

# THE EXISTENCE OF AN INDECOMPOSABLE MINIMAL GENUS TWO LEFSCHETZ FIBRATION

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## Abstract

It was shown by Usher that any fiber sum of Lefschetz fibrations over  $S^2$  is minimal, which was conjectured by Stipsicz. We prove that the converse does not hold by showing that there exists a genus-2 indecomposable minimal Lefschetz fibration (IMLF for short).

## 1. Introduction

Lefschetz fibrations play an important role in 4-manifold topology. It was shown by Donaldson that, after some blow-ups, any closed symplectic 4-manifold admit a Lefschetz fibration over  $\mathbb{S}^2$  [12]. Conversely, Gompf showed that the total space of a Lefschetz fibration admits a symplectic structure, provided the fibers are non-trivial in homology [17], generalizing an earlier work of Thurston in [38].

For a closed, connected, oriented smooth 4-manifold  $X$ , a smooth map  $f : X \rightarrow \mathbb{S}^2$  is called a genus- $g$  *Lefschetz fibration* if a regular fiber of  $f$  is diffeomorphic to a closed oriented surface  $\Sigma_g$  of genus  $g$  and for each critical point  $p$  and  $f(p)$  there are complex local coordinate charts agreeing with the orientations of  $X$  and  $\mathbb{S}^2$  on which  $f$  is of the form  $f(z_1, z_2) = z_1 z_2$ . We suppose that  $f$  is injective on the set of critical points  $C$  and *relatively minimal*, i.e., no fiber contains a  $(-1)$ -sphere. We say that  $f$  is *minimal* if its total space  $X$  is symplectically minimal. Note that from a basic fact proved using Taubes' SW theory [37, 23, 21], a symplectic 4-manifold is symplectically minimal if and only if it is smoothly minimal.

The *fiber sum* is one of the important and natural operations to construct new Lefschetz fibrations. For  $i = 1, 2$ , let  $f_i : X_i \rightarrow \mathbb{S}^2$  be two genus- $g$  Lefschetz fibrations. We remove a fibered neighborhood of a regular fiber  $F_i$  from each fibration and glue the resulting 4-manifolds along their boundaries using a fiber-preserving and orientation-reversing diffeomorphism  $\phi : F_1 \times \mathbb{S}^1 \rightarrow F_2 \times \mathbb{S}^1$ . The result is a new genus- $g$  Lefschetz fibration  $f$  on  $X := X_1 \#_{\phi} X_2$  called the *fiber sum* of  $f_1$  and  $f_2$ . A Lefschetz fibration is called *indecomposable* if it cannot be expressed as a fiber sum.

Stipsicz [35] showed that every Lefschetz fibration with  $(-1)$ -sections is indecomposable (see also [33]). Note that Lefschetz fibrations with  $(-1)$ -sections are nonminimal. Under this, Stipsicz conjectured that *if a Lefschetz fibration is decomposable, then it is minimal* (see Conjecture 2.3 [35]) This was proved by Usher [39] (see also [29, 5]). Our main result is the following.

**Theorem 1.** *There is a genus-2 indecomposable minimal Lefschetz fibration (IMLF for short), i.e., the converse of Stipsicz's conjecture is false.*

## 2. Positive factorizations and Proofs

For a genus- $g$  Lefschetz fibration, any fiber containing a critical point is called *singular fiber*, which is obtained by collapsing a simple closed curve, called the *vanishing cycle*, in the nearby regular fiber to a point. We call a singular fiber *separating* (resp. *nonseparating*) if the corresponding vanishing cycle is separating (resp. nonseparating) curve on the regular fiber.

