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EXPONENTIAL CONCENTRATION IN TERMS OF GROMOV–LEDOUX'S EXPANSION COEFFICIENTS ON A METRIC MEASURE SPACE AND ITS UPPER DIAMETER BOUND SATISFYING VOLUME DOUBLING

Dedicated to Dr. Naoto Abe on his Koki (70*th birthday*)

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Abstract

To investigate a concentration of measure phenomena on metric measure spaces in terms of Gromov–Ledoux's expansion coefficients on this space as well as Ledoux's per se, we studied a concentration function in concert with their expansion coefficients. Further investigation into an exponential concentration in terms of Ledoux's expansion coefficient on a bounded and volume doubling metric measure space enables us to derive an upper bound for its diameter in terms of both the Ledoux's expansion coefficient and doubling constant, provided that Ledoux's expansion coefficient > 1. In this study, we let Ledoux's expansion coefficient > 1 on a metric measure space, which is ensured by adopting Poincaré inequality. We demonstrated that on a metric measure space, Gromov–Ledoux's expansion coefficients with Ledoux's expansion coefficient > 1 give rise to an exponential concentration in terms of themselves. We further showed that on a bounded and volume doubling metric measure space, a Ledoux's expansion coefficient of order bounded from above in terms of both the doubling constant > 1 and its diameter is bounded from above in terms of the doubling constant per se. We applied this upper diameter bound to a closed smooth Riemannian manifold with non-negative Ricci curvature. This upper bound is described in terms of both the spectral gap and dimension.

Contents

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1. Introduction

First, we briefly review relevant concepts used herein; in this study, we call a metric space endowed with a Borel probability measure a metric measure space. To oscillate this measure dynamically makes us capable of grasping the structure of a metric measure space. On a metric measure space, this dynamic association is exactly a concentration of measure phenomenon; in fact, pertinence to this appellation of a concentration of measure phenomenon is attributed to a concentration inequality for metric measure spaces (see Proposition 2.2). Specifically, a concentration of measure phenomenon on metric measure spaces indicates the behaviour of an enlargement (see Subsection 2.2) with respect to which concentration of measure is evaluated; furthermore, letting an enlargement or the dimension of metric measure spaces high, we observe a rapid decrease in concentration function.

1.1. Background. We briefly recall a concentration of measure phenomenon on metric measure spaces based on Berger [6, p. 336]: 'As early as 1919 Paul Lévy studied the so-called *concentration* phenomenon of spheres S^n: most of the measure of the sphere is concentrated around an equator, and this effect becomes more pronounced as the dimension gets large'; furthermore regarding its historical aspect, a concentration of measure phenomenon was most vigorously put forward by V.D. Milman in the local theory of Banach spaces to study Dvoretzky's theorem on almost Euclidean sections of convex bodies; we refer readers to Ledoux [14] and references therein and Ledoux and Talagrand [18]. From a contemporary perspective of Chapter $3\frac{1}{2}_+$ of Gromov's 'Green Book': Gromov [10, Chapter $3\frac{1}{2}_+$]

described groundbreaking results regarding a concentration of measure phenomenon on metric measure spaces. In addition, focusing on a concentration function, whose definition was first introduced in Amir and Milman [2], Ledoux [16] presented a concentration of measure phenomenon on metric measure spaces. The present study was motivated by Ledoux [16] and Gromov [10, Chapter $3\frac{1}{2}_{+}$].

We say that a metric measure space has an exponential concentration if its concentration function decreases exponentially to zero (see Definition 2.4). In this study, we concern ourselves with an exponential concentration. Since the early 21st century, numerous previously conducted studies have been concerned with an exponential concentration; based on one of them, under which complete oriented connected smooth Riemannian manifolds are those whose Ricci curvature is non-negative, an exponential concentration provides us with some significant concepts, such as Cheeger's isoperimetric inequality and Poincaré inequality; for detailed accounts, we refer readers to Milman [21], for which readers are referred to Ledoux [16, Proposition 1.8] and Barthe [4] and references therein. As will be referred to in Subsection 4.2 subsequent, Ledoux's expansion coefficient on an expander graph may be regarded as being akin to the so-called Cheeger constant; we refer readers to Cheeger [7] for its original literature. Ledoux [16, Proposition 1.13] presented a sufficient condition for a metric measure space to have an exponential concentration; one can further see that an expander graph satisfying Cheeger's isoperimetric inequality has an exponential concentration; we refer readers to Ledoux [16, pp. 31–32] for more detailed accounts. The scenario of the proof of the abovementioned sufficient condition refers to that of Gromov and Milman [11, Theorem 4.1]; slightly refining the abovementioned Ledoux's result leads to a sufficient condition for an exponential concentration in terms of Ledoux's expansion coefficient > 1 (see Corollary 5.3).

Now, it is significant to investigate the diameter of a bounded metric measure space as well as an exponential concentration; from the perspective of a closed smooth Riemannian manifold with Ricci curvature bounded below, we refer readers to (Bonnet-)Myers's theorem and Cheng's maximal diameter theorem (see Theorem 7.2) and Ledoux's upper diameter bound theorem (see Theorem 7.3). Their crucial results on Riemannian manifolds with positive curvature motivated this study (see Section 7.2).

1.2. Objective. To investigate a concentration of measure phenomenon on metric measure spaces, much research interest has been devoted to estimate a concentration function from above, an expansion coefficient on a bounded and volume doubling metric measure space and a bound for its diameter.

To the best of our knowledge, the concept of an expansion coefficient on metric measure spaces has two proposers: one demonstrated by M. Gromov and the other demonstrated by M. Ledoux; subsequently, these two expansion coefficients will be referred to as Gromov and Ledoux's expansion coefficients, respectively. We investigated their expansion coefficients; in fact, from Section 4 onwards, we demonstrated that their expansion coefficients give rise to an exponential concentration in terms of themselves.

Onward Section 6 is devoted principally to lower and upper bounds for Ledoux's expansion coefficient on a bounded and volume doubling metric measure space. To present our results for geometric objects, such as closed smooth Riemannian manifolds with non-negative Ricci curvature, we restricted our concern to the sufficient conditions presented below for establishing our results regarding Ledoux's expansion coefficient: $1 <$ Ledoux's expansion coefficient $< \infty$; more precisely, the lower and upper bounds are described in terms of both of the constant attributed to Poincaré inequality and the doubling constant with respect to a doubling measure.

1.3. Overview of Principal Results. Before outlining our results, we briefly describe this paper. The remaining body of this paper is divided into seven sections: Sections 2–4 in which our principal results in Sections 5–7 are based.

Consider the aforementioned enlargement $\geq \varepsilon$ ($= 1/2$ in Ledoux's argument); see Definition 2.3 (Definition 2.2), respectively. In this study, an overview of the four principal results under this enlargement as per above is as follows.

1. In Theorem 5.5, we demonstrated that on a metric measure space, Gromov–Ledoux's expansion coefficients give rise to an exponential concentration in terms of themselves, provided that Ledoux's expansion coefficient > 1.
2. In Theorem 6.7, we described a sufficient condition for Ledoux's expansion coefficient on a bounded and volume doubling metric measure space to be bounded above.
3. In Theorem 7.1, we showed a novel upper bound for the diameter of a bounded and volume doubling metric measure space with Ledoux's expansion coefficient > 1.
4. In Theorem 7.4, we showed a novel upper bound for the diameter of a closed smooth Riemannian manifold with non-negative Ricci curvature.

2. A Concentration of Measure Phenomenon on Metric Measure Spaces

In this section, we formulate the concept of a concentration of measure phenomenon on metric measure spaces in terms of a concentration inequality.

2.1. Setup. We now define the concept of a metric measure space in the sense of Ledoux [16]; we also refer readers to Gromov [10] for pioneering studies regarding this field.

Definition 2.1 (Metric measure space). A *metric measure space* is a metric space (X, d_X) equipped with a Borel probability measure μ_X on X. Let a triplet (X, d_X, μ_X) denote a metric measure space, as called an mm space.

Subsequently, we use an identical letter X to denote a metric space or a metric measure space whenever no confusion can arise.

Shioya [24, p. vii] remarked that measures on metric measure spaces are not necessarily probability measures.

2.2. A Concentration Function. The concept of a concentration of measure phenomenon is attributed to an isoperimetric inequality. A concentration function indicates a concentration of measure phenomena on metric measure spaces. A concentration of measure phenomena involves two main components: a finite measure, such as a probability measure, and an enlargement with respect to which a measure concentration is evaluated. For all non-empty Borel subsets A of X and for all $r \geq 0$, let A_r denote a closed r-neighbourhood of A with respect to d_X, i.e. $A_r := \{ x \in X \mid d_X(x, A) \leq r \}$; in this study, by following an appellation of A_r due to Ledoux [16, Section 1.2], A_r is referred to as an *enlargement* of order r of A with respect to d_X.

Moreover, without referring to 'concentration', the concept of a concentration function originated from Amir and Milman [2] as follows.

DEFINITION 2.2 (VID., E.G. LEDOUX [16, p. 3]). Let X be a metric measure space.

$$\alpha_X(r) := \sup\{\, 1-\mu_X(A_r) \mid X \supset A \text{ is an arbitrary Borel set such that } \mu_X(A) \geq 1/2 \,\}, r \geq 0.$$

Subsequently, to address Gromov's expansion coefficient (see Definition 4.1), we are concerned principally with the generalisation of Definition 2.2 with respect to a lower bound for $\mu_X(A)$ as per above (see Definition 2.3).

Throughout this paper, let ε be such that $0 < \varepsilon < 1$.

DEFINITION 2.3 (CONCENTRATION FUNCTION; VID., LEDOUX [16, p. 5]). A *concentration function* for a metric measure space X, denoted by $\alpha^{\varepsilon}{}_X$, is defined as

$$\alpha^{\varepsilon}{}_X(r) := \sup\{\, 1 - \mu_X(A_r) \mid X \supset A \text{ is an arbitrary Borel set such that } \mu_X(A) \geq \varepsilon \,\}, r \geq 0. \tag{1}$$

In this study, for simplicity, we write $\alpha^{\varepsilon}{}_X$ with $\varepsilon = 1/2$ as α_X.

