Favard space of order 1, cf. Airault & Föllmer [1, p. 320-322]; see (7) below for the definition of the Favard space. In this section, we will show the following Dynkin-type formula for sublinear Markov semigroups

$$-\int_0^t T_s A f(x)\, ds \leq T_t f(x) - f(x) \leq \int_0^t T_s A f(x)\, ds, \qquad x \in \mathbb{R}^d,\ t \geq 0,$$

for $f \in \mathcal{D}(A)$, see Theorem 3.4 below. In general, the inequalities are strict. For the particular case that $(T_t)_{t\geq 0}$ is a Markov semigroup of linear operators, this gives the classical Dynkin formula (6). For the proof of the sublinear Dynkin formula, we need some auxiliary statements, see also Miyadera [17, Section 3.1] for some related results.

Lemma 3.1. *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with Favard space F_1 of order 1, i.e.*

$$f \in F_1 \iff f \in \mathcal{H},\ L(f) := \sup_{t>0} \frac{\|T_t f - f\|_\infty}{t} < \infty. \tag{7}$$

Then:

(i) $T_t(F_1) \subseteq F_1$ *for all* $t \geq 0$,
(ii) $\varphi(t) := L(T_t f)$ *is non-increasing for each* $f \in F_1$,

(iii) $t \mapsto T_t f(x)$ *is globally Lipschitz continuous with Lipschitz constant* $L(f)$ *for all* $f \in F_1$ *and* $x \in \mathbb{R}^d$.

Proof. Fix $t > 0$ and $f \in F_1$. Since T_t is sublinear and monotone, we have

$$T_s T_t f - T_t f = T_t T_s f - T_t f \leq T_t(T_s f - f) \leq \|T_s f - f\|_\infty$$

and

$$-(T_s T_t f - T_t f) \leq T_t(f - T_s f) \leq \|T_s f - f\|_\infty.$$

Hence, $\varphi(t) = L(T_t f) \leq L(f) = \varphi(0)$. In particular, $T_t f \in F_1$ and

$$\|T_{t+s} f - T_t f\|_\infty \leq L(T_t f)|s| \leq \varphi(0)|s|, \qquad s \geq 0. \qquad \square$$

Since $\mathcal{D}(A) \subseteq F_1$, Lemma 3.1 shows, in particular, that $t \mapsto T_t f(x)$ is Lipschitz continuous for all $f \in \mathcal{D}(A)$. It follows from Rademacher's theorem that there exists for each $x \in \mathbb{R}^d$ some Lebesgue null set $N = N(x, f) \subseteq [0, \infty)$ such that the limit

$$\lim_{s \to 0} \frac{T_{t+s} f(x) - T_t f(x)}{s}$$

exists for all $t \in [0, \infty) \backslash N$. For *linear* strongly continuous Markov semigroups $(T_t)_{t \geq 0}$, it can be easily verified that the limit exists for *all* $t \geq 0$ uniformly in $x \in \mathbb{R}^d$, and so $T_t(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$. This is no longer true for sublinear semigroups: there may be functions $f \in \mathcal{D}(A)$ such that $T_t f \in \mathcal{D}(A)$ fails to hold for $t > 0$.

EXAMPLE 3.2. The family of operators

$$T_t f(x) := \sup_{|s| \leq t} f(x + s) = \sup_{b \in [-1,1]} f(x + bt), \qquad t \geq 0,$$

defines a strongly continuous sublinear Markov semigroup on $\mathcal{H} := C_b^{\mathrm{uc}}(\mathbb{R})$. Moreover, it follows from Taylor's formula that

$$\lim_{t \to 0} \frac{T_t f(x) - f(x)}{t} = |f'(x)|$$

for all $f \in C_b^2(\mathbb{R})$, and the convergence is uniformly in $x \in \mathbb{R}$. Thus, $C_b^2(\mathbb{R}) \subseteq \mathcal{D}(A)$ and $Af = |f'|$ for $f \in C_b^2(\mathbb{R})$. Take a function $f \in C_b^2(\mathbb{R}) \subseteq \mathcal{D}(A)$ such that $f(x) \in [0, 1]$ for all $x \in [-1, 1]$, $f(x) = -x$ for $x \in [-2, -1]$ and $f(x) = 1$ for $1/2 \leq x \leq 2$. Then

$$T_t f(0) = \begin{cases} 1, & t \in [1/2, 1], \\ t, & t \in [1, 2], \end{cases}$$

and so $t \mapsto T_t f(0)$ is not differentiable at $t = 1$, i.e. $T_1 f \notin \mathcal{D}(A)$.

Let us remark that Denk et al. [8] showed very recently that the generator of a sublinear semigroup (on a "nice" space) can be extended in such a way that the domain of the extended generator is invariant under T_t. If the semigroup is continuous from above, then the extended generator uniquely characterizes the semigroup.

Though $(T_{t+s} f - T_t f)/s$ does not necessarily converge as $s \to 0$, we can show that difference quotient is bounded from above (resp. below) by $T_t A f$ (resp. $-T_t(-Af)$). This bound for the slope of $t \mapsto T_t f$ is the key for the proof of Dynkin's formula.

Proposition 3.3. *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with strong infinitesimal generator $(A, \mathcal{D}(A))$. If $f \in \mathcal{D}(A)$, then*

$$-T_t(-Af) \leq \liminf_{s\to 0} \frac{T_{t+s} - T_t f}{s} \leq \limsup_{s\to 0} \frac{T_{t+s} f - T_t f}{s} \leq T_t A f \tag{8}$$

for all $t \geq 0$.

Proof. Fix $f \in \mathcal{D}(A)$ and $t \geq 0$. By the semigroup property and subadditivity of $(T_t)_{t\geq 0}$, we have

$$\frac{T_{t+s}f - T_t f}{s} - T_t A f \leq T_t\left(\frac{T_s f - f}{s} - Af\right).$$

Since T_t is monotone and $T_t 1 \leq 1$, this gives

$$\frac{T_{t+s}f - T_t f}{s} - T_t A f \leq \left\|\frac{T_s f - f}{s} - Af\right\|_\infty \xrightarrow{s\to 0} 0.$$

Hence,

$$\limsup_{s\to 0} \frac{T_{t+s}f - T_t f}{s} \leq T_t A f.$$

On the other hand, it follows from the subadditivity of T_t that

$$T_t f \leq T_t(f - T_s f) + T_t T_s f,$$

and so

$$\frac{T_{t+s}f - T_t f}{s} + T_t(-Af) \geq -\frac{T_t(f - T_s f)}{s} + T_t(-Af).$$

Using

$$T_t\left(\frac{f - T_s f}{s}\right) \leq T_t\left(\frac{f - T_s f}{s} + Af\right) + T_t(-Af)$$

we find that

$$\frac{T_{t+s}f - T_t f}{s} + T_t(-Af) \geq -T_t\left(\frac{f - T_s f}{s} + Af\right) \geq -\left\|\frac{f - T_s f}{s} + Af\right\|_\infty \xrightarrow{s\to 0} 0. \qquad \square$$

