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REAL HYPERSURFACES WITH KILLING STRUCTURE JACOBI OPERATOR IN THE COMPLEX HYPERBOLIC QUADRIC

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Abstract

First we introduce the notion of Killing structure Jacobi operator for real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$. Next we give a complete classification of real hypersurfaces in $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ with Killing structure Jacobi operator.

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1. Introduction

In case of Hermitian symmetric space of rank 1, we say a complex projective space $\mathbb{C}P^m$ and a complex hyperbolic space $\mathbb{C}H^m$. In the complex projective space $\mathbb{C}P^m$, a full classification of real hypersurfaces with isometric Reeb flow was obtained by Okumura in [16]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. Moreover, Takagi [41] gave a complete classification of homogeneous hypersurfaces in $\mathbb{C}P^m$ and Kimura and etc., [7] considered the notion GTW Reeb parallel shape operator. In the complex hyperbolic space $\mathbb{C}H^m$, Montiel and Romero [13] have given a complete classification of real hypersurface with isometric Reeb flow.

As another kind of Hermitian symmetric space with rank 2 of non-compact type different from the above ones, we can give the example of complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$. By using the method given in Kobayashi and Nomizu [12], Chapter XI, Example 10.6, the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ can be immersed in indefinite complex hyperbolic space CH^{m+1}_1 as a space-like complex hypersurface (see Montiel and Romero [15] and Suh [34]). The complex hyperbolic quadric Q^{m*} is the non-compact Hermitian symmetric space $SO^0_{2,m}/SO_2SO_m$ of rank 2 and also can be regarded as a kind of real Grassmann manifold of all oriented space-like 2-dimensional subspaces in indefinite flat Riemannian space $\mathbb{R}^{m+2}_2$(see Montiel and Romero [14] and [15]). Accordingly, the complex hyperbolic quadric admits both a complex conjugation structure A and a Kähler structure J, which anti-commutes with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^{m*}, J, g) is a Hermitian symmetric space of noncompact type with rank 2 and its minimal sectional curvature is equal to -4 (see Klein [8] and Reckziegel [22]).

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Now let us consider a real hypersurface in the complex hyperbolic quadric Q^{m*} with isometric Reeb flow. Then from the view of the previous results a natural expectation might be the totally geodesic $Q^{m-1^*} \subset Q^{m*}$. But, suprisingly, in the complex hyperbolic quadric Q^{m*} the situation is quite different from the above ones. Recently, Suh [34] has introduced the following result:

Theorem A. *Let M be a real hypersurface of the complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_mSO_2$, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

Jacobi fields along geodesics of a given Riemannian manifold (M, g) satisfy a well known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if R denotes the curvature operator of M, and X is tangent vector field to M, then the Jacobi operator $R_X \in End(T_xM)$ with respect to X at $x \in M$, defined by $(R_XY)(x) = (R(Y, X)X)(x)$ for any $Y \in T_xM$, becomes a self adjoint endomorphism of the tangent bundle TM of M. Thus, each tangent vector field X to M provides a Jacobi operator R_X with respect to X. In particular, for the Reeb vector field ξ, the Jacobi operator R_ξ is said to be a *structure Jacobi operator*.

Recently Ki, Pérez, Santos and Suh [5] have investigated the Reeb parallel structure Jacobi operator in the complex space form $M_m(c)$, $c \neq 0$ and have used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, Pérez, Jeong and Suh [20] have investigated real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M. Jeong, Suh and Woo [4] and Pérez and Santos [18] have generalized such a notion to the recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain 1-form β and any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$. Moreover, Pérez, Santos and Suh [19] have further investigated the property of the Lie ξ-parallel structure Jacobi operator in complex projective space $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$.

The Reeb vector field ξ is *Killing* on M in Q^{m*} if and only if $g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$ for any vector fields X and Y on M. As a generalization of such a Killing vector field first Yano [42] defined the notion of *Killing tensor* as follows:

A skew symmetric tensor $T_{i_1 \cdots i_r}$ is called a *Killing tensor* of order r if it satisfies

$$\nabla_{i_1} T_{i_2 \cdots i_{r+1}} + \nabla_{i_2} T_{i_1 \cdots i_{r+1}} = 0.$$

Next Blair [2] has applied the notion of Killing tensor to a tensor field of T type $(1, 1)$ on a Riemannian manifold and a geodesic γ on M. If we denote by γ' the tangent vector of the geodesic γ, then $T\gamma'$ is parallel along the geodesic γ for the Killing tensor field T. Geometrically, this means that $(\nabla_{\gamma'} T)\gamma' = 0$ along a geodesic γ on M. If this is the case for any geodesic on M, we have

$$(\nabla_X T)X = 0 \quad \text{or equivalently} \quad (\nabla_X T)Y + (\nabla_Y T)X = 0$$

for any vector fields X and Y on M. In this case we say that the tensor T is a *Killing tensor field of type* $(1, 1)$.

Now we consider such a situation to the structure Jacobi operator R_ξ, which is a tensor field of type $(1,1)$ on a real hypersurface M in Q^{m*}. The structure Jacobi operator R_ξ of M in Q^m is said to be *Killing* if the structure Jacobi operator R_ξ satisfies

$$(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$$

for any $X, Y \in T_zM$, $z \in M$. The equation is equivalent to $(\nabla_X R_\xi)X = 0$ for any $X \in T_zM$, $z \in M$, because of polarization. Moreover, we can give the geometric meaning of the Killing Jacobi operator as follows:

When we consider a geodesic γ with initial conditions such that $\gamma(0) = z$ and $\dot{\gamma}(0) = X$. Then the transformed vector field $R_\xi \dot{\gamma}$ is Levi-Civita *parallel* along the geodesic γ of the vector field X (see Blair [2] and Tachibana [40]).

In addition to the complex structure J there is another distinguished geometric structure on Q^{m*}, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*}. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*} as follows:

$$\mathcal{Q} = \{X \in T_zM \mid AX \in T_zM \quad \text{for all} \quad A \in \mathfrak{A}\}.$$

Recall that a nonzero tangent vector $W \in T_{[z]}Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*}. There are two types of singular tangent vectors for the complex hyperbolic quadric Q^{m*}:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic

where $V(A) = \{X \in T_{[z]}Q^{m*} | AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^{m*} | AX = -X\}$, $[z] \in Q^{m*}$, are the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$.

In the study of real hypersurfaces in the complex quadric Q^m we considered the notion of parallel Ricci tensor, that is, $\nabla \mathrm{Ric} = 0$ (see Suh [31]). But from the assumption of Ricci parallel, it was difficult for us to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [31] we gave a classification with the further assumption of $\mathfrak{A}$-isotropic. But fortunately, if we consider a Hopf real hypersurfaces, which is defined by $S\xi = \alpha\xi$ for the Reeb function $\alpha = g(S\xi, \xi)$ and the shape operator S, in the complex hyperbolic quadric Q^{m*} with Killing structure Jacobi operator, we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal as follows:

Main Theorem 1. *Let M be a Hopf real hypersurface in Q^{m*}, $m \geq 3$, with Killing structure Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

When we consider a hypersurface M in the complex hyperbolic quadric Q^{m*}, the unit normal vector field N of M in Q^{m*} can be divided into two cases : N is $\mathfrak{A}$-isotropic or

$\mathfrak{A}$-principal (see [34], [35] and [27]). In the first case where M has an $\mathfrak{A}$-isotropic unit normal N, we have asserted in [34] and [35] that M is locally congruent to a tube over a totally geodesic complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k^*} or a horosphere with $\mathfrak{A}$-isotropic unit normal vector field centered at the infinity. In the second case when N is $\mathfrak{A}$-principal we have proved that M is locally congruent to a tube over a totally geodesic and totally real submanifold Q^{m-1^*} in Q^{m*} (see [34], [36] and [38]).

