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NESTED OPEN BOOKS AND THE BINDING SUM

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Abstract

We introduce the notion of a *nested open book*, a submanifold equipped with an open book structure compatible with an ambient open book, and describe in detail the special case of a *push-off* of the binding of an open book. This enables us to explicitly describe a natural open book decomposition of a fibre connected sum of two open books along their bindings, provided they are diffeomorphic and admit an open book structure themselves. Furthermore, we apply the results to contact open books, showing that the natural open book structure of a contact fibre connected sum of two adapted open books along their contactomorphic bindings is again adapted to the resulting contact structure.

1. Introduction

In 1923 Alexander [1] proved that every closed oriented 3-manifold admits a so-called *open book decomposition*. In fact, combining the work of Winkelnkemper, Lawson, and Quinn from the 1970s, this statement remains true for odd-dimensional manifolds in general (see [28], [17], and [24] respectively). The existence problem in even dimensions is also solved in these works but is more involved. Arguably the most prominent appearance of open books in recent history is their surprisingly deep connection with *contact structures*. As was observed by Giroux [12] in 2002, contact structures in dimension 3 are of purely topological nature: he established a one-to-one correspondence between isotopy classes of contact structures and open book decompositions up to positive stabilisation. This correlation remains partly intact in higher dimensions. According to Giroux and Mohsen [14] any contact structure on a closed manifold of dimension at least 3 admits a *compatible* open book decomposition. In particular, its binding inherits an induced contact structure and thus possesses a compatible open book itself.

In this paper we investigate how the *fibre connected sum* performed along diffeomorphic binding components of an open book, henceforth called *binding sum*, affects an underlying open book structure. We discuss both regular fibre sums and fibre sums of contact manifolds. While, at first glance, the binding sum destroys the open book structure, we show that this is in fact not the case and generalise a previous result of the second author [15] to higher dimensions. The main results can be summarized as follows:

I. Existence: The fibre connected sum along diffeomorphic binding components of an open book admits a *natural* open book structure, provided the binding components admit open book structures themselves (see Theorem 5.1 for a detailed statement).

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II. Compatibility: Our construction can be adapted to an underlying contact structure and again produces a compatible open book (see Theorem 6.1).

The open book is *natural* in the sense that it will be described in terms of the original open book, and a fixed open book decomposition of the binding components (see Theorem 5.1 for details). The idea of the construction is to form the fibre connected sum not along the binding components themselves but along slightly isotoped copies, called the *push-offs*, realising them as *nested open books*. A nested open book is a submanifold inheriting an open book structure from the ambient manifold and is thus a natural generalisation of a *spinning* as discussed in contact topology by Mori [21] and Martínez Torres [19]. A nice survey on topological spinnings is [9].

Remark 1.1. The first result remains true if the two binding components admit fibrations over the circle (that is, informally, if they admit similar open book decompositions with binding the empty set). In particular, the statement holds in the 3-dimensional case, where the binding components are circles.

Note that, according to the above mentioned work of Winkelnkemper, Lawson, and Quinn, the condition on the binding components to admit open books themselves is not a restriction in odd dimensions. For the same reasons the sheer existence of an open book of the fibre sum in odd dimensions is already answered as well. According to Giroux and Mohsen the same is true for the case of adapted open books and contact structures. However, constructions of particular instances of those open books and their relationship to the original open book decomposition have rarely been discussed in the literature yet. One notable exception is Mori's construction of contact structures and leaf-wise symplectic foliations on $S^4 \times S^1$ arising as the fibre connected sum of two copies of S^5 ([22], see Section 6.2). Besides, very few constructions of open books supporting contact structures in higher dimensions are known. Along these lines we explain how binding sums can be utilised to describe fibrations over the circle whose fibres are convex hypersurfaces in the sense of Giroux, and manifolds that admit the higher-dimensional analogue of *Giroux torsion* introduced by Massot, Niederkrüger and Wendl [18] (cf. Section 6.2). Our construction thus yields open book decompositions for both of these two classes of contact manifolds.

2. Preliminaries

2.1. Open book decompositions. An **open book decomposition** of an n-dimensional manifold M is a pair (B, π), where B is a co-dimension 2 submanifold in M, called the **binding** of the open book, and $\pi\colon\ M \setminus B \to S^1$ is a (smooth, locally trivial) fibration such that each fibre $\pi^{-1}(\theta)$, $\theta \in S^1$, corresponds to the interior of a compact hypersurface $\Sigma_\theta \subset M$ with $\partial\Sigma_\theta$ equal to B, and the binding has a tubular neighbourhood which is trivialised by θ. The hypersurfaces Σ_θ, $\theta \in S^1$, are called the **pages** of the open book. We will say that M admits an **open book structure**.

The question of which data is relevant to remodel the ambient manifold and the underlying open book structure up to diffeomorphism leads us to the following notion.