Proposition 2.1 enables us to compare subsequent results regarding $\alpha^{\varepsilon}{}_X$ with those of α_X:

Proposition 2.1. *We have*

$$\alpha^{\varepsilon}{}_X(r) \leq \alpha^{1-\varepsilon}{}_X(r) \quad \textit{for all } r \geq 0,$$

provided that $\varepsilon \geq 1/2$*, and vice versa.*

Proof. This claim readily follows from the definition of a concentration function (1). □

Two significant classes of metric measure spaces share the upper bounds of an exponent and a Gaussian kernel for a concentration function. In this study, we focus on an upper bound of being exponential for a concentration function as follows.

DEFINITION 2.4 (EXPONENTIAL CONCENTRATION; CF., LEDOUX [16, p. 4]). We say that a metric measure space X has an *exponential concentration* if there exist constants written as $C_i > 0, i = 1, 2$, and $r_0 \geq 0$ such that

$$\alpha^{\varepsilon}{}_X(r) \leq C_1 \exp(-C_2 r), \quad r \geq r_0. \tag{2}$$

The constraint r_0 to r in Definition 2.4 is expedient; more precisely, in Theorem 5.5, we will show an exponential concentration with a positive constraint r_0.

One says that a metric measure space has a normal concentration if an upper bound for a concentration function is given in terms of a Gaussian kernel.

We turn to estimate the diameter of a bounded metric measure space; if a metric space X is bounded, then an enlargement in the definition of a concentration function (1) ranges up to the *diameter* of X, which is denoted by diam(X), i.e.

$$\operatorname{diam}(X) := \sup\{\, d_X(x, y) \mid x, y \in X \,\}.$$

It follows readily that a concentration function tends to zero as an enlargement goes to the diameter. In Section 7, we will show an upper bound for the diameter.

2.3. A Concentration Inequality. In this subsection, we establish a concentration inequality of a Lipschitz function around its quantile with a concentration function to formalise a concentration of measure phenomena on metric measure spaces.

DEFINITION 2.5 (QUANTILE OF ORDER ε, PERCENTILE). Let f be a real-valued measurable function on a probability measure space. Define a real number m_f of f for a probability measure μ such that

$$\mu(\{ f(x) \leq m_f \}) \geq \varepsilon, \quad \mu(\{ f(x) \geq m_f \}) \geq 1 - \varepsilon. \tag{3}$$

Let f be regarded as a random variable. Then, m_f defined above is referred to as a *quantile of order* ε of f for μ or the 100εth *percentile* of f for μ; in particular, if $\varepsilon = 1/2$, then m_f coincides exactly with the so-called *Lévy mean* or *median* of f for μ. This median may not be unique.

REMARK 2.1. A median of a Lipschitz function for a canonical Gaussian probability measure on a Euclidean space is unique; we refer readers to Ledoux and Talagrand [18, p. 21] for more detailed accounts.

Up to Section 3, we concern ourselves with a Lipschitz property on a metric measure space. As we will show below, a Lipschitz property involving a Lipschitz function and its Lipschitz constant enables one to observe a concentration of measure phenomena on metric measure spaces (see Definition 2.7), which yields the concept of an observable diameter (see Section 3).

DEFINITION 2.6 ((1-)LIPSCHITZ AND LOCALLY LIPSCHITZ). A map f from a metric space (X, d_X) to a metric space (Y, d_Y) is called *Lipschitz* if

$$\sup_{x,y \in X; x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} < \infty,$$

which is referred to as a *Lipschitz constant* of f and denoted by $\|f\|_{\mathrm{Lip}}$; in particular, we say that f is 1-*Lipschitz* if its Lipschitz constant $\|f\|_{\mathrm{Lip}} \leq 1$. A map on a metric space is called *locally Lipschitz* if every point in the metric space has a neighbourhood such that its restriction to this neighbourhood is Lipschitz.

We are now ready to establish a concentration inequality.

Proposition 2.2 (Concentration inequality)**.** *Let f be a Lipschitz function on a metric measure space X and m_f its quantile of order ε for μ_X. Then,*

$$\mu_X(\{ |f(x) - m_f| > r \}) \leq \alpha^{\varepsilon}{}_X(r/\|f\|_{\mathrm{Lip}}) + \alpha^{1-\varepsilon}{}_X(r/\|f\|_{\mathrm{Lip}}) \quad \text{for all } r \geq 0; \tag{4}$$

in particular, if f is 1*-Lipschitz, then inequality* (4) *is reduced to the following:*

$$\mu_X(\{ |f(x) - m_f| > r \}) \leq \alpha^{\varepsilon}{}_X(r) + \alpha^{1-\varepsilon}{}_X(r) \quad \text{for all } r \geq 0. \tag{5}$$

Proof. Set $A := \{ f(x) \leq m_f \}$. The inequality $\mu_X(A) \geq \varepsilon$ follows from the definition of a quantile of order ε of (3). Let $r \geq 0$. We see that

$$\mu_X(A_r) \le \mu_X(\{ f(x) \le m_f + \|f\|_{\mathrm{Lip}} r \}); \tag{6}$$

indeed, take $x \in A_r$. It then follows that $f(x) \le f(a) + \|f\|_{\mathrm{Lip}} d_X(x, a)$ for all $a \in X$. By taking the infimum over $a \in A$, $f(x) \le m_f + \|f\|_{\mathrm{Lip}} r$ follows, i.e. $x \in \{ f(x) \le m_f + \|f\|_{\mathrm{Lip}} r \}$; hence, from inequality (6) we have

$$\mu_X(\{ f(x) > m_f + \|f\|_{\mathrm{Lip}} r) \}) \le 1 - \mu_X(A_r). \tag{7}$$

By combining inequality (7) with the definition of a concentration function (1), it follows that

$$\mu_X(\{ f(x) > m_f + r \}) \le \alpha^{\varepsilon}{}_X(r/\|f\|_{\mathrm{Lip}}), \tag{8}$$

which is referred to as a *deviation inequality*; for further accounts, we refer readers to Ledoux [16, p. 6] in which $\varepsilon = 1/2$.

We consider the abovementioned argument again when A is replaced with $\{ -f(x) \le -m_f \}$ to obtain $\mu_X(A_r) \le \mu_X(\{ f(x) \ge m_f - \|f\|_{\mathrm{Lip}} r \})$; the reasoning for this deviation inequality yields

$$\mu_X(\{ f(x) < m_f - r \}) \le \alpha^{1-\varepsilon}{}_X(r/\|f\|_{\mathrm{Lip}}). \tag{9}$$

By combining inequalities (8) and (9), we obtain the desired inequality (4); in particular, if $\|f\|_{\mathrm{Lip}} \le 1$, a concentration function decreases with respect to r; then, inequality (4) is reduced to inequality (5) as desired. Thus, the proposition is proven. □

Definition 2.7 (Concentration inequality and concentration of measure phenomena; cf., Ledoux [16, Section 1.3]). Inequality (4) (inequality (5)) is called a *concentration inequality* of a (1-)Lipschitz function f around its quantile of order ε for μ_X with rate $\alpha^{\varepsilon}{}_X$; that which a concentration inequality implies is just a *concentration of measure phenomena* on metric measure spaces.

3. An Observable Diameter

Before defining the concept of an observable diameter, we first defined a partial diameter; we refer readers to Gromov [10, Chapter $3\frac{1}{2}$.20] as the original literature concerning an observable diameter as well as Berger [6, pp. 336–337], Ledoux [16, Section 1.4] and Shioya [24].

Throughout this paper, let κ be such that $0 < \kappa < 1$.

3.1. A Partial Diameter.

Definition 3.1 (Partial diameter; vid., e.g. Shioya [24, Definition 2.13]). A *partial diameter* of a metric measure space X with respect to μ_X, denoted by $\mathrm{PartDiam}_{\mu_X}(X; 1-\kappa)$, is defined as the infimum of $\mathrm{diam}(A)$, where $X \supset A$ runs over all Borel subsets such that $\mu_X(A) \ge 1 - \kappa$, i.e.

$$\begin{aligned}&\mathrm{PartDiam}_{\mu_X}(X; 1-\kappa)\\ &\qquad := \inf\{ \mathrm{diam}(A) \mid X \supset A \text{ is an arbitrary Borel set such that } \mu_X(A) \ge 1 - \kappa \}.\end{aligned}$$

3.2. An Observation Device for Diameter. What is not obvious is that a partial diameter may dramatically decrease under all 1-Lipschitz maps from a metric measure space to a certain metric space; the target metric space is referred to as a *screen*. That a screen is a 1-dimensional Euclidean space $\mathbb{R}$ provides us with a more geometric view of concentration. A geometric observation device of 1-Lipschitz functions to a screen $\mathbb{R}$ yields the concept of an observable diameter defined below; more precisely, an observable diameter enables one to describe the diameter of a metric measure space viewed through a given Borel probability measure on this metric measure space.

DEFINITION 3.2 (OBSERVABLE DIAMETER; VID., E.G. GROMOV [10, Chapter $3\frac{1}{2}$.20]). An (κ-)*observable diameter* of a metric measure space X with respect to μ_X, denoted by $\mathrm{ObsDiam}(X; -\kappa)$, is defined as the supremum of $\mathrm{PartDiam}_{f_*\mu_X}(\mathbb{R}; 1-\kappa)$ over all 1-Lipschitz functions f on X, where $f_*\mu_X$ is a push-forward measure of μ_X in terms of f, i.e.

$$\mathrm{ObsDiam}(X; -\kappa) := \sup\{\, \mathrm{PartDiam}_{f_*\mu_X}(\mathbb{R}; 1-\kappa) \mid f\colon X \to \mathbb{R} \text{ is 1-Lipschitz} \,\}.$$

It follows readily from the definition of a partial diameter (Definition 3.1) that $\mathrm{PartDiam}_{\mu_X}(X; 1-\kappa)$ is monotone decreasing with respect to κ; accordingly, so is $\mathrm{ObsDiam}(X; -\kappa)$ from the definition of an observable diameter (Definition 3.2). Compared with an observable diameter and the diameter per se, it is obvious that $\mathrm{ObsDiam}(X; -\kappa) \le \mathrm{diam}(X)$.