Let  $\mathcal{M}_g$  be the mapping class group of  $\Sigma_g$ , which is the group of isotopy classes of orientation-preserving diffeomorphisms of  $\Sigma_g$ . A genus- $g$  Lefschetz fibration over  $\mathbb{S}^2$  is determined by its monodromy representation  $\pi_1(\mathbb{S}^2 - f(C)) \rightarrow \mathcal{M}_g$ , where  $C$  is the set of critical points. The monodromy of a genus- $g$  Lefschetz fibration  $f : X \rightarrow \mathbb{S}^2$  comprises a factorization of  $\text{id} \in \mathcal{M}_g$ , called a *positive factorization*, as

$$t_{v_1} t_{v_2} \cdots t_{v_m} = \text{id},$$

where  $v_1, \dots, v_m$  are the vanishing cycles of the singular fibers and  $t_{v_i}$  is the right handed Dehn twist along  $v_i$ . Conversely, the above positive factorization in  $\mathcal{M}_g$  gives a genus- $g$  Lefschetz fibration over  $\mathbb{S}^2$  with vanishing cycles  $v_1, \dots, v_m$ . More details can be found in [24, 17].

In this article, we focus on genus-2 Lefschetz fibrations over  $\mathbb{S}^2$ . For abbreviation, a genus-2 Lefschetz fibration  $f : X \rightarrow \mathbb{S}^2$  is called *of type*  $(n, s)$  if  $f$  has  $n$  nonseparating and  $s$  separating singular fibers. Note that if  $f$  of type  $(n, s)$  is a fiber sum of  $f_1$  of type  $(n_1, s_1)$  and  $f_2$  of type  $(n_2, s_2)$ , then we have  $(n, s) = (n_1 + n_2, s_1 + s_2)$ .

**Lemma 2.** *For a genus-2 Lefschetz fibration  $X \rightarrow \mathbb{S}^2$  of type  $(n, s)$ , the pair  $(n, s)$  satisfies the followings:*

- $n + 2s \equiv 0 \pmod{10}$  (see Section 5 [24]),
- $2n - 5 \geq s$ ,
- If  $n + 2s = 10$ , then  $s \geq 2$  (see Remark 5.1 [30]).

*Proof.* We only prove the second inequality. In Lemma 5 of [8], it was shown that  $2n - s \geq 3$ . When we set  $n + 2s = 10k$  (see the first equality), we have  $20k - 5s \geq 3$ , so  $4k - s \geq 3/5$ . Since  $k$  and  $s$  are integers, we get  $4k - s \geq 1$  or equivalently  $2n - s \geq 5$  (This inequality can also be obtained from Theorem 5 below and Corollary 9 in [27]).  $\square$

**Proposition 3.** *A genus-2 Lefschetz fibration of type  $(n, 2n - 5)$  is indecomposable.*

*Proof.* Suppose that a genus-2 Lefschetz fibration  $f$  of type  $(n, 2n - 5)$  is a fiber sum of  $f_1$  and  $f_2$ , where  $f_i$  is of type  $(n_i, s_i)$  for  $i = 1, 2$ . We see that  $(n, 2n - 5) = (n_1 + n_2, s_1 + s_2)$ . By the second inequality in Lemma 2, we have  $s_i \leq 2n_i - 5$ . This gives

$$2n - 5 = s_1 + s_2 \leq 2(n_1 + n_2) - 10 = 2n - 10,$$

a contradiction. This finishes the proof.  $\square$

The following theorem is a rough version of the result given by Sato [30].

**Theorem 4** (Theorem 5.1 [30]). *Suppose that a genus-2 Lefschetz fibration of type  $(n, s)$  is non-minimal. Then, the following holds:*

- If  $b_2^+ > 1$ , then the possible pairs  $(n, s)$  are  $(14, 3)$ ,  $(16, 2)$ ,  $(28, 1)$ ,  $(30, 0)$  and  $(40, 0)$ ,
- If  $b_2^+ = 1$ , then  $(n, s)$  satisfies either  $n + 2s = 10$  or  $n + 2s = 20$ .

We present a signature formula for genus-2 Lefschetz fibrations given by Matsumoto [24], which was generalized by Endo [13] to genus- $g$  hyperelliptic Lefschetz fibrations.