An **abstract open book** is a pair (Σ, ϕ), where Σ is a compact manifold with non-empty boundary $\partial\Sigma$, called the **page**, and $\phi\colon\ \Sigma \to \Sigma$ is a diffeomorphism equal to the identity near

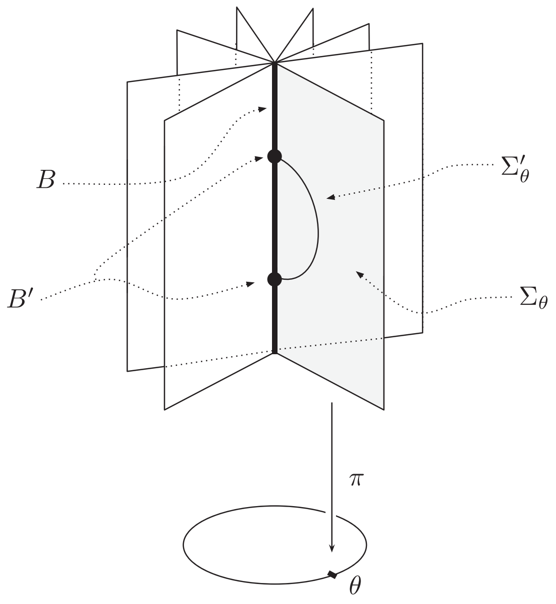

Fig. 1. Schematic picture of an open book. A single page of a nested open book and its nested binding is indicated.

$\partial\Sigma$, called the **monodromy** of the open book. Let $\Sigma(\phi)$ denote the mapping torus of ϕ, that is, the quotient space obtained from $\Sigma \times [0, 2\pi]$ by identifying $(x, 2\pi)$ with $(\phi(x), 0)$ for each $x \in \Sigma$. Then the pair (Σ, ϕ) determines a closed manifold $M_{(\Sigma,\phi)}$ defined by

$$M_{(\Sigma,\phi)} := \Sigma(\phi) \cup_{\mathrm{id}} (\partial\Sigma \times D^2), \tag{2.1}$$

where we identify $\partial\Sigma(\phi) = \partial\Sigma \times S^1$ with $\partial(\partial\Sigma \times D^2)$ using the identity map. Let $B \subset M_{(\Sigma,\phi)}$ denote the embedded submanifold $\partial\Sigma \times \{0\}$. Then we can define a fibration $\pi\colon\ M_{(\Sigma,\phi)} \setminus B \to S^1$ by

$$\left.\begin{array}{l} [x, \theta] \\ [x', re^{i\theta}] \end{array}\right\} \mapsto [\theta],$$

where we understand $M_{(\Sigma,\phi)} \setminus B$ as decomposed as in (2.1) and $[x, \theta] \in \Sigma(\phi)$ or $[x', re^{i\theta}] \in \partial\Sigma \times D^2 \subset \partial\Sigma \times \mathbb{C}$. Clearly, (B, π) defines an open book decomposition of $M_{(\Sigma,\phi)}$.

On the other hand, an open book decomposition (B, π) of some n-manifold M defines an abstract open book as follows: identify a neighbourhood of B with $B \times D^2$ such that $B = B \times \{0\}$ and such that the fibration on this neighbourhood is given by the angular coordinate, θ say, on the D^2-factor. We can define a 1-form α on the complement $M \setminus (B \times D^2)$ by pulling back $d\theta$ under the fibration π, where this time we understand θ as the coordinate on the target space of π. The vector field ∂_θ on $\partial(M \setminus (B \times D^2))$ extends to a nowhere vanishing vector field X which we normalise by demanding it to satisfy $\alpha(X) = 1$. Let ϕ denote the time-2π map of the flow of X. Then the pair (Σ, ϕ), with $\Sigma = \pi^{-1}(0)$, defines an abstract open book such that $M_{(\Sigma,\phi)}$ is diffeomorphic to M.

Nice surveys on open books and their applications are Winkelnkemper's appendix to Ranicki's book [25] and Giroux [13]. More detailed material, in particular on the relation of open books and contact structures can be found in [8, 10, 26].

2.2. The fibre connected sum and the binding sum. We will briefly introduce the fibre connected sum, which is a method to construct manifolds using embedded submanifolds; for details see [10, Section 7.4]. Let M' and M be closed oriented manifolds and let j_0 and j_1 be embeddings of M' into M with disjoint images. Assume that there exists a bundle isomorphism Ψ of the corresponding normal bundles N_0 and N_1 over $j_1 \circ j_0^{-1}|_{j_0(M')}$ that reverses the fibre orientation. Picking a bundle metric on N_0 and choosing the induced metric on N_1 turns Ψ into a bundle isometry. We furthermore identify open disjoint neighbourhoods of the $j_i(M')$ with the normal bundles N_i.

The **fibre connected sum** is the quotient manifold

$$\#_\Psi M := \Big(M \setminus \big(j_0(M') \cup j_1(M')\big)\Big)/_{\sim},$$

where $v \in N_0$ with $0 < \|v\| < \epsilon$ is identified with $\frac{\sqrt{\epsilon^2 - \|v\|^2}}{\|v\|}\Psi(v)$. It is worth noting that the construction can be adapted to work in the symplectic and contact setting if the dimensions of M and M' differ by two, see Section 7.2 in [20] and Theorem 7.4.3 in [10] for details.

Let M be a (not necessarily connected) smooth n-dimensional manifold with open book decomposition (Σ, ϕ) whose binding B contains two diffeomorphic components B_0, B_1 with diffeomorphic open book decompositions (Σ', ϕ'). Their normal bundles νB_0 and νB_1 admit trivializations induced by the pages of the open book decomposition of M. Let Ψ denote the fibre orientation reversing diffeomorphism of $B \times D^2 \subset B \times \mathbb{C}$ sending (b, z) to $(b, \bar{z})$. Hence, we can perform the fibre connected sum along B_0 and B_1 with respect to the above trivializations of the normal bundles and the map induced by Ψ and denote the result by

$$\#_{B_0,B_1} M.$$

We call it the **binding sum** of M along B_0 and B_1.