3.3. Duality Between a Concentration Function and an Observable Diameter. In this subsection, we show that the concept of an observable diameter is dual to that of a concentration function (see Corollary 3.2 and Proposition 3.6).

Proposition 3.1. *We have*

$$\mathrm{ObsDiam}(X; -\kappa) \le 2\inf\{\, r > 0 \mid \alpha^{\varepsilon}{}_X(r) + \alpha^{1-\varepsilon}{}_X(r) \le \kappa \,\}. \tag{10}$$

Proof. Let f be a 1-Lipschitz function on X. To establish claim (10), we estimate a partial diameter of X with respect to $f_*\mu_X$. For all $r \ge 0$, we measure $\{\, |f(x) - m_f| \le r \,\}$ with respect to $f_*\mu_X$; in fact,

$$\begin{aligned} f_*\mu_X(\{\, t \in \mathbb{R} \mid |t - m_f| \le r \,\}) &= \mu_X(\{\, |f(x) - m_f| \le r \,\}) \\ &\ge 1 - (\alpha^{\varepsilon}{}_X(r) + \alpha^{1-\varepsilon}{}_X(r)), \end{aligned} \tag{11}$$

in which the last inequality sign follows from a concentration inequality of a 1-Lipschitz function (5). Put $r_{\varepsilon,\kappa} := \inf\{\, r > 0 \mid \alpha^{\varepsilon}{}_X(r) + \alpha^{1-\varepsilon}{}_X(r) \le \kappa \,\}$. It follows from inequality (11) with $r = r_{\varepsilon,\kappa}$ that

$$f_*\mu_X(\{\, t \in \mathbb{R} \mid |t - m_f| \le r_{\varepsilon,\kappa} \,\}) \ge 1 - \kappa. \tag{12}$$

It remains to evaluate the diameter of $\{\, t \in \mathbb{R} \mid |t - m_f| \le r_{\varepsilon,\kappa} \,\}$ as per above, which is given as follows:

$$\mathrm{diam}(\{\, t \in \mathbb{R} \mid |t - m_f| \le r_{\varepsilon,\kappa} \,\}) \le (m_f + r_{\varepsilon,\kappa}) - (m_f - r_{\varepsilon,\kappa}) = 2r_{\varepsilon,\kappa}. \tag{13}$$

Both the push-forward measure evaluation (12) and diameter evaluation (13) give us an upper bound for a partial diameter:

(14) $$\mathrm{PartDiam}_{f_*\mu_X}(\mathbb{R}; 1-\kappa) \le 2r_{\varepsilon,\kappa}.$$

In virtue of this upper partial diameter bound (14), we obtain the claim. Thus, the proposition is proven. □

Corollary 3.2. *Let $\varepsilon \le 1/2$. Then, we have*

(15) $$\mathrm{ObsDiam}(X; -\kappa) \le 2\inf\{\, r > 0 \mid \alpha^{\varepsilon}{}_X(r) \le \kappa/2 \,\}.$$

Proof. Combining Proposition 3.1 with Proposition 2.1, we obtain

$$\mathrm{ObsDiam}(X; -\kappa) \le 2\inf\{\, r > 0 \mid \alpha^{\varepsilon}{}_X(r) \le \kappa/2 \,\} \quad \text{if } \varepsilon \le 1/2;$$
$$\mathrm{ObsDiam}(X; -\kappa) \le 2\inf\{\, r > 0 \mid \alpha^{1-\varepsilon}{}_X(r) \le \kappa/2 \,\} \quad \text{if } \varepsilon \ge 1/2,$$

of which each is obviously just our claim of duality. □

Corollary 3.2 gives more, namely an upper observable diameter bound is made explicit if X has an exponential concentration.

Proposition 3.3. *Suppose that X has an exponential concentration with $\varepsilon \le 1/2$. Let κ be such that $2C_1 > \kappa$, where C_1 is a positive constant attributed to an exponential concentration* (2). *Then, we have*

$$\mathrm{ObsDiam}(X; -\kappa) \le \frac{2}{C_2} \ln \frac{2C_1}{\kappa}.$$

Proof. This claim readily follows from an upper observable diameter bound (15). □

The remainder of this section will be devoted to the abovementioned duality. To this end, we briefly review the concept of a separation distance.

DEFINITION 3.3 (SEPARATION DISTANCE; VID., E.G. SHIOYA [24, Definition 2.24]). A *separation distance* of a metric measure space X, denoted by $\mathrm{Sep}(X; \kappa_1, \dots, \kappa_n)$, $n \ge 2$, is defined as

$$\mathrm{Sep}(X; \kappa_1, \dots, \kappa_n)$$
$$:= \sup\{\, \min_{i \ne j} d_X(A_i, A_j) \mid X \supset A_i \text{ is each Borel set such that } \mu_X(A_i) \ge \kappa_i, i = 1, \dots, n \,\};$$

if $\kappa_i > 1$ for some i, then we define $\mathrm{Sep}(X; \kappa_1, \dots, \kappa_n) := 0$; accordingly, if $\sum_{i=1}^n \kappa_i > 1$, then $\mathrm{Sep}(X; \kappa_1, \dots, \kappa_n) = 0$.

$\mathrm{Sep}(X; \kappa_1, \dots, \kappa_n)$ is monotone decreasing with respect to each κ_i and n.
Proposition 3.4 enables us to show the abovementioned dual observation whenever $n = 2$:

Proposition 3.4 (vid., e.g. Shioya [24, Proposition 2.26])**.** *Let κ_i, $i = 1, 2$, be such that $0 < \kappa_1 < \kappa_2 < 1$. Then, we have*

(16) $$\mathrm{ObsDiam}(X; -2\kappa_2) \le \mathrm{Sep}(X; \kappa_2, \kappa_2) \le \mathrm{ObsDiam}(X; -\kappa_1).$$

We now review Shioya's result regarding the duality for a concentration function with $\varepsilon = 1/2$:

Proposition 3.5 (vid., Shioya [24, Remark 2.28 (2)]; cf., Naor et al. [23, Subsection 1.3])**.**

$$\alpha_X(r) \le \sup\{\, \kappa > 0 \mid \mathrm{ObsDiam}(X; -\kappa) \ge r \,\} \quad \textit{for all } r \ge 0.$$

The duality as per above is still true whenever $\varepsilon \ge 1/2$:

Proposition 3.6. *Let $\varepsilon \ge 1/2$. Then, we have*

$$\alpha^{\varepsilon}{}_X(r) \le \sup\{\, \kappa > 0 \mid \mathrm{ObsDiam}(X; -\kappa) \ge r \,\} \quad \textit{for all } r \ge 0.$$

Proof. For completeness, we indicate its proof, which is in a fashion similar to that of Proposition 3.5. Fix $r \ge 0$. It follows from the definition of a concentration function (1) that, for an arbitrary $\delta > 0$, there exists some Borel subset A of X such that $\mu_X(A) \ge \varepsilon$ and $\alpha^{\varepsilon}{}_X(r) - \delta < \mu_X({A_r}^c)$; hence,

$$\alpha^{\varepsilon}{}_X(r) - \delta < 1 - \varepsilon \le \varepsilon. \tag{17}$$

It turns out that $\mathrm{ObsDiam}(X; -(\alpha^{\varepsilon}{}_X(r) - 2\delta)) \ge r$; indeed,

from that $d_X(A, {A_r}^c) \ge r$, the definition of a separation distance (Definition 3.3) gives us that

$$r \le \mathrm{Sep}(X; \varepsilon, \alpha^{\varepsilon}{}_X(r) - \delta)$$

to which applying inequality (17) yields

$$\begin{aligned} &\le \mathrm{Sep}(X; \alpha^{\varepsilon}{}_X(r) - \delta, \alpha^{\varepsilon}{}_X(r) - \delta) \\ &\le \mathrm{ObsDiam}(X; -(\alpha^{\varepsilon}{}_X(r) - 2\delta)), \end{aligned} \tag{18}$$

in which the last inequality sign is due to the rightmost sign of inequality (16). Therefore, we obtain our claim of duality; indeed, it follows from inequality (18) that $\alpha^{\varepsilon}{}_X(r) - 2\delta \le \sup\{\, \kappa > 0 \mid \mathrm{ObsDiam}(X; -\kappa) \ge r \,\}$. Now, $\delta > 0$ being arbitrary, this is just what we have desired. Thus, the proposition is proven. □

4. The Expansion Coefficients

M. Gromov and M. Ledoux independently introduced the two expansion coefficients; Gromov's expansion coefficient was defined in Gromov [10, Chapter $3\frac{1}{2}$.35] and Ledoux's is due to Gromov and Milman [11, the proof of Theorem 4.1]. In this study, Gromov and Ledoux's expansion coefficients are identified by their initials: Exp_G and Exp_L, respectively. Notably, in their original papers, their definitions are incorrect (see Definitions 4.1 and 4.2).

4.1. Gromov's Expansion Coefficient and Its Properties.

DEFINITION 4.1 (GROMOV'S EXPANSION COEFFICIENT; VID., GROMOV [10, Chapter $3\frac{1}{2}$.35]). *Gromov's expansion coefficient* of μ_X on a metric measure space X of order ρ (> 0), denoted by $\mathrm{Exp}_G(X; \varepsilon, \rho)$, is defined as

$$\begin{aligned} &\mathrm{Exp}_G(X; \varepsilon, \rho) \\ &\quad := \sup\{\, e \ge 1 \mid \mu_X(A_\rho) \ge e\varepsilon \text{ for an arbitrary Borel set } A \subset X \text{ such that } \mu_X(A) \ge \varepsilon \,\}. \end{aligned} \tag{19}$$

We refer readers to Shioya [24, Definition 8.10] as well.

