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THE QUATERNIONIC/HYPERCOMPLEX-CORRESPONDENCE

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Abstract

Given a quaternionic manifold M with a certain U(1)-symmetry, we construct a hypercomplex manifold M' of the same dimension. This construction generalizes the quaternionic Kähler/hyper-Kähler-correspondence. As an example of this construction, we obtain a compact homogeneous hypercomplex manifold which does not admit any hyper-Kähler structure. Therefore our construction is a proper generalization of the quaternionic Kähler/hyper-Kähler-correspondence.

1. Introduction

Let us recall that there exist constructions due to Andriy Haydys, called the QK/HK-correspondence and the HK/QK-correspondence, which relate quaternionic Kähler manifolds to hyper-Kähler manifolds of the same dimension [12]. These constructions have been generalized to include possibly indefinite metrics [2, 1]. In this way the supergravity c-map metric and a one-parameter deformation thereof have been described as an application of the HK/QK-correspondence with indefinite initial hyper-Kähler data. Many complete quaternionic Kähler manifolds can be obtained in this way, see for instance [8] for co-homogeneity one examples.

The main result of this paper, see Theorem 6.4, is a construction of a hypercomplex manifold from a quaternionic manifold with a U(1)-action, which we may call the *quaternionic/hypercomplex-correspondence* (Q/H-correspondence for short). This construction generalizes the QK/HK-correspondence.

In [22, 14, 21], it is shown that with every quaternionic manifold M one can associate an $\mathbb{H}^*/\{\pm 1\}$-bundle over M and a hypercomplex structure on the total space of the bundle. More precisely [21], there exists a one-parameter family of $\mathbb{H}^*/\{\pm 1\}$-bundles such that, given a quaternionic connection on M, each of the bundles is endowed with an almost hypercomplex structure. For a particular choice of the parameter, the almost hypercomplex structure is integrable and independent of the connection. Here we will adopt a different point of view. Instead of a one-parameter family of bundles, we will define a single principal $\mathbb{H}^*/\{\pm 1\}$-bundle, which we call the *Swann bundle*, endowed with a one-parameter family of almost hypercomplex structures (still depending on a quaternionic connection). Again we find that, for a particular choice of the parameter, namely $c = -4(n+1)$, the almost hypercomplex structure is always integrable and independent of the connection, see Proposition 3.3. Here $4n = \dim M$. For all other values of the parameter, we show that the almost hypercomplex structure is integrable if and only if all $I \in Q$, where Q denotes the quaternionic structure,

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are skew-symmetric with respect to the skew-symmetric part of the Ricci-curvature, see Theorem 3.6.

Now we briefly explain how we obtain the Q/H-correspondence. Given an infinitesimal automorphism X of a quaternionic manifold (M, Q, ∇) endowed with a quaternionic connection ∇, we show that the natural lift $\hat{X}$ of X to the Swann bundle $\hat{M}$ preserves each member of the one-parameter family of almost hypercomplex structures. The next step is to perform a hypercomplex reduction with respect to $\hat{X}$. Recall that hypercomplex reduction was introduced by Dominic Joyce in [14]. It is defined as the quotient of a level set of a moment map by the group action. The construction is based on the notion of a moment map in this context as defined in [14]. Here we define the moment map for the infinitesimal automorphism $\hat{X}$ by the equation (5.4) and analyse Joyce's conditions in Proposition 5.8. Assuming that $\hat{X}$ generates a free U(1)-action, we can finally perform the reduction obtaining a hypercomplex manifold M'. Otherwise, we can construct the hypercomplex structure on a submanifold transversal to the foliation defined by $\hat{X}$ (on some open submanifold of $\hat{M}$).

Note that one can find a related construction of a hypercomplex manifold from a quaternionic manifold (endowed in addition with a complex structure and a compatible S^1-action) in Proposition 5.5 in [6], in which the authors study the quaternionic Feix-Kaledin (qFK) construction. The qFK construction is a generalization of the original one in [16, 9]. In Theorem 5 of [7], it is shown that a quaternionic Kähler manifold obtained from the qFK construction and a hyper-Kähler manifold from the original Feix-Kaledin construction are related by the HK/QK-correspondence. This is analogous to the relation between the supergravity c-map and the rigid c-map [2, 1].

Examples of our Q/H-correspondence include compact homogeneous hypercomplex manifolds. Indeed, starting with a homogeneous quaternionic Hopf manifold

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}},$$

we obtain a homogeneous hypercomplex Hopf manifold

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$$

by the Q/H-correspondence, see Example 7.8. Note that this hypercomplex manifold does not admit any hyper-Kähler structure for topological reasons. Therefore our construction is a proper generalization of the QK/HK-correspondence.

2. Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and without boundary and maps are assumed to be smooth unless otherwise mentioned. The space of sections of a vector bundle $E \to M$ is denoted by $\Gamma(E)$.

We say that M is a *quaternionic manifold* with the quaternionic structure Q if Q is a subbundle of $\mathrm{End}(TM)$ of rank 3 which at every point $x \in M$ is spanned by endomorphisms $I_1, I_2, I_3 \in \mathrm{End}(T_xM)$ satisfying

$$I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}, \quad I_1I_2 = -I_2I_1 = I_3, \tag{2.1}$$

and there exists a torsion-free connection ∇ on M such that ∇ preserves Q, that is, $\nabla_X\Gamma(Q) \subset$

$\Gamma(Q)$ for all $X \in \Gamma(TM)$. Note that we use the same letter ∇ for the connection on $\mathrm{End}(TM)$ induced by ∇ if there is no confusion. Such a torsion-free connection ∇ is called a *quaternionic connection* and the triplet (I_1, I_2, I_3) is called an *admissible frame* of Q at x. The dimension of a quaternionic manifold M is denoted by $4n$. Note that a quaternionic connection is not unique, in fact, there is the following result [10, 5].

Lemma 2.1. *Let ∇^1 and ∇^2 be quaternionic connections on (M, Q). Then there exists a 1-form ξ on M such that*

$$\nabla^2_X Y = \nabla^1_X Y + S^\xi_X Y \tag{2.2}$$

for all X, $Y \in \Gamma(TM)$, where S^ξ is defined by

$$S^\xi_X Y = \xi(X)Y + \xi(Y)X - \xi(I_1X)I_1Y - \xi(I_1Y)I_1X \\ - \xi(I_2X)I_2Y - \xi(I_2Y)I_2X - \xi(I_3X)I_3Y - \xi(I_3Y)I_3X.$$

Conversely, for a given quaternionic connection ∇^1, the connection ∇^2 given by the equation above is also a quaternionic connection.

An *almost hypercomplex manifold* is defined to be a manifold M endowed with 3 almost complex structures I_1, I_2, I_3 satisfying the quaternionic relations (2.1). If I_1, I_2, I_3 are integrable, then M is called a *hypercomplex manifold*. There exists a unique torsion-free connection on a hypercomplex manifold for which the hypercomplex structures are parallel. It is called the *Obata connection* [18]. Obviously, hypercomplex manifolds are quaternionic manifolds with $Q = \langle I_1, I_2, I_3\rangle$.

3. The canonical family of almost hypercomplex structures on the Swann bundle $\hat{M}$

In this section we will define a principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-bundle $\hat{M} \to M$ over a quaternionic manifold (M, Q) equipped with a quaternionic connection ∇ and endow $\hat{M}$ with a one-parameter family of almost hypercomplex structures depending on the quaternionic connection ∇. Then we will study the integrability of the hypercomplex structure and its dependence (or independence) on the choice of ∇ for different values of the parameter.

3.1. The principal bundle $\hat{M} \to M$. Let S be the principal $\mathrm{SO}(3)$-bundle of admissible frames (I_1, I_2, I_3) over a quaternionic manifold (M, Q). The principal action τ of $g \in \mathrm{SO}(3)$ is given by $\tau(s, g) = sg^\varepsilon$ for $s = (I_1, I_2, I_3) \in S$, where $\varepsilon = 1$ (resp. $\varepsilon = -1$) if S is considered as a right (resp. left)-principal bundle. The bundle projection of S is denoted by π_S. We take a basis (e_1, e_2, e_3) of $\mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H} \cong \mathfrak{sp}(1) \cong \mathfrak{so}(3)$ so that

$$[e_\alpha, e_\beta] = 2e_\gamma$$

for any cyclic permutation (α, β, γ). Hereafter (α, β, γ) will be always a cyclic permutation, whenever the three letters appear in an expression. A quaternionic connection induces a principal connection $\theta : TS \to \mathfrak{so}(3)$ and we denote $\theta = \sum \theta^\alpha e_\alpha$. Moreover we consider the principal $\mathbb{R}^{>0}$-bundle $S_0 := (\Lambda^{4n}(T^*M)\backslash\{0\})/\{\pm 1\}$ over M, where $\mathbb{R}^{>0} = \{a \in \mathbb{R} \mid a > 0\}$. The principal $\mathbb{R}^{>0}$-action τ_0 on S_0 is given by scalar multiplication $\tau_0(\rho, a) := \rho a^\varepsilon$ $(\varepsilon = \pm 1)$ for $\rho \in S_0$ and $a \in \mathbb{R}^{>0}$. The bundle projection of S_0 is denoted by π_{S_0}. A quaternionic connection induces also a principal connection $\theta_0 : TS_0 \to \mathbb{R} = \mathrm{Lie}\,(\mathbb{R}^{>0})$. The product $S_0 \times S$ is a principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-bundle over $M \times M$ whose principal action is $\tau_0 \times \tau$. The

$\mathbb{R}^4$ $(\cong \mathbb{R}\oplus\mathfrak{so}(3))$-valued 1-form $(\theta_0\circ pr_{TS_0}, \theta\circ pr_{TS}) = (\theta_0\circ pr_{TS_0}, \theta_1\circ pr_{TS}, \theta_2\circ pr_{TS}, \theta_3\circ pr_{TS})$ is a principal connection on $S_0\times S$, where pr_{TS_0} (resp. pr_{TS}) is the projection from $T(S_0\times S)\cong TS_0\times TS$ onto TS_0 (resp. TS). Note that the Lie group $\mathbb{R}^{>0}\times \mathrm{SO}(3) = \mathbb{H}^*/\{\pm 1\}$.

Let $\triangle : M\to M\times M$ be the diagonal map defined by $\triangle(x) = (x,x)$ for each $x\in M$. The pullback bundle

$$\hat{M} := \triangle^*(S_0\times S) = \{(x,(\rho,s))\in M\times(S_0\times S)\mid x = \pi_{S_0}(\rho) = \pi_S(s)\}$$

is a principal $\mathbb{R}^{>0}\times\mathrm{SO}(3)$-bundle over M and $\bar{\theta} := \triangle_{\#}^*(\theta_0\circ pr_{TS_0}, \theta\circ pr_{TS})$ is a principal connection on $\hat{M}$, where $\triangle_{\#} : \hat{M}\to S_0\times S$ is the canonical bundle map. The bundle projection of $\hat{M}$ onto M is denoted by $\hat{\pi}$. Using the bundle projections $\hat{\pi}$, π_{S_0}, π_S and the principal connections $\bar{\theta}$, θ_0, θ, we have the decomposition

$$T\hat{M} = \bar{\mathcal{V}}\oplus\bar{\mathcal{H}},\ TS_0 = \mathcal{V}_0\oplus\mathcal{H}_0,\ TS = \mathcal{V}\oplus\mathcal{H}, \tag{3.1}$$

where $\bar{\mathcal{V}} = \operatorname{Ker}\hat{\pi}_*$, $\bar{\mathcal{H}} = \operatorname{Ker}\bar{\theta}$ and so on. It holds that $(\triangle_{\#})_*(\bar{\mathcal{V}}_{(x,(\rho,s))}) = (\mathcal{V}_0)_\rho\times\mathcal{V}_s$ and $(\triangle_{\#})_*(\bar{\mathcal{H}}_{(x,(\rho,s))})\subset(\mathcal{H}_0)_\rho\times\mathcal{H}_s$ for each $(x,(\rho,s))\in\hat{M}$. Set $\triangle_S := pr_{TS}\circ(\triangle_{\#})_*$ and $\triangle_{S_0} := pr_{TS_0}\circ(\triangle_{\#})_*$. The principal actions on $\hat{M}$, $S_0\times S$, S_0 and S induce fundamental vector fields. We denote by $\widetilde{A}$ the fundamental vector field corresponding to a Lie algebra element A, irrespective of the manifold on which the vector field is defined, and set $Z_\alpha = \widetilde{e_\alpha}$ $(\alpha = 1,2,3)$. Note that $[Z_\alpha, Z_\beta] = 2\varepsilon Z_\gamma$.

3.2. The canonical family of almost hypercomplex structures. Let (M,Q) be a quaternionic manifold, ∇ a quaternionic connection and $\hat{\pi} : \hat{M}\to M$ the principal $\mathbb{R}^{>0}\times\mathrm{SO}(3)$-bundle with connection $\bar{\theta}$ constructed in the previous subsection. In this subsection, we define a canonical family of almost hypercomplex structures on $\hat{M}$ and consider their integrability.

Set $e_0 := 1\in\mathbb{R}\,(\cong T_1\mathbb{R}^{>0})$ and $Z_0^c := c\,\widetilde{e_0}$ for a nonzero real number c. We denote the horizontal lifts relative to the connections $\bar{\theta}$, θ, θ_0 by $(\,\cdot\,)^{\bar{h}}$, $(\,\cdot\,)^{h}$, $(\,\cdot\,)^{h_0}$, respectively. An almost hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ on $\hat{M}$ is defined by

$$\hat{I}_\alpha^{\bar{\theta},c}Z_0^c = -Z_\alpha,\quad \hat{I}_\alpha^{\bar{\theta},c}Z_\alpha = Z_0^c,\quad \hat{I}_\alpha^{\bar{\theta},c}Z_\beta = Z_\gamma,\quad \hat{I}_\alpha^{\bar{\theta},c}Z_\gamma = -Z_\beta$$

and

$$(\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X) = (I_\alpha(\hat{\pi}_*X))^{\bar{h}}_{(x,(\rho,s))}$$

for all horizontal vector X at $(x,(\rho,s))\in\hat{M}$, where $s = (I_1,I_2,I_3)$. Note that the triple $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ depends on the connection form $\bar{\theta}$ and c.