In this paper we consider the case that the structure Jacobi operator R_ξ of M in Q^{m*} is Killing , that is, $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$ for any tangent vector field X and Y on M, and we prove the following

Main Theorem 2. *There does not exist a Hopf hypersurface in Q^{m*}, $m \geq 3$ with Killing stucture Jacobi operator and $\mathfrak{A}$-principal unit normal vector field.*

Now it remains to prove the case that the unit normal vector field is $\mathfrak{A}$-isotropic. Then by our Main Theorems 1 and 2, we give a classification of real hypersurfaces in Q^{m*} with Killing structure Jacobi operator as follows:

Main Theorem 3. *Let M be a Hopf hypersurface in Q^{m*}, $m \geq 3$ with Killing stucture Jacobi operator. If the Reeb function is constant along the Reeb direction, then M has 4 distinct constant principal curvatures*

$$\alpha, \quad \beta = 0, \quad \lambda_1 \quad \lambda_2.$$

Here the corresponding eigen spaces $\xi \in T_\alpha$, $T_\beta = \mathcal{Q}^\perp$, and $T_{\lambda_1} \oplus T_{\lambda_2} = \mathcal{Q}$, where the principal curvatures λ_1 and λ_2 are two distinct constants given by

$$\lambda_1 = \frac{\alpha(\alpha^2-1) + \alpha\sqrt{(\alpha^2-1-2\sqrt{2})(\alpha^2-1+2\sqrt{2})}}{4}$$

and

$$\lambda_2 = \frac{\alpha(\alpha^2-1) - \alpha\sqrt{(\alpha^2-1-2\sqrt{2})(\alpha^2-1+2\sqrt{2})}}{4}.$$

with multiplicities $(m-2)$ respectively and $\alpha^2 > 2\sqrt{2}+1$.

Remark 1.1. In [29] Suh has proved that the Reeb function $\alpha = g(S\xi, \xi)$ is constant for real hypersurfaces with singular normal vector field in the complex quadric Q^m . But in the complex hyperbolic quadric Q^{m*} the Reeb function α is constant only if the unit normal vector field N is $\mathfrak{A}$-principal (see Suh, Pérez and Woo [39]). Until now it does not known to us whether the Reeb function α is constant for real hypersurfaces in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-isotropic unit normal vector field.

The subbundle $\mathcal{Q}$ mentioned in Main Theorem 3 is the maximal invariant subbundle of T_zM, $z \in M$, such that $\mathcal{Q} \oplus \mathcal{Q}^\perp = [\xi]^\perp$, where $\mathcal{Q}^\perp = \text{Span}\{A\xi, AN\}$ and $[\xi]^\perp$ denotes the orthogonal complement of the Reeb vector field ξ in T_zM, $z \in M$, in Q^{m*}.

When we consider a parallel structure Jacobi operator on M in Q^{m*}, we know that $(\nabla_X R_\xi)Y = 0$ for any vector fields X and Y on M. This gives a condition stronger than the notion of Killing structure Jacobi operator. So naturally it satisfies the assumptions of Killing in Main Theorems 1, 2 and 3. For the case of isotropic unit normal N, it can be easily checked that the results in our Main Theorem 3 do not satisfy the strong assumption of parallel structure Jacobi operator. So we also conclude the following

Corollary (see [39]). *There does not exist a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with parallel stucture Jacobi operator.*

2. The complex hyperbolic quadric

In this section, let us introduce a new known result of the complex hyperbolic quadric Q^{m*} different from the complex quadric Q^m. This section is due to Klein and Suh [10].

The m-dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m-dimensional complex quadric Q^m, which is a kind of Hermitian symmetric space of non-compact type with rank 2 (see Besse [1], and Helgason [3]).

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [24, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m, see [8] and [22]. Another related result by Smyth, [24, Theorem 1], which states that any complex hypersurface $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO^0_{2,m}/SO_2SO_m$. As Q^{1*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO^0_{1,2}/SO_2$, and Q^{2*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO^0_{2,m}$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the "origin" $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(m+2, \mathbb{R}) \,\big|\, X^t \cdot s = -s \cdot X\}$$

(see [11, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} ,$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of the dimension 2×2, $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \middle| \; X_{11}^t = -X_{11},\; X_{12}^t = X_{21},\; X_{22}^t = -X_{22} \right\} .$$

The linearisation $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} =& \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} \,|\, sXs^{-1} = X\} \\ =& \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \middle| \; X_{11}^t = -X_{11},\; X_{22}^t = -X_{22} \right\} \\ \cong& \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} \,|\, sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \middle| \; X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2}\,\mathrm{tr}(Y^t \cdot X) = \mathrm{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

g is clearly $\mathrm{Ad}(K)$-invariant, and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for} \quad X \in \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K .$$

Because j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant, and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication with the unit complex number i with the application of the linear map J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As mentioned for the complex quadric (again compare [8], [9], and [22]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures. The situation here differs from that of the complex quadric in that for Q^{m*}, the real structures in $\mathfrak{A}$ cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays an important role in the description of the geometry of Q^{m*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} .$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\, SO_m$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant, and therefore defines an $\mathbb{R}$-linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{m*}$), and hence a real structure on $T_{p_0}Q^{m*}$. But A_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathbb{R}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \mathrm{Ad}(g) = \mu^{-2} \cdot \mathrm{Ad}(g) \circ A_0 .$$

This equation shows that the object which is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the “circle of real structures”

$$\mathfrak{A}_{p_0} := \{\lambda A_0 | \lambda \in S^1\} .$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant, and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*}. For any $A \in \mathfrak{A}$, the tangent line to the fibre of $\mathfrak{A}$ through A is spanned by JA.

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_pQ^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ resp. $JV(A)$ are the $(+1)$-eigenspace resp. the (-1)-eigenspace of A. For every unit vector $W \in T_pQ^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$W = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [22, Proposition 3]. Here t is uniquely determined by W. The vector W is singular, i.e. contained in more than one Cartan subalgebra of $\mathfrak{m}$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If W is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by W.

The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $R_W = R(\cdot, W)W$ for a singular unit tangent vector W.

1. If W is an $\mathfrak{A}$-principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of R_W are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$ and $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$, respectively.
2. If W is an $\mathfrak{A}$-isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of R_W are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}W \oplus \mathbb{C}(JX + Y)$, $T_zQ^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$ and $\mathbb{R}JW$, respectively.

Like for the complex quadric, the Riemannian curvature tensor $\bar{R}$ of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$. In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $\bar{R}(X,Y)Z$ corresponds to $-[[X,Y],Z]$ for $X,Y,Z \in \mathfrak{m}$, see [12, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned}\bar{R}(X,Y)Z \;=\; & -g(Y,Z)X + g(X,Z)Y \\ & - g(JY,Z)JX + g(JX,Z)JY + 2g(JX,Y)JZ \\ & - g(AY,Z)AX + g(AX,Z)AY \\ & - g(JAY,Z)JAX + g(JAX,Z)JAY\end{aligned}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [22, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

3. Some general equations

Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} and denote by (ϕ,ξ,η,g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define the maximal $\mathfrak{A}$-invariant subspace of T_zM, $z{\in}M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1 (see [29])**.** *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0,\pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX+Y)$.*

We now assume that M is a Hopf hypersurface. Then for the Reeb vector field ξ the shape operator S becomes

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi,\xi)$ on M. When we consider a transform JX of the Kaehler structure J on the complex hyperbolic quadric Q^{m*} for any vector field X on M in Q^{m*}, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M.

Then we now consider the Codazzi equation

$$g((\nabla_X S)Y - (\nabla_Y S)X, Z) = -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) - g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z) - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z). \tag{3.1}$$

Putting $Z = \xi$ we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = 2g(\phi X, Y) & \\ - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) & \\ + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi). & \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = \; & g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ = \; & (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi S X, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.2}$$

Reinserting this into the previous equation yields

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = \; & 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & -2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ & +\alpha g((\phi S + S\phi)X, Y) - 2g(S\phi S X, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 \; = \; & 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ & -g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ & +g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\ & -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & +2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [22]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned}$$

This implies $g(\xi, AN) = 0$ and hence

$$0 \; = \; 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y)$$

$$
\begin{aligned}
&-g(X,AN)g(Y,A\xi)+g(Y,AN)g(X,A\xi)\\
&+g(X,A\xi)g(JY,A\xi)-g(Y,A\xi)g(JX,A\xi)\\
&+2g(X,AN)g(\xi,A\xi)\eta(Y)-2g(Y,AN)g(\xi,A\xi)\eta(X).
\end{aligned}
$$

We have $JA\xi=-AJ\xi=-AN$, and inserting this into the previous equation implies

Lemma 3.2. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*} with (local) unit normal vector field N. For each point $z\in M$ we choose $A\in\mathfrak{A}_z$ such that $N_z=\cos(t)Z_1+\sin(t)JZ_2$ holds for some orthonormal vectors $Z_1,Z_2\in V(A)$ and $0\le t\le\frac{\pi}{4}$. Then*

$$
\begin{aligned}
0 \;=\;& 2g(S\phi SX,Y)-\alpha g((\phi S+S\phi)X,Y)+2g(\phi X,Y)\\
&-2g(X,AN)g(Y,A\xi)+2g(Y,AN)g(X,A\xi)\\
&-2g(\xi,A\xi)\{g(Y,AN)\eta(X)-g(X,AN)\eta(Y)\}
\end{aligned}
$$

holds for all vector fields X and Y on M.