An interesting case occurs when M is disconnected and decomposes into two manifolds M_0 and M_1 with open books admitting diffeomorphic bindings, denoted by B say. In this case we denote the binding sum along the receptive copies of B in M_0 and M_1 by

$$M_0 \#_B M_1.$$

Note that $M_0 \#_B M_1$ admits the structure of a fibration over the circle with fibre given by

$$(-\Sigma_0) \cup_B \Sigma_1,$$

where Σ_0 and Σ_1 are the pages of the open book for M_0 and M_1 respectively. In the contact setting each fibre defines a *convex* hypersurface, i.e. there exists a contact vector field on $(M_0, \xi_0) \#_B (M_1, \xi_1)$ which is transverse to the fibres. Furthermore for each fibre $(-\Sigma_0) \cup_B \Sigma_1$ the contact vector field is tangent to the contact structure exactly over B (cf. Section 6.2.4).

As stated in the introduction, this paper is concerned with the question whether the binding sum admits again the structure of an open book and how it is related to the original open books.

3. Nested open books

In this section we turn our attention to a special class of submanifolds and introduce the notion of a *nested open book*, i.e. a submanifold carrying an open book structure compatible with the open book structure of the ambient manifold. We also discuss fibre connected sums in this context.

Let M be an n-dimensional manifold supported by an open book decomposition (B, π). Let $M' \subset M$ be a k-dimensional submanifold which on its part is supported by an open book decomposition (B', π') such that

$$\pi|_{M' \setminus B'} = \pi'.$$

Note that B' necessarily defines a $(k-2)$-dimensional submanifold in B. We will always assume that M' intersects the binding B transversely. We refer to M', as well as to (B', π'), as a **nested open book** of (B, π).

Let (Σ, ϕ) be an abstract open book and $\Sigma' \subset \Sigma$ a properly embedded submanifold intersecting $\partial\Sigma$ exactly in its boundary $\partial\Sigma'$ with the intersection being transverse. We call $(\Sigma', \phi|_{\Sigma'})$ an **abstract nested open book** if Σ' is invariant under the monodromy ϕ. The equivalence of the two definitions follows analogously to the equivalence of abstract and non-abstract open books. If not indicated otherwise, we will assume the normal bundle of any nested open book used in the present paper to be trivial.

EXAMPLE 3.1. Consider a k-disc $D^k \subset D^n$ inside an n-disc D^n coming from the natural inclusion $\mathbb{R}^k \subset \mathbb{R}^n$. This realises $S^{k+1} \cong (D^k, \mathrm{id})$ as a nested open book of $S^{n+1} \cong (D^n, \mathrm{id})$. The case $k = 1$ and $n = 2$ is depicted in Figure 1. For $k = n-2$, the nested S^{n-1} is a *push-off*, as will be defined in Section 4, of the binding of (D^n, id).

3.1. Fibre sums along nested open books. For the remainder of the section, assume the co-dimension of the nested open books to be two. We will show that the fibre connected sum operation of two open books along diffeomorphic nested open books carries an open book structure with page a fibre connected sum of the original pages along the nested bindings.

Let M' be an $(n-2)$-dimensional manifold supported by an open book (π', B'), and let $j_0, j_1 \colon M' \hookrightarrow M$ be two disjoint embeddings defining nested open books of M such that their images admit isomorphic normal bundles N_i. We denote by $M'_i := j_i(M')$ the embedded copies of the nested open book M' and by $B'_i := j_i(B')$ their respective bindings. Finally let $\pi'_i := \pi' \circ j_i^{-1}$ denote the induced open book fibration on $M'_i \setminus B'_i$ and let $(\Sigma'_i)_\theta$ denote their pages.

Given an orientation reversing bundle isomorphism Ψ of the normal bundles $\nu M'_i$, we can perform the fibre connected sum $\#_\Psi M$. We only have to ensure that the fibres of the normal bundles of M'_0 and M'_1 lie within the pages of (π, B). In particular, we require the fibres over the nested bindings to lie within the binding of M. Moreover, we require the isomorphism Ψ of the normal bundle to respect the open book structure of M (which implies that it is compatible with the nested open book structures of M'_0 and M'_1 as well), i.e. Ψ to satisfies $\pi \circ \Psi = \pi$. Now, an open book structure of $\#_\Psi M$ is given as follows.