It follows from the definition of Exp_G of (19) that

$$\mu_X(A_\rho) \geq \mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon, \tag{20}$$

from which it follows that $\mathrm{Exp}_G(X;\varepsilon,\rho)$ is bounded; indeed,

$$(1 \leq)\,\mathrm{Exp}_G(X;\varepsilon,\rho) \leq 1/\varepsilon. \tag{21}$$

Proposition 4.1. *We have a comparison with* Exp_G *with respect to* ε *and its order* ρ*:*

1. *If* $\varepsilon_1 < \varepsilon_2$*, then*
$$\frac{\mathrm{Exp}_G(X;\varepsilon_2,\rho)}{\mathrm{Exp}_G(X;\varepsilon_1,\rho)} > \frac{\varepsilon_1}{\varepsilon_2}.$$
2. *If* $\rho_1 < \rho_2$*, then*
$$\mathrm{Exp}_G(X;\varepsilon,\rho_1) \leq \mathrm{Exp}_G(X;\varepsilon,\rho_2).$$

Proof. 1. Let $A \subset X$ be such that $\mu_X(A) \geq \varepsilon_1$, so that inequality (20) with $\varepsilon = \varepsilon_1$ follows. For all $A \subset X$ such that $\mu_X(A) \geq \varepsilon_2 (> \varepsilon_1)$, the inequality still holds. It being rewritten as $\mu_X(A_\rho) \geq (\mathrm{Exp}_G(X;\varepsilon_1,\rho)\varepsilon_1/\varepsilon_2)\varepsilon_2$, one deduces that
$$\mathrm{Exp}_G(X;\varepsilon_2,\rho) > \mathrm{Exp}_G(X;\varepsilon_1,\rho)\varepsilon_1/\varepsilon_2$$
as desired.

2. This claim readily follows. □

Shioya [24, Proposition 8.12] remarked an application of a lower bound for Gromov's expansion coefficient to the concept of dissipation, which is the opposite of a concentration; we refer readers to his book for more detailed accounts.

4.2. Ledoux's Expansion Coefficient and Its Properties.

DEFINITION 4.2 (LEDOUX'S EXPANSION COEFFICIENT; CF., LEDOUX [16, Section 1.5]). *Ledoux's expansion coefficient* of μ_X on a metric measure space X of order ρ (> 0), denoted by $\mathrm{Exp}_L(X;\varepsilon,\rho)$, is defined as

$$\begin{aligned}&\mathrm{Exp}_L(X;\varepsilon,\rho)\\ &:= \sup\{\, e \geq 1 \mid \mu_X(B_\rho) \geq e\mu_X(B) \text{ for an arbitrary Borel set } B \subset X \text{ such that } \mu_X(B_\rho) \leq \varepsilon \,\}.\end{aligned} \tag{22}$$

Without Borel sets such as those in the definition of Exp_L of (22), we formally define $\mathrm{Exp}_L(X;\varepsilon,\rho)$ as ∞.

Ledoux [16, Section 1.5] originally proposed this expansion coefficient with $\varepsilon = 1/2$.

It follows from the definition of Exp_L of (22) that

$$\mu_X(B_\rho) \geq \mathrm{Exp}_L(X;\varepsilon,\rho)\mu_X(B). \tag{23}$$

If B is such that $\mu_X(B_{k\rho}) \leq \varepsilon$ for some integer $k \in \mathbb{N}$, then inequality (23) inductively gives

$$\varepsilon \geq \mu_X(B_{k\rho}) \geq (\mathrm{Exp}_L(X;\varepsilon,\rho))^k\,\mu_X(B); \tag{24}$$

one can see from inequality (24) that if $\mathrm{Exp}_L(X;\varepsilon,\rho) > 1$, then B has an extremely small measure.

Proposition 4.2. *There are monotone properties of* Exp_L *with respect to* ε *and its order* ρ*:*

1. *If* $\varepsilon_1 < \varepsilon_2$*, then*
$$\mathrm{Exp}_L(X;\varepsilon_2,\rho) \le \mathrm{Exp}_L(X;\varepsilon_1,\rho).$$
2. *If* $\rho_1 < \rho_2$*, then*
$$\mathrm{Exp}_L(X;\varepsilon,\rho_1) \le \mathrm{Exp}_L(X;\varepsilon,\rho_2).$$

Proof. 1. Let $B \subset X$ be such that $\mu_X(B_\rho) \le \varepsilon_2$, so that inequality (23) with $\varepsilon = \varepsilon_2$ follows. For all $B \subset X$ such that $\mu_X(B_\rho) \le \varepsilon_1 (< \varepsilon_2)$, this inequality still holds. Thus, the claim follows.

2. Let $B \subset X$ be such that $\mu_X(B_{\rho_1}) \le \varepsilon$, so that inequality (23) with $\rho = \rho_1$ follows; moreover, it follows from the assumption that

$$\mu_X(B_{\rho_2}) \ge \mathrm{Exp}_L(X;\varepsilon,\rho_1)\mu_X(B). \tag{25}$$

For all $B \subset X$ such that $(\mu_X(B_{\rho_1}) \le)\mu_X(B_{\rho_2}) \le \varepsilon$, inequality (25) still holds. Thus, the claim follows. □

Remark 4.1. Contrary to Gromov's expansion coefficient, Ledoux's generally does not possess its universal upper bound; cf., a universal upper Exp_G bound (21). Onward Section 6, we will be concerned principally with which Exp_L satisfies $1 < \mathrm{Exp}_L < \infty$.

We conclude this section with another argument concerning Ledoux's expansion coefficient. Notably, $\mathrm{Exp}_L(X;1/2,\rho)$ may be regarded as an analogy to the Cheeger constant if X is an expander graph (see Subsection 1.1). According to Ledoux's argument, the Cheeger constant amounts to Ledoux's expansion coefficient; we refer readers to Ledoux [16, pp. 31–32] for more detailed accounts.

5. An Exponential Concentration in Terms of the Expansion Coefficients

In this section, assuming that Ledoux's expansion coefficient > 1, we show that Ledoux's expansion coefficient and Gromov–Ledoux's give rise to an exponential concentration in terms of itself and themselves, respectively.

5.1. Ledoux's Expansion Coefficient. From this subsection onwards, as far as we are concerned with a complete connected smooth Riemannian manifold with finite volume with respect to the Riemannian measure, we let such a Riemannian manifold be a metric measure space endowed with the Riemannian distance and its normalised measure. Let M be a complete connected smooth Riemannian manifold with finite volume, which is designated as $\mathrm{vol}_M(M) < \infty$; write the normalised measure as $\mu_M := \mathrm{vol}_M / \mathrm{vol}_M(M)$. A smooth Riemannian manifold is said to be *closed* if it is compact, connected and without boundary.

Ledoux [13] and [16, Theorem 3.1] rephrased Gromov and Milman [11, Theorem 4.1] in terms of an exponential concentration with $\varepsilon = 1/2$. Let us restate this result.

Theorem 5.1 (cf., Gromov and Milman [11, Theorem 4.1]). *Let M be a closed smooth Riemannian manifold. M has an exponential concentration:*

$$\alpha^{\varepsilon}{}_M(r) \le (1-\varepsilon^2)\exp(-\sqrt{\lambda_1(M)}\ln(1+\varepsilon)r) \quad \textit{for all } r \ge 0. \tag{26}$$

Notably, $\lambda_1(M) > 0$ *(see Definition 6.5).*

Corollary 5.2. *Let M be as in Theorem 5.1. Let $\varepsilon \le 1/2$ and κ be such that* $2(1-\varepsilon^2) > \kappa$*. Then, we have*

$$\operatorname{ObsDiam}(M; -\kappa) \le \frac{2}{\sqrt{\lambda_1(M)}\ln(1+\varepsilon)} \ln \frac{2(1-\varepsilon^2)}{\kappa}.$$

Proof. Apply the exponential concentration in terms of $\lambda_1(M)$ of (26) to Proposition 3.3. □

Shioya [24, Section 2.5] indicates some upper bounds for an observable diameter of closed smooth Riemannian manifolds with Ricci curvature bounded from below in terms of a positive constant.

We now focus on a metric measure space X. M. Ledoux studied a case in which $\varepsilon = 1/2$, although his statement is not in terms of $\operatorname{Exp}_L(X; 1/2, \rho)$ but e ($\le \operatorname{Exp}_L(X; 1/2, \rho)$); we refer readers to Ledoux [16, Proposition 1.13] for more detailed accounts. He designed the scenario of the proof of Gromov and Milman [11, Theorem 4.1] to demonstrate that X has an exponential concentration in terms of Ledoux's expansion coefficient; Corollary 5.3 is that of Ledoux [16, Proposition 1.13]:

Corollary 5.3 (Exponential concentration in terms of Ledoux's expansion coefficient)**.** *We have*

$$\alpha^{\varepsilon}{}_X(r) \le (1-\varepsilon)\operatorname{Exp}_L(X; 1-\varepsilon, \rho)(\operatorname{Exp}_L(X; 1-\varepsilon, \rho))^{-r/\rho} \quad \textit{for all } r \ge 0;$$

in particular, if $\operatorname{Exp}_L(X; 1-\varepsilon, \rho) > 1$ *for some ε and ρ, then X has an exponential concentration:*

$$\alpha^{\varepsilon}{}_X(r) \le (1-\varepsilon)\operatorname{Exp}_L(X; 1-\varepsilon, \rho)\exp(-(\ln \operatorname{Exp}_L(X; 1-\varepsilon, \rho))r/\rho) \quad \textit{for all } r \ge 0. \tag{27}$$

Proof. This claim actually follows from a slight variant of the proof of Ledoux [16, Proposition 1.13]. □

This exponential concentration in terms of Ledoux's expansion coefficient (Corollary 5.3) gives more, namely an observable diameter of such a metric measure space is bounded from above in terms of Ledoux's:

Corollary 5.4. *Assume that* $\operatorname{Exp}_L(X; 1-\varepsilon, \rho) > 1$ *for some $\varepsilon \le 1/2$ and ρ. Let κ be such that* $2(1-\varepsilon)\operatorname{Exp}_L(X; 1-\varepsilon, \rho) > \kappa$*. Then, we have*

$$\operatorname{ObsDiam}(X; -\kappa) \le \frac{2\rho}{\ln \operatorname{Exp}_L(X; 1-\varepsilon, \rho)} \ln \frac{2(1-\varepsilon)\operatorname{Exp}_L(X; 1-\varepsilon, \rho)}{\kappa}.$$

Proof. Apply this exponential concentration in terms of Ledoux's expansion coefficient (27) to Proposition 3.3. □

5.2. Gromov–Ledoux's Expansion Coefficients.

Theorem 5.5 (Exponential concentration in terms of Gromov–Ledoux's expansion coefficients)**.** *Assume that* $1 > \mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon$ *for some* ε *and* ρ*. Then, we have*

$$\alpha^{\varepsilon}{}_X(r) \leq (1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^2(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{-r/\rho} \text{ for all } r \geq \rho; \tag{28}$$

in particular, if $\mathrm{Exp}_L(X;1-\varepsilon,\rho) > 1$ *for some* ε *and* ρ*, then* X *has an exponential concentration:*

$$\alpha^{\varepsilon}{}_X(r) \leq (1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^2 \exp(-(\ln \mathrm{Exp}_L(X;1-\varepsilon,\rho))r/\rho) \quad \text{for all } r \geq \rho. \tag{29}$$

Proof. The scenario of the proof is in a fashion similar to that of an exponential concentration in terms of Ledoux's expansion coefficient (Corollary 5.3). We interpolate an arbitrary $r \geq 0$ between $(k-1)\rho$ and $k\rho$ for some $k \in \mathbb{N}$. Let $A \subset X$ be such that $\mu_X(A) \geq \varepsilon$. Put $B := A_{k\rho}{}^c$. The following claim is requisite:

CLAIM. Let A be a subset of a metric space X. Then, $(A_{k\rho}{}^c)_{(k-1)\rho} \subset A_\rho{}^c$ for each $k \in \mathbb{N}$ and for all $\rho \geq 0$, where $A_0 := A$ if $k = 1$.

We verify this claim. For all $x \in (A_{k\rho}{}^c)_{(k-1)\rho}$, there exists some $y \in A_{k\rho}{}^c$ such that $d_X(x,y) \leq (k-1)\rho$. Now, consider a 1-Lipschitz function f on X defined by $f(z) := d_X(z, A_\rho)$, $z \in X$. For such a y, it follows from $(A_\rho)_{(k-1)\rho} \subset A_{k\rho}$ for each $k \in \mathbb{N}$ that $f(y) > (k-1)\rho$; hence, $x \in A_\rho{}^c$. Indeed, it follows that $f(x) \geq f(y) - d_X(x,y) > (k-1)\rho - d_X(x,y) \geq 0$. Thus, the claim is verified.

Now, it follows that

$$\mu_X(B_{(k-1)\rho}) \leq 1-\mu_X(A_\rho) \leq 1-\mu_X(A) \leq 1-\varepsilon, \tag{30}$$

in which the leftmost inequality sign is due to the abovementioned claim. Having $\mu_X(B_{(k-1)\rho}) \leq 1-\varepsilon$ from inequality (30), to adopt inequality (24) makes it possible to estimate the leftmost-side of inequality (30) from below:

$$\begin{aligned} \mu_X(B_{(k-1)\rho}) &\geq (\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{k-1}\mu_X(B) \\ &= (\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{k-1}(1-\mu_X(A_{k\rho})). \end{aligned} \tag{31}$$

Combining inequalities (30) and (31), we obtain

$$1-(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{k-1}(1-\mu_X(A_{k\rho})) \geq \mu_X(A_\rho) \geq \mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon,$$

in which the rightmost inequality sign follows from inequality (20); hence, we obtain

$$\begin{aligned} \mu_X(A_{k\rho}) &\geq ((\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{k-1} - (1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon))(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{-(k-1)} \\ &> 0, \end{aligned} \tag{32}$$

in which the last inequality sign of positivity follows from an assumption on Exp_G. In fact, this assumption is just a universal upper Exp_G bound (21) with strictness.

We are now ready to show the conclusion. Adopting the abovementioned interpolation for all $r \geq 0$, we see that

$$1-\mu_X(A_{r+\rho}) \le 1-\mu_X(A_{k\rho})$$

by using inequality (32)

$$\begin{aligned}
&\le (1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)\,\mathrm{Exp}_L(X;1-\varepsilon,\rho)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{-k}\\
&< (1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)\,\mathrm{Exp}_L(X;1-\varepsilon,\rho)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{-r/\rho}\\
&= (1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^2(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^{-(r+\rho)/\rho},
\end{aligned}$$

which implies the desired result (28), from which an exponential concentration in terms of Gromov–Ledoux's expansion coefficients (29) readily follows if $\mathrm{Exp}_L(X;1-\varepsilon,\rho) > 1$ as desired. Thus, the theorem is proven. □

Remark 5.1. One can further observe that an exponential concentration in terms of Gromov–Ledoux's expansion coefficients yields an exponential concentration in terms of Ledoux's as long as the exponential concentration in terms of Gromov–Ledoux's holds.

This exponential concentration in terms of Gromov–Ledoux's expansion coefficients (Theorem 5.5) gives more, namely an observable diameter of such a metric measure space is bounded from above in terms of themselves:

Corollary 5.6. *Assume that* $1 > \mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon$ *and* $\mathrm{Exp}_L(X;1-\varepsilon,\rho) > 1$ *for some* $\varepsilon \le 1/2$ *and* ρ. *Let* κ *be such that* $2(1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^2 > \kappa$. *Then, we have*

$$\mathrm{ObsDiam}(X;-\kappa) \le \frac{2\rho}{\ln \mathrm{Exp}_L(X;1-\varepsilon,\rho)} \ln \frac{2(1-\mathrm{Exp}_G(X;\varepsilon,\rho)\varepsilon)(\mathrm{Exp}_L(X;1-\varepsilon,\rho))^2}{\kappa}.$$

Proof. In a manner identical to the proof of Corollary 5.4, we obtain the claim. □

6. Ledoux's Expansion Coefficient on a Metric Measure Space Satisfying Poincaré Inequality and Volume Doubling

A metric measure space satisfying Poincaré inequality with a doubling measure provides us with a conspicuous result; in fact, in Cheeger [8], he demonstrates that real-valued Lipschitz functions defined on such a metric measure space satisfy properties akin to Rademacher's differentiability theorem; in particular, this metric measure space possesses a differentiable structure with which Lipschitz functions can be differentiated almost everywhere. The aforementioned summary of Cheeger's work is due to Keith [12]; we refer readers to their papers for more detailed accounts.

In this study as well, insomuch as we are concerned with which Exp_L satisfies $1 < \mathrm{Exp}_L < \infty$, onward this section, we restrict our argument to a metric measure space satisfying Poincaré inequality with a doubling measure. To this end, we begin with reviewing Poincaré inequality and a doubling measure.

6.1. Review of Poincaré Inequality. In this study, we work with Poincaré inequality on entire metric measure spaces. In this subsection, to refer to Poincaré inequality, referring to Ledoux [16, Section 3.1] and [17, Section 2] and Shioya [24, Section 7.4], we gather relevant concepts.

DEFINITION 6.1 (VARIANCE). Let f be a real-valued locally Lipschitz continuous function on a metric measure space X. A *variance* of f with respect to μ_X, denoted by Var_{μ_X}, is defined as

$$\mathrm{Var}_{\mu_X}(f) := \int_X \left(f - \int_X f \, d\mu_X \right)^2 d\mu_X. \tag{33}$$

DEFINITION 6.2 (LENGTH OF GRADIENT). Let f be a real-valued locally Lipschitz continuous function on a metric space X. A *length of a gradient* of f, denoted by $|\nabla f|$, at $x \in X$ is defined as

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{d_X(x, y)} := \lim_{r \downarrow 0} \sup_{\substack{y \in X; \\ 0 < d_X(x,y) < r}} \frac{|f(x) - f(y)|}{d_X(x, y)}. \tag{34}$$

$|\nabla f|$ is written as $|\operatorname{grad} f|$ as well.

DEFINITION 6.3 (ENERGY). Let f be a real-valued locally Lipschitz continuous function on a metric measure space X. An *energy* of f, denoted by $\mathcal{E}$, is defined as

$$\mathcal{E}(f) := \int_X |\nabla f|^2 \, d\mu_X. \tag{35}$$

We are now ready to define Poincaré inequality:

DEFINITION 6.4 (POINCARÉ INEQUALITY). Let f be a real-valued locally Lipschitz continuous function on a metric measure space X. We say that X satisfies *Poincaré inequality* if there exists some universal constant $C > 0$ such that

$$C \, \mathrm{Var}_{\mu_X}(f) \le \mathcal{E}(f). \tag{36}$$

6.2. $\mathrm{Exp}_L > 1$**.** In this subsection, we show that Poincaré inequality enables us to demonstrate that $\mathrm{Exp}_L > 1$ (see Theorem 6.1). Ledoux [16, Corollary 3.2] made this claim without its proof. Nonetheless, the scenario of the proof being identical to that of Gromov and Milman [11, Theorem 4.1] for closed smooth Riemannian manifolds, for completeness, we reformulate their result in terms of Exp_L and give all details of its proof:

Theorem 6.1. *Let a metric measure space X satisfy Poincaré inequality. Then, we have a lower bound for Ledoux's expansion coefficient of μ_X on X of order ρ:*

$$\mathrm{Exp}_L\,(X; 1 - \varepsilon, \rho) \ge 1 + C\varepsilon\rho^2,$$

where $C > 0$ is a constant attributed to Poincaré inequality.