**Theorem 5** ([24]). *Let  $f : X \rightarrow \mathbb{S}^2$  be a genus-2 Lefschetz fibration of type  $(n, s)$ . Then the signature  $\sigma(X)$  of  $X$  is*

$$\sigma(X) = -(3n + s)/5.$$

**Lemma 6.** *There is a genus-2 Lefschetz fibration of type  $(14, 13)$ .*

*Proof.* Let us consider a genus-2 Lefschetz fibration  $f$  of type  $(4, 3)$  with a positive factorization  $t_{a_1}t_{a_2}t_{a_3}t_{a_4}t_{a_5}t_{a_6}t_{a_7} = \text{id}$  in  $\mathcal{M}_2$  (The existence of such a fibration and very explicit algebro-geometric construction is given in [40]. Also, the explicit monodromy of a fibration of type  $(4, 3)$  was presented [8]). By applying cyclic permutations, we may assume that  $a_1$  is nonseparating. From the relation  $t_{a_2}t_{a_3}t_{a_4}t_{a_5}t_{a_6}t_{a_7} = t_{a_1}^{-1}$ , we obtain the following two positive factorizations

$$\begin{aligned} (t_{a_2}t_{a_3}t_{a_4}t_{a_5}t_{a_6}t_{a_7})^2 t_{a_1}^2 &= \text{id}, \\ t_{a_1}^2 (t_{a_2}t_{a_3}t_{a_4}t_{a_5}t_{a_6}t_{a_7})^2 &= \text{id}. \end{aligned}$$

Since  $a_1$  is a nonseparating curve on  $\Sigma_2$ , there is a nonseparating curve  $b_1$  on  $\Sigma_2$  which is disjoint from  $a_1$  and not homologous to  $a_1$ . Moreover, there is a diffeomorphism  $\phi$  such that  $\phi(a_1) = b_1$ . Therefore, by simultaneous conjugation to the above second relation by  $\phi$ , we obtain

$$t_{b_1}^2 (t_{b_2}t_{b_3}t_{b_4}t_{b_5}t_{b_6}t_{b_7})^2 = \text{id},$$

where  $b_i = \phi(a_i)$  for  $i = 1, 2, \dots, 7$ . From the above arguments, we get the following positive factorization

$$(t_{a_2}t_{a_3}t_{a_4}t_{a_5}t_{a_6}t_{a_7})^2 t_{a_1}^2 t_{b_1}^2 (t_{b_2}t_{b_3}t_{b_4}t_{b_5}t_{b_6}t_{b_7})^2 = \text{id}.$$

Here, we consider a sphere  $S$  with four boundary components  $a, b, c, d$ . By the lantern relation [11, 19], there are three simple closed curves  $x, y, z$  on  $S$  such that

$$t_a t_b t_c t_d = t_x t_y t_z$$

Since the genus of  $\Sigma_2$  is two, and the two nonseparating curves  $a_1$  and  $b_1$  are disjoint and not homologous,  $S$  can be embedded in  $\Sigma_2$  in such a way that  $a$  and  $b$  are  $a_1$ ,  $c$  and  $d$  are  $b_1$ ,  $x$  and  $z$  are nonseparating and  $y$  is separating. This gives the following positive factorization

$$(t_{a_2}t_{a_3}t_{a_4}t_{a_5}t_{a_6}t_{a_7})^2 t_x t_y t_z (t_{b_2}t_{b_3}t_{b_4}t_{b_5}t_{b_6}t_{b_7})^2 = \text{id}.$$

Since three of  $a_2, \dots, a_7$  (resp.  $b_2, \dots, b_7$ ) are nonseparating and the rest are separating curves, we obtain a genus-2 Lefschetz fibration of type  $(14, 13)$ , and the proof is complete.  $\square$

REMARK 7. The operation using the lantern relation in the above proof is called the *lantern substitution*. In [14], it was shown that the lantern substitution means the rational blowing down process, which was discovered in [16], along a  $(-4)$ -sphere. This was generalized in [15]. The lantern substitution preserves the minimality of symplectic 4-manifolds (cf [2]).