Lemma 3.1. *For any horizontal lift $X^{\bar{h}}\in\bar{\mathcal{H}}_{(x,(\rho,s))}$ at $(x,(\rho,s))\in\hat{M}$, we have $(\triangle_{\#})_*X^{\bar{h}} = (X_\rho^{h_0}, X_s^{h})$. In particular, it holds $(\triangle_{\#})_*((\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}})) = ((I_\alpha X)_\rho^{h_0}, (I_\alpha X)_s^{h})$, where $s = (I_1,I_2,I_3)$. As a consequence, the horizontal lift $X^{\bar{h}}$ of a vector field X on M is $\triangle_{\#}$-related to the vector field (X^{h_0}, X^{h}), which is the horizontal lift of (X,X):*

$$(\triangle_{\#})_*X^{\bar{h}} = (X^{h_0}, X^{h})\circ\triangle_{\#}.$$

Proof. $(\triangle_{\#})_*X^{\bar{h}}$ and (X^{h_0}, X^{h}) are horizontal vectors of $S_0\times S$, since applying the connection form $(\theta_0\circ pr_{TS_0}, \theta\circ pr_{TS})$ on both vectors gives zero. On the other hand, applying $(\pi_{S_0}\times\pi_S)_*$ on both vectors gives (X,X) because $(\pi_{S_0}\times\pi_S)\circ\triangle_{\#} = \triangle\circ\hat{\pi}$. This proves

$(\triangle_\#)_* X^{\bar h} = (X^{h_0}, X^h) \circ \triangle_\#$. Now it is easy to obtain $(\triangle_\#)_*((\hat I_\alpha^{\bar\theta,c})_{(x,(\rho,s))}(X^{\bar h})) = ((I_\alpha X)_\rho^{h_0}, (I_\alpha X)_s^h)$ using that $(\hat I_\alpha^{\bar\theta,c})_{(x,(\rho,s))}(X^{\bar h}) = (I_\alpha X)^{\bar h}$. □

Lemma 3.2. *Let* ∇^1 *and* $\nabla^2 = \nabla^1 + S^\xi$ *be quaternionic connections on* (M, Q)*, where* $\xi \in \Gamma(T^*M)$*. We denote the almost hypercomplex structure defined above with respect to* ∇^i $(i = 1, 2)$ *and* $c \neq 0$ *by* $\hat I_\alpha^{i,c}$ $(\alpha = 1, 2, 3)$*. Then we have*

$$\hat I_\alpha^{1,c} - \hat I_\alpha^{2,c} = \varepsilon\left(1 + \frac{4(n+1)}{c}\right)\Big((\hat\pi^*\xi) \otimes Z_\alpha + ((\hat\pi^*(\xi \circ I_\alpha)) \otimes Z_0^c\Big)$$

at each point $(x, (\rho, s)) \in \hat M$*, where* $s = (I_1, I_2, I_3)$*.*

Proof. We consider any point $(x, (\rho, s)) \in \hat M$, $s = (I_1, I_2, I_3)$, and omit the reference point in the proof. The corresponding connection forms induced by ∇^i are denoted by $\bar\theta^i$, $\theta^i = (\theta_1^i, \theta_2^i, \theta_3^i)$, θ_0^i $(i = 1, 2)$, respectively. The tangent bundle $T\hat M$ is decomposed into $T\hat M = \bar{\mathcal V} \oplus \bar{\mathcal H}^1 = \bar{\mathcal V} \oplus \bar{\mathcal H}^2$, where $\bar{\mathcal H}^i = \operatorname{Ker} \bar\theta^i$. We express any tangent vector X of $\hat M$ as

$$X = Y^{\bar h_i} + \sum_{\delta=1}^3 a_\delta^i Z_\delta + b^i Z_0^c,$$

where $Y \in TM$. By the definition of $\hat I_\alpha^{i,c}$, we see

$$\hat I_\alpha^{i,c}(X) = (I_\alpha Y)^{\bar h_i} + a_\alpha^i Z_0^c + a_\beta^i Z_\gamma - a_\gamma^i Z_\beta - b^i Z_\alpha.$$

Since

$$\begin{aligned}\bar\theta^1(X) &= \sum_{\delta=1}^3 a_\delta^1 e_\delta + cb^1 e_0 = \bar\theta^1(Y^{\bar h_2}) + \sum_{\delta=1}^3 a_\delta^2 e_\delta + cb^2 e_0 \\ &= \sum_{\delta=1}^3 \theta_\delta^1(\triangle_S Y^{\bar h_2}) e_\delta + \theta_0^1(\triangle_{S_0} Y^{\bar h_2}) e_0 + \sum_{\delta=1}^3 a_\delta^2 e_\delta + cb^2 e_0,\end{aligned}$$

we have $b^1 = b^2 + (1/c)\theta_0^1(\triangle_{S_0} Y^{\bar h_2})$ and $a_\delta^1 = a_\delta^2 + \theta_\delta^1(\triangle_S Y^{\bar h_2})$ $(\delta = 1, 2, 3)$. Therefore it holds

$$\begin{aligned}\hat I_\alpha^{1,c}(X) &= (I_\alpha Y)^{\bar h_1} + a_\alpha^1 Z_0^c + a_\beta^1 Z_\gamma - a_\gamma^1 Z_\beta - b^1 Z_\alpha \\ &= (I_\alpha Y)^{\bar h_1} + (a_\alpha^2 + \theta_\alpha^1(\triangle_S Y^{\bar h_2}))Z_0^c + (a_\beta^2 + \theta_\beta^1(\triangle_S Y^{\bar h_2}))Z_\gamma \\ &\quad - (a_\gamma^2 + \theta_\gamma^1(\triangle_S Y^{\bar h_2}))Z_\beta - (b^2 + (1/c)\theta_0^1(\triangle_{S_0} Y^{\bar h_2}))Z_\alpha \\ &= (I_\alpha Y)^{\bar h_1} - (I_\alpha Y)^{\bar h_2} + \hat I_\alpha^{2,c}(X) \\ &\quad + \theta_\alpha^1(\triangle_S Y^{\bar h_2})Z_0^c + \theta_\beta^1(\triangle_S Y^{\bar h_2})Z_\gamma - \theta_\gamma^1(\triangle_S Y^{\bar h_2})Z_\beta - (1/c)\theta_0^1(\triangle_{S_0} Y^{\bar h_2})Z_\alpha.\end{aligned}$$

Let $s_0 : U \to S_0$ and $s : U \to S$ be local sections defined on an open set U in M. Then $\bar s := (s_0, s) \circ \triangle$ is a local section of $\hat M$. The pull backs of θ^i, θ_0^i to U are denoted by $\theta^{i,U}$ and $\theta_0^{i,U}$. If we define the one forms $\theta_\alpha^{i,U}$ by $\theta^{i,U} = s^*\theta^i = (1/2)\sum(\theta_\alpha^{i,U})e_\alpha$. From Lemma 2.1 and

$$\nabla^i I_\alpha = \varepsilon(\theta_\gamma^{i,U} \otimes I_\beta - \theta_\beta^{i,U} \otimes I_\gamma) \quad (i = 1, 2),$$

one can check that

$$\theta_\delta^{2,U} - \theta_\delta^{1,U} = -2\varepsilon(\xi \circ I_\delta) \quad (\delta = 1, 2, 3), \tag{3.2}$$

$$\theta_0^{2,U} - \theta_0^{1,U} = 4\varepsilon(n+1)\xi.$$

It is easy to see that

$$\begin{aligned}
Y^{\bar h_1} - Y^{\bar h_2} &= \bar s_*(Y) - v_1(\bar s_*(Y)) - \bar s_*(Y) + v_2(\bar s_*(Y)) \\
&= -\sum_{\delta=1}^{3}(\theta_\delta^1(s_*Y) - \theta_\delta^2(s_*Y))Z_\delta - (1/c)(\theta_0^1(s_{0*}Y) - \theta_0^2(s_{0*}Y))Z_0^c \\
&= -\frac{1}{2}\sum_{\delta=1}^{3}(\theta_\delta^{1,U}(Y) - \theta_\delta^{2,U}(Y))Z_\delta - (1/c)(\theta_0^{1,U}(Y) - \theta_0^{2,U}(Y))Z_0^c \\
&\overset{(3.2)}{=} -\varepsilon\sum_{\delta=1}^{3}\xi(I_\delta Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(Y)Z_0^c, \qquad (3.3)
\end{aligned}$$

where $v_i : T\hat M \to \bar{\mathcal V}$ is the projection with respect to $\bar\theta^i$ $(i = 1, 2)$. Finally we obtain

$$\begin{aligned}
&\hat I_\alpha^{1,c}(X) - \hat I_\alpha^{2,c}(X) \\
&= -\varepsilon\sum_{\delta=1}^{3}\xi(I_\delta I_\alpha Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(I_\alpha Y)Z_0^c \\
&\quad + \theta_\alpha^1(\triangle_S Y^{\bar h_2})Z_0^c + \theta_\beta^1(\triangle_S Y^{\bar h_2})Z_\gamma - \theta_\gamma^1(\triangle_S Y^{\bar h_2})Z_\beta - (1/c)\theta_0^1(\triangle_{S_0} Y^{\bar h_2})Z_\alpha \\
&\overset{(*)}{=} -\varepsilon\sum_{\delta=1}^{3}\xi(I_\delta I_\alpha Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(I_\alpha Y)Z_0^c \\
&\quad + \varepsilon\xi(I_\alpha Y)Z_0^c + \varepsilon\xi(I_\beta Y)Z_\gamma - \varepsilon\xi(I_\gamma Y)Z_\beta + \frac{4\varepsilon(n+1)}{c}\xi(Y)Z_\alpha \\
&= \left(\varepsilon\xi(Y) + \frac{4(n+1)\varepsilon}{c}\xi(Y)\right)Z_\alpha + \left(\varepsilon\xi(I_\alpha Y) + \frac{4(n+1)\varepsilon}{c}\xi(I_\alpha Y)\right)Z_0^c,
\end{aligned}$$

where in the step $(*)$ of the calculation we have computed

$$\theta_\alpha^1(\triangle_S Y^{\bar h_2}) = \theta_\alpha^1(Y^{h_2}) = \theta_\alpha^1(Y^{h_2} - Y^{h_1}) \overset{(3.3)}{=} \varepsilon\xi(I_\alpha Y)$$

and similarly for the other terms. □

The following proposition is an immediate consequence of Lemma 3.2, cf. the result with [21, Proposition 3.3].

Proposition 3.3. *The almost hypercomplex structure is independent of the choice of quaternionic connection if and only if $c = -4(n+1)$.*

Next we investigate transformation properties of the structures $\hat I_\alpha^{\bar\theta,c}$ $(\alpha = 1, 2, 3)$ under the principal action.

Lemma 3.4. *We have $L_{Z_0}\hat I_\alpha^{\bar\theta,c} = L_{Z_\alpha}\hat I_\alpha^{\bar\theta,c} = 0$, $L_{Z_\alpha}\hat I_\beta^{\bar\theta,c} = 2\varepsilon\hat I_\gamma^{\bar\theta,c}$ and $L_{Z_\alpha}\hat I_\gamma^{\bar\theta,c} = -2\varepsilon\hat I_\beta^{\bar\theta,c}$.*

Proof. Note first that the principal action generated by the vector fields Z_a, $a = 0, \ldots, 3$, preserves the horizontal and vertical distributions. Moreover, the central vector field Z_0 commutes with the principal action and thus preserves the three canonical almost complex structures $\hat I_\alpha^{\bar\theta,c}$.

Next we observe that it is easy to check the above equations on the vertical distribution by evaluating them on $Z_0^c, \ldots, Z_3$. So it only remains to check them on the horizontal distribution. Let $\{\phi_t\}_{t\in\mathbb{R}}$ be the flow of Z_1. Since

$$\phi_t((x,(\rho,s))) = (x,(\rho,(I_1,(\cos 2\varepsilon t)I_2 + (\sin 2\varepsilon t)I_3, (-\sin 2\varepsilon t)I_2 + (\cos 2\varepsilon t)I_3)))$$

for $(x,(\rho,s)) \in \hat{M}$, where $s = (I_1, I_2, I_3)$ and the horizontal lift of any vector field or tangent vector of M is invariant under ϕ_t, we have

$$\begin{aligned}
(L_{Z_1}\hat{I}_2^{\bar{\theta},c})_{(x,(\rho,s))}(Y^h) &= [Z_1, \hat{I}_2^{\bar{\theta},c}Y^h]_{(x,(\rho,s))} \\
&= \frac{d}{dt}\phi_{t*}^{-1}((\hat{I}_2^{\bar{\theta},c}Y^h)_{\phi_t((x,(\rho,s)))})\Big|_{t=0} \\
&= \frac{d}{dt}\phi_{t*}^{-1}((\cos 2\varepsilon t)(I_2Y)^h_{\phi_t((x,(\rho,s)))} + (\sin 2\varepsilon t)(I_3Y)^h_{\phi_t((x,(\rho,s)))})\Big|_{t=0} \\
&= \frac{d}{dt}\Big((\cos 2\varepsilon t)(I_2Y)^h_{(x,(\rho,s))} + (\sin 2\varepsilon t)(I_3Y)^h_{(x,(\rho,s))}\Big)\Big|_{t=0} \\
&= 2\varepsilon(I_3Y)^h_{(x,(\rho,s))} = 2\varepsilon(\hat{I}_3^{\bar{\theta},c})_{(x,(\rho,s))}Y^h
\end{aligned}$$

and similarly $L_{Z_1}\hat{I}_1^{\bar{\theta},c} = 0$, which imply $L_{Z_1}\hat{I}_3^{\bar{\theta},c} = -2\varepsilon\hat{I}_2^{\bar{\theta},c}$. □