Lemma 3.2. *The original fibration $\pi\colon M \setminus B \to S^1$ descends to a fibration*

$$\Pi\colon \#_\Psi M \setminus \#_{\Psi|_{\nu B'_0}} B \to S^1.$$

In particular, the new binding is given by the fibre connected sum $\#_{\Psi|_{\nu B'_0}} B$ of the binding along the nested bindings (with respect to the isomorphism of $\nu B'_i \subset TB$ induced by Ψ), and the pages of the open book are given by the (relative) fibre sum of the original page along *the nested pages (with respect to the isomorphism of* $\nu\pi_i'^{-1}(\theta) \subset T\pi^{-1}(\theta)$ *induced by* Ψ*), i.e.* $\overline{\Pi^{-1}(\theta)} = \#_{\Psi|_{\nu(\Sigma_0')_\theta}}\Sigma_\theta$. □

In the following we are going to extract the remaining information to express $\#_\Psi M$ in terms of an abstract open book, that is we describe a recipe to find the monodromy. Let X be a vector field transverse to the interior of the ambient pages, vanishing on the binding, and normalised by $\pi^* d\theta(X) = 1$. Recall from Section 2.1 that the time-2π map ϕ of the flow of X yields the monodromy of the ambient open book. Furthermore, if we assume that X is tangent to the submanifolds M_i', we obtain abstract nested open book descriptions (Σ_i, ϕ_i) of M_i' within the abstract ambient open book (Σ, ϕ). Moreover, by adapting the vector field if necessary, we can choose embeddings of the normal bundles of M_i' such that the fibres are preserved under the flow of X. The normal bundles of M_0' and M_1' being isomorphic translates into the condition that the normal bundles $\nu\Sigma_i'$ of the induced (abstract) nested pages in the ambient (abstract) page Σ are ϕ-equivariantly isomorphic.

For the remaining part of the section we identify $\nu M_i'$ with the quotient

$$(\nu\Sigma_i' \times [0, 2\pi])/\sim_\phi .$$

Now let Ψ_0 be the ϕ-equivariant fibre-orientation reversing isomorphism of $\nu\Sigma_i'$ induced by the restriction of Ψ. Moreover we define

$$\Psi_t := \Psi|_{\nu\Sigma_0' \times \{t\}}.$$

Note that each Ψ_t is isotopic to Ψ_0, the whole family $\{\Psi_t\}_t$ however defines an (*a priori*) non-trivial loop of maps $\nu\Sigma_0' \to \nu\Sigma_1'$ based at Ψ_0. By choosing suitable bundle metrics, this loop yields an (*a priori*) non-trivial loop $\{\mathcal{D}_t\}_t$ of maps $\Sigma' \to S^1$ based at the identity via

$$\mathcal{D}_t(x) \cdot \Psi_0(q) := \Psi_t(q),$$

for $x \in \Sigma'$ and $q \neq 0$ a non-trivial point in the normal-fibre over x. With this in hand we can define a monodromy-like map of $\nu\Sigma_1'$ which is the identity in a neighbourhood of the zero section and outside the unit-disc bundle by

$$\mathcal{D}(q) := \mathcal{D}_{\boldsymbol{r}(x)} \cdot q,$$

where $\boldsymbol{r}$ is a radial cut-off function in the fibre which is 1 on the zero section and vanishes away from it. We call it the **twist map** induced by ϕ and Ψ. Given this map we can now give an abstract description of the open book in Lemma 3.2. Recall that we already identified the page as the fibre sum of the original page along the nested pages.

Lemma 3.3. *Let* Ψ_0*,* ϕ *and* $\mathcal{D}$ *be the maps described in the above paragraph. Then the monodromy of the open book in* Lemma 3.2 *is given by* $\phi \circ \mathcal{D}$*, and the page is* $\#_{\Psi_0}\Sigma$. □

4. The push-off

In this section we describe a *push-off* of the binding of an open book which realises it as a nested open book. The push-off construction will enable us to describe a natural open book structure on the fibre connected sum of two open books along their diffeomorphic bindings. We will first describe how the binding is being pushed away from itself and then introduce a natural framing of the pushed-off copy in Subsection 4.1, which will be equivalent to the canonical *page framing* of the binding. In Subsection 4.2 we show that the push-off can be realised as an *abstract nested open book*.

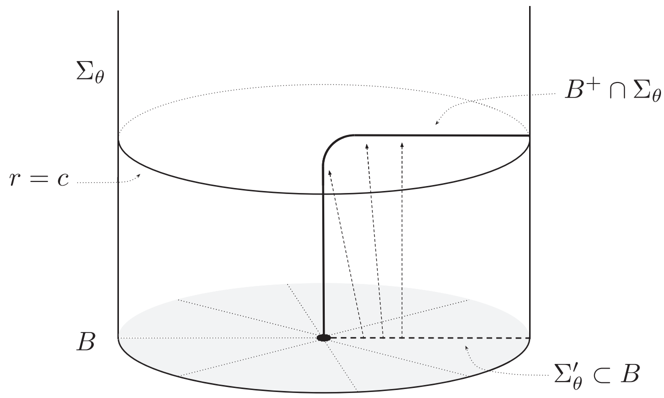

Fig. 2. The page Σ'_θ pushed into Σ_θ

Let M be a manifold with open book decomposition (Σ, ϕ) and binding B which also admits an open book decomposition (Σ', ϕ'). We denote the fibration maps by $\pi\colon\ M\backslash B \to S^1$ and $\pi'\colon\ B \setminus B' \to S^1$, respectively. Our aim is to define a push-off B^+ of the binding B in such a way that each page Σ'_θ of the binding open book is pushed into Σ_θ, the page corresponding to the same angle θ in the ambient open book. As all our constructions are local in a neighbourhood of the binding B, we can assume, without loss of generality, that ϕ is the identity.