We can write for any vector field Y on M in Q^{m*}

$$AY=BY+\rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y)=g(AY,N)$.

If N is $\mathfrak{A}$-prinicipal, that is, $AN=N$, we have $\rho=0$, because $\rho(Y)=g(Y,AN)=g(Y,N)=0$ for any tangent vector field Y on M in Q^{m*}. So we have $AY=BY$ for any tangent vector field Y on M in Q^{m*}. Otherwise we can use Lemma 3.1 to calculate $\rho(Y)=g(Y,AN)=g(Y,AJ\xi)=-g(Y,JA\xi)=-g(Y,JB\xi)=-g(Y,\phi B\xi)$ for any tangent vector field Y on M in Q^{m*}. From this, together with Lemma 3.2, we have proved

Lemma 3.3. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m\ge 3$. Then we have*

$$(2S\phi S-\alpha(\phi S+S\phi)+2\phi)X=2\rho(X)(B\xi-\beta\xi)+2g(X,B\xi-\beta\xi)\phi B\xi,$$

where the function β is given by $\beta=g(\xi,A\xi)=-g(N,AN)$.

If the unit normal vector field N is $\mathfrak{A}$-principal, we can choose a real structure $A\in\mathfrak{A}$ such that $AN=N$. Then we have $\rho=0$ and $\phi B\xi=-\phi\xi=0$, and therefore

$$2S\phi S-\alpha(\phi S+S\phi)=-2\phi. \tag{3.3}$$

If N is not $\mathfrak{A}$-principal, we can choose a real structure $A\in\mathfrak{A}$ as in Lemma 3.1 and get

$$
\begin{aligned}
&\rho(X)(B\xi-\beta\xi)+g(X,B\xi-\beta\xi)\phi B\xi\\
&=-g(X,\phi(B\xi-\beta\xi))(B\xi-\beta\xi)+g(X,B\xi-\beta\xi)\phi(B\xi-\beta\xi)\\
&=\|B\xi-\beta\xi\|^2\{g(X,U)\phi U-g(X,\phi U)U\}\\
&=\sin^2(2t)\{g(X,U)\phi U-g(X,\phi U)U\},
\end{aligned}
\tag{3.4}
$$

which is equal to 0 on $\mathcal{Q}$ and equal to $\sin^2(2t)\phi X$ on $\mathcal{C}\ominus\mathcal{Q}$. Altogether we have proved:

Lemma 3.4. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\beta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where $\beta = g(A\xi, \xi) = -\cos 2t$ as in section 3.

Then from the equation of Gauss the curvature tensor R of M in complex quadric Q^{m*} is defined so that

$$\begin{aligned} R(X,Y)Z &= -g(Y,Z)X + g(X,Z)Y - g(\phi Y,Z)\phi X + g(\phi X,Z)\phi Y + 2g(\phi X,Y)\phi Z \\ &\quad -g(AY,Z)(AX)^T + g(AX,Z)(AY)^T - g(JAY,Z)(JAX)^T \\ &\quad +g(JAX,Z)(JAY)^T + g(SY,Z)SX - g(SX,Z)SY, \end{aligned}$$

where $(AX)^T$ and S denote the tangential component of the vector field AX and the shape operator of M in Q^{m*} respectively.

From this, putting $Y = Z = \xi$ and using $g(A\xi, N) = 0$, the structure Jacobi operator is defined by

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= -X + \eta(X)\xi - g(A\xi,\xi)(AX)^T + g(AX,\xi)A\xi \\ &\quad +g(X,AN)(AN)^T + g(S\xi,\xi)SX - g(SX,\xi)S\xi. \end{aligned}$$

Then we may put the following

$$(AY)^T = AY - g(AY,N)N.$$

Now let us denote by ∇ and $\bar{\nabla}$ the covariant derivative of M and the covariant derivative of Q^{m*} respectively. Then by using the Gauss and Weingarten formulas we can assert the following

Lemma 3.5. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}. Then*

$$\begin{aligned} \nabla_X(AY)^T =& q(X)JAY + A\nabla_X Y + g(SX,Y)AN \\ &- g(\{q(X)JAY + A\nabla_X Y + g(SX,Y)AN\}, N)N \\ &+ g(AY,N)SX. \end{aligned} \tag{3.5}$$

Proof. First let us use the Gauss formula to $(AY)^T = AY - g(AY,N)N$. Then it follows that

$$\begin{aligned}\nabla_X(AY)^T =&\bar{\nabla}_X(AY)^T - \sigma(X,(AY)^T)\\ =&\bar{\nabla}_X\{AY - g(AY,N)N\} - g(SX,(AY)^T)N\\ =&(\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - g((\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y, N)N - g(AY,\bar{\nabla}_X N)N\\ &- g(AY,N)\bar{\nabla}_X N - g(SX,(AY)^T)N,\end{aligned}$$

where σ denotes the second fundamental form and N the unit normal vector field on M in Q^{m*}. Then from this, if we use Weingarten formula $\bar{\nabla}_X N = -SX$, then we get the above formula. □

By puting $Y = \xi$ and using $g(A\xi,N) = 0$, we have

$$\begin{aligned}\nabla_X(A\xi) =&q(X)JA\xi + A\phi SX + \alpha\eta(X)AN\\ &- \{q(X)g(JA\xi,N) + g(A\phi SX,N) + \alpha\eta(X)g(AN,N)\}N.\end{aligned}\tag{3.6}$$

Moreover, let us also use Gauss and Weingarten formula to $(AN)^T = AN - g(AN,N)N$. Then it follows that

$$\begin{aligned}\nabla_X(AN)^T =&\bar{\nabla}_X(AN)^T - \sigma(X,(AN)^T)\\ =&\bar{\nabla}_X\{AN - g(AN,N)N\} - \sigma(X,(AN)^T)\\ =&(\bar{\nabla}_X A)N + A\bar{\nabla}_X N - g((\bar{\nabla}_X A)N + A\bar{\nabla}_X N, N)\\ &- g(AN,\bar{\nabla}_X N)N - g(AN,N)\bar{\nabla}_X N - \sigma(X,(AN)^T)\\ =&q(X)JAN - ASX - g(q(X)JAN - ASX,N)N + g(AN,N)SX.\end{aligned}\tag{3.7}$$

On the other hand, we know that

$$\begin{aligned}X\beta =&X(g(A\xi,\xi))\\ =&g((\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi,\xi) + g(A\xi,\bar{\nabla}_X\xi)\\ =&g(q(X)JA\xi + A\phi SX + g(SX,\xi)AN,\xi) + g(A\xi,\phi SX + g(SX,\xi)N)\\ =&2g(A\phi SX,\xi).\end{aligned}\tag{3.8}$$

4. Some Important Lemmas and Proof of Theorem 1

The curvature tensor $R(X,Y)Z$ for a Hopf real hypersurface M in the complex hyperbolic quadric Q^{m*} induced from the curvature tensor of Q^{m*} is given in section 3. Now the structure Jacobi operator R_ξ can be rewritten as follows:

$$\begin{aligned}R_\xi(X) =&R(X,\xi)\xi\\ =&-X + \eta(X)\xi - \beta(AX)^T + g(AX,\xi)A\xi + g(AX,N)(AN)^T\\ &+ \alpha SX - g(SX,\xi)S\xi,\end{aligned}\tag{4.1}$$

where we have put $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$, because we assume that M is Hopf. The Reeb vector field $\xi = -JN$ and the anti-commuting property $AJ = -JA$ gives that the function β becomes $\beta = -g(AN,N)$. When this function $\beta = g(A\xi,\xi)$ identically vanishes, we say that a real hypersurface M in Q^{m*} is $\mathfrak{A}$-isotropic as in section 1.

Here let us differentiate the structure Jacobi operator R_ξ along any direction X on M in the complex hyperbolic quadric Q^{m*}. Then (4.1), together with (3.5), (3.6), (3.7), give that

$$
\begin{aligned}
\nabla_X R_\xi(Y) &= \nabla_X(R_\xi(Y)) - R_\xi(\nabla_X Y) \\
&= g(\phi SX, Y)\xi + \eta(Y)\phi SX - (X\beta)(AY)^T \\
&\quad - \beta\Big[q(X)JAY + A\nabla_X Y + g(SX, Y)AN \\
&\quad - g(\{q(X)JAY + A\nabla_X Y + g(SX, Y)AN\}, N)N \\
&\quad + g(AY, N)SX\Big] \\
&\quad + g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN, Y)A\xi \\
&\quad + g(AY, \xi)\Big[g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN \\
&\quad - \{q(X)g(JA\xi, N) + g(A\phi SX, N) + \alpha\eta(X)g(AN, N)\}N\Big] \\
&\quad + \Big[g(q(X)JAN - ASX + g(AN, N)SX, Y)(AN)^T \\
&\quad + g(Y, (AN)^T)\{q(X)JAN - ASX + g(AN, N)SX \\
&\quad - g(q(X)JAN - ASX, N)N\}\Big] \\
&\quad + (X\alpha)SY + \alpha(\nabla_X S)Y - X(\alpha^2)\eta(Y)\xi \\
&\quad - \alpha^2(\nabla_X\eta)(Y)\xi - \alpha^2\eta(Y)\nabla_X\xi,
\end{aligned}
\tag{4.2}
$$

where we have used $g(A\xi, N) = 0$, and N the unit normal to M in Q^{m*}.