Proof of Theorem 1. We show that at least one of the following statements is true:

- (a) There is a genus-2 IMLF of type  $(6, 7)$ ,
- (b) There is a genus-2 IMLF of type  $(8, 11)$ ,
- (c) There is a genus-2 IMLF of type  $(10, 10)$ ,
- (d) There is a genus-2 IMLF of type  $(14, 13)$ ,

Let us consider a genus-2 Lefschetz fibration of type  $(14, 13)$ . Such a fibration is guaranteed to exist by Lemma 6 and minimal from Theorem 4. If there is an indecomposable one, then it is the required fibration of Theorem 1. Therefore, we suppose that any genus-2 Lefschetz fibrations of type  $(14, 13)$  are a fiber sum of Lefschetz fibrations of types  $(n_1, s_1)$  and  $(n_2, s_2)$ . Then, the following pairs  $(n, s)$  satisfy the conditions of Lemma 2 for  $n < 14$  and  $s < 13$ :

- If  $n + 2s = 10$ , then  $(n, s) = (6, 2), (4, 3)$ ,
- If  $n + 2s = 20$ , then  $(n, s) = (12, 4), (10, 5), (8, 6), (6, 7)$ ,
- If  $n + 2s = 30$ , then  $(n, s) = (12, 9), (10, 10), (8, 11)$ ,

Note that there is no such pair  $(n, s)$  for  $n + 2s = 10k$  and  $k \geq 4$ . Hence, we see that the possible pairs  $(n_i, s_i)$  are the following:

- (1)  $(n_1, s_1) = (6, 7)$  and  $(n_2, s_2) = (8, 6)$ ,
- (2)  $(n_1, s_1) = (8, 11)$  and  $(n_2, s_2) = (6, 2)$ ,
- (3)  $(n_1, s_1) = (10, 10)$  and  $(n_2, s_2) = (4, 3)$ .

We first look at the case (1). Then, a genus-2 Lefschetz fibration of type  $(6, 7)$  is indecomposable and minimal, and therefore, it is the required one of Theorem 1. The proof is as follows. The indecomposability immediately follows from Proposition 3. Assume that there is a non-minimal genus-2 Lefschetz fibration  $f : X \rightarrow \mathbb{S}^2$  of type  $(6, 7)$ . Then, from Theorem 4, we have  $b_2^+(X) = 1$ . Since  $\sigma(X) = b_2^+(X) - b_2^-(X) = -5$  by Theorem 5, we obtain  $b_2^-(X) = 6$ . On the other hand, we have  $b_2^-(X) \geq 7$  since every separating singular fiber contains a torus of negative self-intersection, and all of them are linearly independent in homology, a contradiction.

Next, we consider the case (2). By Theorem 4 and Proposition 3 we see that a genus-2 Lefschetz fibration of type  $(8, 11)$  is indecomposable and minimal, and it is the required fibration.

Finally, we deal with the case (3). The minimality of a Lefschetz fibration of type  $(10, 10)$  follows from Theorem 4. If there is an indecomposable one, we obtain the claimed fibration. We suppose that every Lefschetz fibration of type  $(10, 10)$  is a fiber sum of Lefschetz fibrations of types  $(n_3, s_3)$  and  $(n_4, s_4)$ . The possible pairs are  $(n_3, s_3) = (6, 7)$  and  $(n_4, s_4) = (4, 3)$  from the above pairs  $(n, s)$  satisfying the conditions of Lemma 2 for  $n < 14$  and  $s < 13$ . From the case (1), we obtain the required fibration of type  $(6, 7)$ .

This finishes the proof. □

REMARK 8. Strictly speaking, for a genus- $g(\geq 2)$  Lefschetz fibration on  $X$  over a closed surface with  $s$  separating singular fibers, we have  $b_2^-(X) \geq s + 1$  (see, for example, Lemma 2.4 in [22]) as follows. Every separating singular fibers contains a surface of self-intersection  $-1$ . Since all of the surfaces are linearly independent in  $H_2(X)$  and independent of the class of a smooth fiber, which has selfintersection 0, we obtain  $b_2^-(X) \geq s + 1$ .