Identify a neighbourhood of the binding $B' \subset B$ of the open book of the binding B with $B' \times D^2$ with coordinates (b', r', θ') such that (r', θ') are polar coordinates on the D^2-factor and θ' corresponds to the fibration π' – these are standard coordinates for a neighbourhood of a binding of an open book. We will also use Cartesian coordinates x', y' on the D^2-factor. Analogously, we have coordinates (b, r, θ) in a neighbourhood of $B \subset M$ with the corresponding properties. Combining these, we get two sets of coordinates on $(B' \times D^2) \times D^2 \subset M$:

$$(b', r', \theta', r, \theta) \text{ and } (b', x', y', x, y).$$

First, we will describe the geometric idea of the push-off by considering just a single page Σ_θ of the open book before defining it rigorously afterwards, see Figure 2. The page Σ' of the binding open book corresponding to angle θ is pushed into the page Σ of the ambient open book corresponding to the same angle θ. Restricted to a single page of the binding open book, the push-off depends on the radial direction r' only and is invariant in the B'-component. In particular, the boundary of the page Σ' stays fixed. We divide the collar neighbourhood in Σ' into four parts by the collar parameter r'. The outermost one consisting of points in Σ' with $r' \leq \epsilon_1$ is mapped to run straight into the r-direction of the ambient page Σ. The innermost part consisting of points with $r' \geq \epsilon_3$ is translated by a constant c into the r-direction. This translation is extended over the whole of Σ'. On the rest of the collar the push-off is an interpolation between these innermost and outermost parts. This is done such that points with $\epsilon_1 \leq r' \leq \epsilon_2$ are used to interpolate in r-direction and points with $\epsilon_2 \leq r' \leq \epsilon_3$ in r'-direction.

Let $f, h\colon \mathbb{R}_0^+ \to \mathbb{R}$ be the smooth functions described in Figure 3, i.e. they have the following properties:

- $f(r) = 0$ for $r \leq \epsilon_1$ and $f(r) = r$ for $r \geq \epsilon_3$,
- $f'(r) > 0$ for $r > \epsilon_1$,
- $h(0) = 0$, $h'(0) = 1$ and $h(r) \equiv c$ for $r \geq \epsilon_2$,
- $h(r) > 0$ for $r > 0$ and $h'(r) > 0$ for $0 < r < \epsilon_2$.

By choosing f small between ϵ_1 and ϵ_2, the "curved part" of the push-off can be realised in arbitrarily small. Recall that B can be decomposed as $(B' \times D^2) \cup \Sigma'(\phi')$. Let $g\colon B \to B \times D^2 \subset M$ be the embedding defined by

$$g(b) = \begin{cases} ((b', f(r') \cdot e^{i\theta'}), h(r') \cdot e^{i\theta'}) & \text{for } b = (b', r'e^{i\theta'}) \in B' \times D^2 \\ ([x', \theta'], c \cdot e^{i}\theta') & \text{for } b = [x', \theta'] \in \Sigma'(\phi'). \end{cases}$$

Observe that g is well-defined and a smooth embedding.

Definition 4.1. We define the **push-off B^+ of B** as the image of the embedding g defined above, i.e. we define

$$B^+ := g(B).$$

Observe that we can easily obtain an isotopy between the binding B and the push-off B^+ by parametrising f and h.

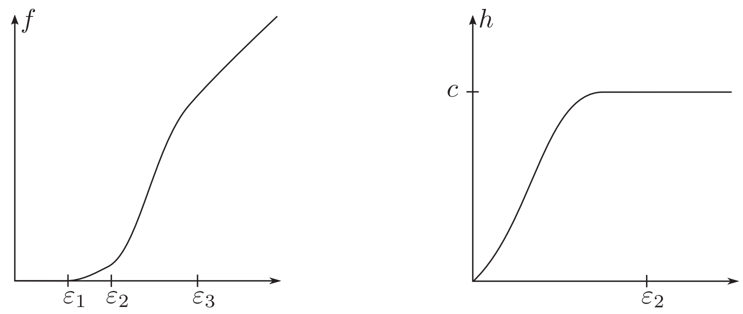

Fig. 3. The functions f and h

Remark 4.2. We call the submanifold B^+ a *push-off* of B although it is not a push-off in the usual sense since B and B^+ are not disjoint. However, it generalizes the notion of a push-off of a *transverse* knot in a contact manifold. Note that $B^+ \cap \Sigma_\theta \cap \{r = c\}$ is a copy of the interior of the page Σ'_θ of the binding and $B^+ \cap \Sigma_\theta \cap \{r = r_0\} \cong B' \times \{r_0\}$ with $r_0 < c$.

4.1. Framings. The fibre connected sum explained in Section 2.2 requires the submanifolds to have isomorphic normal bundles and explicitly uses a given bundle isomorphism. The binding of an open book has trivial normal bundle. Hence it is sufficient to specify a framing, i.e. a trivialisation of its normal bundle, to be able to perform a fibre connected sum along the binding. Note that because the codimension of the binding is two, this can be done by specifying a push-off, or equivalently a non-zero vector field along the submanifold that is nowhere tangent, by considering the normal bundle as a complex line bundle.

A natural framing of the binding $B \subset M$ of an open book is the **page framing** obtained by pushing B into one fixed page of the open book. We denote the page framing given by ∂_x by F_0, i.e.

$$F_0 := \partial_x.$$

Next we are going to define a framing for the push-off B^+. Let $\widetilde{u}\colon\ M \to \mathbb{R}$ be a smooth function such that

- $\widetilde{u} \equiv 0$ near B and on $B' \times \{r' \leq \epsilon\} \times D^2_{c-\epsilon}$,
- $\widetilde{u} \equiv 1$ on $B' \times \{r' \geq \epsilon\} \times \{r = c\}$ and outside $\{r \leq c + \epsilon\}$,
- $\widetilde{u}$ is monotone in r'- and r-direction.