Here let us assume that the structure Jacobi operator is Killing, that is, $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$ for any tangent vector fields X and Y on M in Q^{m*}. Then from this, together with (4.1), we have the following

$$
\begin{aligned}
0 &= \nabla_X R_\xi(Y) + \nabla_Y R_\xi(X) \\
&= \{g(\phi SX, Y) + g(\phi SY, X)\}\xi + \eta(Y)\phi SX + \eta(X)\phi SY \\
&\quad - (X\beta)(AY)^T - (Y\beta)(AX)^T \\
&\quad - \beta\Big[q(X)JAY + q(Y)JAX + A(\nabla_X Y + \nabla_Y X) + 2g(SX, Y)AN \\
&\quad - g(\{q(X)JAY + q(Y)JAX + A(\nabla_X Y + \nabla_Y X) + 2g(SX, Y)AN\}, N)N \\
&\quad + g(AY, N)SX + g(AX, N)SY\Big] \\
&\quad + \Big[g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN, Y) \\
&\quad + g(q(Y)JA\xi + A\phi SY + \alpha\eta(Y)AN, X)\Big]A\xi \\
&\quad + g(AY, \xi)\Big[q(X)JA\xi + A\phi SX + \alpha\eta(X)AN \\
&\quad - \{q(X)g(JA\xi, N) + g(A\phi SX, N) + \alpha\eta(X)g(AN, N)\}N\Big] \\
&\quad + g(AX, \xi)\Big[q(Y)JA\xi + A\phi SY + \alpha\eta(Y)AN \\
&\quad - \{q(Y)g(JA\xi, N) + g(A\phi SY, N) + \alpha\eta(Y)g(AN, N)\}N\Big] \\
&\quad + \Big[\{g(q(X)JAN - ASX + g(AN, N)SX, Y)
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&+ g(q(Y)JAN - ASY + g(AN,N)SY, X)\}(AN)^T \\
&+ g(Y,(AN)^T)\{q(X)JAN - ASX - g(q(X)JAN - ASX, N)N \\
&+ g(AN,N)SX\} \\
&+ g(X,(AN)^T)\{q(Y)JAN - ASY - g(q(Y)JAN - ASY, N)N \\
&+ g(AN,N)SY\}\Big] \\
&+ (X\alpha)SY + (Y\alpha)SX + \alpha\{(\nabla_X S)Y + (\nabla_Y S)X\} \\
&- X(\alpha^2)\eta(Y)\xi - (Y\alpha^2)\eta(X)\xi - \alpha^2\{(\nabla_X\eta)(Y)\xi + (\nabla_Y\eta)(X)\xi\} \\
&- \alpha^2\{\eta(Y)\nabla_X\xi + \eta(X)\nabla_Y\xi\}.
\end{aligned}
$$

From this, by taking the inner product of (4.3) with the Reeb vector field ξ, we have

$$
\begin{aligned}
0 =& g((\phi S - S\phi)X, Y) - (X\beta)g(AY,\xi) - (Y\beta)g(AX,\xi) \\
&- \beta\{q(X)g(JAY,\xi) + q(Y)g(JAX,\xi) + g(A(\nabla_X Y + \nabla_Y X),\xi) \\
&+ g(AY,N)g(SX,\xi) + g(AX,N)g(SY,\xi)\} \\
&+ \{g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN, Y) \\
&+ g(q(Y)JA\xi + A\phi SY + \alpha\eta(Y)AN, X)\}g(A\xi,\xi) \\
&+ g(AY,\xi)g(A\phi SX,\xi) + g(AX,\xi)g(A\phi SY,\xi) \\
&+ g(Y,(AN)^T)\{g(q(X)JAN,\xi) - g(ASX,\xi) + g(AN,N)g(SX,\xi)\} \\
&+ g(X,(AN)^T)\{g(q(Y)JAN,\xi) - g(ASY,\xi) + g(AN,N)g(SY,\xi)\} \\
&+ \alpha(X\alpha)\eta(Y) + \alpha(Y\alpha)\eta(X) \\
&+ \alpha\{g((\nabla_X S)Y,\xi) + g((\nabla_Y S)X,\xi)\} \\
&- X(\alpha^2)\eta(Y) - Y(\alpha^2)\eta(X) - \alpha^2(\nabla_X\eta)(Y) - \alpha^2(\nabla_Y\eta)(X).
\end{aligned}
$$

Then, first, by putting $Y = \xi$ and using $g(A\xi, N) = 0$, we have

$$
\begin{aligned}
0 =& -(X\beta)g(A\xi,\xi) - \beta g(A\phi SX,\xi) + \beta g(A\phi SX,\xi) + \beta g(A\phi SX,\xi) \\
&- (\xi\beta)g(AX,\xi) - \beta\{q(\xi)g(JAX,\xi) + g(A\nabla_\xi X,\xi) + \alpha g(AX,N)\} \\
&+ \{g(q(\xi)JA\xi + A\phi S\xi + \alpha AN, X)\}g(A\xi,\xi) \\
&+ g(X,AN)(q(\xi) - 2\alpha)\beta \\
=& -\beta\{g(A\phi SX,\xi) + g(A\nabla_\xi X,\xi) - (q(\xi) - 2\alpha)g(X,AN)\}.
\end{aligned} \tag{4.4}
$$

Here if the function $\beta = g(A\xi,\xi) = -\cos 2t = 0$, we have $t = \frac{\pi}{4}$. Then the unit normal vector field N becomes

$$
N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)
$$

for $Z_1, Z_2 \in V(A)$ as in section 3, that is, the unit normal N is $\mathfrak{A}$-isotropic .

Now hereafter, from (4.4) let us consider the following case

$$
0 = \{g(A\phi SX,\xi) + g(A\nabla_\xi X,\xi) - (q(\xi) - 2\alpha)g(X,AN)\}. \tag{4.5}
$$

On the other hand, by using (3.1) for any tangent vector field $X \perp A\xi$, we have

(4.6)
$$\begin{aligned} g(A\nabla_\xi X,\xi) =& g(\nabla_\xi X, A\xi) = -g(X,\nabla_\xi(A\xi)) \\ =& -g(q(\xi)JA\xi+\alpha AN, X) = (q(\xi)-\alpha)g(AN,X). \end{aligned}$$

Then from (4.5) and (4.6) we have the following for any tangent vector field X orthogonal to $A\xi$

(4.7)
$$\begin{aligned} 0 =& g(A\phi SX,\xi) + (q(\xi)-\alpha)g(AN,X) - (q(\xi)-2\alpha)g(AN,X) \\ =& g(A\phi SX,\xi) + \alpha g(AN,X) \\ =& g(SAN+\alpha AN, X). \end{aligned}$$

So it follows that

(4.8)
$$g(S(AN)^T,(AN)^T) = -\alpha(1-\beta^2),$$

where $g((AN)^T,(AN)^T) = g(AN-g(AN,N)N, AN-g(AN,N)N) = 1-g(AN,N)^2 = 1-\beta^2$.

On the other hand, by using (3.3) to the second term of (4.5) for $X=(AN)^T$, we have

(4.9)
$$\begin{aligned} g(A\nabla_\xi(AN)^T,\xi) =& g(q(\xi)\xi - S\xi + \alpha g(AN,N)A\xi,\xi) \\ =& q(\xi)-\alpha-\alpha\beta^2, \end{aligned}$$

where we have used $A^2=I$ and $g(AN,N)=-g(A\xi,\xi)=-\beta$.