Proof. Let A and B be two arbitrary Borel subsets of X such that $\mu_X(A) > 0$ and $\mu_X(B) > 0$, respectively and $d_X(A, B) > 0$. Consider a real-valued function f on X defined by

$$f(x) := \frac{1}{\mu_X(A)} - \left(\frac{1}{\mu_X(A)} + \frac{1}{\mu_X(B)} \right) \frac{\min\{d_X(x, A), d_X(A, B)\}}{d_X(A, B)}, \quad x \in X. \tag{37}$$

It is straightforward that f is bounded; indeed,

$$-1/\mu_X(B) \le f(x) \le 1/\mu_X(A) \quad \text{for all } x \in X. \tag{38}$$

In particular, $f(x)$ is constantly equal to $1/\mu_X(A)$ and $-1/\mu_X(B)$ on A and B, respectively. It follows readily that f is Lipschitz; indeed,

$$\|f\|_{\mathrm{Lip}} \leq \left(\frac{1}{\mu_X(A)} + \frac{1}{\mu_X(B)}\right)\frac{1}{d_X(A,B)}. \tag{39}$$

Combining inequality (39) and that

$$|\nabla f|(x) \leq \|f\|_{\mathrm{Lip}} \quad \text{for all } x \in X, \tag{40}$$

which follows from the definitions of both Lipschitz (Definition 2.6) and a length of a gradient (34), we obtain

$$|\nabla f|(x) \leq \left(\frac{1}{\mu_X(A)} + \frac{1}{\mu_X(B)}\right)\frac{1}{d_X(A,B)} \quad \text{for all } x \in X. \tag{41}$$

In particular, inequality (40) yields

$$|\nabla f|(x) \equiv 0 \quad \text{on } A \sqcup B, \tag{42}$$

so that combining the definition of energy (35) with inequality (41) and equation (42) with regard to a length of a gradient of f, we obtain

$$\mathcal{E}(f) \leq \left(\frac{1}{\mu_X(A)} + \frac{1}{\mu_X(B)}\right)^2 \frac{1-\mu_X(A)-\mu_X(B)}{d_X(A,B)^2}. \tag{43}$$

We now estimate $\mathrm{Var}_{\mu_X}(f)$ from below. It follows from the definition of a variance (33) that

$$\mathrm{Var}_{\mu_X}(f) \geq \int_A \left(f - \int_X f\, d\mu_X\right)^2 d\mu_X + \int_B \left(f - \int_X f\, d\mu_X\right)^2 d\mu_X,$$

for which we note from inequality (38) that $\int_X f\, d\mu_X$ is finite,

$$\begin{aligned} &= \frac{1}{\mu_X(A)} + \left(\int_X f\, d\mu_X\right)^2 \mu_X(A) + \frac{1}{\mu_X(B)} + \left(\int_X f\, d\mu_X\right)^2 \mu_X(B) \\ &\geq \frac{1}{\mu_X(A)} + \frac{1}{\mu_X(B)}. \end{aligned} \tag{44}$$

Combining Poincaré inequality (36) with an upper $\mathcal{E}(f)$ bound (43) and a lower $\mathrm{Var}_{\mu_X}(f)$ bound (44), we obtain

$$\begin{aligned} C &\leq \left(\frac{1}{\mu_X(A)} + \frac{1}{\mu_X(B)}\right)\frac{1-\mu_X(A)-\mu_X(B)}{d_X(A,B)^2} \\ &\leq \frac{1-\mu_X(A)-\mu_X(B)}{\mu_X(A)\mu_X(B)d_X(A,B)^2}, \end{aligned}$$

from which it follows immediately that

$$1-\mu_X(A) \geq (1 + C\mu_X(A)d_X(A,B)^2)\mu_X(B). \tag{45}$$

In particular, let B be such that $\mu_X(B_\rho) \leq 1-\varepsilon$, for which we let $A = {B_\rho}^c$, so that $\mu_X(A) \geq \varepsilon$ and $d_X(A,B) \geq \rho$. Then, it follows readily from inequality (45) that

$$\mu_X(B_\rho) \geq (1 + C\varepsilon\rho^2)\mu_X(B),$$

which implies our claim. Thus, the theorem is proven. □

6.2.1. Closed Smooth Riemannian Manifold. The remainder of this subsection is devoted to a closed smooth Riemannian manifold. Let Ric_M denote the Ricci curvature of a Riemannian manifold M.

DEFINITION 6.5 (SPECTRAL GAP). Let M be a closed smooth Riemannian manifold and Δ be the Laplacian on M. $\lambda_1(M)$ (> 0) denotes the first non-trivial eigenvalue of $-\Delta$, which is called the *spectral gap* of M.

Theorem 6.2 (vid., Ledoux [16, Proof of Theorem 3.1])**.** *Let M be a closed smooth Riemannian manifold. Then, we have a lower bound for Ledoux's expansion coefficient of μ_M on M of order ρ:*

$$\mathrm{Exp}_L(M; 1-\varepsilon, \rho) \geq 1 + \lambda_1(M)\varepsilon\rho^2. \tag{46}$$

Proof. Recall Poincaré inequality (36) on M; in fact, we have for an arbitrary real-valued smooth function f on M

$$\lambda_1(M)\,\mathrm{Var}_{\mu_M}(f) \leq \mathcal{E}(f).$$

Let f on M be as in the function (37). As shown in the proof of Theorem 6.1, f is Lipschitz on M, Rademacher's theorem enables us to show that f is differentiable almost everywhere; we refer readers to, e.g. Villani [25, Theorem 10.8], therefore $|\nabla f|(x)$ is defined μ_M-a.e. $x \in M$. Therefore, almost everywhere on M, the remainder of our proof actually runs as in Theorem 6.1; accordingly, it turns out that a lower Exp_L bound to be derived is just what we have desired. Thus, the theorem is proven. □

To give corollaries to Theorem 6.2, we restrict our concern to works due to Lichnerowicz [20] and Yang [26]; we refer readers to Li and Yau [19] and references therein for detailed accounts:

Corollary 6.3. *Let M be a closed smooth n-dimensional Riemannian manifold with* $\mathrm{Ric}_M \geq (n-1)K > 0$ *for some constant K. Then, we have*

$$\mathrm{Exp}_L(M; 1-\varepsilon, \rho) \geq 1 + nK\varepsilon\rho^2.$$

Proof. Thanks to Lichnerowicz' result:

$$\lambda_1(M) \geq nK;$$

we refer readers to Lichnerowicz [20]; hence, we apply Lichnerowicz' to the lower Exp_L bound (46). □

The lower bound for the spectral gap due to A. Lichnerowicz results from the Bochner formula; we refer readers to Lichnerowicz [20], and Bérard-Meyer [5] for a different manner of the proof of Lichnerowicz'. D. Yang showed that if the diameter of a closed smooth n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq (n-1)K \geq 0$ for some constant K is small, then his result is better than Lichnerowicz'; more specifically, we show a lower Exp_L bound.

Corollary 6.4. *Let M be a closed smooth n-dimensional Riemannian manifold with* $\mathrm{Ric}_M \geq (n-1)K \geq 0$ *for some constant K. Then, we have*

$$\mathrm{Exp}_L(M; 1-\varepsilon, \rho) \geq 1 + \left(\frac{\pi^2}{\mathrm{diam}(M)^2} + \frac{1}{4}(n-1)K \right) \varepsilon \rho^2.$$

Proof. Thanks to Yang's result:

$$\lambda_1(M) \geq \frac{\pi^2}{\mathrm{diam}(M)^2} + \frac{1}{4}(n-1)K;$$

we refer readers to Yang [26]; hence, in a manner identical to the proof of Corollary 6.3, we obtain the claim. □

6.3. Review of Volume Doubling. In this subsection, referring to Villani [25, Chapter 18], we give a brief exposition of doubling property: 'Controlling the volume of balls is a universal problem in geometry. This means of course controlling the volume from above when the radius increases to infinity; but also controlling the volume from below when the radius decreases to 0. The doubling property is useful in both situations'. Subsequently, let $B(x, r)$ denote a closed ball of a metric space X of centre $x \in X$ and radius $r > 0$.

Definition 6.6 (Doubling, doubling constant and (locally) volume doubling; vid., e.g. Ambrosio and Tilli [1, Definition 5.2.1] and Villani [25, Chapter 18]). Let $\mathcal{B}(X)$ denote the σ-algebra of all Borel subsets of a metric measure space X. A Borel measure $\mu_X \colon \mathcal{B}(X) \to [0, +\infty]$ is said to be *doubling* if μ_X is finite on bounded sets and there exists a constant $C_{\mu_X} \geq 1$ with respect to μ_X such that for all $x \in X$ and $r > 0$,

$$\mu_X(B(x, 2r)) \leq C_{\mu_X} \mu_X(B(x, r)). \tag{47}$$

It is reasonable to designate the best constant C_{μ_X} in the doubling (47) as the *doubling constant* with respect to a doubling measure μ_X. We call a metric measure space of doubling (47) of the doubling constant *volume doubling*. A Borel measure μ_X is said to be *locally volume doubling* if for all fixed closed balls $B(z, R) \subset X$, there exists a constant $C_{\mu_X} = C_{\mu_X}(z, R)$ such that for all $x \in B(z, R)$ and $r \in (0, R)$ doubling (47) holds.

Remark 6.1. In particular, if $C_{\mu_X} = 1$, then X is a one-point set, i.e. $\# X = 1$.

Following Shioya [24, p. 6], we define the support of μ_X, denoted by $\mathrm{supp}\, \mu_X$, as

$$\mathrm{supp}\, \mu_X := \{\, x \in X \mid \mu_X(U) > 0 \text{ for an arbitrary open neighbourhood } U \text{ of } x \,\}.$$

Proposition 6.5 (vid., e.g. Villani [25, Proposition 18.4])**.** *Let X be locally volume doubling. Then,* $\mathrm{supp}\, \mu_X = X$.

A doubling measure μ_X is characterised in terms of the volume comparison theorem for a metric measure space. The theorem provides us with an upper bound for the growth of $r \mapsto \mu_X(B(x, r))$ for μ_X.