With this in hand we define a framing of the push-off B^+ by

$$F_1 := -(1 - \widetilde{u})\partial_{x'} - \widetilde{u} \cdot (\sin^2\theta\partial_{x'} - \cos\theta\partial_r).$$

One easily checks that this is indeed nowhere tangent to B^+. The push-off B^+ with the framing F_1 is in fact equivalent to the binding B with its natural page framing F_0.

Lemma 4.3. *The framed submanifolds* (B, F_0) *and* (B^+, F_1) *are isotopic.*

Proof. This is a direct calculation, see [5] for details. □

4.2. The push-off as an abstract nested open book. The push-off B^+ is clearly an embedded nested open book of $M = (B, \pi)$. The aim of this section is to obtain a description of the push-off as an *abstract* nested open book, and ultimately as a *framed* abstract nested open book by altering the monodromy of the abstract open book (Σ, ϕ).

Identify a neighbourhood of $B' \subset B$ with $B' \times D^2$ as above, i.e. the pages are defined by the angular coordinate θ'. Also denoting the coordinate on S^1 by θ', we can define a non-vanishing 1-form on $B \setminus B'$ by the pull–back of $d\theta'$ under the fibration map $\pi'\colon\ B \setminus B' \to S^1$. With the help of this 1-form we can extend the vector field $\partial_{\theta'}$ to a vector field X' on B by prescribing the condition $(\pi')^*d\theta'(X') = 1$. The vector field X' can furthermore be extended trivially to a neighbourhood $B \times D^2$ of B in M. Likewise, we obtain an abstract open book description of M by regarding the time-2π map of a suitable vector field on $M \setminus B$. Let X denote the vector field that recovers the abstract open book (Σ, ϕ). Let $u\colon\ \mathbb{R}^+_0 \to \mathbb{R}$ be the smooth function depicted in Figure 4 with c as in the definition of the push-off B^+. Then $\widetilde{X} := X + u(r)X'$ on $M \setminus B$ defines a vector field on $M \setminus B$.

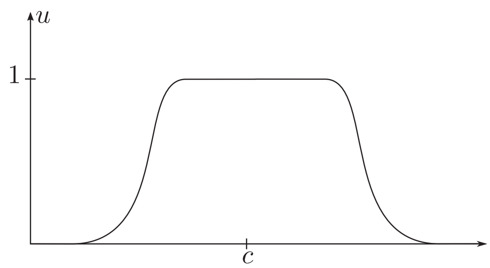

Fig.4. The function u

We claim that $\widetilde{X}$ realises the push-off B^+ as an abstract nested open book of an abstract open book description of M. Observe that, as X' is tangent to the pages of (B, π), the condition $\pi^* d\theta(\widetilde{X}) = 1$ is satisfied and that $\widetilde{X}$ and X coincide near the binding B. Thus, the vector field $\widetilde{X}$ does indeed yield an abstract open book description of M. Furthermore, the vector field $\widetilde{X}$ is tangent to the push-off B^+, which means that it realises B^+ as an abstract nested open book of the abstract ambient open book. The monodromy is given by the time-2π flow of $\widetilde{X}$. However, we want to give a description that better encodes the change of the monodromy ϕ of the ambient open book we started with in terms of the monodromy of the binding.

We denote the flow of X' on B by ϕ'_t and use it to define diffeomorphisms ψ_t of a neighbourhood $B \times D^2$ of B in M:

$$\psi_t(b, r, \theta) = (\phi'_t(b), r, \theta).$$

Definition 4.4. Define a diffeomorphism $\psi\colon M \to M$ via $\psi := \psi_{2\pi u(r)}$ and refer to it as **Chinese burn** along B. By abuse of notation, its restriction to a single page Σ_θ is also denoted by ψ.

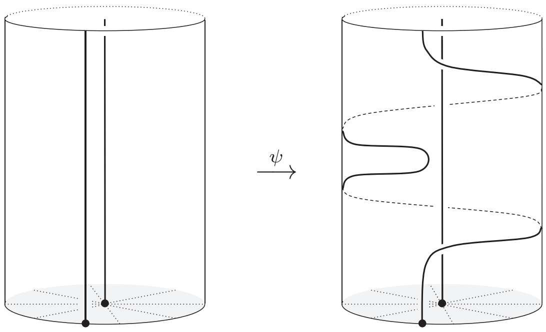

Fig. 5. A Chinese burn along a boundary component.

Observe that the monodromy of the abstract open book obtained from the vector field $\widetilde{X}$ is $\phi \circ \psi$, i.e. the push-off B^+ yields an an abstract nested open book of $(\Sigma, \phi \circ \psi)$. We thus proved the following statement.