Then by putting $X=(AN)^T$ in (4.5) and using (4.8) and (4.9), we have

(4.10)
$$\begin{aligned} 0 =& g(A\phi S(AN)^T,\xi) + g(A\nabla_\xi(AN)^T,\xi) - (q(\xi)-2\alpha)g((AN)^T,(AN)^T) \\ =& -\alpha(1-\beta^2) + q(\xi)-\alpha-\alpha\beta^2 - (q(\xi)-2\alpha)(1-\beta^2) \\ =& (q(\xi)-2\alpha)\beta^2, \end{aligned}$$

where we have used $g(A\phi S(AN)^T,\xi) = g(S(AN)^T,(AN)^T) = -\alpha(1-\beta^2)$. Here we note that $\xi\beta=0$, because we can calculate the following

$$\begin{aligned} \xi\beta =& \xi g(A\xi,\xi) \\ =& g((\bar\nabla_\xi A)\xi + A\bar\nabla_\xi\xi,\xi) + g(A\xi,\bar\nabla_\xi\xi) \\ =& g(q(\xi)JA\xi,\xi) \\ =& -q(\xi)g(A\xi,N) \\ =& 0. \end{aligned}$$

Now we consider an open subset $\mathcal{U}=\{p\in M|\beta(p)\neq 0\}$ in M. Then by (4.10), we have

Lemma 4.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$. Then*

$$q(\xi)=2\alpha$$

holds on $\mathcal{U}$ on M in Q^{m*}.

Now hereafter unless otherwise stated, on such an open subset $\mathcal{U}$ let us prove that the unit vector field N in the complex hyperbolic quadric Q^{m*} is $\mathfrak{A}$-principal. Then by Lemma 4.1 and (4.4), we have the following for any tangent vector field X on M

$$g(A\phi SX,\xi) + g(A\nabla_\xi X,\xi) = 0.$$

From this, by putting $X = A\xi$ and using $g(A\xi, A\xi) = 1$, we know that

$$0 = g(A\phi SA\xi, \xi) = g(SA\xi, (AN)^T). \tag{4.11}$$

Moreover, for any $X \perp A\xi$ the second term in the left side of the above equation becomes

$$g(A\nabla_\xi X, \xi) = -g(X, \nabla_\xi A\xi) = \alpha g((AN)^T, X),$$

where in the third equality we have used Lemma 4.1. Consequently, for any tangent vector field $X \perp A\xi$ we conclude

$$\begin{aligned} 0 =& g(A\phi SX, \xi) + g(A\nabla_\xi X, \xi) \\ =& g(X, S(AN)^T) + \alpha g((AN)^T, X) \\ =& g(S(AN)^T + \alpha(AN)^T, X). \end{aligned}$$

Moreover, by (4.11) we also know that

$$g(S(AN)^T + \alpha(AN)^T, A\xi) = 0.$$

So these two equations give the following

Lemma 4.2. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then*

$$S(AN)^T = -\alpha(AN)^T$$

holds on $\mathcal{U}$ on M in Q^{m*}.

Now let us differentiate the equation in Lemma 4.2. Then it follows that

$$(\nabla_X S)(AN)^T + S\nabla_X(AN)^T = -(X\alpha)(AN)^T - \alpha\nabla_X(AN)^T.$$

From this, by taking the inner product with the Reeb vector field ξ and using the formulas (3.3), we have

$$\begin{aligned} 0 =& g((AN)^T, (\nabla_X S)\xi) \\ &+ 2\alpha g(q(X)JAN - ASX - g(q(X)JAN - ASX, N)N, \xi) \\ &+ 2\alpha g(AN, N)g(SX, \xi) \\ =& g((AN)^T, \alpha\phi SX - S\phi SX) \\ &+ 2\alpha\{q(X)g(A\xi, \xi) - g(SX, A\xi) + g(AN, N)g(SX, \xi)\}. \end{aligned}$$

Then by putting $X = (AN)^T$ and using Lemma 4.2, we have $\alpha q((AN)^T) = 0$. When the function $\alpha = 0$, in section 3, $\beta g(Y, AN) = 0$ for any tangent vector field Y on M. Then on the open subset $\mathcal{U} = \{p \in M \,|\, \beta(p) \neq 0\}$ in M we conclude

Lemma 4.3. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then either*

$$q((AN)^T) = 0$$

or the unit normal vector field N is $\mathfrak{A}$-principal.

On the other hand, by putting $X = \xi$ in (3.3) and using Lemma 4.1, we have

$$\begin{aligned}\nabla_\xi (AN)^T =&(q(\xi) - \alpha)A\xi + \alpha g(AN, N)\xi \\ =&\alpha(A\xi - \beta\xi).\end{aligned} \tag{4.12}$$

Differentiating the equation in Lemma 4.2 along the Reeb direction ξ and using (4.12) implies

$$\begin{aligned}(\nabla_\xi S)(AN)^T =& - S\nabla_\xi (AN)^T - (\xi\alpha)(AN)^T - \alpha\nabla_\xi (AN)^T \\ =& - \alpha(SA\xi - \alpha\beta\xi) - (\xi\alpha)(AN)^T - \alpha^2(A\xi - \beta\xi).\end{aligned} \tag{4.13}$$

Moreover, differentiating $S\xi = \alpha\xi$ and using Lemma 4.2, we get the following

$$\begin{aligned}(\nabla_{(AN)^T} S)\xi =&\{(AN)^T\alpha\}\xi + \alpha\phi S(AN)^T - S\phi S(AN)^T \\ =&\{(AN)^T\alpha\}\xi - \alpha^2\phi(AN)^T + \alpha S\phi(AN)^T.\end{aligned} \tag{4.14}$$

Then substracting (4.14) from (4.13) and Lemma 4.2 give

$$\begin{aligned}g((\nabla_\xi S)(AN)^T - (\nabla_{(AN)^T} S)\xi, (AN)^T) = -(\xi\alpha)(1 - \beta^2) \\ = - g(\phi(AN)^T, (AN)^T) - g(\xi, A\xi)g(JA(AN)^T, (AN)^T) \\ =0,\end{aligned} \tag{4.15}$$

where in the second equality we have used the equation of Codazzi (3.1) in section 3. Then it follows that

$$\xi\alpha = 0 \quad \text{or} \quad \beta^2 = 1.$$

When the latter part $\beta = \pm 1$ occurs on $\mathcal{U}$, then $AN = \pm N$. So we know that the unit normal vector filed N is $\mathfrak{A}$-principal. When $\xi\alpha = 0$, if we use the derivative formula $Y\alpha$ and $g(\xi, AN) = 0$ in section 3, we have the following

Lemma 4.4. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then either*

$$grad\ \alpha = 2\beta(AN)^T$$

or the unit normal vector field N is $\mathfrak{A}$-principal.

Now let us consider the first formula in Lemma 4.4. Then by differentiating the above formula it follows that

$$\begin{aligned}\nabla_X \text{grad}\ \alpha =&2(X\beta)(AN)^T + 2\beta\nabla_X (AN)^T \\ =&4g(A\phi SX, \xi)(AN)^T + 2\beta\{q(X)JAN - ASX \\ &- g(q(X)JAN - ASX, N)N + g(AN, N)SX\}.\end{aligned} \tag{4.16}$$

Then we have

$$\begin{aligned}g(\nabla_X \text{grad}\ \alpha, Y) =&4g(A\phi SX, \xi)g((AN)^T, Y) + 2\beta\{q(X)g(JAN, Y) - g(ASX, Y)\} \\ &+ 2\beta g(AN, N)g(SX, Y).\end{aligned} \tag{4.17}$$

Since $g(\nabla_X \text{grad}\ \alpha, Y) = g(\nabla_Y \text{grad}\ \alpha, X)$ and Lemma 4.2, we have

$$0 = 2\beta\{q(X)g(JAN,Y) - q(Y)g(JAN,X)\} - 2\beta\{g(ASX,Y) - g(ASY,X)\}. \tag{4.18}$$

So on the open subset $\mathcal{U} = \{p\in M\,|\,\beta(p)\neq 0\}$ in M it follows that

$$q(X)g(JAN,Y) - q(Y)g(JAN,X) = g(ASX,Y) - g(ASY,X).$$

From this, by putting $X = \xi$, we know that

$$SA\xi = -\alpha A\xi + \beta \text{grad}\, q.$$

Then differentiating this formula gives

$$(\nabla_X S)A\xi + S\nabla_X A\xi = -(X\alpha)A\xi - \alpha\nabla_X A\xi + (X\beta)\text{grad}\, q + \beta\nabla_X \text{grad}\, q. \tag{4.19}$$