Theorem 3.4 (Dynkin's formula)**.** *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with strong infinitesimal generator $(A, \mathcal{D}(A))$. If $f \in \mathcal{D}(A)$, then*

$$-\int_0^t T_s(-Af)\,ds \leq T_t f - f \leq \int_0^t T_s(Af)\,ds \qquad \textit{for all } t \geq 0. \tag{9}$$

Proof. Fix $f \in \mathcal{D}(A)$. By Lemma 3.1, $t \mapsto T_t f(x)$ is globally Lipschitz continuous for all $x \in \mathbb{R}^d$, and therefore it follows from Rademacher's theorem that there exists a mapping $g(t, x)$ such that

$$T_t f(x) - f(x) = \int_0^t g(s, x)\,ds$$

and

$$\frac{d}{dt}T_t f(x) = g(t, x)$$

for Lebesgue almost every $t \geq 0$ (the exceptional null set may depend on $x \in \mathbb{R}^d$). Dynkin's formula (9) is now an immediate consequence of Proposition 3.3. □

REMARK 3.5. (i) If $(T_t)_{t\geq 0}$ is a Markov semigroup of *linear* operators, then we recover the classical Dynkin formula:

$$T_t f - f = \int_0^t T_s A f \, ds, \quad f \in \mathcal{D}(A), \, t \geq 0.$$

(ii) In general, the inequalities in (9) are strict. Consider, for instance,

$$T_t f(x) = \sup_{b\in[-1,1]} f(x + bt) \qquad Af = |f'|$$

(see Example 3.2), then

$$T_t f(x) - f(x) = \sup_{b\in[-1,1]} \int_0^{bt} f'(x + r) \, dr$$

is, in general, strictly smaller than

$$\int_0^t T_s A f(x) \, ds = \int_0^t \sup_{b\in[-1,1]} |f'(x + bs)| \, ds,$$

e.g. if f' has strict maximum in x.

(iii) For another variant of Dynkin's formula for sublinear semigroups see [8, Theorem 4.5].

(iv) In Section 4 we will identify $A|_{C_c^\infty(\mathbb{R}^d)}$ under the assumption that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$. In combination with Dynkin's formula (9), this gives a useful tool to establish probability estimate for Markov processes on sublinear spaces, e.g. estimates for fractional moments.

As a direct consequence of Dynkin's formula we obtain the following corollary; see e.g. [20, Lemma 2.3] for the counterpart in the framework of linear semigroups.

Corollary 3.6. *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with strong infinitesimal generator $(A, \mathcal{D}(A))$. If $f \in \mathcal{D}(A)$, then*

$$\|Af\|_\infty = \sup_{t>0} \frac{\|T_t f - f\|_\infty}{t}. \tag{10}$$

In particular,

$$\|T_t f - T_s f\|_\infty \leq \|Af\|_\infty |t - s|, \qquad s, t \geq 0. \tag{11}$$

Proof. Since T_t is monotone for each $t \geq 0$, it follows from Dynkin's formula (9) that

$$\|T_t f - f\|_\infty \leq t\|Af\|_\infty.$$

On the other hand, the very definition of A gives

$$\|Af\|_\infty = \lim_{t\to 0} \frac{\|T_t f - f\|_\infty}{t},$$

and this proves (10). From Lemma 3.1(ii), we get the Lipschitz estimate (11). □

4. Courrège-von Waldenfels theorem for sublinear operators

Let $A : \mathcal{D}(A) \to \mathcal{B}_b(\mathbb{R}^d)$ be a linear operator satisfying the positive maximum principle, i.e.

$$f \in \mathcal{D}(A),\ f(x_0) = \sup_{x \in \mathbb{R}^d} f(x) \geq 0 \implies Af(x_0) \leq 0. \tag{12}$$

If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, then $A|_{C_c^\infty(\mathbb{R}^d)}$ has a representation of the form

$$\begin{aligned} Af(x) = &-c(x)f(x) + b(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q(x) \cdot \nabla^2 f(x)) \\ &+ \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|))\ \nu(x, dy) \end{aligned}$$

where $c(x) \geq 0$, $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d \times d}$ is positive semidefinite and $\nu(x, dy)$ is a measure such that $\int_{y \neq 0} \min\{1, |y|^2\}\, \nu(x, dy) < \infty$; this result is due to Courrège [5] and von Waldenfels [22, 23]. Since infinitesimal generators of Markov processes satisfy the positive maximum principle, this gives immediately a representation formula for infinitesimal generators with a sufficiently rich domain, cf. [4, Theorem 2.21] or [13]. Recently, the result by Courrège and von Waldenfels was generalized to Lie groups, cf. [2]. In this section, we establish the following Courrège-von Waldenfels theorem for *sub*linear operators satisfying the positive maximum principle.

Theorem 4.1. *Let $A : \mathcal{D}(A) \to \mathcal{B}_b(\mathbb{R}^d)$ be a sublinear operator with $\mathcal{D}(A) \subseteq \mathcal{B}_b(\mathbb{R}^d)$. Assume that A satisfies the positive maximum principle* (12). *If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, then there exist an index set I and a family $(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x, dy))$, $\theta \in I$, $x \in \mathbb{R}^d$, of characteristics such that*

$$Af(x) = \sup_{\theta \in I} A_\theta f(x), \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d, \tag{13}$$

where

$$\begin{aligned} A_\theta f(x) := &-c_\theta(x)f(x) + b_\theta(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q_\theta(x) \cdot \nabla^2 f(x)) \\ &+ \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|))\, \nu_\theta(x, dy). \end{aligned}$$

The supremum is attained, i.e. for any $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ there exists $\theta = \theta(f, x)$ such that $Af(x) = A_\theta f(x)$. For each $x \in \mathbb{R}^d$ the family $(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x, dy))$, $\theta \in I$, is uniformly bounded, i.e.

$$\sup_{\theta \in I} \left(|c_\theta(x)| + |b_\theta(x)| + |Q_\theta(x)| + \int_{y \neq 0} \min\{1, |y|^2\}\, \nu_\theta(x, dy) \right) < \infty. \tag{14}$$

Since the generator of a sublinear Markov semigroup satisfies the positive maximum principle, Theorem 4.1 gives, in particular, a representation formula for sublinear generators whose domains contain $C_c^\infty(\mathbb{R}^d)$, see Corollary 4.3 below. If A is a *linear* operator, then the index set I consists of a single element, and we recover the classical Courrège-von

Waldenfels theorem.

To prove Theorem 4.1, we need a representation result for sublinear functionals. We say that a functional $B : \mathcal{D} \to \mathbb{R}$ defined on a subspace $\mathcal{D}$ of functions $f : \mathbb{R}^d \to \mathbb{R}$ *satisfies the positive maximum principle in* $x_0 \in \mathbb{R}^d$ if $f \in \mathcal{D}$, $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geq 0$ implies $Bf \leq 0$.

Lemma 4.2. *Let $B : \mathcal{D} \to \mathbb{R}$ be a sublinear functional on a linear space $\mathcal{D} \subseteq \mathcal{B}_b(\mathbb{R}^d)$. If B satisfies the positive maximum principle in some point $x_0 \in \mathbb{R}^d$, then there exists a family $(B_\theta)_{\theta\in\Theta}$ of linear functionals on $\mathcal{D}$ satisfying the positive maximum principle in x_0 such that*

$$Bf = \sup_{\theta\in\Theta} B_\theta f, \qquad f \in \mathcal{D}.$$

The supremum is attained, i.e. for every $f \in \mathcal{D}$ there exists some $\theta = \theta(f) \in \Theta$ such that $Bf = B_\theta f$.