Lemma 4.5. *The push-off B^+ induces an abstract nested open book of $(\Sigma, \phi \circ \psi)$ with page diffeomorphic to Σ' (more concretely, the page is $g|_{\Sigma'_0}(\Sigma'_0) \cong \Sigma'$), where ψ is a Chinese burn along B.*

We constructed the push-off B^+ inside the manifold $M(\Sigma, \phi)$ and equipped it with a natural framing F_1 corresponding to the page framing. In particular, the push-off is a *framed* nested open book, i.e. a nested open book with a specified framing. The previous lemma shows that the push-off also defines an *abstract* nested open book of $(\Sigma, \phi \circ \psi)$. However, the framing F_1 does *a priori* not give a framing in the abstract setting since it is not invariant under the monodromy. We call an abstract nested open book with a framing which is invariant under the monodromy a **framed abstract nested open book**.

Remark 4.6. Given two diffeomorphic framed nested open books with pages $\Sigma'_0, \Sigma'_1 \subset (\Sigma, \phi)$ and isomorphic normal bundles, it is easy to obtain an open book structure of their fibre connected sum. The new page is

$$\widetilde{\Sigma} := (\Sigma \setminus (\nu\Sigma'_0 \cup \nu\Sigma'_1))/\sim$$

with the identification induced by the given framings, and the old monodromy ϕ restricts to the new monodromy.

A natural framing of the push-off in the abstract setting is the following: We define the **constant framing** F_2 as

$$F_2\colon\ \widetilde{u}\partial_r - (1-\widetilde{u})\partial_{x'},$$

where $\widetilde{u}$ is the restriction of the function $\widetilde{u}\colon\ M \to \mathbb{R}$ defined in Section 4.1 to a page Σ.

To realise the push-off as a framed abstract nested open book with framing F_2, we have to alter the monodromy of the ambient abstract open book. We will change the monodromy by a certain diffeomorphism of the page fixing the push-off, the so-called *twist map*. Let $\sigma\colon\ \mathbb{R} \to \mathbb{R}$ be a cut-off function with $\sigma(0) = 0$ and $\sigma(\epsilon) = 1$ and

$$\tau_{\pm}^{\sigma(r)}\colon\ D^2 \to D^2$$

a smoothened Dehn twist of the disc. That is, a diffeomorphism with the qualitative behaviour of

$$(s,\theta) \mapsto (s, \theta \pm 2\pi\sigma(r)(1-s)),$$

smoothened near the boundary and the origin, such that the origin is an isolated fixed point and a neighbourhood of the boundary is fixed. This can be achieved by constructing it as the flow of an appropriate vector field. Recall that, by construction, the intersection of the push-off of the binding B with a single page Σ of the ambient open book is a copy of a page Σ' of the open book of the binding. In particular, we can identify a tubular neighbourhood of $B^+ \cap \Sigma_0$ in Σ_0 with $\Sigma' \times D^2 \subset \Sigma$. Observe that the r-coordinate can be regarded as a collar parameter on Σ'. We define a diffeomorphism of a neighbourhood of the collar by

$$\begin{array}{ccc} \partial\Sigma' \times [0,\epsilon] \times D^2 & \to & \partial\Sigma' \times [0,\epsilon] \times D^2 \\ (b', r, p) & \mapsto & (b', r, \tau_{-}^{\sigma(r)}(p)). \end{array}$$

We can extend this to a diffeomorphism $\Sigma' \times D^2 \to \Sigma' \times D^2$ of the whole tubular neighbourhood of Σ' via $\mathrm{id}_{\Sigma'} \times \tau_{-}^{1}$. Furthermore, this map can be extended to a self-diffeomorphism $\mathcal{D}\colon\ \Sigma \to \Sigma$ of the page Σ via the identity.

Definition 4.7. The diffeomorphism $\mathcal{D}\colon\ \Sigma \to \Sigma$ is called **twist map**.

The twist map $\mathcal{D}$ is isotopic to the identity, so we have $M_{(\Sigma,\phi\circ\psi)} \cong M_{(\Sigma,\phi\circ\psi\circ\mathcal{D})}$.

Lemma 4.8. *The push-off B^+ with its induced framing F_1 corresponds to the framed abstract nested open book of $(\Sigma, \phi \circ \psi \circ \mathcal{D})$ with page $g|_{\Sigma'_0}(\Sigma'_0) \cong \Sigma'$ framed by the natural framing F_2.*

Proof. Again, this is a straightforward calculation, see [5] for details. □

5. The main result

Combining the results of the previous sections, in particular Lemma 4.3, which states that the binding is isotopic to its push-off, and Lemma 3.2, which yields a natural open book structure on the fibre connected sum along push-offs, we are now able to give an explicit open book decomposition for the binding sum operation and thus state our main result, or, more concretely, the following theorem.

Theorem 5.1. *Let M be a (not necessarily connected) smooth manifold with open book decomposition (Σ, ϕ) whose binding B contains two diffeomorphic components B_0, B_1 with diffeomorphic open book decompositions (Σ', ϕ'). Then the fibre connected sum of M performed along B_0 and B_1 with respect to the page framings admits an open book decomposition naturally adapted to the construction. The new page is the fibre connected sum of the page Σ along the nested pages Σ'_i induced by the push-offs of the bindings components B_i. The new binding is given by a fibre connected sum of B_0 and B_1 along their respective bindings $B' = \partial\Sigma$. The monodromy remains unchanged outside of a neighbourhood of the original binding components B_i, and over the remaining part it restricts to $\psi \circ \mathcal{D}$, where ψ is a* Chinese burn *along B_i (see* Definition 4.4*) and $\mathcal{D}$ the* twist map *(see* Definition 4.7*).* □

It is worth noting that the new page can be obtained from the old one by two consecutive generalised 1-handle attachments of type Σ', where a generalized 1-handle in the sense of [16] is of the form $H_{\Sigma'} = D^1 \times (\Sigma' \times D^1)$. In case Σ can be endowed with an appropriate exact symplectic form, generalised 1-handles can be adapted to the contact setting, and naturally extends the symplectic handle constructions due to Eliashberg [7] and Weinstein [27].