First let us take the inner product of (4.19) with Y and make the skew-symmetric part with respect X and Y. Next we use $g(\nabla_X \text{grad}\, q, Y) = g(\nabla_Y \text{grad}\, q, X)$ to the obtained equation. Then finally by putting $X = \xi$, we have

$$\begin{aligned} &g((\nabla_\xi S)A\xi, Y) - g((\nabla_Y S)A\xi, \xi) + g(S(\nabla_\xi A\xi), Y) - g(S(\nabla_Y A\xi), \xi)\\ &\quad = -(\xi\alpha)g(A\xi, Y) + (Y\alpha)g(A\xi, \xi)\\ &\qquad - \alpha\{g(\nabla_\xi A\xi, Y) - g(\nabla_Y A\xi, \xi)\} + (\xi\beta)q(Y) - (Y\beta)q(\xi). \end{aligned} \tag{4.20}$$

In this equation (4.20), we want to use the following formulas

$$q(\xi) = 2\alpha, \quad \xi\alpha = 0, \quad \xi\beta = 0,$$

$$\begin{aligned} \nabla_\xi(A\xi) &= 2\alpha JA\xi + \alpha AN - \{2\alpha g(JA\xi, N) + \alpha g(AN, N)\}N\\ &= -\alpha AN - \alpha\beta N\\ &= -\alpha(AN)^T, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} g(\nabla_Y(A\xi), \xi) &= q(Y)g(JA\xi, \xi) + g(A\phi SY, \xi)\\ &= g(SY, AN) = -\alpha g((AN)^T, Y). \end{aligned} \tag{4.22}$$

Then by the help of (4.21) and (4.22), the equation (4.20) can be reformed as

$$\begin{aligned} &g((\nabla_\xi S)A\xi, Y) - g((\nabla_Y S)A\xi, \xi) + 2\alpha^2 g((AN)^T, Y)\\ &\quad = (Y\alpha)\beta - 2\alpha(Y\beta). \end{aligned} \tag{4.23}$$

On the other hand, if we use the equation of Codazzi (3.1) in the first term of (4.23), we have

$$\begin{aligned} g((\nabla_\xi S)A\xi, Y) &= g((\nabla_\xi S)Y, A\xi) = g((\nabla_Y S)\xi, A\xi)\\ &\quad - g(\phi Y, A\xi) + g(Y, AN)g(A\xi, A\xi) - g(\xi, A\xi)g(JAY, A\xi). \end{aligned} \tag{4.24}$$

Then substituting (4.24) into the first term of (4.23) gives

$$\begin{aligned} &-g(\phi Y, A\xi) + g(Y, AN)g(A\xi, A\xi) - g(\xi, A\xi)g(JAY, A\xi) + 2\alpha^2 g((AN)^T, Y)\\ &\quad = (Y\alpha)\beta - 2\alpha(Y\beta) \end{aligned} \tag{4.25}$$

$$=2\beta^2 g(Y, AN) + 4\alpha^2 g(Y, (AN)^T),$$

where in the second equality we have used $\xi\alpha = 0$ in (3.2) of section 3, Lemma 4.2 and (3.8) in the following formula

$$\begin{aligned} Y\beta &= 2g(A\phi SY, \xi) = 2g(SY, AJ\xi) \\ &= 2g(SY, (AN)^T) = -2\alpha g(Y, (AN)^T). \end{aligned}$$

In (4.25) the first two terms of the left side cancelled out each other and the third term vanishes identically. The fourth term $2\alpha^2 g((AN)^T, Y)$ can be deleted with the second term in the right side of (4.25). So (4.25) implies $2(\alpha^2 + \beta^2)g(Y, AN) = 0$ for any tangent vector field Y on M, which means that on the open subset $\mathcal{U} = \{p \in M \,|\, \beta(p) \neq 0\}$ the unit normal vector field N is $\mathfrak{A}$-principal $AN = g(AN, N)N$.

Summing up the above discussions, we can prove our Main Theorem 1 in the introduction.

By virtue of Main Theorem 1, we can distinguish two classes of real hypersurfaces in the complex hyperbolic quadric Q^{m*} with Killing structure Jacobi operator : those that have $\mathfrak{A}$-principal unit normal, and those that have $\mathfrak{A}$-isotropic unit normal vector field N. We treat the respective cases in sections 5 and 6.

5. Killing structure Jacobi operator with $\mathfrak{A}$-principal normal

In this section we consider a real hypersurface M in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal unit normal vector field. Then the unit normal vector field N satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$. Naturally, we have also the following

$$A\xi = -\xi, \quad \text{and} \quad JA\xi = -J\xi = -N.$$

Then the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = -X + 2\eta(X)\xi + AX + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \tag{5.1}$$

Since we assume that M is Hopf, (5.1) becomes

$$R_\xi(X) = -X + 2\eta(X)\xi + AX + \alpha SX - \alpha^2\eta(X)\xi. \tag{5.2}$$

By the assumption of the Killing structure Jacobi operator R_ξ, the derivative of R_ξ along any tangent vector field Y on M is given by

$$\begin{aligned} (\nabla_Y R_\xi)(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= 2\{(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi\} + (\nabla_Y A)X + (Y\alpha)SX \\ &\qquad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\ &\qquad - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \end{aligned} \tag{5.3}$$

We can write

$$AY = BY + \rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y) = g(AY, N) = g(Y, AN) = g(Y, N) = 0$. So for any tangent vector field Y on M the vector field $AY(= BY)$ also becomes a tangent vector field on M in Q^{m*}. Then it follows

$$\begin{aligned}(\nabla_Y A)X =&\nabla_Y(AX) - A\nabla_Y X\\ =&\bar{\nabla}_Y(AX) - \sigma(Y, AX) - A\nabla_Y X\\ =&(\bar{\nabla}_Y A)X + A\bar{\nabla}_Y X - \sigma(Y, AX) - A\nabla_Y X\\ =&q(Y)JAX + A\sigma(Y, X) - \sigma(Y, AX)\\ =&q(Y)JAX + g(SX, Y)AN - g(SY, AX)N,\end{aligned} \tag{5.4}$$

where we have used the equation of Gauss in the second equality and the Weingarten formula in the fifth equality. From this, together with (5.3) and using that $\mathfrak{A}$-principal, the Killing structure Jacobi operator gives

$$\begin{aligned}0 =&(\nabla_Y R_\xi)(X) + (\nabla_X R_\xi)(Y)\\ =&(2+\alpha^2)\{(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi\}\\ &+ (2+\alpha^2)\{(\nabla_X\eta)(Y)\xi + \eta(Y)\nabla_X\xi\}\\ &+ \{q(Y)JAX + g(SX, Y)N - g(SY, AX)N\}\\ &+ \{q(X)JAY + g(SY, X)N - g(SX, AY)N\}\\ &+ (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi\\ &+ (X\alpha)SY + \alpha(\nabla_X S)Y - (X\alpha^2)\eta(Y)\xi.\end{aligned} \tag{5.5}$$

From this, taking the inner product of (5.5) with the Reeb vector field ξ, and using the constancy of the Reeb function α in Lemma 3.2, we have

$$\begin{aligned}0 =&(2+\alpha^2)\{g(\phi SY, X) + g(\phi SX, Y)\} + \alpha g((\nabla_Y S)X + (\nabla_X S)Y, \xi)\\ =&2g((\phi S - S\phi)Y, X)\end{aligned} \tag{5.6}$$

where we have used $g(JAX, \xi) = -g(AX, N) = -g(X, AN) = -g(X, N) = 0$ for any tangent vector field X on M in Q^{m*} and $(\nabla_X S)\xi = \alpha\phi SX - S\phi SX$. The formula (5.6) means that the shape operator S commutes with the structure tensor ϕ. Then by Theorem A in the introduction, M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular. That is, the Reeb flow on M is isometric.

On the other hand, we want to introduce the following proposition (see Suh [34]).

Proposition 5.1. *Let M be a real hypersurface in Q^{m*}, $m \geq 3$, with isometric Reeb flow. Then the unit normal vector field N is $\mathfrak{A}$-isotropic everywhere.*

By Proposition 5.1, we know that the unit normal vector field N of M is $\mathfrak{A}$-isotropic, not $\mathfrak{A}$-principal. This rules out the existence of an $\mathfrak{A}$-principal unit normal N together with Killing structure Jacobi operator. So we give the proof of our Main Theorem 2 with $\mathfrak{A}$-principal unit normal N.