The Nijenhuis tensor for $\hat{I}_\alpha^{\bar{\theta},c}$ is given by

$$N^\alpha(U,V) = [U,V] + \hat{I}_\alpha^{\bar{\theta},c}[\hat{I}_\alpha^{\bar{\theta},c}U,V] + \hat{I}_\alpha^{\bar{\theta},c}[U,\hat{I}_\alpha^{\bar{\theta},c}V] - [\hat{I}_\alpha^{\bar{\theta},c}U,\hat{I}_\alpha^{\bar{\theta},c}V]$$

for $U, V \in \Gamma(T\hat{M})$. Let $\bar{\Omega}$ (resp. Ω) be the curvature form of $\bar{\theta}$ (resp. θ). Take a local section $s : U \to S$ defined on an open set U of M. The pull back of $\Omega = \sum_{\alpha=1}^3 \Omega_\alpha e_\alpha$ by s is denoted by Ω^U. Since the curvature form is horizontal, we have

$$\varepsilon\Omega|_{s(U)} = \pi_S^*\Omega^U|_{s(U)}. \tag{3.4}$$

If we define the two-forms Ω_α^U by $\Omega^U = (1/2)\sum \Omega_\alpha^U e_\alpha$ and denote by $\bar{\nabla}$ the connection on Q induced by ∇, then we have

$$R_{X,Y}^{\bar{\nabla}}I_\alpha = [R_{X,Y}^{\nabla}, I_\alpha] = \left[\frac{1}{2}\sum \Omega_\delta^U(X,Y)I_\delta, I_\alpha\right] = \Omega_\gamma^U(X,Y)I_\beta - \Omega_\beta^U(X,Y)I_\gamma,$$

which implies

$$\Omega_\alpha^U(X,Y) = -\frac{1}{2n}\mathrm{Tr}I_\alpha R_{X,Y}^{\nabla} \tag{3.5}$$

for X, $Y \in TM$. In fact, multiplying the equation $R_{X,Y}^{\nabla}\circ I_\alpha - I_\alpha \circ R_{X,Y}^{\nabla} = \Omega_\gamma^U(X,Y)I_\beta - \Omega_\beta^U(X,Y)I_\gamma$ with I_β, we obtain

$$I_\beta \circ R_{X,Y}^{\nabla}\circ I_\alpha + I_\gamma \circ R_{X,Y}^{\nabla} = -\Omega_\gamma^U(X,Y)\mathrm{id} - \Omega_\beta^U(X,Y)I_\alpha.$$

Taking the trace proves (3.5). Let Ric^∇ be the Ricci curvature of ∇ and its symmetric (resp. anti-symmetric) part is denoted by $(Ric^\nabla)^s$ (resp. $(Ric^\nabla)^a$). The Nijenhuis tensors of the canonical almost complex structures on the bundle $\hat{M}$ over the quaternionic manifold (M, Q, ∇) are computed in the next lemma.

Lemma 3.5. *If $n > 1$ or Q is anti-self-dual provided $n = 1$, we have*

$$N^\alpha(Z_0^c, Z_i) = 0 \ \textit{for}\ 1 \le i \le 3, \tag{3.6}$$

(3.7) $$N^\alpha(Z_i, Z_j) = 0 \ \textit{for} \ 1 \le i, j \le 3,$$

(3.8) $$N^\alpha(Z_0^c, X^{\bar h}) = 0,$$

(3.9) $$N^\alpha(Z_i, X^{\bar h}) = 0 \ \textit{for} \ 1 \le i \le 3,$$

(3.10)
$$\begin{aligned}&\bar\theta(N^\alpha(X^{\bar h}, Y^{\bar h})_{(x,(\rho,s))})\\ &= \frac{4\varepsilon(n+1)+\varepsilon c}{2(n+1)}\Big((Ric^\nabla)^a(X,Y) - (Ric^\nabla)^a(I_\alpha X, I_\alpha Y)\Big)e_0\\ &- \frac{4\varepsilon(n+1)+\varepsilon c}{2c(n+1)}\Big((Ric^\nabla)^a(X, I_\alpha Y) + (Ric^\nabla)^a(I_\alpha X, Y)\Big)e_\alpha \ \textit{and}\end{aligned}$$

(3.11) $$\hat\pi_*(N^\alpha(X^{\bar h}, Y^{\bar h})) = 0 \ \textit{for} \ X, Y \in \Gamma(TM),$$

where $(x,(\rho,s)) \in \hat M$ $(s = (I_1, I_2, I_3))$.

Proof. We write $\hat I_\alpha = \hat I_\alpha^{\bar\theta,c}$ for simplicity in the proof of this lemma. It is easy to see that (3.6–3.9) hold by the definition of the almost hypercomplex structure on $\hat M$ and Lemma 3.4. In fact, for example, we have $N^\alpha(Z_\beta, Z_\gamma) = [Z_\beta, Z_\gamma] + [Z_\gamma, Z_\beta] = 0$ and $N^\alpha(Z_\beta, X^{\bar h}) = \hat I_\alpha[Z_\beta, \hat I_\alpha X^{\bar h}] - [\hat I_\alpha Z_\beta, \hat I_\alpha X^{\bar h}] = \hat I_\alpha(-2\varepsilon \hat I_\gamma)(X^{\bar h}) - 2\varepsilon \hat I_\beta(X^{\bar h}) = 0$. The other equations are proved similarly. Next we show (3.10). It holds $(\theta_i \circ \hat I_\alpha)(Z_0^c) = -\delta_{i\alpha}$, $(\theta_i \circ \hat I_\alpha)(Z_\alpha) = c\delta_{i0}$, $(\theta_i \circ \hat I_\alpha)(Z_\beta) = \delta_{i\gamma}$, $(\theta_i \circ \hat I_\alpha)(Z_\gamma) = -\delta_{i\beta}$. Using this and Lemma 3.1, we have

$$\begin{aligned}&\bar\theta(\hat I_\alpha[\hat I_\alpha X^{\bar h}, Y^{\bar h}])\\ =&(\bar\theta \circ \hat I_\alpha)(\sum_{i=1}^3 \theta_i(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])Z_i + \frac{1}{c}\theta_0(\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}])Z_0^c)\\ =&\sum_{j=0}^3(\sum_{i=1}^3 \theta_i(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])(\theta_j \circ \hat I_\alpha)(Z_i)e_j + \frac{1}{c}\theta_0(\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}])(\theta_j \circ \hat I_\alpha)(Z_0^c)e_j)\\ =&\sum_{j=0}^3(\theta_\alpha(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])c\delta_{j0}e_j + \theta_\beta(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])\delta_{j\gamma}e_j\\ &- \theta_\gamma(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])\delta_{j\beta}e_j + \frac{1}{c}\theta_0(\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}])(-\delta_{j\alpha})e_j)\end{aligned}$$

As a consequence of Lemma 3.1, we have $\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}] = [(I_\alpha X)^{h_0}, Y^{h_0}]|_{\triangle\#(\hat M)}$ and $\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}] = [(I_\alpha X)^h, Y^h]|_{\triangle\#(\hat M)}$. By $\bar\Omega = \Omega + d\theta_0 = \sum_{\delta=1}^3 \Omega_\delta e_\delta + (d\theta_0)e_0$, it holds

$$\begin{aligned}\bar\theta(\hat I_\alpha[\hat I_\alpha X^{\bar h}, Y^{\bar h}]) =& - c\Omega_\alpha((I_\alpha X)^h, Y^h)e_0 - \Omega_\beta((I_\alpha X)^h, Y^h)e_\gamma\\ &+ \Omega_\gamma((I_\alpha X)^h, Y^h)e_\beta + \frac{1}{c}d\theta_0((I_\alpha X)^{h_0}, Y^{h_0})e_\alpha.\end{aligned}$$

Defining

$$\begin{aligned}A_\alpha(X^{\bar h}, Y^{\bar h}) =& - \Omega_\beta(\triangle_S X^{\bar h}, \triangle_S Y^{\bar h}) + \Omega_\beta(\triangle_S \hat I_\alpha X^{\bar h}, \triangle_S \hat I_\alpha Y^{\bar h})\\ &+ \Omega_\gamma(\triangle_S \hat I_\alpha X^{\bar h}, \triangle_S Y^{\bar h}) + \Omega_\gamma(\triangle_S X^{\bar h}, \triangle_S \hat I_\alpha Y^{\bar h})\end{aligned}$$

for $X, Y \in TM$, we obtain

$$
\begin{aligned}
&\bar{\theta}(N^{\alpha}(X^{\bar{h}}, Y^{\bar{h}})) \\
=&(-d\theta_0(X^{h_0}, Y^{h_0}) + d\theta_0((I_\alpha X)^{h_0}, (I_\alpha Y)^{h_0}) - c\Omega_\alpha(X^h, (I_\alpha Y)^h) - c\Omega_\alpha((I_\alpha X)^h, Y^h)e_0 \\
&(-\Omega_\alpha(X^h, Y^h) + \Omega_\alpha((I_\alpha X)^h, (I_\alpha Y)^h) + \frac{1}{c}d\theta_0((I_\alpha X)^{h_0}, Y^{h_0}) + \frac{1}{c}d\theta_0(X^{h_0}, (I_\alpha Y)^{h_0}))e_\alpha \\
&+ A_\alpha(X^{\bar{h}}, Y^{\bar{h}})e_\beta + A_\alpha((I_\alpha X)^{\bar{h}}, Y^{\bar{h}})e_\gamma.
\end{aligned}
$$

Next we show that the coefficients of e_0 and e_α can be described by the Ricci tensor of ∇ and that the other components vanish thanks to the integrability of the almost complex structure on the twistor space of M [22]. Set

$$
B := \frac{1}{4(n+1)}(Ric^{\nabla})^a + \frac{1}{4n}(Ric^{\nabla})^s - \frac{1}{2n(n+2)}\Pi_h(Ric^{\nabla})^s, \tag{3.12}
$$

where $\Pi_h(Ric^{\nabla})^s$ is the Q-hermitian $(0,2)$-tensor defined by

$$
(\Pi_h(Ric^{\nabla})^s)(X, Y) = \frac{1}{4}\left((Ric^{\nabla})^s(X, Y) + \sum_{i=1}^{3}(Ric^{\nabla})^s(I_iX, I_iY)\right)
$$

for $X, Y \in TM$. By [5], we have

$$
\Omega^U_\alpha(X, Y) = 2(B(X, I_\alpha Y) - B(Y, I_\alpha X)). \tag{3.13}
$$

Then it holds

$$
\Omega^U_\alpha(I_\alpha X, Y) + \Omega^U_\alpha(X, I_\alpha Y) = -\frac{1}{n+1}\left((Ric^{\nabla})^a(X, Y) - (Ric^{\nabla})^a(I_\alpha X, I_\alpha Y)\right).
$$

Since $\varepsilon d\theta^U_0(X, Y) = \mathrm{Tr}R^{\nabla}_{X,Y} = -Ric^{\nabla}(X, Y) + Ric^{\nabla}(Y, X) = -2(Ric^{\nabla})^a(X, Y)$ and $\Omega_\alpha(X^h, Y^h) = (1/2)\varepsilon\Omega^U_\alpha(X, Y)$ for all tangent vector X, Y on M, to prove (3.10), it is sufficient to check $A_\alpha = 0$. This is related to the integrability of the almost complex structure on the twistor space $\mathcal{Z}$ of the quaternionic manifold (M, Q) as we explain now. Recall that $\mathcal{Z} = \{A \in Q \mid A^2 = -\mathrm{id}\}$. We set

$$
R^{\nabla(0,2)I}_{X,Y} := \frac{1}{4}(R^{\nabla}_{X,Y} + IR^{\nabla}_{IX,Y} + IR^{\nabla}_{X,IY} - R^{\nabla}_{IX,IY})
$$

for $X, Y \in TM$ and $I \in \mathcal{Z}$. Then

$$
[R^{\nabla(0,2)I}_{X,Y}, I] = 0 \tag{3.14}
$$

for any $I \in \mathcal{Z}$ if $n > 1$. In the case of $\dim M = 4$, (3.14) holds if and only if Q is anti-self-dual. See [3] for example. By (3.5) and (3.14), we have $[R^{\nabla(0,2)I_\alpha}_{X,Y}, I_\alpha]I_\gamma = 0$ and thus

$$
\begin{aligned}
0 =&2\mathrm{Tr}[R^{\nabla(0,2)I_\alpha}_{X,Y}, I_\alpha]I_\gamma \\
=&\mathrm{Tr}(-I_\beta R^{\nabla}_{X,Y} + I_\beta R^{\nabla}_{I_\alpha X, I_\alpha Y} + I_\gamma R^{\nabla}_{I_\alpha X,Y} + I_\gamma R^{\nabla}_{X,I_\alpha Y}) \\
=&2n(\Omega^U_\beta(X, Y) - \Omega^U_\beta(I_\alpha X, I_\alpha Y) - \Omega^U_\gamma(I_\alpha X, Y) - \Omega^U_\gamma(X, I_\alpha Y)) \\
=& - 4n\varepsilon A_\alpha(X^{\bar{h}}, Y^{\bar{h}})
\end{aligned}
$$

for all $X, Y \in TM$. This proves that $A_\alpha = 0$.

Since ∇ is torsion-free, we have (3.11) by the similar calculation for the Nijenhuis tensor of the almost complex structure on the twistor space. □

From Lemma 3.5 (and Proposition 3.3) we obtain the following result.

Theorem 3.6. *Let (M, Q) be a quaternionic manifold and ∇ a quaternionic connection. Let $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ be the almost hypercomplex structure on $\hat{M}$. We assume that Q is anti-self-dual when $n = 1$. If $c = -4(n+1)$, then the almost hypercomplex structure is integrable (and independent of ∇). When $c \neq -4(n+1)$, the almost hypercomplex structure is integrable if and only if $(Ric^{\nabla})^a$ is Q-hermitian, that is, it is hermitian with respect to I for all $I \in \mathcal{Z}$, where $\mathcal{Z}$ is the twistor space of (M, Q).*

We call $\hat{M}$ the *Swann bundle* of M, although the terminology "Swann bundle" is also used for the quotient space $\hat{M}/\mathbb{Z}$ with $c = -4(n+1)$ in [20]. From now on we will only consider the case that $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ is a hypercomplex structure, i.e. integrable. We note that, for each fixed quaternionic connection, $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c}) \neq (\hat{I}_1^{\bar{\theta},c'}, \hat{I}_2^{\bar{\theta},c'}, \hat{I}_3^{\bar{\theta},c'})$ if $c \neq c'$. Although it is obvious from the definition, we can also see it by considering the Obata connection. From Lemma 3.4, it follows that $\hat{\nabla}^c_{\widetilde{e}_0}\widetilde{e}_0 = (1/c)\hat{\nabla}^1_{\widetilde{e}_0}\widetilde{e}_0$, where $\hat{\nabla}^c$ is the Obata connection for the hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$.