It was shown in [12, Theorem 3.5] that any sublinear functional B on a linear space has a representation of the form $Bf = \sup_{\theta\in\Theta} B_\theta f$ for a family of linear functionals. For our application it is crucial to have the positive maximum principle for B_θ.

Proof of Lemma 4.2. Set $\Theta := \{\theta : \mathcal{D} \to \mathbb{R}; \theta \text{ is linear and } \theta \leq B\}$ and $B_\theta := \theta$ for $\theta \in \Theta$. An application of the Hahn-Banach theorem shows that

$$Bf = \sup_{\theta\in\Theta} B_\theta f, \qquad f \in \mathcal{D},$$

and the supremum is attained, see [12, Proof of Theorem 3.5] for details. Now let $f \in \mathcal{D}$ such that $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geq 0$. By assumption,

$$0 \geq Bf = \sup_{\theta\in\Theta} B_\theta f,$$

and so $B_\theta f \leq 0$ for all $\theta \in \Theta$, i.e. B_θ satisfies the positive maximum principle in $x_0 \in \mathbb{R}^d$. □

Proof of Theorem 4.1. Throughout this first part of the proof, we fix $x \in \mathbb{R}^d$. On a linear subspace $\mathcal{D}$ of $\mathcal{D}(A)$ define a sublinear operator $B : \mathcal{D} \to \mathbb{R}$ by $Bf := Af(x)$. From the positive maximum principle for A, it is immediate that B satisfies the positive maximum principle in x. Applying Lemma 4.2, we find that there exist an index set $\Theta = \Theta(x)$ and a family $(A_\theta)_{\theta\in\Theta}$ of linear functionals on $\mathcal{D}$ satisfying the positive maximum principle in $x \in \mathbb{R}^d$ such that

$$Af(x) = \sup_{\theta\in\Theta} A_\theta f, \qquad f \in \mathcal{D}. \tag{15}$$

By assumption, we can choose $\mathcal{D} := C_c^\infty(\mathbb{R}^d)$. Since the operators A_θ are linear, the classical Courrège-von Waldenfels theorem shows that there exist $c_\theta \geq 0$, $b_\theta \in \mathbb{R}^d$, a positive semidefinite matrix $Q_\theta \in \mathbb{R}^{d\times d}$ and a measure ν_θ with $\int_{y\neq 0} \min\{1, |y|^2\}\, \nu_\theta(dy) < \infty$ such that

$$A_\theta f = -c_\theta f(x) + b_\theta \cdot \nabla f(x) + \frac{1}{2}\operatorname{tr}(Q_\theta \cdot \nabla^2 f(x)) + \int_{y\neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|))\, \nu_\theta(dy)$$

for all $f \in C_c^\infty(\mathbb{R}^d)$, see also [2, Theorem 3.4]. Next we prove that the family $(c_\theta, b_\theta, Q_\theta, \nu_\theta)$, $\theta \in \Theta$, is bounded, i.e.

$$(\star)\qquad \sup_{\theta\in\Theta}\left(c_\theta + |b_\theta| + |Q_\theta| + \int_{y\neq 0} \min\{1,|y|^2\}\,\nu_\theta(dy)\right) < \infty.$$

For $r>0$ pick $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{B(x,r/2)} \leq \chi \leq \mathbb{1}_{B(x,r)}$. From $\chi(x)=1$, $\nabla\chi(x)=0$ and $\nabla^2\chi(x)=0$, we find that

$$A_\theta(-\chi) = c_\theta + \int_{y\neq 0} (1-\chi(x+y))\,\nu_\theta(dy) \geq c_\theta + \int_{|y|\geq r} \nu_\theta(dy).$$

Hence,

$$\sup_{\theta\in\Theta}\left(c_\theta + \int_{|y|\geq r} \nu_\theta(dy)\right) \leq \sup_{\theta\in\Theta} A_\theta(-\chi) = A(-\chi)(x) < \infty.$$

For fixed $j\in\{1,\dots,d\}$ consider the mapping $f(y) := (y^{(j)}-x^{(j)})^2\chi(y)$, then

$$A_\theta f = Q_\theta^{(j,j)} + \int_{y\neq 0} (y^{(j)})^2\chi(y+x)\,\nu_\theta(dy) \geq Q_\theta^{(j,j)} + \int_{0<|y|<r/2} (y^{(j)})^2\,\nu_\theta(dy)$$

for all $\theta\in\Theta$. Since $A_\theta f$ is bounded from above by $Af(x)$, this implies that the right-hand side is bounded uniformly in $\theta\in\Theta$. In a similar fashion, we consider $y\mapsto (y^{(j)}-x^{(j)})\chi(y)$ and $y\mapsto (y^{(j)}-x^{(j)})(y^{(i)}-x^{(i)})\chi(y)$ to obtain that

$$\sup_{\theta\in\Theta}\left(\sum_{j=1}^d |b_\theta^{(j)}| + \sum_{i\neq j} |Q_\theta^{(i,j)}|\right) < \infty.$$

Combining the above estimates, we get $(\star)$. Consequently, there exist for each $x\in\mathbb{R}^d$ an index set $\Theta=\Theta(x)$ and a uniformly bounded family $(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x,dy))$ of characteristics such that

$$\begin{aligned} Af(x) = \sup_{\theta\in\Theta(x)}\Big(&-c_\theta(x)f(x) + b_\theta(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q_\theta(x)\cdot\nabla^2 f(x)) \\ &+ \int_{y\neq 0} (f(x+y)-f(x)-y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu_\theta(x,dy)\Big) \end{aligned}$$

for all $f\in C_c^\infty(\mathbb{R}^d)$. If we define an index set I by $I := \bigcup_{x\in\mathbb{R}^d}\Theta(x)$ and set

$$(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x,dy)) := (c_{\theta'}(x), b_{\theta'}(x), Q_{\theta'}(x), \nu_{\theta'}(x,dy)), \qquad \theta\in I\backslash\Theta(x)$$

for some fixed $\theta'\in\Theta(x)$, then we obtain the representation

$$\begin{aligned} Af(x) = \sup_{\theta\in I}\Big(&-c_\theta(x)f(x) + b_\theta(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q_\theta(x)\cdot\nabla^2 f(x)) \\ &+ \int_{y\neq 0} (f(x+y)-f(x)-y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu_\theta(x,dy)\Big). \end{aligned}$$