6. Killing structure Jacobi operator with $\mathfrak{A}$-isotropic normal

In this section we assume that the unit normal vector field N is $\mathfrak{A}$-isotropic and the Reeb function $\alpha = g(S\xi, \xi)$ is constant along the Reeb direction ξ, that is, $\xi\alpha = 0$. Then the normal vector field N can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a (+1)-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2),\ AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2),\ \text{and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

By virtue of these formulas for $\mathfrak{A}$-isotropic unit normal, the structure Jacobi operator can be given as follows:

$$\begin{aligned} R_\xi(X) =& R(X, \xi)\xi \\ =& - X + \eta(X)\xi + g(AX, \xi)A\xi + g(JAX, \xi)JA\xi \\ & + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \end{aligned} \tag{6.1}$$

On the other hand, we know that $JA\xi = -JAJN = AJ^2N = -AN$, and $g(JAX, \xi) = -g(AX, J\xi) = -g(AX, N)$. Then the structure Jacobi operator R_ξ can be rearranged as follows:

$$\begin{aligned} R_\xi(X) =& - X + \eta(X)\xi + g(AX, \xi)A\xi + g(X, AN)AN \\ & + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned} \tag{6.2}$$

Then by differentiating (6.2), we obtain

$$\begin{aligned} \nabla_Y R_\xi(X) =& \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ =& (\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi + g(X, \nabla_Y(A\xi))A\xi \\ & + g(X, A\xi)\nabla_Y(A\xi) + g(X, \nabla_Y(AN))AN + g(X, AN)\nabla_Y(AN) \\ & + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\ & - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \end{aligned} \tag{6.3}$$

Here let us consider the equation of Gauss. It is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

for any vector fields X and Y on M in Q^{m*}, where $\nabla_X Y = (\bar{\nabla}_X Y)^T$ and $\sigma(X, Y)$ respectively denote the tangential and normal component on T_zM of $\bar{\nabla}_X Y$ in T_zQ^{m*}, $z \in M$. The Weingarten formula is given by

$$\bar{\nabla}_X N = -SX$$

for an $\mathfrak{A}$-isotropic unit normal vector field N. Here S denotes the shape operator of M in the complex hyperbolic quadric Q^{m*} derived from the unit normal N. Then by using these two equations to some terms in (6.3), we have the following :

$$\begin{aligned}\nabla_Y(A\xi) =&\bar{\nabla}_Y(A\xi) - \sigma(Y, A\xi)\\ =&(\bar{\nabla}_Y A)\xi + A\bar{\nabla}_Y\xi - \sigma(Y, A\xi)\\ =&q(Y)JA\xi + A\{\phi SY + \eta(SY)N\} - g(SY, A\xi)N\end{aligned}$$

and

$$\begin{aligned}\nabla_Y(AN) =&\bar{\nabla}_Y(AN) - \sigma(Y, AN)\\ =&(\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N - \sigma(Y, AN)\\ =&q(Y)JAN - ASY - g(SY, AN)N.\end{aligned}$$

Substituting these formulas into (6.3) and using the assumption of Killing structure Jacobi operator, we have

$$\begin{aligned}0 =&\nabla_Y R_\xi(X) + \nabla_X R_\xi(Y)\\ =&g(\phi SY, X)\xi + \eta(X)\phi SY\\ &+ g(\phi SX, Y)\xi + \eta(Y)\phi SX\\ &+ \{q(Y)g(A\xi, X) + g(A\phi SY, X) + g(SY, \xi)g(AN, X)\}A\xi\\ &+ \{q(X)g(A\xi, Y) + g(A\phi SX, Y) + g(SX, \xi)g(AN, Y)\}A\xi\\ &+ g(X, A\xi)\{q(Y)JA\xi + A\phi SY + g(SY, \xi)AN - g(SY, A\xi)N\}\\ &+ g(Y, A\xi)\{q(X)JA\xi + A\phi SX + g(SX, \xi)AN - g(SX, A\xi)N\}\\ &+ \{q(Y)g(X, AN) - g(X, ASY)\}AN\\ &+ \{q(X)g(Y, AN) - g(Y, ASX)\}AN\\ &+ g(X, AN)\{q(Y)JAN - ASY - g(SY, AN)N\}\\ &+ g(Y, AN)\{q(X)JAN - ASX - g(SX, AN)N\}\\ &+ (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi\\ &+ (X\alpha)SY + \alpha(\nabla_X S)Y - (X\alpha^2)\eta(Y)\xi\\ &- \alpha^2 g(\phi SY, X)\xi - \alpha^2\eta(X)\phi SY\\ &- \alpha^2 g(\phi SX, Y)\xi - \alpha^2\eta(Y)\phi SX.\end{aligned} \tag{6.4}$$

Taking the inner product of (6.4) with the unit normal N and using the properties of $\mathfrak{A}$-isotropic, that is, $g(A\xi, \xi) = 0$, $g(AN, N) = 0$, it follows that

$$\begin{aligned}0 =&g(X, A\xi)g(A\phi SY, N) - g(X, A\xi)g(SY, A\xi)\\ &+ g(Y, A\xi)g(A\phi SX, N) - g(Y, A\xi)g(SX, A\xi)\\ &- g(X, AN)g(ASY, N) - g(X, AN)g(SY, AN)\\ &- g(Y, AN)g(ASX, N) - g(Y, AN)g(SX, AN).\end{aligned} \tag{6.5}$$

From this, putting $X = AN$ and using that N is $\mathfrak{A}$-isotropic and $A\xi = \phi AN$, we have

$$0 = -2g(ASY, N) - 2g(Y, AN)g(SAN, AN) + 2g(Y, A\xi)g(A\phi SAN, N).$$

By putting $Y = AN$, we get $g(SAN, AN) = 0$. Then the above equation reduces to

$$g(ASY, N) = g(Y, A\xi)g(A\phi SAN, N).$$

So it follows that

$$\begin{aligned} SAN =& g(A\phi SAN, N)A\xi \\ =& -g(SAN, \phi AN)A\xi \\ =& -g(SAN, A\xi)A\xi, \end{aligned}$$

where we have used $A\xi = \phi AN$. Then this gives that $g(SAN, A\xi) = 0$, which implies

$$SAN = 0 \quad \text{and} \quad S\phi A\xi = 0. \tag{6.6}$$

Then (6.5) reduces to the following

$$\begin{aligned} 0 =& g(X, A\xi)g(A\phi SY, N) - g(X, A\xi)g(SY, A\xi) \\ &+ g(Y, A\xi)g(A\phi SX, N) - g(Y, A\xi)g(SX, A\xi). \end{aligned} \tag{6.7}$$

By putting $X = A\xi$ in (6.7) and using $A\xi = \phi AN$, it follows that

$$g(SY, A\xi) + g(Y, A\xi)g(SA\xi, A\xi) = 0$$

for any vector field Y on M in Q^{m*}. This gives

$$SA\xi = -g(SA\xi, A\xi)A\xi.$$

Then by taking the inner product with $A\xi$, we know $g(SA\xi, A\xi) = 0$. From this, together with the above equation, we have

$$SA\xi = 0 \quad \text{and} \quad S\phi AN = 0. \tag{6.8}$$

Putting $X = \xi$ into (6.4), and using (6.8) and the $\mathfrak{A}$-isotropic property $g(A\xi, \xi) = 0$, we have

$$\begin{aligned} 0 =& \phi SY + \{q(\xi)g(A\xi, Y) + \alpha g(AN, Y)\}A\xi \\ &+ g(Y, A\xi)\{q(\xi)A\xi + \alpha AN - g(S\xi, A\xi)N\} \\ &+ \{q(\xi)g(Y, AN) - \alpha g(Y, A\xi)\}AN + g(Y, AN)\{q(\xi)AN - \alpha A\xi\} \\ &+ (Y\alpha)\alpha\xi + \alpha(\nabla_Y S)\xi - (Y\alpha^2)\xi - \alpha^2\phi SY \\ &+ (\xi\alpha)SY + \alpha(\nabla_\xi S)Y - (\xi\alpha^2)\eta(Y)\xi \\ =& \phi SY + 2q(\xi)g(A\xi, Y)A\xi + 2q(\xi)g(Y, AN)AN \\ &- \alpha S\phi SY + (\xi\alpha)SY - (\xi\alpha^2)\eta(Y)\xi + \alpha(\nabla_\xi S)Y. \end{aligned} \tag{6.9}$$

On the other hand, $SA\xi = 0$ implies $(\nabla_\xi S)A\xi + S\nabla_\xi(A\xi) = 0$. By the equation of Gauss, the following holds

$$\begin{aligned} \nabla_\xi(A\xi) =& \bar{\nabla}_\xi(A\xi) - \sigma(\xi, A\xi) \\ =& q(\xi)JA\xi + g(S\xi, \xi)AN - g(S\xi, A\xi)N \\ =& q(\xi)JA\xi + \alpha AN. \end{aligned}$$