REMARK 9. After writing the first draft of the paper, the second author was informed by Inanc Baykur that he also obtained the similar proof of the minimality and the indecomposability of type (6, 7) and that his former student Kai Nakamura has studied the geography of genus-2 Lefschetz fibrations and produced some examples of new Lefschetz fibrations in his undergraduate thesis [25]. In [25, 26], an example of type (10, 10) was given using similar methods to [4, 1, 2], which are applying lantern and daisy substitutions to a word obtained from a twisted fiber sum of Lefschetz fibrations. More precisely, the articles [4, 1, 2] constructed new examples of Lefschetz fibrations by applying the lantern and daisy substitutions to the monodromies of well known genus two Lefschetz fibrations on  $K3\#2\overline{\mathbb{C}P}^2$  in [4], the higher genus ( $g \geq 3$ ) Lefschetz fibrations in [1], and the twisted fiber sums of Matsumoto's genus two Lefschetz fibration on  $\mathbb{T}^2 \times \mathbb{S}^2\#4\overline{\mathbb{C}P}^2$ , respectively. The additional examples via this approach are constructed in our preprint with S. Sakallı [3]

It is not known whether the example constructed in Lemma 6 is decomposable. Therefore, the following question arises.

QUESTION 10. Is the example of type (14, 13) in Lemma 6 decomposable?

The types (4, 3), (6, 7) and (12, 19), which satisfy  $s = 2n - 5$ , were constructed in [40]. On the other hand, to the best of our knowledge, there is no example of type (8, 11). This observation leads to the following geography question for genus-2 Lefschetz fibrations of type  $(n, s)$ .

QUESTION 11. For which pairs of integers  $(n, s)$ , is there a genus-2 Lefschetz fibration of type  $(n, s)$ ?

Since the Chern numbers  $c_1^2$  and  $c_2$  (or equivalently, the signature and the Euler characteristic) of the total space of genus-2 Lefschetz fibration of type  $(n, s)$  are determined by  $n$  and  $s$ , we can rewrite this question as follows: *Which pairs of integers are realized as  $(c_1^2, c_2)$  of the total space of a genus-2 Lefschetz fibration?* The answer of Question 11 gives the symplectic version of Persson's result [28] that all allowed lattice points  $(c_1^2, c_2)$  except finitely many lying in the region  $(c_2 - 36)/5 \leq c_1^2 \leq 2c_2$  are realized a complex surface admitting a genus-2 Lefschetz fibration over  $\Sigma_h$ .

It is natural to ask the following question.

QUESTION 12. How many indecomposable minimal genus- $g$  Lefschetz fibrations do there exist for  $g \geq 2$ ?

REMARK 13. Stipsicz also conjectured that every indecomposable Lefschetz fibration has a  $(-1)$ -section (see Conjecture 2.4 [35]). There are nonminimal counterexamples to this conjecture, i.e., indecomposable nonminimal genus- $g$  Lefschetz fibrations with no  $(-1)$ -sections

( $g = 2$  [31],  $g = 2, 3$  [6] and  $g \geq 2$  [7]). Our result shows that a minimal counterexample exists.

REMARK 14. For  $g \geq 3$  and  $h = 1, 2$  we find that genus- $g$  Lefschetz fibrations over  $\Sigma_h$  constructed in [20, 18, 36] are indecomposable and minimal. The minimality follows from the result of [34], and the indecomposability follows from the number of singular fibers and the lower bounds on the number of singular fibers of Lefschetz fibrations over  $\mathbb{S}^2$  (see [10]) and  $T^2$  (see [36]). In [9], it was shown that a genus- $g$  Lefschetz fibration over  $\Sigma_h$  with a “maximal section” (see [9] for the definition) is indecomposable (as a fiber sum of two genus- $g$  Lefschetz fibrations with a section) if  $h \geq 1$ . Note that a maximal section means that a  $(-1)$ -section for  $h = 0$ . If  $g \geq 5$ , then we can show that the fibrations given in [20, 18, 36] has a maximal section from the constructions of [20, 36] and Theorem 13 and the technique in Section 3.3 of [9]. On the other hand, our indecomposable minimal genus-2 Lefschetz fibration over  $\mathbb{S}^2$  does not admit any maximal sections (i.e.  $(-1)$ -sections).

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