4. A quaternionic vector field and its natural lift

A vector field X on (M, Q) is called *quaternionic* if its (local) flow φ_t satisfies

$$\varphi_{-t}^* I := \varphi_{t*} \circ I \circ \varphi_{t*}^{-1} \in Q$$

for all $I \in Q$ and for all t. For a connection ∇ and $X \in \Gamma(TM)$, we define

$$(L_X\nabla)_Y Z := L_X(\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y(L_X Z), \tag{4.1}$$

where $Y, Z \in \Gamma(TM)$. Note that $L_X\nabla$ is a tensor. In this paper, we study (M, Q) with a quaternionic vector field X which is also affine, that is $L_X\nabla = 0$. So we start by studying the condition $L_X\nabla = 0$. We define the Hessian H^{∇} with respect to ∇ by

$$H^{\nabla}_{Y,Z} X = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$$

for $X, Y, Z \in \Gamma(TM)$. By similar arguments as in [4], we have the following.

Lemma 4.1. *Let ∇ be a quaternionic connection of (M, Q) and X a quaternionic vector field. Then the following conditions are equivalent each other.*
(1) $L_X\nabla = 0$,
(2) $R^{\nabla}_{X,Y} Z = -H^{\nabla}_{Y,Z} X$ for all $Y, Z \in TM$,
(3) $Ric^{\nabla}(X, Z) = \mathrm{Tr} H^{\nabla}_{(\,\cdot\,),Z} X$ for all $Z \in TM$.

Proof. Since X is a quaternionic vector field, $\varphi_{-t}^*\nabla$ is a quaternionic connection with respect to Q, where $\varphi_{-t}^*\nabla$ is the connection defined by

$$(\varphi_{-t}^*\nabla)_Y Z = \varphi_{t*}(\nabla_{\varphi_{t*}^{-1} Y} \varphi_{t*}^{-1} Z)$$

for $Y, Z \in \Gamma(TM)$. Therefore there exists a one form ξ_t such that $\varphi_t^*\nabla - \nabla = S^{\xi_t}$ by Lemma 2.1. Then we have

$$L_X\nabla = \frac{d}{dt}\varphi_t^*\nabla\bigg|_{t=0} = \frac{d}{dt}S^{\xi_t}\bigg|_{t=0} = S^{\xi_X}, \tag{4.2}$$

where $\xi_X = (d/dt)\xi_t|_{t=0}$. On the other hand, by a straightforward calculation, we have

$(L_X\nabla)_Y Z = R^{\nabla}_{X,Y}Z + H^{\nabla}_{Y,Z}X$ for all $Y, Z \in TM$. Therefore, we have

$$(L_X\nabla)_Y Z = R^{\nabla}_{X,Y}Z + H^{\nabla}_{Y,Z}X = S^{\xi_X}_Z Y.$$

It follows that (1) $\Rightarrow$ (2). It is also easy to see that (2) $\Rightarrow$ (3) by taking a trace. Since $\mathrm{Tr}S^{\xi_X}_Z = 4(n+1)\xi_X(Z)$, we have

$$-Ric^{\nabla}(X,Z) + \mathrm{Tr}H^{\nabla}_{(\,\cdot\,),Z}X = \mathrm{Tr}S^{\xi_X}_Z = 4(n+1)\xi_X(Z).$$

If (3) holds, then we have (1). □

We consider the normalizer

$$N(Q) := \{A \in \mathrm{End}(TM) \mid [A, I] \in Q \ \text{ for all } \ I \in Q\}$$

and the centralizer

$$Z(Q) := \{A \in \mathrm{End}(TM) \mid [A, I] = 0 \ \text{ for all } \ I \in Q\}.$$

Then we see $N(Q) = Q + Z(Q) = Q + \mathbb{R}\cdot\mathrm{id} + Z_0(Q)$, where $Z_0(Q)$ is the subspace of $Z(Q)$ of trace-free tensors [5]. Let ∇ be a quaternionic connection and X a quaternionic vector field. Since $L_X I_\alpha = \nabla_X I_\alpha + [I_\alpha, (\nabla X)]$, ∇X is an element of $N(Q)$. We write $\nabla X = T + T_0$, where $T \in \Gamma(Q + \mathbb{R}\cdot\mathrm{id})$ and $T_0 \in \Gamma(Z_0(Q))$. Note that, by [5], we have explicitly

$$\begin{aligned}\nabla X = &-\frac{1}{4n}\sum_{\alpha=1}^{3}\mathrm{Tr}((\nabla X)\circ I_\alpha)I_\alpha \ \ (\in \Gamma(Q))\\ &+\frac{1}{4n}(\mathrm{Tr}\nabla X)\mathrm{id} \ (\in C^\infty(M)\mathrm{id})\\ &+\frac{1}{4}((\nabla X) - \sum_{\alpha=1}^{3} I_\alpha(\nabla X)I_\alpha) - \frac{1}{4n}(\mathrm{Tr}\nabla X)\mathrm{id} \ (\in \Gamma(Z_0(Q))).\end{aligned}$$

So it holds

$$T = \frac{1}{4n}\sum_{\alpha=0}^{3}\varepsilon_\alpha \mathrm{Tr}((\nabla X)\circ I_\alpha)I_\alpha, \tag{4.3}$$

where $I_0 = \mathrm{id}$ and $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$.

Proposition 4.2. *Let ∇ be a quaternionic connection and X a quaternionic vector field. Then $L_X\nabla = 0$ if and only if $2(Ric^{\nabla})^a(X,\,\cdot\,) = d(\mathrm{Tr}(\nabla X))$.*

Proof. By the Bianchi identity, it holds

$$\begin{aligned}R^{\nabla}_{X,Y}Z &= -R^{\nabla}_{Z,X}Y - R^{\nabla}_{Y,Z}X\\ &= -R^{\nabla}_{Z,X}Y - H^{\nabla}_{Y,Z}X + H^{\nabla}_{Z,Y}X\\ &= -R^{\nabla}_{Z,X}Y - H^{\nabla}_{Y,Z}X + (\nabla_Z T)(Y) + (\nabla_Z T_0)(Y)\end{aligned}$$

for all $Y, Z \in TM$. Then we have $-Ric^{\nabla}(X,Z) = -\mathrm{Tr}R^{\nabla}_{Z,X} - \mathrm{Tr}H^{\nabla}_{(\cdot),Z}X + \mathrm{Tr}(\nabla_Z T)$, since T_0 and $\nabla_Z T_0$ are trace-free. Therefore, by Lemma 4.1, we see that $L_X\nabla = 0$ if and only if $2(Ric^{\nabla})^a(Z,X) + \mathrm{Tr}(\nabla_Z T) = 0$. Finally, because $T \in \Gamma(Q + \mathbb{R}\cdot\mathrm{id})$, we obtain $\mathrm{Tr}(\nabla_Z T) = Z\mathrm{Tr}(\nabla X)$ by (4.3). This implies the conclusion. □

Recall that every Killing vector field is affine with respect to the Levi-Civita connection. This means that every quaternionic Killing vector field X on a quaternionic Kähler manifold (M, g, Q) is an example of an affine quaternionic vector field. This can be seen also by Proposition 4.2.

Corollary 4.3. *Let X be a quaternionic vector field on a quaternionic manifold (M, Q). If there exists a volume element ν on M such that $L_X\nu = 0$, then there exists a quaternionic connection ∇ such that $L_X\nabla = 0$.*

Proof. We can find a quaternionic connection ∇ such that $\nabla\nu = 0$ by [5, Theorem 2.4]. Then Ric^∇ is symmetric. Because $L_X = \nabla_X - (\nabla X)$ and $L_X\nu = 0$, we have $\mathrm{Tr}(\nabla X) = 0$. Now the conclusion follows from Proposition 4.2. □

If X is a quaternionic vector field with the flow $\{\varphi_t\}$, then X can be lifted to $\hat{X}$ on $\hat{M}$ as follows. We define $\hat{\varphi}_t : \hat{M} \to \hat{M}$ by

$$\hat{\varphi}_t((x, (\rho, s))) = (\varphi_t(x), (\varphi_{-t}^*\rho, (\varphi_{-t}^*I_1, \varphi_{-t}^*I_2, \varphi_{-t}^*I_3)))$$

for $(x, (\rho, s)) \in \hat{M}$, where $s = (I_1, I_2, I_3)$ and define

$$\hat{X}_{(x,(\rho,s))} = \frac{d}{dt}\hat{\varphi}_t((x, (\rho, s)))\Big|_{t=0}.$$

The vector field $\hat{X}$ on $\hat{M}$ is called the *natural lift* of X. Since $\hat{X}$ is invariant by the principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-action, we have the following.

Lemma 4.4. *Let $\hat{X}$ be the natural lift of a quaternionic vector field X. We have $[\hat{X}, \widetilde{B}] = 0$ for $B \in \mathbb{R} \oplus \mathfrak{so}(3)$.*

Existence of $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$ (see Corollary 4.3) is related to the following condition for $\hat{X}$.

Lemma 4.5. *Let X be a quaternionic vector field X on (M, Q). The following conditions are equivalent :*
(1) *there exists $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$,*
(2) *there exists a trivialization $S_0 \cong M \times \mathbb{R}^{>0}$ such that $\hat{X}_p \in T_sS \subset T_p\hat{M} \cong T_sS \oplus \mathbb{R}$ for all $p = (x, (\rho, s)) \in \hat{M}$.*

Proof. At first, assume that (1) holds. Then ν gives a trivialization $S_0 \cong M \times \mathbb{R}^{>0}$ and $\hat{M} = S \times \mathbb{R}^{>0}$. We denote the component of $\hat{X}$ tangent to the second factor by $\hat{X}^{\mathbb{R}}$. For any point $(x, (\rho, s)) \in \hat{M}$, we see that $\hat{X}^{\mathbb{R}}_{(x,(\rho,s))} = 0 \iff \hat{X}^{\mathbb{R}}_{(x,(\nu(x),s))} = 0 \iff (L_X\nu)_x = 0$ by Lemma 4.4. Conversely, we can obtain the desired section $\nu \in \Gamma(S_0)$ by $\nu(x) = \Phi^{-1}(x, 1)$ for each $x \in M$, where $\Phi : S_0 \to M \times \mathbb{R}^{>0}$ is a trivialization satisfying (2). □

If there exists $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$, we may assume that $\hat{X}$ is a tangent vector field on S by Lemma 4.5. From now on we will assume that the quaternionic vector field X generates a free U(1)-action. Since U(1) is compact, there exists a volume form ν invariant under the group action. This also implies that there exists a quaternionic connection ∇ such that $L_X\nabla = 0$ by Corollary 4.3.

5. The hypercomplex moment map

In this section, we consider a hypercomplex moment map on the Swann bundle. In [14], a hypercomplex moment map is defined as follows.

Definition 5.1 [14]. Let M be a hypercomplex manifold with hypercomplex structure I_1, I_2, I_3 and F a compact Lie group acting smoothly and freely on M preserving I_i $(i = 1, 2, 3)$. F acts on $\mathfrak{F} = \mathrm{Lie}\, F$ by the adjoint action. A vector field on M induced by $f \in \mathfrak{F}$ is denoted by X_f. If a triple $\mu = (\mu_1, \mu_2, \mu_3)$ of F-equivariant maps $\mu_i : M \to \mathfrak{F}^*$ $(i = 1, 2, 3)$ satisfies

$$d\mu_1 \circ I_1 = d\mu_2 \circ I_2 = d\mu_3 \circ I_3 \tag{5.1}$$

and

$$(d\mu_1 \circ I_1)(X_f) \text{ does not vanish on } M \text{ for any non-zero } f \in \mathfrak{F}, \tag{5.2}$$

then μ is called the *hypercomplex moment map* of F. The equations (5.1) are called the CR (Cauchy-Riemann) equations and the condition (5.2) is called the transversality condition.

A hypercomplex moment map produces another hypercomplex manifold by a quotient (Proposition 3.1 in [14]). Let (M, Q) be a quaternionic manifold with a quaternionic connection ∇ and an affine quaternionic vector field X. The following lemmas hold.

Lemma 5.2. *If X is an affine quaternionic vector field on (M, Q, ∇) and $\bar{\theta}$ is the principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-connection on $\hat{M}$ induced by ∇, then $L_{\hat{X}}\bar{\theta} = 0$ and $L_{\hat{X}}\hat{I}_\alpha^{\bar{\theta},c} = 0$.*

Proof. The first equation follows from the fact that $\hat{\varphi}_t$ preserves the horizontal distribution, because $\hat{\varphi}_t$ is induced by a local flow φ_t of affine transformations preserving the quaternionic structure. Since the almost hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ is canonically associated with the data (Q, ∇) on M, it is also invariant under $\hat{\varphi}_t$, which implies the second equation. □

From now on we assume that there exists $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$. Then we can identify $S_0 = M \times \mathbb{R}^{>0}$, $\hat{M} = S \times \mathbb{R}^{>0}$ and $\hat{X}$ is a tangent vector field on S by Lemma 4.5. In the next lemma, we identify S with the $\hat{\varphi}_t$-invariant submanifold $S \times \{1\} \subset \hat{M} = S \times \mathbb{R}^{>0}$.