Theorem 6.6 (*Volume comparison theorem for a metric measure space*; vid., e.g. Ambrosio and Tilli [1, Theorem 5.2.2])**.** *Let $\mu_X \colon \mathcal{B}(X) \to [0, +\infty]$ be a finite measure on bounded sets. Then, μ_X is doubling if and only if there exists a constant $C_{\mu_X} \geq 1$ such that*

$$\frac{\mu_X(B(x_2,r_2))}{\mu_X(B(x_1,r_1))} \le C_{\mu_X}^2 \left(\frac{r_2}{r_1}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}}$$

for all $r_1, r_2 > 0$ *such that* $r_1 \le r_2$ *and for all* $x_1, x_2 \in X$ *such that* $x_1 \in B(x_2, r_2)$.

6.4. $\mathrm{Exp}_L < \infty$**.**

Theorem 6.7. *Let* X *be a bounded and volume doubling metric measure space. Suppose that for some* $x \in X$ *and* ρ *there exists a closed ball of* X *of centre* x *and radius* 2ρ *such that for some* ε,

$$(0 <)\ \mu_X(B(x, 2\rho)) \le 1 - \varepsilon. \tag{48}$$

Then, we have

$$\mathrm{Exp}_L(X; 1-\varepsilon, \rho) \le C_{\mu_X}$$

whose order ρ *is such that*

$$0 < \rho < \min\{1/2, 2(1-\varepsilon)^{\frac{\ln 2}{\ln C_{\mu_X}}}\}\,\mathrm{diam}(X), \quad C_{\mu_X} > 1. \tag{49}$$

Proof. Supposition (48) gives us an a priori upper bound for ρ:

$$\rho < \mathrm{diam}(X)/2. \tag{50}$$

Further argument enables us to bound ρ from above in terms of ε, C_{μ_X} and $\mathrm{diam}(X)$; indeed, supposition (48), in which we note from Proposition 6.5 that $\mu_X(B(x, 2\rho)) > 0$, equivalently indicates a lower bound for a volume comparison as follows:

$$\frac{1}{1-\varepsilon} \le \frac{\mu_X(B(x, \mathrm{diam}(X)))}{\mu_X(B(x, 2\rho))}$$

to which applying the volume comparison theorem for a metric measure space (Theorem 6.6), we obtain a further upper bound for the volume comparison:

$$\le C_{\mu_X}^2 \left(\frac{\mathrm{diam}(X)}{2\rho}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}}.$$

Notably, supposition (48) implies that $\#X \ge 2$; hence, it follows from Remark 6.1 that $C_{\mu_X} > 1$. Therefore, we obtain an upper bound for ρ as desired:

$$\rho \le 2(1-\varepsilon)^{\frac{\ln 2}{\ln C_{\mu_X}}}\,\mathrm{diam}(X). \tag{51}$$

Consequently, both upper bounds (50) and (51) for ρ yield conclusion (49).

We now focus on handling $\mathrm{Exp}_L(X; 1-\varepsilon, \rho)$ of its order $\rho = \rho(\varepsilon, C_{\mu_X}, \mathrm{diam}(X))$ desired above. Applying a closed ball satisfying supposition (48) to inequality (23), we conclude that

$$\begin{aligned}\mathrm{Exp}_L(X; 1-\varepsilon, \rho) &\le \frac{\mu_X(B(x, 2\rho))}{\mu_X(B(x, \rho))}\\ &\le C_{\mu_X},\end{aligned}$$

in which the last inequality sign is in virtue of doubling (47). Thus, the theorem is proven. □

7. An Upper Bound for the Diameter of a Bounded and Volume Doubling Metric Measure Space and Its Application to a Closed Smooth Riemannian Manifold with Non-Negative Ricci Curvature

7.1. An Upper Bound for the Diameter of a Bounded and Volume Doubling Metric Measure Space.

Theorem 7.1. *Let X be a bounded and volume doubling metric measure space with* $\mathrm{Exp}_L(X;\varepsilon,\rho) > 1$ *for some ε and ρ. Then, the diameter of X is bounded from above in terms of both Ledoux's expansion coefficient of a doubling measure μ_X on X of order ρ and the doubling constant with respect to μ_X:*

$$\begin{aligned}(52)\qquad &\mathrm{diam}(X)\\ &\leq 3\rho \max\left\{\frac{\ln(C_{\mu_X}^4(1-\varepsilon)\,\mathrm{Exp}_L(X;1-\varepsilon,\rho)\varepsilon^{-1})}{\ln \mathrm{Exp}_L(X;1-\varepsilon,\rho)}, \frac{2\ln(3^{\frac{\ln C_{\mu_X}}{\ln 2}} C_{\mu_X}^3 \varepsilon\,\mathrm{Exp}_L(X;\varepsilon,\rho))}{\ln \mathrm{Exp}_L(X;\varepsilon,\rho)}\right\}.\end{aligned}$$

Proof. The exponential concentration in terms of Ledoux's expansion coefficient (Corollary 5.3) enables us to show that for all Borel sets $A \subset X$ such that $\mu_X(A) \geq \varepsilon$,

$$\begin{aligned}(53)\qquad &1-\mu_X(A_r)\\ &\leq (1-\varepsilon)\,\mathrm{Exp}_L(X;1-\varepsilon,\rho)\exp(-(\ln \mathrm{Exp}_L(X;1-\varepsilon,\rho))r/\rho) \quad \text{for all } r \geq 0.\end{aligned}$$

Let $r > 0$ be sufficiently small and fixed. Now, considering a positive parameter τ (≤ 1) for $\mathrm{diam}(X)$, we observe a closed ball of X centred at an arbitrary $x \in X$ and with a radius $\tau\,\mathrm{diam}(X)$. Let $z \in X$ be distinct from $x \in X$ such that

$$(0 <) d_X(x,z) = \tau\,\mathrm{diam}(X) + 2r (\leq \mathrm{diam}(X)) \tag{54}$$

for some τ such that $\tau\,\mathrm{diam}(X) \leq r$; hence,

$$(0 <)\tau \leq 1/3. \tag{55}$$

In the remainder of this proof, for such a closed ball $B(x,\tau\,\mathrm{diam}(X))$, we will put $\mu_X(B(x,\tau\,\mathrm{diam}(X)))$ at ε from above and below; first, we observe

$$\mu_X(B(x,\tau\,\mathrm{diam}(X))) \geq \varepsilon.$$

In inequality (53), put $A = B(x,\tau\,\mathrm{diam}(X))$. Then, it follows from the definition of z of (54) that

$$A \subset B(z, 2(\tau\,\mathrm{diam}(X) + r)) \tag{56}$$

and for all $\varsigma \in (0,1)$

$$\begin{aligned}\mu_X(B(z,\varsigma r)) &\leq 1 - \mu_X(B(x,\tau\,\mathrm{diam}(X)+r))\\ &\leq 1-\mu_X(A_r), \qquad (57)\end{aligned}$$

in which we note from Proposition 6.5 that $\mu_X(B(z,\varsigma r)) > 0$; hence, it is allowable to apply the volume comparison theorem for a metric measure space (Theorem 6.6) to $B(z, 2(\tau\,\mathrm{diam}(X) + r))$ and $B(z,\varsigma r)$. We obtain the following volume comparison: observe from Remark 6.1 that $C_{\mu_X} > 1$,

$$\frac{\mu_X(B(z, 2(\tau\operatorname{diam}(X)+r)))}{\mu_X(B(z,\varsigma r))} \le C_{\mu_X}^2\left(\frac{2(\tau\operatorname{diam}(X)+r)}{\varsigma r}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}}$$
$$= C_{\mu_X}^3\left(\frac{\tau\operatorname{diam}(X)+r}{\varsigma r}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}};$$

hence,

$$\mu_X(B(z,\varsigma r)) \ge \frac{\mu_X(B(z, 2(\tau\operatorname{diam}(X)+r)))}{C_{\mu_X}^3}\left(\frac{\varsigma r}{\tau\operatorname{diam}(X)+r}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}}$$
$$\ge \frac{\mu_X(A)}{C_{\mu_X}^3}\left(\frac{\varsigma r}{\tau\operatorname{diam}(X)+r}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}} \quad \text{with an inclusion relation (56)}$$

(58)
$$\ge \frac{\varepsilon}{C_{\mu_X}^3}\left(\frac{\varsigma r}{\tau\operatorname{diam}(X)+r}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}}.$$

Combining inequalities (57) and (58) with inequality (53) results in

(59)
$$\frac{\varepsilon}{C_{\mu_X}^3}\left(\frac{\varsigma r}{\tau\operatorname{diam}(X)+r}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}} \le (1-\varepsilon)\operatorname{Exp}_L(X;1-\varepsilon,\rho)\exp(-(\ln\operatorname{Exp}_L(X;1-\varepsilon,\rho))r/\rho).$$

Hence, because r is arbitrary, letting $r = \tau\operatorname{diam}(X)$ in inequality (59) and letting $\varsigma \uparrow 1$, we obtain an upper bound for $\operatorname{diam}(X)$:

(60)
$$\operatorname{diam}(X) \le \frac{\rho\ln(C_{\mu_X}^4(1-\varepsilon)\operatorname{Exp}_L(X;1-\varepsilon,\rho)\varepsilon^{-1})}{\tau\ln\operatorname{Exp}_L(X;1-\varepsilon,\rho)}.$$

Based on the following observation, we let

(61)
$$\mu_X(B(x,\tau\operatorname{diam}(X))) < \varepsilon.$$

Replace A with $B(x,\tau\operatorname{diam}(X))^c$, so that condition (61) is rewritten as

(62)
$$\mu_X(A) > 1-\varepsilon.$$

Let us observe an enlargement of order $\tau\operatorname{diam}(X)/2$ of A with respect to d_X. It follows that

(63)
$$\mu_X(B(x,\tau\operatorname{diam}(X)/2)) \le 1 - \mu_X(A_{\tau\operatorname{diam}(X)/2});$$

indeed, for all $p \in A_{\tau\operatorname{diam}(X)/2}$ take $q \in A$ such that $d_X(p,q) = d_X(p,A)$. Then, the triangle inequality yields

$$d_X(x,p) \ge -d_X(p,q) + d_X(x,q)$$
$$> -\tau\operatorname{diam}(X)/2 + \tau\operatorname{diam}(X)$$
$$= \tau\operatorname{diam}(X)/2,$$

from which $p \in B(x,\tau\operatorname{diam}(X)/2)^c$ follows; therefore, we obtain inequality (63).