□

Corollary 4.3. *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with pointwise infinitesimal generator $(A^{(p)}, \mathcal{D}(A^{(p)}))$. If $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{D}(A^{(p)})$, then there exists a family $q_\theta(x,\cdot)$, $\theta\in I$, $x\in\mathbb{R}^d$, of continuous negative definite functions such that*

$$(16)\qquad A^{(p)}f(x) = \sup_{\theta\in I}(-q_\theta(x,D)f(x)) \qquad \textit{for all } f\in C_c^\infty(\mathbb{R}^d),\ x\in\mathbb{R}^d,$$

cf. (4). *Equivalently,*

$$A^{(p)}f(x) = \sup_{\theta\in I}\Big(- c_\theta(x)f(x) + b_\theta(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q_\theta(x)\cdot\nabla^2 f(x)) + \int_{y\neq 0}(f(x+y) - f(x) - y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu_\theta(x,dy)\Big) \tag{17}$$

where $(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x,\cdot))$ is the characteristics associated via the Lévy–Khintchine representation (2) *with $q_\theta(x,\cdot)$. The family $(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x,dy))$, $\theta\in I$, is uniformly bounded for each $x\in\mathbb{R}^d$:*

$$\sup_{\theta\in I}\left(c_\theta(x) + |b_\theta(x)| + |Q_\theta(x)| + \int_{y\neq 0}\min\{1,|y|^2\}\,\nu_\theta(x,dy)\right) < \infty.$$

Proof. By Theorem 4.1 and the positive maximum principle for sublinear generators, cf. Lemma 2.1, the generator $A^{(p)}$ has a representation of the form (13). Equivalently,

$$A^{(p)}f(x) = \sup_{\theta\in I}(-q_\theta(x,D)f)(x), \qquad f\in C_c^\infty(\mathbb{R}^d),$$

where $q_\theta(x,\cdot)$ is the continuous negative definite function which is associated via the Lévy–Khintchine representation with $(c_\theta(x), b_\theta(x), Q_\theta(x), \nu_\theta(x,dy))$, cf. (2) and (3). □

REMARK 4.4. (i) The representation (16) is, in general, not unique. For instance, the operator

$$Af(x) = |f'(x)| = \sup\{f'(x), -f'(x)\}$$

(see Example 3.2) has at least two representations of the form (16):

$$Af(x) = \sup_{\theta\in[-1,1]}(\theta f'(x)) = \sup_{\theta\in\{-1,1\}}(\theta f'(x)).$$

(ii) If $(T_t)_{t\geq 0}$ is a conservative sublinear Markov semigroup, then $q_\theta(x,0) = 0$ and $c_\theta(x) = 0$ for all $\theta\in I$ and $x\in\mathbb{R}^d$, see Corollary 4.9 below.
(iii) In the statement of Corollary 4.3 we may replace the pointwise generator $A^{(p)}$ by the strong generator A. This follows from the fact that $(A^{(p)}, \mathcal{D}(A^{(p)}))$ extends $(A, \mathcal{D}(A))$.
(iv) Sufficient conditions which ensure that $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{D}(A^{(p)})$ were obtained in [16, 12, 18].

For translation-invariant Markov semigroups, i.e. for Markov semigroups satisfying $(T_tf)(x) = (T_tf(x+\cdot))(0)$, the representation (16) simplifies since the family of continuous negative definite functions does not depend on $x\in\mathbb{R}^d$.

Corollary 4.5. *Let $(T_t)_{t\geq 0}$ be a translation invariant sublinear Markov semigroup on $\mathcal{H}$ with pointwise generator $(A^{(p)}, \mathcal{D}(A^{(p)}))$. If $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{D}(A)$, then there exists a family $(\psi_\theta)_{\theta\in I}$ of continuous negative definite functions such that*

$$A^{(p)}f(x) = \sup_{\theta\in I}(-\psi_\theta(D)f(x)) \qquad \textit{for all } f\in C_c^\infty(\mathbb{R}^d),\ x\in\mathbb{R}^d.$$

The associated family $(c_\theta, b_\theta, Q_\theta, \nu_\theta)$, $\theta\in I$, of triplets satisfies

$$\sup_{\theta\in I}\left(c_\theta + |b_\theta| + |Q_\theta| + \int_{y\neq 0}\min\left\{1,|y|^2\right\}\nu_\theta(dy)\right) < \infty.$$

Proof. By Corollary 4.3, there exists a family $q_\theta(x,\cdot)$ of continuous negative definite functions such that

$$A^{(p)}f(x) = \sup_{\theta\in I}(-q(x,D)f)(x), \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

Since the semigroup is translation invariant, it follows from the definition of the generator that $A^{(p)}f(x) = (A^{(p)}f(x+\cdot))(0)$ for all $f \in \mathcal{D}(A^{(p)})$, $x \in \mathbb{R}^d$. Thus,

$$A^{(p)}f(x) = \sup_{\theta\in I}(-\psi_\theta(D)f)(x), \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

for $\psi_\theta := q_\theta(0,\cdot)$. The uniform boundedness of the associated triplets is evident from Corollary 4.3. □

Remark 4.6. If $(T_t)_{t\geq 0}$ is a *linear* translation invariant Markov semigroup, say on $C_b(\mathbb{R}^d)$, then $(T_t)_{t\geq 0}$ is the semigroup of a Lévy process, and so the assumption $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$ in Corollary 4.5 is automatically satisfied, see e.g. [14, Lemma 6.3]. This is no longer true for sublinear semigroups. For instance, $T_0 := \mathrm{id}$,

$$T_t f(x) := \|f\|_\infty = \sup_{y\in\mathbb{R}^d}|f(y)|, \qquad t > 0,$$

defines a translation invariant sublinear Markov semigroup, but $C_c^\infty(\mathbb{R}^d)$ is not contained in the domain $\mathcal{D}(A^{(p)})$ of the pointwise generator. In fact, even pointwise convergence $T_t f(x) \xrightarrow{t\to 0} f(x)$ fails to hold for $f \in C_c^\infty(\mathbb{R}^d)$.

In view of this example, it is natural to ask whether $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$ holds under the additional assumption that $(T_t)_{t\geq 0}$ is strongly continuous (on a sufficiently large domain). This seems to be an open question.

For many applications it would be useful to have the representation $A^{(p)}f = \sup_\theta(-q_\theta(x,D)f)$ from Corollary 4.3 not only for $f \in C_c^\infty(\mathbb{R}^d)$, but for a larger class of functions. In the linear framework, one typically invokes the closedness of the infinitesimal generator to extend the representation formula e.g. to $f \in C_c^2(\mathbb{R}^d)$, cf. [21, Corollary 3.8] or [4, Theorem 2.37]. The situation is more complicated for sublinear infinitesimal generators. In our next result we give an extension to smooth functions with bounded derivatives. It will play a crucial role when we study viscosity solutions to non-linear Cauchy problems, see Corollary 5.1.

Corollary 4.7. *Let $(T_t)_{t\geq 0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with pointwise infinitesimal generator $(A^{(p)}, \mathcal{D}(A^{(p)}))$. Assume that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$. By* Corollary 4.3*,*

$$A^{(p)}f(x) = \sup_{\theta\in I}(-q_\theta(x,D)f)(x), \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d, \tag{18}$$

for a family of continuous negative definite functions $q_\theta(x,\cdot)$ with characteristics $(0, b_\theta(x), Q_\theta(x), \nu_\theta(x,\cdot))$, $\theta \in I$.

(i) *If the family $\nu_\theta(x,\cdot)$, $\theta \in I$, is tight for each $x \in \mathbb{R}^d$, then $C_b^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$.*

(ii) *If $\mathcal{D} \subseteq C_b^\infty(\mathbb{R}^d)$ is a linear subspace such that $\mathcal{D} \subseteq \mathcal{D}(A^{(p)})$, then* (18) *holds for any $f \in \mathcal{D}$.*

REMARK 4.8. (i) If $(T_t)_{t\geq 0}$ is a linear Markov semigroup, then the index set I consists of a single element, and therefore the tightness of $\nu_\theta(x,\cdot)$, $\theta \in I$, is automatically satisfied for each fixed $x \in \mathbb{R}^d$.