Lemma 5.3. *Under the above assumption, $L_{\hat{X}}\theta = 0$. Moreover $\bar{\mathcal{H}}|_S = \mathcal{H}$ if and only if $\nabla\nu = 0$.*

Proof. The projection from $\mathbb{R} \oplus \mathfrak{so}(3)$ onto $\mathbb{R}$ (resp. $\mathfrak{so}(3)$) is denoted by $pr_{\mathbb{R}}$ (resp. $pr_{\mathfrak{so}(3)}$). The first statement follows from the previous lemma, since $pr_{\mathfrak{so}(3)}\bar{\theta}|_S = \theta$. The second statement follows from $pr_{\mathbb{R}}\bar{\theta}|_S = (\nu \circ \pi_S)^*\theta_0$, since $\nabla\nu = (\nu^*\theta_0) \otimes \nu$. □

For $1 \in \mathbb{R} \cong T_1\mathbb{R}^{>0}$, at $\rho \in S_0$, we have

$$(\widetilde{e_0})_\rho = \widetilde{1}_\rho = \frac{d}{dr}\rho \exp(\varepsilon t)\Big|_{t=0} = \varepsilon\rho = \varepsilon r \frac{\partial}{\partial r}\Big|_\rho,$$

where r is the standard coordinate on $\mathbb{R}^{>0}$. Let ∇ be a quaternionic connection on (M, Q). We define 1-forms $\hat{\theta}_\alpha^c$ on $\hat{M}$ $(\alpha = 1, 2, 3)$ by

$$\hat{\theta}_\alpha^c|_{TS} := Ar^{\frac{2}{c}}\theta_\alpha \quad \text{and} \quad \hat{\theta}_\alpha^c(Z_0^c) = 0$$

where $A \in \mathbb{R}$ is a constant. A symmetric tensor $\langle\theta,\theta\rangle$ is defined by

$$\langle\theta,\theta\rangle(Y,Z) = \sum_{i=1}^{3} \theta_i(Y)\theta_i(Z)$$

for Y and $Z \in TS$ and we set

$$G^c_\alpha := -Ar^{\frac{2}{c}}\Omega_\alpha(\,\cdot\,, \hat{I}^{\bar{\theta},c}_\alpha\,\cdot\,) + 2\varepsilon A r^{\frac{2}{c}}\langle\theta,\theta\rangle + \frac{2\varepsilon A}{c^2} r^{\frac{2}{c}-2}(dr\otimes dr). \tag{5.3}$$

Note that $G^c_1|_{\mathcal{V}\times\mathcal{V}} = G^c_2|_{\mathcal{V}\times\mathcal{V}} = G^c_3|_{\mathcal{V}\times\mathcal{V}}$, that is, the vertical components of G^c_α are independent of α.

Lemma 5.4. *We have $d\hat{\theta}^c_\alpha(Y,Z) = G^c_\alpha(Y, \hat{I}^{\bar{\theta},c}_\alpha Z)$ for Y, $Z \in T\hat{M}$.*

Proof. Put $f(r) = Ar^{\frac{2}{c}}$. Then

$$G^c_\alpha(Y,Z) = -f(r)\Omega_\alpha(Y, \hat{I}^{\bar{\theta},c}_\alpha Z) + 2\varepsilon f(r)\langle\theta,\theta\rangle(Y,Z) + \frac{2\varepsilon}{c^2}\frac{f(r)}{r^2}(dr\otimes dr)(Y,Z)$$

for $Y, Z \in T\hat{M}$. Since $Z^c_0 = \varepsilon c r\frac{\partial}{\partial r}$, we obtain

$$d\hat{\theta}^c_\alpha(Z^c_0, Z_\alpha) = \varepsilon c r f'(r) = \varepsilon c r \cdot \frac{2A}{c} r^{\frac{2}{c}-1} = 2\varepsilon f(r),$$

$$G^c_\alpha(Z^c_0, \hat{I}^{\bar{\theta},c}_\alpha(Z_\alpha)) = G^c_\alpha(Z^c_0, Z^c_0) = c^2r^2 \cdot \frac{2\varepsilon}{c^2}\frac{f(r)}{r^2} = 2\varepsilon f(r)$$

and

$$G^c_\alpha(Z_\alpha, \hat{I}^{\bar{\theta},c}_\alpha(Z^c_0)) = -G^c_\alpha(Z_\alpha, Z_\alpha) = -2\varepsilon f(r).$$

Moreover we have

$$d\hat{\theta}^c_\alpha(Z_\beta, Z_\gamma) = -\hat{\theta}^c_\alpha([Z_\beta, Z_\gamma]) = -2\varepsilon f(r)$$

and

$$G^c_\alpha(Z_\beta, \hat{I}^{\bar{\theta},c}_\alpha(Z_\gamma)) = -G^c_\alpha(Z_\beta, Z_\beta) = -2\varepsilon f(r),$$

similarly $G^c_\alpha(Z_\gamma, \hat{I}^{\bar{\theta},c}_\alpha(Z_\beta)) = 2\varepsilon f(r)$. Finally, we see

$$d\hat{\theta}^c_\alpha(Y^h, Z^h) = f(r)(d\theta^c_\alpha)(Y^h, Z^h) = f(r)\Omega_\alpha(Y^h, Z^h)$$

and

$$G^c_\alpha(Y^h, \hat{I}^{\bar{\theta},c}_\alpha Z^h) = f(r)\Omega_\alpha(Y^h, Z^h).$$

For other combinations of tangent vectors on $\hat{M}$, both tensors $d\hat{\theta}_\alpha$, G_α vanish. □

We define $\mu^c : \hat{M} \to \mathbb{R}^3$ by

$$\mu^c(x) = \hat{\theta}^c(\hat{X}_x) = (\hat{\theta}^c_1(\hat{X}_x), \hat{\theta}^c_2(\hat{X}_x), \hat{\theta}^c_3(\hat{X}_x)) \tag{5.4}$$

for $x \in \hat{M}$. We calculate some formulae which will be used later to determine sufficient conditions for μ^c to be a hypercomplex moment map.

Lemma 5.5. *If X is an affine quaternionic vector field on (M, Q, ∇), then we have*

$$d\mu^c_\alpha = -\iota_{\hat{X}} d\hat{\theta}^c_\alpha.$$

Proof. By Lemmas 4.4 and 5.3, $L_{\hat{X}}\theta_\alpha = 0$ and $[\hat{X}, Z^c_0] = 0$. It follows $L_{\hat{X}}\hat{\theta}^c_\alpha = 0$ and $d\mu^c_\alpha = d\iota_{\hat{X}}\hat{\theta}^c_\alpha = L_{\hat{X}}\hat{\theta}^c_\alpha - \iota_{\hat{X}}d\hat{\theta}^c_\alpha = -\iota_{\hat{X}}d\hat{\theta}^c_\alpha$. □

For the CR-condition for μ^c, we have

Lemma 5.6. *If X is an affine quaternionic vector field on (M, Q, ∇), then we have*

$$(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Z^c_0) = 0, \tag{5.5}$$

$$(d\mu^c_1 \circ \hat{I}^{\bar{\theta},c}_1)(\widetilde{B}) = (d\mu^c_2 \circ \hat{I}^{\bar{\theta},c}_2)(\widetilde{B}) = (d\mu^c_3 \circ \hat{I}^{\bar{\theta},c}_3)(\widetilde{B}) \textit{ for any } B \in \mathfrak{so}(3), \tag{5.6}$$

$$(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Y) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha Y), \tag{5.7}$$

for all horizontal vector Y.

Proof. By Lemmas 5.4 and 5.5, we have $d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha = G^c_\alpha(\hat{X}, \cdot)$. Then it is easy to see $(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Z^c_0) = G^c_\alpha(\hat{X}, Z^c_0) = 0$. Since $G^c_1 = G^c_2 = G^c_3$ on $\bar{\mathcal{V}} \times T\hat{M}$, we obtain (5.6). Finally for horizontal vector Y we have $(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Y) = G^c_\alpha(\hat{X}, Y) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha Y)$. □

For the transversality condition for μ^c, we state the next lemma, which follows from the equation $d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha = G^c_\alpha(\hat{X}, \cdot)$.

Lemma 5.7. *We have*

$$(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(\hat{X}) = G^c_\alpha(\hat{X}, \hat{X}) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha \hat{X}) + 2\varepsilon Ar^{\frac{2}{c}}\langle\theta, \theta\rangle(\hat{X}, \hat{X}).$$

By Lemma 5.6, μ^c satisfies CR equations (5.1) if and only if

$$\Omega_1(\hat{X}, \hat{I}^{\bar{\theta},c}_1 Y) = \Omega_2(\hat{X}, \hat{I}^{\bar{\theta},c}_2 Y) = \Omega_3(\hat{X}, \hat{I}^{\bar{\theta},c}_3 Y)$$

holds for all horizontal vector Y. On the other hand, from the equations (3.12) and (3.13), Ω_α satisfies

$$\begin{aligned} 2\varepsilon\Omega_\alpha(Y^h, \hat{I}^{\bar{\theta},c}_\alpha Z^h) &= \Omega^U_\alpha(Y, I_\alpha Z) \\ &= \frac{1}{2(n+1)}\Big((Ric^\nabla)^a(I_\alpha Y, I_\alpha Z) - (Ric^\nabla)^a(Y, Z)\Big) \\ &\quad -\frac{1}{2n}\Big((Ric^\nabla)^s(I_\alpha Y, I_\alpha Z) + (Ric^\nabla)^s(Y, Z)\Big) \\ &\quad +\frac{2}{n(n+2)}(\Pi_h(Ric^\nabla)^s)(Y, Z) \end{aligned} \tag{5.8}$$

for tangent vectors Y and Z on M. In particular, if Ric^∇ is Q-hermitian, then $\Omega_\alpha(\,\cdot\,, \hat{I}^{\bar{\theta},c}_\alpha \,\cdot\,)$ does not depend on α. We see that μ^c satisfies the CR equations (5.1) if $Ric^\nabla(X, Y) = Ric^\nabla(IX, IY)$ for all $Y \in TM$ and $I \in \mathcal{Z}$. Moreover if the vector field X on M is affine quaternionic and μ^c satisfies the CR equations, then

$$\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha \hat{X})(= \Omega_\beta(\hat{X}, \hat{I}^{\bar{\theta},c}_\beta \hat{X}) = \Omega_\gamma(\hat{X}, \hat{I}^{\bar{\theta},c}_\gamma \hat{X})) = -\frac{\varepsilon}{2(n+2)}(Ric^\nabla)(X, X) \circ \hat{\pi}.$$

The following statements can be obtained for the CR equations and the transversality condition.

Proposition 5.8. *Let M be a quaternionic manifold with a quaternionic connection ∇ and X an affine quaternionic vector field on (M, Q, ∇). Assume that there exists $\nu \in \Gamma(S_0)$ such that $L_X \nu = 0$. If*

$$Ric^{\nabla}(X, Y) = Ric^{\nabla}(IX, IY)$$

for all $Y \in TM$, $I \in \mathcal{Z}$ and

$$(Ric^{\nabla})(X, X) \circ \hat{\pi} + 4(n + 2)\langle\theta, \theta\rangle(\hat{X}, \hat{X})$$

does not vanish on $\hat{M}$, then the map $\mu^c = Ar^{\frac{2}{c}}\theta(\hat{X}) : \hat{M} \to \mathbb{R}^3$ $(A \neq 0)$ satisfies the CR equations and the transversality condition for any $c \neq 0$.

Also we have

Corollary 5.9. *Let M be a quaternionic manifold with a quaternionic connection ∇ and X an affine quaternionic vector field on (M, Q, ∇). Assume that there exists $\nu \in \Gamma(S_0)$ such that $L_X \nu = 0$. If Ric^{∇} is Q-hermitian and*

$$(Ric^{\nabla})(X, X) \circ \hat{\pi} + 4(n + 2)\langle\theta, \theta\rangle(\hat{X}, \hat{X})$$

does not vanish on $\hat{M}$, then the map $\mu^c = Ar^{\frac{2}{c}}\theta(\hat{X}) : \hat{M} \to \mathbb{R}^3$ $(A \neq 0)$ satisfies the CR equations and the transversality condition for any $c \neq 0$.

6. The proof of the main result

In this section, we give the proof of our main result. Using the hypercomplex quotient in [14], we can obtain a hypercomplex manifold M' with certain properties. To show it, the following lemmas are needed.

Lemma 6.1. *We have $L_{\widetilde{B}}\theta = -\varepsilon[B, \theta]$. Moreover $L_{Z_\alpha}\theta_\alpha = 0$, $L_{Z_\alpha}\theta_\beta = 2\varepsilon\theta_\gamma$ and $L_{Z_\alpha}\theta_\gamma = -2\varepsilon\theta_\beta$.*

Proof. We see that $(L_{\widetilde{B}}\theta)(\widetilde{C}) = -\theta([\widetilde{B}, \widetilde{C}]) = -\varepsilon[B, C] = -\varepsilon[B, \theta(\widetilde{C})]$ and $(L_{\widetilde{B}}\theta)(Y^h) = -\theta([\widetilde{B}, Y^h]) = 0(= -[B, \theta(Y^h)])$. For the latter statements, we compute

$$\sum (L_{Z_\alpha}\theta_i)e_i = L_{Z_\alpha}\theta = -\varepsilon[e_\alpha, \sum \theta_i e_i] = -\varepsilon[e_\alpha, \theta_\beta e_\beta] - \varepsilon[e_\alpha, \theta_\gamma e_\gamma] = -2\varepsilon\theta_\beta e_\gamma + 2\varepsilon\theta_\gamma e_\beta. \quad \square$$

By Lemma 6.1, we have

Lemma 6.2. *It holds that $L_{Z_\alpha}\hat{\theta}^c_\alpha = 0$, $L_{Z_\alpha}\hat{\theta}^c_\beta = 2\varepsilon\hat{\theta}^c_\gamma$ and $L_{Z_\alpha}\hat{\theta}^c_\gamma = -2\varepsilon\hat{\theta}^c_\beta$.*

From Lemma 5.3, it holds

Lemma 6.3. *If $L_X\nabla = 0$, then $L_{\hat{X}} d\hat{\theta}^c_\alpha = 0$ for $\alpha = 1, 2, 3$.*

We can now prove the main theorems in this paper.