Furthermore, we apply the volume comparison theorem for a metric measure space (Theorem 6.6) to $B(x,\tau\operatorname{diam}(X)/2)$ to estimate $\mu_X(B(x,\tau\operatorname{diam}(X)/2))$ (> 0) as follows:

$$\begin{aligned}\frac{1}{\mu_X(B(x,\tau\,\mathrm{diam}(X)/2))} &= \frac{\mu_X(B(x,\mathrm{diam}(X)))}{\mu_X(B(x,\tau\,\mathrm{diam}(X)/2))}\\ &\le C_{\mu_X}^2\left(\frac{\mathrm{diam}(X)}{\tau\,\mathrm{diam}(X)/2}\right)^{\frac{\ln C_{\mu_X}}{\ln 2}}\\ &= C_{\mu_X}^3\tau^{-\frac{\ln C_{\mu_X}}{\ln 2}},\end{aligned}$$

from which we obtain

$$\tau^{\frac{\ln C_{\mu_X}}{\ln 2}}/C_{\mu_X}^3 \le \mu_X(B(x,\tau\,\mathrm{diam}(X)/2)). \tag{64}$$

Combining inequalities (63) and (64) into inequality (53) under condition (62) results in

$$\tau^{\frac{\ln C_{\mu_X}}{\ln 2}}/C_{\mu_X}^3 \le \varepsilon\,\mathrm{Exp}_L(X;\varepsilon,\rho)\exp(-(\ln \mathrm{Exp}_L(X;\varepsilon,\rho))\tau\,\mathrm{diam}(X)/2\rho);$$

hence, we obtain an upper bound for $\mathrm{diam}(X)$:

$$\mathrm{diam}(X) \le \frac{2\rho\ln(\tau^{-\frac{\ln C_{\mu_X}}{\ln 2}}C_{\mu_X}^3\varepsilon\,\mathrm{Exp}_L(X;\varepsilon,\rho))}{\tau\ln \mathrm{Exp}_L(X;\varepsilon,\rho)}. \tag{65}$$

Finally, we are now in a position to evaluate τ (see inequality (55)) appeared in both upper diameter bounds (60) and (65). It turns out that $\tau = 1/3$ makes them actually reasonable. Therefore, we obtain the desired upper bound (52) for the diameter of X.

Now, there remains a concern with an anti-logarithm that appeared in upper diameter bound (65) under condition (61) with $\tau = 1/3$. We have

$$3^{\frac{\ln C_{\mu_X}}{\ln 2}}C_{\mu_X}^3\varepsilon\,\mathrm{Exp}_L(X;\varepsilon,\rho) > 1;$$

indeed,

$$\begin{aligned}\varepsilon &> \mu_X(B(x,\mathrm{diam}(X)/3))\\ &\ge \frac{\mu_X(B(x,2\,\mathrm{diam}(X)/3))}{C_{\mu_X}}, &&\mu_X \text{ being volume doubling}\\ &\ge \frac{\mu_X(B(x,4\,\mathrm{diam}(X)/3))}{C_{\mu_X}^2}, &&\mu_X \text{ being volume doubling}\\ &= \frac{1}{C_{\mu_X}^2}.\end{aligned}$$

Therefore, we can conclude the desired upper diameter bound to be constantly positive. Thus, the theorem is proven. □

As a preceding work related to Theorem 7.1, to the best of our knowledge, Naor et al. were the first to study an upper bound for the diameter of a certain bounded metric measure space with the doubling constant (> 3), which is described in terms of its observable diameter; cf., a comparison with an observable diameter and a diameter per se; we refer readers to Naor et al. [23, Theorem 1.7].

7.2. An Upper Bound for the Diameter of a Closed Smooth Riemannian Manifold with Non-Negative Ricci Curvature. We conclude this section by briefly outlining some previously conducted studies on the diameter of a complete connected smooth Riemannian manifold and applying Theorem 7.1 to a closed smooth Riemannian manifold with non-negative Ricci curvature.

The most well-known theorem on an upper bound for the diameter of a complete connected smooth Riemannian manifold with Ricci curvature bounded from below in terms of a positive constant, which can be traced back to the work of Sumner B. Myers (1941), building on earlier work due to Ossian Bonnet, Heinz Hopf and John L. Synge, is currently referred to as (Bonnet-)Myers's theorem; for some contemporary studies, we refer readers to Bakry and Ledoux [3] and Villani [25, p. 378]. We now review (Bonnet-)Myers's theorem.

Theorem 7.2 ((Bonnet-)Myers's theorem due to Myers [22]). *Let M be a complete connected smooth n-dimensional Riemannian manifold with* $\mathrm{Ric}_M \geq (n-1)K > 0$. *Then, we have*

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{K}}; \tag{66}$$

furthermore, M is compact and its fundamental group is finite.

Cheng [9, Theorem 3.1] proved that the equality of the upper diameter bound (66) holds if and only if M is isometric to an n-dimensional Euclidean sphere, which is referred to as the generalised Toponogov sphere theorem and Cheng's maximal diameter theorem.

In Ledoux [15, pp. 120–216], [16, Section 3.2, Notes and Remarks] and [17, Section 3], he surveyed a relationship between the spectral gap and diameter of a closed smooth Riemannian manifold. We review his upper diameter bound, which is based on Cheng [9]:

Theorem 7.3 (vid., Ledoux [16, Theorem 3.5] and [17, Theorem 3.1]). *Let M be a complete connected smooth n-dimensional Riemannian manifold without boundary with finite volume and Ricci curvature bounded below. Then, M is compact provided*

$$\liminf_{r\to\infty} \frac{\ln(1-\mu_M(B(p,r)))}{r} = -\infty$$

for some (or all) $p \in M$. We have then $\lambda_1(M) > 0$. In particular, if $\mathrm{Ric}_M \geq 0$, *then*

$$\mathrm{diam}(M) \leq \frac{C_n}{\sqrt{\lambda_1(M)}},$$

where C_n is a positive constant depending only on n.

We are now in a position to state our upper bound for the diameter of a closed smooth Riemannian manifold with non-negative Ricci curvature.

Theorem 7.4. *Let M be a closed smooth n-dimensional Riemannian manifold with* $\mathrm{Ric}_M \geq 0$. *Suppose that for some $p \in M$ and ρ there exists a closed metric ball of M of centre p and radius 2ρ such that for some ε,*

$$(0 <)\mu_M(B(p,2\rho)) \leq 1-\varepsilon.$$

Then, we have

$$\mathrm{diam}(M) \leq 3\rho \max\left\{ \frac{\ln(2^{5n}(1-\varepsilon)\varepsilon^{-1})}{\ln(1+\lambda_1(M)\varepsilon\rho^2)}, \frac{2\ln(3^n 2^{4n}\varepsilon)}{\ln(1+\lambda_1(M)(1-\varepsilon)\rho^2)} \right\} \tag{67}$$

whose order ρ is such that

$$0 < \rho < \min\{1/2, 2(1-\varepsilon)^{\frac{1}{n}}\}\,\mathrm{diam}(M). \tag{68}$$

Proof. First, we observe that condition (49) for M is actually given by that of (68) because $C_{\mu_M} = 2^n$, which results from Bishop–Gromov's volume comparison theorem. In virtue of Theorems 6.2 and 6.7, it turns out that the Ledoux's expansion coefficient of μ_M on M of order ρ bounded above (68) is bounded as per below:

$$1 + \lambda_1(M)\varepsilon\rho^2 \le \mathrm{Exp}_L(M; 1-\varepsilon, \rho) \le 2^n. \tag{69}$$

Combining our upper diameter bound (52) with the upper and lower bounds for $\mathrm{Exp}_L(M; 1-\varepsilon, \rho)$ of (69) and C_{μ_M}, we obtain the desired upper bound for $\mathrm{diam}(M)$ of (67). Thus, the theorem is proven. □

Remark 7.1. For a compact connected smooth n-dimensional Riemannian manifold with $\mathrm{Ric}_M \ge -(n-1)K$, $K \ge 0$, under the supposition of Theorem 7.4, one can demonstrate that Bishop–Gromov's volume comparison theorem enables us to show the result akin to the upper bound for Ledoux's expansion coefficient (69). This upper bound is described in terms of the local volume doubling constant with respect to the Riemannian measure; in fact, for some ε,

$$\mathrm{Exp}_L(M; 1-\varepsilon, \rho) \le 2^n \exp(2(n-1)\sqrt{K}\rho)$$

for some $\rho = \rho(\varepsilon, n, \mathrm{diam}(M), K) > 0$. In this paper, we refrain from detailing this result.

8. Conclusions

In this study, we investigated an exponential concentration in terms of Gromov–Ledoux's expansion coefficients (Theorem 5.5), an upper bound for Ledoux's expansion coefficient on a bounded and volume doubling metric measure space (Theorem 6.7) and an upper bound for the diameter of a bounded and volume doubling metric measure space (Theorem 7.1) and of a closed smooth n-dimensional Riemannian manifold with $\mathrm{Ric}_M \ge 0$ (Theorem 7.4). We assumed that Ledoux's expansion coefficient > 1 while dealing with an exponential concentration in terms of Ledoux's expansion coefficient. We demonstrated that a metric measure space satisfying Poincaré inequality yields one with Ledoux's expansion coefficient > 1 (Theorem 6.1).

We conclude this study by referring to Gromov's 'Green Book', in which he proposed the following problem regarding an expansion coefficient (Definition 4.1) and an observable diameter (Definition 3.2) of a metric measure space; we refer readers to Gromov [10, Chapter $3\frac{1}{2}$.35]:

Exercise. Bound $\mathrm{Exp}_G(X; \varepsilon, \rho)$ from below in terms of $\mathrm{ObsDiam}(X; -\kappa)$.

Numerous efforts should be made to study Gromov's problem mentioned above. From a fundamentally novel perspective of the procedure for our results, we will study this problem later.

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