(ii) If the family $\nu_\theta(x,dy)$, $\theta \in I$, is tight, then we can choose $\mathcal{D} := C_b^\infty(\mathbb{R}^d)$ in Corollary 4.7(ii). Moreover, tightness implies

$$\lim_{r\to 0} \sup_{\theta\in I} \sup_{|\xi|\leq r} |q_\theta(x,\xi)| = 0,$$

cf. [16, Lemma A.2]. Conversely, the family $\nu_\theta(x,\cdot)$, $\theta \in I$, cannot be tight if the above limit does not equal zero. For instance, for $q_\theta(x,\xi) := 1-\cos(\theta\xi)$, $\theta \in I := \mathbb{N}$, tightness fails to hold. See Proposition 4.10 for an equivalent characterization of tightness in terms of the generator.

Proof of Corollary 4.7. Set $Lf(x) := \sup_{\theta\in I}(-q_\theta(x,D)f)(x)$. Note that, by (4) and (5), Lf is well defined for any $f \in C_b^2(\mathbb{R}^d)$. Throughout this proof, $\chi \in C_c^\infty(\mathbb{R}^d)$ is such that $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$, and $\chi_r(x) := \chi(x/r)$ for $r > 0$.

(i) For fixed $x \in \mathbb{R}^d$ and $f \in C_b^\infty(\mathbb{R}^d)$ define $f_n(y) := f(y)\chi_n(y-x)$. Since $f_n(x) = f(x)$ and $T_t(f) \leq T_t(f-f_n) + T_t(f_n)$, we have

$$\frac{T_tf(x)-f(x)}{t} - Lf(x) \leq \frac{T_t(f-f_n)(x)}{t} + \frac{T_tf_n(x)-f_n(x)}{t} - Lf(x).$$

On the other hand, $T_tf_n \leq T_t(f_n - f) + T_tf$ gives

$$Lf(x) - \frac{T_tf(x)-f(x)}{t} \leq Lf(x) - \frac{T_tf_n(x)-f_n(x)}{t} + \frac{T_t(f_n-f)(x)}{t}.$$

By definition, $f_n \in C_c^\infty(\mathbb{R}^d)$, and so $A^{(p)}f_n = Lf_n$. Using $|f_n - f| \leq 2\|f\|_\infty(1-\chi_n(\cdot - x))$, we get

$$\limsup_{t\to 0}\left|\frac{T_tf(x)-f(x)}{t} - Lf(x)\right| \leq 2\|f\|_\infty \limsup_{t\to 0} \frac{1+T_t(-\chi_n(\cdot - x))(x)}{t} + |Lf_n(x) - Lf(x)|.$$

As $\chi_n(\cdot - x) \in C_c^\infty(\mathbb{R}^d)$, this gives

$$\limsup_{t\to 0}\left|\frac{T_tf(x)-f(x)}{t} - Lf(x)\right| \leq 2\|f\|_\infty L(-\chi_n(\cdot - x))(x) + |Lf_n(x) - Lf(x)|.$$

It remains to show that the right-hand side converges to 0 as $n \to \infty$. Since $\chi_n = 1$ on $B(0,n)$, it follows from the definition of L and the tightness of the Lévy measures that

$$L(-\chi_n(\cdot - x))(x) = \sup_{\theta\in I}\int_{y\neq 0}(1-\chi_n(y))\,\nu_\theta(x,dy) \leq \sup_{\theta\in I}\int_{|y|\geq n}\nu_\theta(x,dy) \xrightarrow{n\to\infty} 0.$$

For the second term, we use the elementary inequality

$$\sup_i a_i - \sup_i b_i \leq \sup_i (a_i - b_i),$$

the estimate (5) and the fact that $f_n = f$ on $B(x,n)$ to deduce that

$$\begin{aligned} Lf(x) - Lf_n(x) &\leq \sup_{\theta\in I}\left(-q_\theta(x,D)f(x) + q_\theta(x,D)f_n\right)(x) \\ &\leq \left(\|f_n\|_{C_b^2(\mathbb{R}^d)} + \|f\|_{C_b^2(\mathbb{R}^d)}\right)\sup_{\theta\in I}\int_{|y|\geq n}\nu_\theta(x,dy). \end{aligned}$$

Interchanging the roles of f and f_n, we get

$$|Lf(x) - Lf_n(x)| \leq c\|f\|_{C_b^2(\mathbb{R}^d)} \sup_{\theta \in I} \int_{|y|\geq n} \nu_\theta(x, dy) \xrightarrow{n\to\infty} 0.$$

(ii) Fix $x \in \mathbb{R}^d$. From the proof of Theorem 4.1, cf. (15), we know that there exists a family $(A_\theta)_{\theta\in\Theta}$ of linear functionals on $\mathcal{D}$ satisfying a positive maximum principle in $x \in \mathbb{R}^d$ such that

$$A^{(p)} f(x) = \sup_{\theta\in\Theta} A_\theta f, \qquad f \in \mathcal{D}, \tag{19}$$

and

$$A_\theta f = -q_\theta(x, D) f(x), \qquad f \in C_c^\infty(\mathbb{R}^d). \tag{20}$$

Let $u \in C_b^\infty(\mathbb{R}^d)$ be such that $u = 0$ on $B(x, 2r)$ for some $r > 0$. Since

$$g(y) := \|u\|_\infty(1 - \chi_r(y - x)) - u(y) \geq 0 = g(x),$$

it follows from the positive maximum principle that $A_\theta g \geq 0$, i.e. $A_\theta u \leq \|u\|_\infty A_\theta(1-\chi_r(\cdot - x))$. Replacing u by $-u$, we find that

$$|A_\theta u| \leq \|u\|_\infty |A_\theta(1 - \chi_r(\cdot - x))|. \tag{21}$$

Now let $f \in \mathcal{D} \subseteq C_b^\infty(\mathbb{R}^d)$ and set $f_n(y) := f(y)\chi((y - x)/2n)$. Since $f - f_n = 0$ on $B(x, 2n)$, we get from (20) and (21) that

$$\begin{aligned}
&|A_\theta(f) - q_\theta(x, D) f(x)| \\
&\leq |A_\theta(f - f_n)| + |A_\theta(f_n) - q_\theta(x, D) f_n(x)| + |q_\theta(x, D) f_n(x) - q_\theta(x, D) f(x)| \\
&\leq \|f - f_n\|_\infty |A_\theta(1 - \chi_n(\cdot - x))| + |q_\theta(x, D) f_n(x) - q_\theta(x, D) f(x)|.
\end{aligned}$$

Using the representation of $q_\theta(x, D)$ as an integro–differential operator, cf. (4), it can be easily verified that $q_\theta(x, D) f_n(x) \xrightarrow{n\to\infty} q_\theta(x, D) f(x)$. Moreover, as in (i), we find for fixed $\theta \in \Theta$ that $q_\theta(x, D)(1 - \chi_n(\cdot - x))(x)$ converges to 0 as $n \to \infty$. Consequently we conclude that $A_\theta f = q_\theta(x, D) f(x)$ for any $f \in C_b^\infty(\mathbb{R}^d)$, and by (19) this proves the assertion. □