Theorem 6.4. *Let (M, Q) be a quaternionic manifold. We assume that Q is anti-self-dual when $n = 1$. Moreover assume that* $\mathrm{U}(1)$ *acts freely on M preserving Q. We denote by X the vector field generating the* $\mathrm{U}(1)$*-action. If Q admits a quaternionic connection ∇ such that*

$L_X\nabla = 0$,

$$Ric^{\nabla}(X,Y) = Ric^{\nabla}(IX,IY) \tag{6.1}$$

for all $Y \in TM$, $I \in \mathcal{Z}$ and

$$(Ric^{\nabla})(X,X) \circ \hat{\pi} + 4(n+2)\langle\theta,\theta\rangle(\hat{X},\hat{X}) \tag{6.2}$$

does not vanish on $\hat{M}$, then the natural lift $\hat{X}$ generates a free U(1)*-action with the moment map μ^c defined by* (5.4)*, where $c = -4(n+1)$. Then the corresponding hypercomplex quotient is a hypercomplex manifold $(M', H = (I'_1, I'_2, I'_3))$ with an I'_1-holomorphic vector field Z such that $L_Z I'_2 = 2\varepsilon I'_3$, $L_Z I'_3 = -2\varepsilon I'_2$. Moreover the exact 2-forms $d\hat{\theta}^c_\alpha$ on $\hat{M}$ induce closed 2-forms Θ'_α on M' which satisfy $L_Z\Theta'_1 = 0$, $L_Z\Theta'_2 = 2\varepsilon\Theta'_3$, $L_Z\Theta'_3 = -2\varepsilon\Theta'_2$.*

Proof. Choose $c = -4(n+1)$. Then $\hat{M}$ is a hypercomplex manifold by Theorem 3.6. We can choose a U(1)-invariant volume form ν on M. Then the condition (2) in Lemma 4.5 holds, so $\hat{X}$ is tangent to S, which means that the results in the previous section can be applied. Since Proposition 5.8 and the second statement of Lemma 5.2 hold, $M' = P/\mathrm{U}(1)$ is a hypercomplex manifold with the induced hypercomplex structure I'_1, I'_2, I'_3 by [14, Proposition 3.1], where P is the level set $(\mu^c)^{-1}((1,0,0))$. Based on the proof of [14, Proposition 3.1], take $V = \{v \in TP \mid (d\mu^c_\alpha \circ \hat{I}^{\hat{\theta},c}_\alpha)(v) = 0\}$. Then we see that $TP = V \oplus \langle\hat{X}\rangle$. In particular, $\pi_P^* TM' \cong V$, where $\pi_P : P \to M'$ is the quotient map. The vector field $\hat{I}^{\hat{\theta},c}_1 Z^c_0 = Z_1$ is tangent to P, since

$$Z_1\mu^c_\alpha|_P = (2\varepsilon\delta_{2\alpha}\mu^c_3 - 2\varepsilon\delta_{3\alpha}\mu^c_2)|_P$$

by Lemma 6.2. By Lemma 4.4, Z_1 is a projectable vector field, that is, $Z := \pi_{P*}(Z_1)$ is a vector field on M'. The vector field Z satisfies $(L_Z I'_\alpha)(U) = \pi_{P*}((L_{Z_1}\hat{I}^{\hat{\theta},c}_\alpha)(U_P))$, where $U_P \in \Gamma(V)$ is any projectable vector field and $U = \pi_{P*}(U_P)$ is its projection. In fact, this can be obtained from $I'_\alpha \circ \pi_{P*} \circ pr_V = \pi_{P*} \circ pr_V \circ \hat{I}^{\hat{\theta},c}_\alpha$, where $pr_V : T\hat{M}|_P \to V$ is the projection with respect to the $\hat{I}^{\hat{\theta},c}_\alpha$-invariant decomposition

$$(T\hat{M})|_P = V \oplus \langle\hat{X}, \hat{I}^{\hat{\theta},c}_1\hat{X}, \hat{I}^{\hat{\theta},c}_2\hat{X}, \hat{I}^{\hat{\theta},c}_3\hat{X}\rangle.$$

Therefore, by Lemmas 3.4 and 4.4, we see that Z is a I'_1-holomorphic vector field such that $L_Z I'_2 = 2\varepsilon I'_3$ and $L_Z I'_3 = -2\varepsilon I'_2$. Finally, by Lemma 6.3, we can define 2-forms Θ'_1, Θ'_2, Θ'_3 on M' by $\Theta'_\alpha(U,W) = (d\hat{\theta}^c_\alpha)(U_P, W_P)$ for $U = \pi_{P*}(U_P)$ and $W = \pi_{P*}(W_P)$. It is clear that these forms are closed. Finally we see that these forms satisfy the desired conditions by Lemma 6.2. □

REMARK 6.5. In Theorem 6.4, the same conclusion can be obtained under the assumption that the action induced by $\hat{X}$ is free instead of the assumption that the action induced by X is free.

In the case that Ric^{∇} is Q-hermitian, we have the following theorem.

Theorem 6.6. *Let (M,Q) be a quaternionic manifold. We assume that Q is anti-self-dual when $n = 1$. Moreover assume that* U(1) *acts freely on M preserving Q. We denote by X the vector field generating the* U(1)*-action. If Q admits a quaternionic connection ∇ such that*

$L_X\nabla = 0$, Ric^∇ is Q-hermitian and

$$(Ric^\nabla)(X,X)\circ\hat{\pi} + 4(n+2)\langle\theta,\theta\rangle(\hat{X},\hat{X}) \tag{6.3}$$

does not vanish on $\hat{M}$, then there exists a 1-parameter family $\{(M'^c, H^c = (I_1'^c, I_2'^c, I_3'^c))\}_{c\neq 0}$ of hypercomplex manifolds with an $I_1'^c$-holomorphic vector field Z^c on M'^c such that $L_{Z^c}I_2'^c = 2\varepsilon I_3'^c$, $L_{Z^c}I_3'^c = -2\varepsilon I_2'^c$. Moreover the exact 2-forms $d\hat{\theta}_\alpha^c$ on $\hat{M}$ give a 1-parameter family $\{(\Theta_1'^c, \Theta_2'^c, \Theta_3'^c)\}_{c\neq 0}$ of triplets of closed 2-forms on M'^c such that $L_{Z^c}\Theta_1'^c = 0$, $L_{Z^c}\Theta_2'^c = 2\varepsilon\Theta_3'^c$, $L_{Z^c}\Theta_3'^c = -2\varepsilon\Theta_2'^c$ and

$$\Theta_1'^c(\,\cdot\,, I_1'^c\,\cdot\,) = \Theta_2'^c(\,\cdot\,, I_2'^c\,\cdot\,) = \Theta_3'^c(\,\cdot\,, I_3'^c\,\cdot\,). \tag{6.4}$$

Proof. Since Ric^∇ is Q-hermitian, the almost hypercomplex structures $(\hat{I}_1^{\hat{\theta},c}, \hat{I}_2^{\hat{\theta},c}, \hat{I}_3^{\hat{\theta},c})$ on $\hat{M}$ are integrable for all $c \neq 0$ by Theorem 3.6. Following the same procedure as in the proof of Theorem 6.4, we obtain the claims with exception of the equation (6.4). The latter equation follows from Q-hermitian assumption for Ric^∇ using Lemma 5.4 and (5.8). □

The assumption (6.2) is formulated in terms of objects on the Swann bundle $\hat{M}$. We have the following corollary under assumptions formulated directly on M.

Corollary 6.7. *Let (M, Q) be a quaternionic manifold. We assume that Q is anti-self-dual when $n = 1$. Moreover assume that* U(1) *acts freely on M preserving Q. We denote by X the vector field generating the* U(1)*-action. If Q admits a quaternionic connection ∇ such that $L_X\nabla = 0$, $Ric^\nabla(X,X) > 0$ and (6.1) is satisfied (resp. Ric^∇ is Q-hermitian), then we have the same conclusion as* Theorem 6.4 (*resp.* Theorem 6.6).

We call the correspondence from (M, Q, X) to (M', H, Z) or to $\{(M'^c, H^c, Z^c)\}_{c\neq 0}$ described in Theorems 6.4 and 6.6 the *Quaternionic/Hypercomplex-correspondence* (*Q/H-correspondence* for short).

A relation with Swann's twist construction. Now we explain how M' considered just as a smooth manifold can be related to M by Swann's twist construction. Consider the Lie subgroup $\mathrm{U}(1)_{Z_1} := \{g \in \mathrm{SO}(3) \mid Ad_g e_1 = e_1\}$ of SO(3), which can be identified with U(1). Notice this group is different from the group $\langle\hat{X}\rangle \cong \mathrm{U}(1)$ generated by $\hat{X}$. Then $P = (\mu^c)^{-1}((1,0,0)) = (\mu^c)^{-1}(e_1)$ is a principal $\mathrm{U}(1)_{Z_1}$-bundle over $\hat{\pi}(P)$ with a connection $\iota_P^*(\theta_1)$, where $\iota_P : P \to \hat{M}$ is the inclusion map from P. In fact, the calculation

$$\mu^c(pg) = \hat{\theta}(\hat{X}_{pg}) = \hat{\theta}(R_{g*}\hat{X}_p) = Ad_{g^{-1}}\hat{\theta}(\hat{X}_p) = Ad_{g^{-1}}e_1 = e_1$$

for $p \in P$ and $g \in \mathrm{U}(1)_{Z_1}$ shows that P is invariant under $\mathrm{U}(1)_{Z_1}$. In particular, $P \cap \hat{\pi}^{-1}(x)$ is a union of circles ($\mathrm{U}(1)_{Z_1}$-orbits). Since the functions $\theta_\alpha(\hat{X})|_{\hat{\pi}^{-1}(x)}$ on $\hat{\pi}^{-1}(x) \cong \mathbb{H}^*/\{\pm 1\}$ are linear in the natural coordinates on $\mathbb{H} \cong \mathbb{R}^4$ and $\hat{\theta}_\alpha(\hat{X}) = Ar^{\frac{2}{c}}\theta_\alpha(\hat{X})$, we see that the above intersection $P \cap \hat{\pi}^{-1}(x)$ is a single circle. Recall [24] that Swann's twist construction produces a new manifold M' from a manifold M with the following twist data: a vector field ξ, a two form F and a function a on M. More precisely, ξ generates a U(1)-action, F is an invariant closed 2-form which is the curvature form of a connection form on a principal U(1)-bundle, and a is non-vanishing and satisfies $da = -\iota_\xi F$. It was shown in [17] that the HK/QK-correspondence can be described using the twist construction and a so-called elementary deformation of the metric.

In the setting of the Q/H-correspondence, let $s : U \to P$, $U \subset \hat{\pi}(P)$ be a local section.

Then we define a two form F and a function a on $\hat{\pi}(P)$ by

$$F := s^*(d(\iota_P^*\theta_1)) = s^*(d\theta_1) = s^*\Omega_1,$$
$$a := s^*(\theta_1(\hat{X}) \circ \iota_P)) = \theta_1(\hat{X}) \circ s, = s^*(\theta_1(\hat{X})).$$

Note that both F and a are independent of the choice of s. Then we have

Proposition 6.8. *As a smooth manifold, M' obtained by the Q/H-correspondence is a twist of $\hat{\pi}(P)$ in the sense of* [24] *with the twist data ($\xi = X, F, a$) as above.*

Proof. Since $L_X\nabla = 0$, we have

$$\begin{aligned} da &= s^*(d\iota_{\hat{X}}\theta_1) = -s^*(d\theta_1(\hat{X}, \cdot)) = -\Omega_1(\hat{X}, s_*(\cdot)) \\ &= -\Omega_1(X^h, s_*(\cdot)) = -\Omega_1(s_*(X), s_*(\cdot)) = -F(X, \cdot). \end{aligned}$$

Also we obtain $L_X F = (\iota_X d + d\iota_X)F = -dda = 0$. □

Note that the complex structures I'_α are not $\mathcal{H}$-related to I_α in the sense of [24], because the invariant subbundle $V \subset T\hat{M}$ does not coincides with $\mathcal{H}$ in general.

7. Examples

In this section, we give examples.

QK/HK-correspondence: When M is a possibly indefinite quaternionic Kähler manifold with non zero scalar curvature, we can take the Levi-Civita connection ∇ as a quaternionic connection and if there exists a non-zero quaternionic Killing vector field X on M, then we can take X as the affine quaternionic vector field in the Q/H-correspondence. The tensor field (5.3) gives a (pseudo-)hyper-Kähler metric on $\hat{M}$ and (6.4) gives a (pseudo-)hyper-Kähler metric on M' if $\hat{X}$ is time-like or space-like (see [1]). Therefore our Q/H-correspondence is a generalization of the QK/HK-correspondence. The following example is well-known (see [13, 11, 23] for example).

Example 7.1 The cotangent bundle $T^*\mathbb{C}P^n$ as a hyper-Kähler manifold. Consider the quaternionic (right-)projective space $M = \mathbb{H}P^n$ with the standard quaternionic structure. We can choose the Levi-Civita connection ∇ of the standard quaternionic Kähler metric on M as a quaternionic connection. Then we see the Swann bundle $\hat{M} = (\mathbb{H}^{n+1}\backslash\{0\})/\{\pm 1\} \to M$ as a hypercomplex manifold with the hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$, where $\bar{\theta}$ is the principal connection associated with the Levi-Civita connection. Let X be the vector field on M which generates the U(1)-action on M by quaternionic affine transformations defined by $e^{i\theta} \cdot [z_0, \ldots, z_n] := [e^{i\theta}z_0, \ldots, e^{i\theta}z_n]$ for $e^{i\theta} \in \mathrm{U}(1)$ and $[z_0, \ldots, z_n] \in M$. It holds that the Ricci tensor Ric^∇ is Q-hermitian and $Ric^\nabla(X, X) > 0$. Then we can apply Corollary 6.7. Note that the action induces the well-known hyper-Kähler moment map on $\hat{M}$ when $c = 1$. The hyper-Kähler metric $G^1_\alpha = G^1_\beta = G^1_\gamma$ is given by a constant multiple of the standard Euclidean flat metric on $\mathbb{H}^{n+1}\backslash\{0\}$. Applying the QK/HK-correspondence (which amounts to taking the hyper-Kähler quotient of $\hat{M}$ with respect to the vector field $\hat{X}$) to this example yields Calabi's hyper-Kähler structure on $T^*\mathbb{C}P^n$.