This gives $S(\nabla_\xi(A\xi)) = q(\xi)SJA\xi + \alpha SAN = 0$ from (6.6). From this, together with the above formula, we have

$$(\nabla_\xi S)A\xi = 0. \tag{6.10}$$

By taking the inner product of (6.9) with $A\xi$ and AN respectively, and using (6.6), (6.8) and (6.10), we know that

$$q(\xi)A\xi = 0 \quad \text{and} \quad q(\xi)AN = 0.$$

By virtue of these formulas, (6.9) reduces to the following

$$0 = \phi SY - \alpha S\phi SY + (\xi\alpha)SY - (\xi\alpha^2)\eta(Y)\xi + \alpha(\nabla_\xi S)Y. \tag{6.11}$$

On the other hand, by using the equation of Codazzi, we have

$$\begin{aligned}(\nabla_\xi S)Y =& (\nabla_Y S)\xi - \phi Y + g(AN, Y)A\xi + g(Y, A\xi)\phi A\xi \\ =& (Y\alpha)\xi + \alpha\phi SY - S\phi SY - \phi Y \\ &+ g(AN, Y)A\xi + g(Y, A\xi)\phi A\xi.\end{aligned}$$

Then by the properties of M being Hopf and with $\mathfrak{A}$-isotropic unit normal vector field, we have

$$Y\alpha = g((\nabla_\xi S)Y, \xi) = g((\nabla_\xi S)\xi, Y) = (\xi\alpha)\eta(Y).$$

From this, together with the assumption of $\xi\alpha = 0$ in section 6, it follows that the Reeb function α is constant for a real hypersurface in Q^{m*} with $\mathfrak{A}$-isotropic unit normal. Then the derivative of the shape operator S along the Reeb direction ξ is given by

$$\begin{aligned}-\alpha(\nabla_\xi S)Y =& -\alpha^2\phi SY + \alpha S\phi SY \\ &+ \alpha\phi Y - \alpha g(AN, Y)A\xi - \alpha g(Y, A\xi)\phi A\xi.\end{aligned}$$

From this, by (6.11) and using the constancy of the Reeb function α, we know that

$$\begin{aligned}0 =& \phi SY - 2\alpha S\phi SY + \alpha^2\phi SY \\ &- \alpha\phi Y + \alpha g(AN, Y)A\xi + \alpha g(Y, A\xi)\phi A\xi.\end{aligned} \tag{6.12}$$

Then for any $Y\in\mathcal{Q}$ such that $SY = \lambda Y$, where Y is orthogonal to the vector fields $A\xi$ and AN, (6.12) reduces to the following

$$2\alpha\lambda S\phi Y = (\lambda\alpha^2 - \alpha + \lambda)\phi Y. \tag{6.13}$$

Then (6.13) gives $\alpha\neq 0$.

In fact, if the Reeb function $\alpha = 0$, from (6.13) it follows that $\lambda = 0$. From this, together with (6.6) and (6.8), the shape operator S becomes identically vanishing. That is, M is totally geodesic. Then by the equation of Codazzi in section 3, we have a contradiction.

Naturally we should have $2\alpha\lambda\neq 0$. If the function $\lambda = 0$, then (6.13) also implies that the Reeb function α vanishes. So also the contradiction appears. This fact gives

$$S\phi Y = \frac{\alpha\lambda - 2}{2\lambda - \alpha}\phi Y = \frac{\alpha^2\lambda - \alpha + \lambda}{2\alpha\lambda}\phi Y.$$

It can be written as follows:

$$2\lambda^2 + \alpha(1 - \alpha^2)\lambda + \alpha^2 = 0. \tag{6.14}$$

Then the discriminant of (6.14) is given by

$$D = \alpha^2(1 - \alpha^2)^2 - 8\alpha^2 = \alpha^2\{(\alpha^2 - 1)^2 - 8\}.$$

Then the solution has two roots as follows:

$$\lambda = \frac{-\alpha(1-\alpha^2)\pm\alpha\sqrt{(\alpha^2-1-2\sqrt{2})(\alpha^2-1+2\sqrt{2})}}{4}.$$

When $\alpha^2 > 2\sqrt{2}+1$, we have two distinct roots λ_1 and λ_2 of the equation (6.14).

Now let us consider the case that $\alpha^2 = 2\sqrt{2}+1$. Then we may put $\alpha = \sqrt{2\sqrt{2}+1}$. In this case we have

$$\lambda_1 = \lambda_2 = \frac{-\alpha(1-\alpha^2)}{4} = -\sqrt{\sqrt{2}+\frac{1}{2}}.$$

Here let us put $\delta = -\sqrt{\sqrt{2}+\frac{1}{2}}$. Then the shape operator S has three distinct constant principal curvatures such that

$$\alpha = \sqrt{2\sqrt{2}+1}, \quad \beta = \gamma = 0, \quad \text{and} \quad \delta = -\sqrt{\sqrt{2}+\frac{1}{2}} = -\sqrt{\frac{2\sqrt{2}+1}{2}}.$$

The corresponding eigen spaces are given by $\xi\in T_0$, $A\xi, AN\in T_\beta = \mathcal{Q}^\perp$ and $T_\delta = \mathcal{Q}$ with multiplicities 1, 2 and $2m-4$ respectively.

On the other hand, on the distribution $\mathcal{Q}$ let us introduce an important formula mentioned in section 3 as follows:

$$2S\phi SY - \alpha(\phi S + S\phi)Y = -2\phi Y \tag{6.15}$$

for any tangent vector field Y on M in Q^m (see also [29] , pages 1350050-11). So if $SY = \delta Y$ in (6.15), then $(2\delta-\alpha)S\phi Y = (\alpha\delta-2)\phi Y$, which gives

$$S\phi Y = \frac{\alpha\delta-2}{2\delta-\alpha}\phi Y, \tag{6.16}$$

because if $2\delta-\alpha = 0$, then $\alpha\delta-2 = 0$. This implies $\alpha^2 = 4$, then $\alpha = 2$ and $\delta = 1$. In this case M is locally congruent to a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.

On the other hand, let us consider $S\phi Y = \delta\phi Y$ for $2\delta\neq\alpha$, because $T_\delta = \mathcal{Q}$. From this, together with the above equation, we have

$$\delta^2 - \alpha\delta + 1 = 0.$$

Then $\delta^2+1 = \sqrt{2}+\frac{3}{2}$. But $\delta^2+1 = \alpha\delta = -\sqrt{2\sqrt{2}+1}\sqrt{\frac{2\sqrt{2}+1}{2}} = -\frac{\sqrt{2}}{2}-2$. This gives a contradiction. So this case can not be happened.

Accordingly, the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_2 \end{bmatrix}$$

where the principal curvatures are constants and are given by

$$\lambda_1 = \frac{\alpha(\alpha^2-1) + \alpha\sqrt{(\alpha^2-1-2\sqrt{2})(\alpha^2-1+2\sqrt{2})}}{4}$$

and respectively

$$\lambda_2 = \frac{\alpha(\alpha^2-1) - \alpha\sqrt{(\alpha^2-1-2\sqrt{2})(\alpha^2-1+2\sqrt{2})}}{4}.$$

By virtue of Remark below, we note that the horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular can not be appeared. Then we give a complete proof of our Main Theorem 3.

Remark 6.1. Let us check that a tube around the totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular. Then by Theorem A in the introduction, the tube has a commuting shape operator, that is, $S\phi = \phi S$ and the unit normal N is $\mathfrak{A}$-isotropic and the Reeb curvature $\alpha = g(S\xi, \xi)$ is constant (see Suh [34]). By the $\mathfrak{A}$-isotropic unit normal, the properties $g(A\xi, \xi) = 0$ and $g(AN, N) = 0$ hold on M. Moreover from the expression of this tube we know that $SA\xi = 0$ and $SAN = 0$, by differentiating we also confirm that $(\nabla_\xi S)A\xi = 0$ and $(\nabla_\xi S)AN = 0$.

Now we assume that the tube admits a Killing structure Jacobi operator. Then by the same process as in the proof of our Main Theorem 2, the principal curvature of the tube should satisfies (6.14), that is,

$$2\lambda^2 + \alpha(1-\alpha^2)\lambda + \alpha^2 = 0.$$

Then two roots $\coth r$ and $\tanh r$ of the tube should satisfy $1 = \lambda\mu = \coth r \cdot \tanh r = \frac{\alpha^2}{2}$. Then $2 = \alpha^2 = \coth^2 r + \tanh^2 r + 2$ implies $\coth^2 r + \tanh^2 r = 0$. This makes a contradiction. So the tube does not admit a Killing structure Jacobi operator. Then naturally the tube around the totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or the horosphere does not have a parallel structure Jacobi operator, which is more strong condition than Killing structure Jacobi operator.

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