Corollary 4.9. *Let $(T_t)_{t\geq0}$ be a sublinear Markov semigroup with pointwise generator $(A^{(p)}, \mathcal{D}(A^{(p)})$ satisfying $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$. Denote by $q_\theta(x, \cdot)$, $\theta \in I$, $x \in \mathbb{R}^d$, the family of continuous negative definite functions associated with $(T_t)_{t\geq0}$* via Corollary 4.3. *If $(T_t)_{t\geq0}$ is conservative, then $q_\theta(x, 0) = 0$ for all $\theta \in I$ and $x \in \mathbb{R}^d$.*

Note that $q_\theta(x, 0) = 0$ is equivalent to $c_\theta(x) = 0$ in the representation (17) of $A^{(p)}$ as an integro-differential operator. If $(T_t)_{t\geq0}$ is a *linear* Markov semigroup, then the index set I consists of a single element, i.e. the family $q(x, \cdot)$, $x \in \mathbb{R}^d$, does not depend on an additional parameter θ. Hence, Corollary 4.9 shows that conservativeness of the semigroup implies $q(x, 0) = 0$ for all $x \in \mathbb{R}^d$. This extends [20, Lemma 5.1], see also [4, Lemma 2.32], where the statement was shown for (linear) Feller semigroups under the additional assumption that $x \mapsto q(x, 0)$ is continuous.

Proof of Corollary 4.9. Define a linear space by $\mathcal{D} := \{f + c\mathbb{1}_{\mathbb{R}^d}; f \in C_c^\infty(\mathbb{R}^d), c \in \mathbb{R}\}$. From the conservativeness of $(T_t)_{t\geq0}$, it follows that $T_t(f + c) = T_t(f) + c$ for all $f \in \mathcal{H}$ and

$c \in \mathbb{R}$, and therefore $\mathcal{D} \subseteq \mathcal{D}(A^{(p)})$ and

$$A^{(p)}(f + c\mathbb{1}_{\mathbb{R}^d}) = A^{(p)}(f), \qquad f \in C_c^\infty(\mathbb{R}^d),\ c \in \mathbb{R}^d.$$

Applying Corollary 4.7(ii), we find for $f := 0$ that

$$0 = A^{(p)}(c\mathbb{1}_{\mathbb{R}^d})(x) = \sup_{\theta \in I} q_\theta(x, 0), \qquad x \in \mathbb{R}^d.$$

As $q_\theta(x, 0) \geq 0$, this implies $q_\theta(x, 0) = 0$ for all $x \in \mathbb{R}^d$ and $\theta \in I$. □

We close this section with an equivalent characterization of tightness of the family $\nu_\theta(x, \cdot)$, $\theta \in I$.

Proposition 4.10. *Let $q_\theta(x, \cdot)$, $\theta \in I$, $x \in \mathbb{R}^d$, be a family of continuous negative definite functions with $q_\theta(x, 0) = 0$. Denote by $(b_\theta(x), Q_\theta(x), \nu_\theta(x, \cdot))$ the associated family of triplets and set*

$$Lf(x) := \sup_{\theta \in I}(-q_\theta(x, D)f)(x), \qquad f \in C_b^\infty(\mathbb{R}^d).$$

The following statements are equivalent for each $x \in \mathbb{R}^d$.

(i) *The family $\nu_\theta(x, \cdot)$, $\theta \in I$, is tight.*

(ii) *For any $\varepsilon > 0$ there exists some $\varphi \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \varphi \leq 1$, such that $L(1 - \varphi)(x) \leq \varepsilon$ and $\varphi = 1$ in a neighbourhood of x.*

Proof. Fix $x \in \mathbb{R}^d$ and $\varepsilon > 0$. If $\nu_\theta(x, \cdot)$, $\theta \in I$, is tight, then there exists $R > |x|$ such that

$$\sup_{\theta \in I} \int_{|y|>R} \nu_\theta(x, dy) \leq \varepsilon.$$

Choosing a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi|_{B(0,R)} = 1$ and $\varphi|_{B(0,2R)} = 0$, we find that

$$L(1 - \varphi)(x) = \sup_{\theta \in I} \int_{y \neq 0} (1 - \varphi(y))\, \nu_\theta(x, dy) \leq \sup_{\theta \in I} \int_{|y|>R} \nu_\theta(x, dy) \leq \varepsilon.$$

On the other hand, if φ is a function as in (ii), then

$$\varepsilon \geq L(1 - \varphi)(x) = \sup_{\theta \in I} \int_{y \neq 0} (1 - \varphi(y))\, \nu_\theta(x, dy) \geq \sup_{\theta \in I} \int_{|y|>R} \nu_\theta(x, dy)$$

for $R \gg 1$ sufficiently large, and so the family of measures is tight. □

5. Solutions to non-linear Cauchy problems associated with integro-differential operators

In this section, we apply the Courrège-von Waldenfels theorem to study solutions to the non-linear Cauchy problem

$$\frac{\partial}{\partial t} u(t, x) = \sup_{\theta \in I}(-q_\theta(x, D)u(t, \cdot))(x), \qquad u(0, x) = f(x) \tag{22}$$

associated with a family of pseudo-differential operators $q_\theta(x, D)$, $\theta \in I$, cf. (3). Because of the non-linearity, which is caused by the supremum, there exist, in general, no pointwise solutions to (22). We work with the weaker notion of viscosity solutions which was originally introduced by Crandall & Lions [6] and Evans [10]. The following definition is taken from Hollender [12]. We refer the reader to [12, Chapter 2] and [3] for a discussion of equivalent definitions.

Definition 5.1. Let $L : \mathcal{D}(L) \to \mathbb{R}$ be an operator with domain $\mathcal{D}(L)$ containing the space of smooth functions with bounded derivatives $C_b^\infty(\mathbb{R}^d)$. An upper semicontinuous function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a *viscosity subsolution* to the equation

$$\partial_t u(t,x) - L_x u(t,x) = 0$$

if the inequality $\partial_t \varphi(t,x) - L_x \varphi(t,x) \le 0$ holds for any function $\varphi \in C_b^\infty([0,\infty) \times \mathbb{R}^d)$ such that $u - \varphi$ has a global maximum in $(t,x) \in (0,\infty) \times \mathbb{R}^d$ with $u(t,x) = \varphi(t,x)$. A mapping u is a *viscosity supersolution* if $-u$ is a viscosity subsolution. If u is both a viscosity sub- and supersolution, then u is called *viscosity solution*.

As usual, we write L_x to indicate that L acts with respect to the space variable x. In order to construct a viscosity solution to (22), we use the following fundamental theorem by Hollender [12] which associates with a sublinear semigroup $(T_t)_{t\ge 0}$ an evolution equation.