Hypercomplex manifold with the Obata connection: Let $(M,(I_1,I_2,I_3))$ be a hypercomplex manifold and ∇ its Obata connection on M. We recall the Obata connection is a canonical torsion-free connection preserving the hypercomplex structure [18]. In particular, it is a quaternionic connection with respect to the quaternionic structure $Q=\langle I_1,I_2,I_3\rangle$. Assume that a vector field X with the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on M is given, which generates a free action of $\mathrm{U}(1)=\mathbb{R}/2\pi\mathbb{Z}$ on M such that

$$L_X I_1 = 0,\quad L_X I_2 = 2\varepsilon I_3 \quad\text{and}\quad L_X I_3 = -2\varepsilon I_2. \tag{7.1}$$

Then it holds

$$\varphi_{-t}^* I_1 = I_1,\ \varphi_{-t}^* I_2 = (\cos(2\varepsilon t))I_2 + (\sin(2\varepsilon t))I_3,\ \varphi_{-t}^* I_3 = (-\sin(2\varepsilon t))I_2 + (\cos(2\varepsilon t))I_3.$$

This shows that X is a quaternionic vector field for the quaternionic structure $Q=\langle I_1,I_2,I_3\rangle$. Since $(\varphi_{-t}^*\nabla)$ is the Obata connection for the hypercomplex structure $(\varphi_{-t}^*I_1,\varphi_{-t}^*I_2,\varphi_{-t}^*I_3)$, $(\varphi_{-t}^*\nabla)$ is again a quaternionic connection for Q. By the explicit expression of the Obata connection in [5], we have

$$\frac{d}{dt}(\varphi_{-t}^*\nabla) = 0,$$

and hence $\varphi_{-t}^*\nabla = \nabla$ for all t. It follows that $L_X\nabla = 0$. Because the Ricci curvature of the Obata connection is skew symmetric and Q-hermitian by Corollary 1.6 in [5], we can apply the Q/H-correspondence to (M,Q,∇) obtaining a hypercomplex manifold M'. The manifolds M and M' are related as follows.

Proposition 7.2. *M is a double covering space of M'.*

Proof. The hypercomplex structure is a global section $s : M \to S$

$$x \mapsto s(x) = (I_1(x), I_2(x), I_3(x)),$$

and defines a global trivialization of the principal $\mathrm{SO}(3)$-bundle S. Take a $\mathrm{U}(1)$-invariant volume form. Since

$$\begin{aligned}\hat{M} &= \{(x, s(x)g, r) \mid x\in M,\ g\in \mathrm{SO}(3),\ r>0\}\\ &\cong M\times \mathrm{SO}(3)\times\mathbb{R}^{>0} \supset M\times\mathrm{SO}(3)\times\{1\}\cong M\times\mathrm{SO}(3) = S\end{aligned}$$

and $(I_1,\varphi_{-t}^*I_2,\varphi_{-t}^*I_3) = (I_1,I_2,I_3)g_{\varepsilon t} = (I_1,I_2,I_3)g_t^{\varepsilon}$, where

$$g_t = \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos(2t) & -\sin(2t)\\ 0 & \sin(2t) & \cos(2t)\end{pmatrix}, \tag{7.2}$$

we can write $\hat{\varphi}_t(x,(I_1,I_2,I_3),r) = (\varphi_t(x),(I_1,I_2,I_3)g_{\varepsilon t},r)$ and hence $\hat{X}_{s(x)} = X^h_{s(x)} + (\widetilde{e_1})_{s(x)}$. Therefore we see that $\hat{X}_{s(x)g^{\varepsilon}} = X^h_x + (\widetilde{Ad_{g^{-\varepsilon}}e_1})_{s(x)g^{\varepsilon}}$, where $g\in\mathrm{SO}(3)$. Then the moment map $\mu^c : \hat{M}\to\mathfrak{so}(3)(=\mathbb{R}^3)$ on $\hat{M}$ is given by

$$\mu^c(p) = Ar^{\frac{2}{c}}\theta(\hat{X}_p) = Ar^{\frac{2}{c}}g^{-\varepsilon}e_1g^{\varepsilon}$$

at any point $p=(x,s(x)g^{\varepsilon},r)\in\hat{M}$. The level set $P := (\mu^c)^{-1}(e_1) = (\mu^c)^{-1}((1,0,0))$ is given by

$$\{(x, s(x)g^{\varepsilon}, r) \in \hat{M} \mid x \in M,\ Ar^{\frac{2}{c}} g^{-\varepsilon} e_1 g^{\varepsilon} = e_1\}.$$

Hence we have

$$P = \{(x, s(x)g^{\varepsilon}, A^{-\frac{c}{2}}) \in \hat{M} \mid x \in M,\ g \in \mathrm{U}(1)\} \cong M \times \mathrm{U}(1). \tag{7.3}$$

We obtain a hypercomplex manifold $M' = P/\langle \hat{X} \rangle$, where $\langle \hat{X} \rangle \cong \mathrm{U}(1)$. Define a map $k : M \to M'$ by $k(x) = \pi_P((x, s(x), A^{-\frac{c}{2}}))$ for each $x \in M$, where $\pi_P : P \to M' = P/\langle \hat{X} \rangle$ is the quotient map. Since $k^{-1}(y) = \pi_P^{-1}(y) \cap (s(M) \times \{A^{-\frac{c}{2}}\})$ consists of exactly two points for each $y \in M'$ by (7.2), k is a double covering map. By (5.7) in Lemma 5.6, it holds $V_p = \{v \in T_p P \mid (d\mu_\alpha^c \circ \hat{I}_\alpha^{\hat{\theta},c})(v) = 0\} = \mathcal{H}_p$. It follows that $\pi_P^*(TM') \cong \mathcal{H}|_P$, where $\mathcal{H}$ is the horizontal subbundle with respect to the Obata connection. Since $s_*(Y) = Y^h$ for $Y \in TM$, we have

$$k_*(I_\alpha Y) = \pi_{P*}(s_* I_\alpha Y) = \pi_{P*}((I_\alpha Y)^h) = I'_\alpha(\pi_{P*}(Y^h)) = I'_\alpha(\pi_{P*} s_* Y) = I'_\alpha(k_* Y).$$

Therefore $k : M \to M'$ is a double covering map satisfying $k_* \circ I_\alpha = I'_\alpha \circ k_*$. □

Note that M' is obtained by the twist data $(X, F = 0, a = 1)$.

Example 7.3. For the Swann bundle $\hat{M}$ of a quaternionic manifold (M, Q), we see that $Z_1 = -\hat{I}_1^{\bar{\theta},c} Z_0^c$ satisfies the conditions required above by Lemma 3.4. So $\hat{M}$ is a double covering space of $(\hat{M})'$.

Quaternionic Hopf manifold: Consider $\mathbb{H}^n \cong \mathbb{R}^{4n}$ as a right-vector space over the quaternions. Set $\tilde{M} := \mathbb{H}^n \backslash \{0\}$. The standard hypercomplex structure $\tilde{H} = (\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ on $\tilde{M}$ is defined by $\tilde{I}_1 = R_i, \tilde{I}_2 = R_j, \tilde{I}_3 = -R_k$, where R_q is the right-multiplication by $q \in \mathbb{H}$. The hypercomplex structure $\tilde{H}$ gives a global section $s : \tilde{M} \to S$ ($\cong \tilde{M} \times \mathrm{SO}(3)$) as in the previous example. The corresponding quaternionic structure is denoted by $\tilde{Q} = \langle \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \rangle$. Let $\tilde{g}$ be the standard flat hyper-Kähler metric on $\tilde{M}$, $\mathcal{A} \in \mathrm{Sp}(n)\mathrm{Sp}(1)$ and $\lambda > 1$. Then $\gamma := \lambda \mathcal{A}$ generates a group $\Gamma = \langle \gamma \rangle$ of homotheties which acts freely and properly discontinuously on the simply connected manifold $(\tilde{M}, \tilde{g})$. We can identify $\tilde{M}$ with $\mathbb{R} \times S^{4n-1}$ by means of the diffeomorphism $v \mapsto (t, v/\|v\|)$, where $t = \log\|v\|/\log\lambda$. Under this identification, γ corresponds to the transformation

$$T_{\mathcal{A}} : \mathbb{R} \times S^{4n-1} \to \mathbb{R} \times S^{4n-1},\ \ (t, v) \mapsto (t+1, \mathcal{A}v). \tag{7.4}$$

The quotient $\tilde{M}/\Gamma \cong (\mathbb{R} \times S^{4n-1})/\langle T_{\mathcal{A}} \rangle$ is diffeomorphic to $S^1 \times S^{4n-1}$ and inherits a quaternionic structure Q and a quaternionic connection ∇, both invariant under the centralizer $G^Q := Z_{\mathrm{GL}(n,\mathbb{H})\mathrm{Sp}(1)}(\gamma)$ of γ in $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. (In particular, if $\mathcal{A} \in \mathrm{Sp}(n)$, then $\tilde{M}/\Gamma$ inherits a hypercomplex structure H and its Obata connection ∇, both invariant under the centralizer $G^H := Z_{\mathrm{GL}(n,\mathbb{H})}(\gamma)$ of γ in $\mathrm{GL}(n, \mathbb{H})$.) In fact, the quaternionic structure $\tilde{Q}$ on $\tilde{M}$ is $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$-invariant and induces therefore an almost quaternionic structure Q on $\tilde{M}/\Gamma$, since $\Gamma \subset \mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. Moreover, the Levi-Civita connection $\tilde{\nabla}$ on $(\tilde{M}, \tilde{g})$, which coincides with the Obata connection with respect to $\tilde{H}$, is invariant under all homotheties of $\tilde{M}$. Since Γ acts by homotheties, we see that $\tilde{\nabla}$ induces a torsion-free connection ∇ on $\tilde{M}/\Gamma$, which preserves Q. This means that Q is a quaternionic structure on $\tilde{M}/\Gamma$. The group G^Q acts on $\tilde{M}/\Gamma$ preserving the data (Q, ∇). If $\mathcal{A} \in \mathrm{Sp}(n)$, then Γ preserves the hypercomplex structure $\tilde{H}$ on $\tilde{M}$ and thus induces a hypercomplex structure H and $\tilde{\nabla}$ induces the Obata connection ∇ on $(\tilde{M}/\Gamma, H)$. The centralizer G^H of $\gamma = \lambda\mathcal{A}$ in $\mathrm{GL}(n,\mathbb{H})$ acts on $\tilde{M}/\Gamma$ preserving (H, ∇). We say that $(\tilde{M}/\Gamma, Q)$ (resp. $(\tilde{M}/\Gamma, H)$) is a *quaternionic* (*resp. hypercomplex*) *Hopf manifold*. Note that the hypercomplex Hopf manifolds are sometimes called quaternionic Hopf manifolds (see [19] for example).

Now taking $\mathcal{A} = R_q$ for some unit quaternion $q \neq \pm 1$, we have a quaternionic Hopf manifold $M = \tilde{M}/\Gamma$. Then we see $G^Q = \mathrm{GL}(n,\mathbb{H})\mathrm{U}(1) = \mathbb{R}^{>0} \times \mathrm{SL}(n,\mathbb{H})\mathrm{U}(1)$, where $\mathrm{U}(1)$ denotes the centralizer of q in $\mathrm{Sp}(1)$. Up to an automorphism of $\mathrm{Sp}(1)$, we can assume that

$$\mathrm{U}(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}.$$

We take a subgroup $\mathbb{R}^{>0} \times \mathrm{Sp}(n)\mathrm{U}(1)$ of G^Q, which acts on M transitively. The isotropy subgroup is given by $\langle\lambda\rangle \times \mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}$, where $\triangle_{U(1)}$ is a diagonally embedded subgroup of $\mathrm{Sp}(n)\mathrm{U}(1) \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$ which is isomorphic to $\mathrm{U}(1)$. This has an expression as

$$\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)} = \left\{ \left[\left(\begin{array}{c|ccc} e^{i\theta} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array} \right),\ e^{i\theta} \right] \,\middle|\, A \in \mathrm{Sp}(n-1), e^{i\theta} \in \mathrm{U}(1) \right\}.$$

As described above, we obtain an invariant quaternionic structure on the homogeneous space

$$M = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}}.$$

REMARK 7.4. In particular, for $n = 1$, this yields a left invariant quaternionic structure on $\mathrm{U}(1) \times \mathrm{Sp}(1)$. For $n = 2$, we obtain an invariant quaternionic structure on the homogeneous space

$$\mathrm{U}(1) \times \frac{\mathrm{Sp}(2)\mathrm{U}(1)}{\mathrm{Sp}(1)\triangle_{\mathrm{U}(1)}} = \frac{T^2 \cdot \mathrm{Sp}(2)}{\mathrm{U}(2)}.$$

Note that the homogeneous quaternionic space $T^2 \cdot \mathrm{Sp}(2)/\mathrm{U}(2)$ has a finite covering of the form $(T^2 \times G)/\mathrm{U}(2)$, where G is a compact semisimple Lie group, namely $\mathrm{Sp}(2)$. This presentation is of the form $(T^k \times G)/\mathrm{U}(2)$ as considered in [15].

Consider the $\mathrm{U}(1)$-action on $\tilde{M}$ defined by the right-multiplication by elements of $\mathrm{U}(1)$ ($\subset \mathbb{R}^{>0} \times \mathrm{Sp}(n)\mathrm{U}(1) \subset G^Q$) : $z \mapsto z \cdot e^{\varepsilon i t}$ ($z \in \tilde{M}$). Then the corresponding vector field $\tilde{X}$ satisfies $\tilde{X}_z = \varepsilon z i = \varepsilon \tilde{I}_1 z$ for $z \in \tilde{M}$. Moreover we see that the relations (7.1) in the previous example hold, that is, $L_{\tilde{X}}\tilde{I}_1 = 0$, $L_{\tilde{X}}\tilde{I}_2 = 2\varepsilon\tilde{I}_3$, $L_{\tilde{X}}\tilde{I}_3 = -2\varepsilon\tilde{I}_2$. The $\mathrm{U}(1)$-action preserving the quaternionic structure induces one on M and $\tilde{X}$ induces the vector field X on M generating the latter $\mathrm{U}(1)$-action on M. Considering the hypercomplex moment map on the Swann bundle $\hat{\tilde{M}}$ (resp. $\hat{M}$) of $\tilde{M}$ (resp. M) and the level set $\tilde{P} \subset \hat{\tilde{M}}$ (resp. $P \subset \hat{M}$) of the corresponding moment map, we can obtain a hypercomplex manifold $\tilde{M}'$ (resp. M'). In fact, since $Ric^{\tilde{\nabla}} = 0$ (resp. $Ric^{\nabla} = 0$) and $\hat{\tilde{X}}$ (resp. $\hat{X}$) is not horizontal, the Q/H-correspondence can be applied to $\tilde{M}$ (resp. M), cf (6.2).

Now we consider $\tilde{M}_+ := \tilde{M}/\{\pm 1\}$ and $M_+ := M/\{\pm 1\}$. The quotient maps by the action of the group $\{\pm 1\} \cong \mathbb{Z}_2$ on the manifolds are denoted by $\tilde{\pi}_+ : \tilde{M} \to \tilde{M}_+$ and $\pi_+ : M \to M_+$, respectively. The induced objects on $\tilde{M}_+$ and M_+ are denoted by the same letter. We obtain a hypercomplex isomorphism between $\tilde{M}'$ and $\tilde{M}_+$ as follows. Define $\tilde{f} : \tilde{M}' \to \tilde{M}_+$ by $\tilde{f}(x) = \tilde{\pi}_+(\hat{\tilde{\pi}}(u))$ for any $x \in \tilde{M}'$, where $\hat{\tilde{\pi}} : \hat{\tilde{M}} \to \tilde{M}$ is the bundle projection and $u \in \pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{c}{2}}\})$. Since $\pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{c}{2}}\})$ consists of exactly two points of the form $\{(\pm p, \tilde{H}, A^{-\frac{c}{2}})\}$ as we observed in the proof of Proposition 7.2, $\tilde{f}$ is well-defined. It is easy to see

$$\tilde{f} \circ \pi_{\tilde{P}} = \tilde{\pi}_+ \circ \hat{\tilde{\pi}} \tag{7.5}$$

on $s(\tilde{M}) \times \{A^{-\frac{c}{2}}\}$ by the definition of $\tilde{f}$. Furthermore we have

$$\tilde{f} \circ \tilde{k} = \tilde{\pi}_+, \tag{7.6}$$

where $\tilde{k} : \tilde{M} \to \tilde{M}'$ is the double covering as in Proposition 7.2 for $\tilde{M}$. In fact, from (7.5), it follows that $\tilde{f}(\tilde{k}(x)) = \tilde{f}(\pi_{\tilde{P}}(s(x), A^{-\frac{c}{2}})) = \tilde{\pi}_+(\hat{\tilde{\pi}}(s(x), A^{-\frac{c}{2}})) = \tilde{\pi}_+(x)$ for all $x \in \tilde{M}$.

Lemma 7.5. *The map $\tilde{f} : \tilde{M}' \to \tilde{M}_+$ is an isomorphism of hypercomplex manifolds.*

Proof. To prove that $\tilde{f}$ is injective, let $x_1, x_2 \in \tilde{M}'$ such that $\tilde{f}(x_1) = \tilde{f}(x_2)$. There exists $y_a \in \tilde{M}$ such that $x_a = \pi_{\tilde{P}}(s(y_a), A^{-\frac{c}{2}})$ $(a = 1, 2)$. Since $\tilde{f}(x_1) = \tilde{f}(x_2)$ and (7.5), we have $\hat{\tilde{\pi}}(s(y_1), A^{-\frac{c}{2}}) = \pm\hat{\tilde{\pi}}(s(y_2), A^{-\frac{c}{2}})$, that is, $y_1 = \pm y_2$, or equivalently $y_1 = \varphi_0(y_2)$ or $y_1 = \varphi_\pi(y_2)$. Therefore, we see that $(s(y_1), A^{-\frac{c}{2}}) = (s(\varphi_\delta(y_2)), A^{-\frac{c}{2}})$, where $\delta = 0$ or π. Then we have $x_1 = \pi_{\tilde{P}}(s(y_1), A^{-\frac{c}{2}}) = \pi_{\tilde{P}}(s(y_2), A^{-\frac{c}{2}}) = x_2$, which means $\tilde{f}$ is injective. To show that $\tilde{f}$ is surjective, let $z \in \tilde{M}_+$ and choose $y \in \tilde{M}$ such that $z = \tilde{\pi}_+(y)$. By (7.6), we obtain $z = \tilde{\pi}_+(y) = \tilde{f}(\tilde{k}(y))$. Hence $\tilde{f}$ is surjective. The lift of $v \in T\tilde{M}'$ to $\mathcal{H}$ is denoted by $v^{h'}$. By

$$(I'_\alpha)_x(v) = \pi_{\tilde{P}*u}((\hat{I}_\alpha^{\tilde{\theta},c})_u(v^{h'})) = \pi_{\tilde{P}*u}(((I_\alpha)_{\tilde{f}(x)}(\hat{\tilde{\pi}}_{*u}(v^{h'})))^h),$$

then

$$\begin{aligned}\tilde{f}_{*x}((I'_\alpha)_x(v)) &\overset{(7.5)}{=} \tilde{\pi}_{+*}(\hat{\tilde{\pi}}_{*u}(((I_\alpha)_{\tilde{f}(x)}((\hat{\tilde{\pi}}_{*u})(v^{h'})))^h)) \\ &= (I_\alpha)_{\tilde{f}(x)}(\tilde{\pi}_{+*}(\hat{\tilde{\pi}}_{*u}(v^{h'}))) \overset{(7.5)}{=} (I_\alpha)_{\tilde{f}(x)}(\tilde{f}_{*x}(v))\end{aligned}$$

at each point $x \in \tilde{M}'$, where $u \in \pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{c}{2}}\})$. This shows that the hypercomplex manifolds M' and M are isomorphic. □

Set $F := \tilde{f} \circ \pi_{\tilde{P}} : \tilde{P} \to \tilde{M}_+$. Hereafter we will denote the equivalence class with respect to the action of a group K by $[\,\cdot\,]_K$.

Lemma 7.6. *We have $F(\gamma \cdot y) = \lambda \cdot F(y)$ for all $y \in \tilde{P}$.*

Proof. For any point $y = (p, \tilde{H}g, A^{-\frac{c}{2}})$ $(p \in \tilde{M}$ and $g \in \mathrm{U}(1))$ of $\tilde{P}$, we have $\gamma \cdot y = (\lambda pi, \tilde{H}g\rho(i), A^{-\frac{c}{2}})$ by (7.3), where $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is the standard double covering. Therefore we obtain

$$\gamma \cdot [y]_{\langle\hat{X}\rangle} = [(\lambda pi, \tilde{H}g\rho(i), A^{-\frac{c}{2}})]_{\langle\hat{X}\rangle} = [(\pm\lambda p\hat{g}^{-1}, \tilde{H}, A^{-\frac{c}{2}})]_{\langle\hat{X}\rangle},$$

where $\hat{g} \in \mathrm{Sp}(1)$ such that $\rho(\hat{g}) = g$. Then it holds

$$F(\gamma \cdot y) = \tilde{f}(\gamma \cdot [y]_{\langle\hat{X}\rangle}) = \tilde{f}([(\pm\lambda p\hat{g}^{-1}, \tilde{H}, A^{-\frac{c}{2}})]_{\langle\hat{X}\rangle}) = \tilde{\pi}_+(\pm\lambda p\hat{g}^{-1}) = \lambda\tilde{f}([y]_{\langle\hat{X}\rangle}) = \lambda F(y)$$

from (7.5). □

Note that M' has an induced $\{\pm 1\}$-action, since the lifted action of $\{\pm 1\}$ to the Swann bundle $\hat{\tilde{M}}$ commutes with Γ and $\mathrm{U}(1)$. Let $\pi'_+ : M' \to M'_+$ be the quotient map of the action by $\{\pm 1\}$ on M'. We can define a map

$$\Phi : M'_+ (= \pi'_+(M')) \to \tilde{M}_+/\langle \lambda \rangle$$

as follows. Take any $x \in M'_+$. Then there exists $y \in \tilde{P}$ such that $x = \pi'_+(\pi_P([y]_\Gamma))$ and we set $\Phi(x) := [F(y)]_{\langle \lambda \rangle}$. We shall show that $[F(y)]_{\langle \lambda \rangle}$ is independent of the choice of y. If $(x =)\pi'_+(\pi_P([y_1]_\Gamma)) = \pi'_+(\pi_P([y_2]_\Gamma))$, there exist $\delta \in \{\pm 1\}$, $g \in \mathrm{U}(1)$ and $l \in \mathbb{Z}$ such that $y_1 = \delta \cdot g \cdot \gamma^l \cdot y_2$. By Lemma 7.6 and the definitions of $\tilde{f}$ and $\pi_{\tilde{P}}$, we see

$$F(y_1) = F(\delta \cdot g \cdot \gamma^l \cdot y_2) = \lambda^l F(y_2),$$

which implies $[F(y_1)]_{\langle \lambda \rangle} = [F(y_2)]_{\langle \lambda \rangle}$. Moreover we have

Lemma 7.7. *The map $\Phi : M'_+ \to \tilde{M}_+/\langle \lambda \rangle$ is an isomorphism.*

Proof. To prove that Φ is injective, let $x_1, x_2 \in M'_+ = \pi'_+(M')$ such that $\Phi(x_1) = \Phi(x_2)$. There exists $y_1, y_2 \in \tilde{P}$ such that $x_a = \pi'_+(\pi_P([y_a]_\Gamma))$ $(a = 1, 2)$. Since $[F(y_1)]_{\langle \lambda \rangle} = [F(y_2)]_{\langle \lambda \rangle}$, there exists $l \in \mathbb{Z}$ such that $F(y_1) = \lambda^l \cdot F(y_2) = F(\gamma^l \cdot y_2)$. Then we have $\tilde{f}(\pi_{\tilde{P}}(y_1)) = \tilde{f}(\pi_{\tilde{P}}(\gamma^l \cdot y_2))$, so there exists $g \in \mathrm{U}(1)$ such that $y_1 = g \cdot \gamma^l \cdot y_2$. Therefore we obtain

$$x_1 = \pi'_+(\pi_P([y_1]_\Gamma)) = \pi'_+(\pi_P([g \cdot \gamma^l \cdot y_2]_\Gamma)) = \pi'_+(\pi_P([y_2]_\Gamma)) = x_2.$$

So Φ is injective. Next we shall show that Φ is surjective. Take any $z \in \tilde{M}_+/\langle \lambda \rangle$. There exists $y \in \tilde{P}$ such that $z = [F(y)]_{\langle \lambda \rangle}$. Setting $x = \pi'_+(\pi_P([y]_\Gamma))$, we have $\Phi(x) = z$, which means Φ is surjective. Since the hypercomplex structures are invariant under actions of all groups $\mathrm{U}(1) = \langle \hat{X} \rangle = \langle \hat{\hat{X}} \rangle$, Γ, $\langle \lambda \rangle$ and $\{\pm 1\}$ in the argument, Φ is a hypercomplex isomorphism. □

$$\begin{tikzcd}
 & \tilde{P} \arrow[r, "/\Gamma"] \arrow[dl, "\pi_{\tilde{P}}"'] & P \arrow[dr, "\pi_P"] & \\
\tilde{M}' \arrow[dr, "\tilde{f}"] & \tilde{M} \arrow[l, "\tilde{k}"', "\text{(as in Lemma 7.5)}"] \arrow[u, "s \times A^{-c/2}"'] \arrow[r, "/\Gamma"] \arrow[d, "\tilde{\pi}_+"'] & M = \tilde{M}/\Gamma \arrow[r, dotted, "\text{Q/H-corresp.}"'] \arrow[d, "\pi_+"'] & M' \cong \tilde{M}/\langle \lambda \rangle \arrow[d, "\pi'_+"'] \\
 & \tilde{M}_+ \arrow[r, "/\Gamma"] \arrow[dr, "/\langle \lambda \rangle"] & M_+ = \tilde{M}_+/\Gamma \arrow[r, dotted, "\text{Q/H-corresp.}"'] & M'_+ = \pi'_+(M') \arrow[dl, "\Phi", "\cong"'] \\
 & & \tilde{M}_+/\langle \lambda \rangle &
\end{tikzcd}$$

Therefore, by Lemma 7.7, the hypercomplex manifold M'_+ obtained from M_+ by the Q/H-correspondence is identified with $\tilde{M}_+/\langle \lambda \rangle$. Since M' is the double covering space of M'_+, we have

$$M' \cong \tilde{M}/\langle \lambda \rangle.$$

The centralizer G^H of λ is $\mathrm{GL}(n, \mathbb{H}) = \mathbb{R}^{>0} \times \mathrm{SL}(n, \mathbb{H})$ and take a subgroup $\mathbb{R}^{>0} \times \mathrm{Sp}(n)$ of G^H. As we explained, $\tilde{M}/\langle\lambda\rangle$ can be expressed by the homogeneous space

$$\tilde{M}/\langle\lambda\rangle = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}.$$

Finally, we summarize the discussion as follows.

EXAMPLE 7.8. The hypercomplex manifold

$$M' = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$$

is obtained by the Q/H-correspondence from the quaternionic manifold

$$M = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}}.$$

(Note that we are considering the invariant quaternionic (resp. hypercomplex) structure on M (resp. M') described above.)

We remark that M' does not admit any hyper-Kähler structure for topological reasons, since M' is diffeomorphic to $S^1 \times S^{4n-1}$. Therefore our Q/H-correspondence yields examples which can not appear in the QK/HK-correspondence.

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