Proposition 5.2 ([12, Proposition 4.10])**.** *Let $(T_t)_{t\ge0}$ be a sublinear Markov semigroup on $\mathcal{H}$. Assume that the domain of the pointwise infinitesimal generator $A^{(p)}$ contains the smooth functions with bounded derivatives. If $f \in \mathcal{H}$ is such that $(t,x) \mapsto u(t,x) = T_t f(x)$ is continuous, then u is a viscosity solution to*

$$\frac{\partial}{\partial t} u(t,x) = A_x^{(p)} u(t,x), \qquad u(0,x) = f(x).$$

Combining Proposition 5.2 with the Courrège–Waldenfels theorem, we obtain the following corollary.

Corollary 5.3. *Let $(T_t)_{t\ge0}$ be a sublinear Markov semigroup on $\mathcal{H}$ with pointwise infinitesimal generator $(A^{(p)}, \mathcal{D}(A^{(p)}))$ such that $C_b^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$. If $f \in \mathcal{H}$ is such that $(t,x) \mapsto u(t,x) := T_t f(x)$ is continuous, then u is a viscosity solution to*

$$\frac{\partial}{\partial t} u(t,x) = \sup_{\theta \in I}(-q_\theta(x,D)u(t,\cdot))(x), \qquad u(0,x) = f(x) \tag{23}$$

where $(q_\theta(x,\cdot))$ is the family of continuous negative definite functions from Corollary 4.3.

Proof. By Proposition 5.2, $u(t,x) = T_t f(x)$ is a viscosity solution to

$$\frac{\partial}{\partial t} u(t,x) = A_x^{(p)} u(t,x), \qquad u(0,x) = f(x).$$

Applying Corollary 4.7(ii) and using the very definition of the notion of viscosity solutions, cf. Definition 5.1, we get immediately the assertion. □

Remark 5.4. (i) If the associated family of Lévy measures $\nu_\theta(x,\cdot)$, $\theta \in I$, is tight for each $x \in \mathbb{R}^d$, then the assumption $C_b^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$ can be relaxed to $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$, see Corollary 4.7(i).

(ii) Corollary 5.3 requires that $(t,x) \mapsto T_t f(x)$ is continuous. Typically, continuity with respect to t is much easier to verify than continuity with respect to x. For a large class of sublinear Markov semigroups, it is shown in [16, Theorem 5.3] that $t \mapsto T_t f(x)$ is continuous uniformly for x in a compact set. There seems to be no general result which allows us to deduce continuity with respect to the space variable x, see also the discussion in [12, Remark 4.43]. Translation invariant semigroups $(T_t)_{t\geq 0}$ are one of the few exceptions where continuity with respect to x is easy to obtain; for instance, it is not difficult to see that $T_t f \in C_b^{\text{uc}}(\mathbb{R}^d)$ for any $f \in C_b^{\text{uc}}(\mathbb{R}^d)$.

(iii) For recent existence results for the non-linear Cauchy problem (23) see [18, 7] (via the dynamic programming principle) and [12, 16] (via processes under uncertainty) and the references therein.

6. Lévy processes on sublinear expectation spaces

It is well known that there is a one-to-one correspondence between (classical) Lévy processes and (linear) translation invariant Markov semigroups, see e.g. [4, Section 2.1]. Recently, Denk et. al [7] obtained a similar result in the framework of non-linear semigroups and processes on non-linear expectation spaces. Let us first give the definitions, see also Section 2.

Definition 6.1. We call a family of sublinear operators $T_t : \mathcal{H} \to \mathcal{H}$, $t \geq 0$, a *sublinear Markov convolution semigroup (on $\mathcal{H}$)* if

(i) $(T_t)_{t\geq 0}$ is a sublinear conservative Markov semigroup on $\mathcal{H}$,
(ii) T_t is translation invariant, i.e. $f(x + \cdot) \in \mathcal{H}$ and $(T_t f)(x) = (T_t f(x + \cdot))(0)$ for all $x \in \mathbb{R}^d$ and $f \in \mathcal{H}$,
(iii) $(T_t)_{t\geq 0}$ is strongly continuous at $t = 0$, i.e. $\|T_t f - f\|_\infty \to 0$ as $t \to 0$ for all $f \in \mathcal{H}$.

Definition 6.2. Let $(\Omega, \mathcal{H}, \mathcal{E})$ be a sublinear expectation space, cf. Section 2. A family $X_t : \Omega \to \mathbb{R}^d$, $t \geq 0$, is a *Lévy process for sublinear expectations* (or *sublinear Lévy process*) if

(i) X_t is adapted for all $t \geq 0$, i.e. $f(X_t) \in \mathcal{H}$ for all $f \in C_b^{\text{uc}}(\mathbb{R}^d)$,
(ii) $X_0 \overset{d}{=} 0$ in distribution,
(iii) $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent for any $0 = t_0 < \dots < t_n$, $n \in \mathbb{N}$ (*independent increments*),
(iv) $X_t - X_s \overset{d}{=} X_{t-s}$ for all $s \leq t$ (*stationary increments*),
(v) $X_t \overset{d}{\to} 0$ as $t \downarrow 0$, i.e. $\mathcal{E} f(X_t) \to f(0)$ for any $f \in C_b^{\text{uc}}(\mathbb{R}^d)$.

If $(X_t)_{t\geq 0}$ is a Lévy process for sublinear expectations, then $T_t f(x) := \mathcal{E} f(x + X_t)$ defines a sublinear Markov convolution semigroup on $C_b^{\text{uc}}(\mathbb{R}^d)$, see [7, (Proof of) Theorem 2.3] and also [12, Remark 4-38]. Conversely, sublinear Markov convolution semigroups can be used to construct Lévy processes for sublinear expectations, cf. [7].

Classical Lévy processes can be uniquely characterized (in distribution) by their Lévy triplet (b, Q, ν), and it is natural to ask whether there is an analogous result for sublinear Lévy processes. In this section, we show that the answer is positive for "nice" sublinear Lévy processes, see Theorem 6.4 below. Instead of a single Lévy triplet, we will be dealing with a family of triplets $(b_\theta, Q_\theta, \nu_\theta)$, $\theta \in I$, which is obtained from the Courrège-von Waldenfels theorem.

Definition 6.3. Let $(X_t)_{t\geq 0}$ be a Lévy process for sublinear expectations with semigroup $(T_t)_{t\geq 0}$. If its pointwise infinitesimal generator $(A^{(p)}, \mathcal{D}(A^{(p)}))$ it satisfies $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A^{(p)})$, then we say that (the generator of) $(X_t)_{t\geq 0}$ *has a rich domain.* By the Courrège–Waldenfels theorem, Corollary 4.5, and Corollary 4.9, we can associate a family $(b_\theta, Q_\theta, \nu_\theta)$, $\theta \in I$, with any such process. We call this family *characteristics* of $(X_t)_{t\geq 0}$.

Sublinear Lévy processes can be interpreted as stochastic processes under uncertainty, cf. Hollender [12], and therefore $(b_\theta, Q_\theta, \nu_\theta)$, $\theta \in I$, are sometimes called uncertainty coefficients. The following theorem is the main result in this section.

Theorem 6.4. *Let $(b_\theta, Q_\theta, \nu_\theta)$, $\theta \in I$, be a uniformly bounded family of triplets, i.e.*

$$\sup_{\theta\in I}\left(|b_\theta| + |Q_\theta| + \int_{y\neq 0} \min\{1, |y|^2\}\, \nu_\theta(dy)\right) < \infty.$$

(i) *There exists a Lévy process for sublinear expectations with characteristics $(b_\theta, Q_\theta, \nu_\theta)_{\theta\in I}$.*
(ii) *If ν_θ, $\theta \in I$, is tight, then there exists a unique sublinear Lévy process with characteristics $(b_\theta, Q_\theta, \nu_\theta)_{\theta\in I}$, i.e. any two sublinear Lévy processes with the given characteristics have the same finite-dimensional distributions.*

There are several possibilities to construct a sublinear Lévy process with a given characteristics, e.g. via the dynamic programming principle [7, 18] or as process under uncertainty [12, 16]. Theorem 6.4 tells us, in particular, that for nice triplets (i.e. if tightness holds) these constructions yield the same process, i.e. the constructed processes have the same finite-dimensional distributions and the same semigroup.

For the proof of Theorem 6.4 the following result plays a crucial role. It gives a sufficient condition ensuring that a Markov convolution semigroup is uniquely determined by its pointwise generator restricted to $C_c^\infty(\mathbb{R}^d)$. The recent paper [8] studies in a more general framework under which conditions a sublinear semigroup is uniquely determined by its generator.

Proposition 6.5. *Let $(P_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ be sublinear Markov convolution semigroups on $\mathcal{H} \subseteq C_b(\mathbb{R}^d)$. Assume that the domains of the pointwise generators $(A^{(p)}, \mathcal{D}(A^{(p)}))$ and $(L^{(p)}, \mathcal{D}(L^{(p)}))$ contain $C_c^\infty(\mathbb{R}^d)$ and*

$$\forall f \in C_c^\infty(\mathbb{R}^d)\ :\ A^{(p)}f = L^{(p)}f.$$

Denote by $(b_\theta, Q_\theta, \nu_\theta)$, $\theta \in I$, the associated family of Lévy triplets, cf. Corollary 4.5. *If the family of measures ν_θ, $\theta \in I$, is tight, i.e.*

$$\lim_{R\to\infty} \sup_{\theta\in I} \int_{|y|>R} \nu_\theta(dy) = 0, \tag{24}$$

then $P_t f = T_t f$ for all $t \geq 0$ and $f \in \mathcal{H}$.

Proof. Let $f \in \mathcal{H}$. First we show that the mappings $(t, x) \mapsto T_t f(x)$ and $(t, x) \mapsto P_t f(x)$ are continuous; clearly, it suffices to consider one of the semigroups. For fixed $s \leq t$ the subadditivity and monotonicity of the operators yield

$$T_t - T_s f = T_s T_{t-s} f - T_s f \leq T_s(T_{t-s} f - f) \leq \|T_{t-s} f - f\|_\infty.$$

Interchanging the roles of s and t we get

$$\|T_t f - T_s f\|_\infty \leq \|T_{|t-s|} f - f\|_\infty, \qquad s, t \geq 0.$$

Hence,

$$|T_t f(x) - T_s f(y)| \leq |T_t f(x) - T_t f(y)| + |T_t f(y) - T_s f(y)| \leq |T_t f(x) - T_t f(y)| + \|T_{|t-s|} f - f\|_\infty.$$

Since $T_t f \in \mathcal{H} \subseteq C_b(\mathbb{R}^d)$ is continuous and the semigroup is strongly continuous at $t = 0$, we find that the right-hand side converges to 0 if we let $s \to t$ and $y \to x$. Hence, $(t, x) \mapsto T_t f(x)$ is continuous. By Corollary 5.3 and Remark 5.4(i), $(t, x) \mapsto T_t f(x)$ and $(t, x) \mapsto P_t f(x)$ are viscosity solutions to the evolution equation

$$\frac{\partial}{\partial t} u(t, x) = \sup_{\theta \in I}(-\psi_\theta(D) u(t, \cdot))(x), \qquad u(0, x) = f(x).$$

Because of the tightness condition (24), the Cauchy problem has a unique viscosity solution, cf. [12, Corollary 2.34], and so $P_t f = T_t f$ for all $t \geq 0$. □

After these preparations, we are ready to prove Theorem 6.4.

Proof of Theorem 6.4. It follows from [12, Remark 4.38] and [16, Proposition 4.1, Corollary 4.2] that there exist a measurable space $(\Omega, \mathcal{A})$, a family of probability measures $(\mathbb{P})_{\mathbb{P} \in \mathfrak{P}}$ on $(\Omega, \mathcal{A})$ and a stochastic process $X_t : \Omega \to \mathbb{R}$, $t \geq 0$, with the following properties:

- $\mathcal{E}(Y) := \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}_\mathbb{P}(Y)$ defines a sublinear expectation on $(\Omega, \hat{\mathcal{H}})$ for the space $\hat{\mathcal{H}}$ of bounded random variables $Y : \Omega \to \mathbb{R}$; here $\mathbb{E}_\mathbb{P}$ denotes the expectation with respect to $\mathbb{P}$.
- $X_0 = 0$ and $(X_t)_{t \geq 0}$ has stationary and independent increments on the sublinear expectation space $(\Omega, \hat{\mathcal{H}}, \mathcal{E})$.
- For each $\mathbb{P} \in \mathfrak{P}$, the process $(X_t)_{t \geq 0}$ is a semimartingale on the (classical) probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and its differential characteristics takes values in $\bigcup_{\theta \in I}(b_\theta, Q_\theta, \nu_\theta)$ with probability 1.
- $T_t f(x) := \mathcal{E} f(x + X_t)$ defines a sublinear Markov semigroup on $\mathcal{H} := C_b^{uc}(\mathbb{R}^d)$.
- $C_b^2(\mathbb{R}^d)$ is contained in the domain of the pointwise generator $A^{(p)}$.
- $A^{(p)} f(x) = \sup_{\theta \in I}(-\psi_\theta(D) f)(x)$ for all $f \in C_b^2(\mathbb{R}^d)$.

Moreover, it follows from [16, Theorem 5.3] that $(T_t)_{t \geq 0}$ is strongly continuous on $C_b^{uc}(\mathbb{R}^d)$ and that $T_t f(0) \to f(0)$ for all $f \in C_b(\mathbb{R}^d)$. This proves (i).

Now assume that the family ν_θ, $\theta \in I$, is tight, and let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be two sublinear Lévy processes with rich domains and characteristics $(b_\theta, Q_\theta, \nu_\theta)_{\theta \in I}$. The associated semigroups $(P_t)_{t \geq 0}$ and $(T_t)_{t \geq 0}$ are sublinear Markov convolution semigroups on $C_b^{uc}(\mathbb{R}^d)$, cf. [7, (Proof of) Theorem 2.3]. By assumption, their pointwise generators coincide on $C_c^\infty(\mathbb{R}^d)$. Applying Corollary 4.7 and Proposition 6.5, we find that $P_t f = T_t f$ for any $f \in C_b^{uc}(\mathbb{R}^d)$ and $t \geq 0$. It is easily seen from the independence of the increments that the finite dimensional distributions of a sublinear Lévy process are uniquely determined by its semigroup, and so the assertion follows. □

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Institut für Mathematische Stochastik, Fachrichtung Mathematik
Technische Universität Dresden
01062 Dresden
Germany
e-mail: franziska.kuehn1@tu-dresden.de