6. Application to contact topology

In this section we want to show that the binding sum of contact open books yields an open book and a contact structure that again fit nicely together.

For an introduction to open books in contact topology we kindly refer the reader to [8, 10, 26]. We will always assume contact structures to be coorientable.

A positive contact structure ξ on an oriented manifold M is **supported** by an open book structure (B, π) if it can be written as the kernel of a contact form α inducing a positive contact structure on B and such that $d\alpha$ induces a positive symplectic structure on the fibres of π. Such a contact form α is then called **adapted** to the open book. Note that if the binding B of an open book (B, π) is already assumed to be a contact manifold, then a contact form is adapted to the open book if and only if its Reeb vector field is positively transverse to the fibres of π (cf. [26, Lemma 2.13]).

Theorem 6.1. *The binding sum construction can be made compatible with the underlying contact structures, i.e. the contact structure obtained by the contact fibre connected sum along contactomorphic binding components is supported by the natural open book structure resulting from the sum along the push-offs of the respective binding components.*

We proceed by introducing some further terminology needed in the remainder of the section. Let $(M, \xi = \ker\alpha)$ be a contact manifold supported by an open book (B, π) and denote the pull-back of $d\theta$ under $\pi\colon M \setminus B \to S^1$ also by $d\theta$. A vector field X is called **monodromy vector field** if

- it is transverse to the pages and satisfies $d\theta(X) = 1$ on $M \setminus B$,
- the restriction of $\mathcal{L}_X d\alpha$ to any page vanishes,

- it equals ∂_θ on a neighbourhood $B \times D^2 \subset M$ of the binding, where (r, θ) are polar coordinates on the D^2-factor, and the open book fibration is given by the angular coordinate on D^2.

Given a monodromy field we get an associated abstract contact open book description of (M, ξ), i.e. a triple $(\Sigma, \phi, d\lambda)$ consisting of an exact symplectic page $(\Sigma, d\lambda)$ and a symplectomorphism ϕ such that the Liouville vector field Y defined by $i_Y\omega = \lambda$ is transverse to the boundary $\partial\Sigma$ pointing outwards (and thus induces a contact structure on $\partial\Sigma$), and in turn an identification of (M, ξ) as a generalised Thurston–Winkelnkemper construction (cf. [10, Section 7.3]). In particular, such a vector field always exists if the open book and contact structure comes from a generalized Thurston–Winkelnkemper construction. Since any contact manifold supported by an open book can be realised by a generalised Thurston–Winkelnkemper construction (e.g. see [10, Section 7.3] or [6, Theorem 3.1.22]), we can always assume the existence of a compatible monodromy vector field. An adapted Reeb vector field is not a monodromy vector field as it does not fix the binding point-wise.

Define a **Lutz pair** h_1, h_2 as a pair of smooth functions $h_1\colon [0, 1] \to \mathbb{R}^+$ and $h_2\colon [0, 1] \to \mathbb{R}^+_0$ such that

- $h_1(0) = 1$ and h_2 vanishes like r^2 at $r = 0$,
- $h_1'(r) < 0$ and $h_2'(r) \geq 0$ for $r > 0$,
- all derivates of h_1 vanish at $r = 0$.

We will now define nested open books in the contact world.

Definition 6.2. Let $(\Sigma, d\lambda, \phi)$ be a contact open book and $\Sigma' \subset \Sigma$ a symplectic submanifold with boundary $\partial\Sigma' \subset \partial\Sigma$ such that in a collar neighbourhood $\partial\Sigma \times (-\varepsilon, 0]$ of $\partial\Sigma$ given by an outward-pointing Liouville vector field we have $\Sigma' \cap (\partial\Sigma \times (-\varepsilon, 0]) = \partial\Sigma' \times (-\varepsilon, 0]$. Suppose furthermore that $\phi(\Sigma') = \Sigma'$, i.e. the monodromy leaves Σ' invariant (not necessarily pointwise). Then Σ' is a **contact abstract nested open book** of the contact open book $(\Sigma, d\lambda, \phi)$.

Note that the fibre connected sum construction can be adapted to the contact setting as follows (we will only apply it in the case of trivial normal bundles where the existence of the bundle isomorphism Φ is always granted).

Theorem 6.3 ([10, Theorem 7.4.3])**.** *Let (M, ξ) and (M', ξ') be contact manifolds of dimension $\dim M' = \dim M - 2$, where the contact structures ξ, ξ' are assumed to be cooriented; these cooriented contact structures induce orientations of M and M'. Let $j_0, j_1\colon (M', \xi') \to (M, \xi)$ be disjoint contact embeddings that respect the coorientations, and such that there exists a fibre-orientation-reversing bundle isomorphism $\Phi\colon N_0 \to N_1$ of the normal bundles of $j_0(M')$ and $j_1(M')$. Then the fibre connected sum $\#_\Phi M$ admits a contact structure that coincides with ξ outside tubular neighbourhoods of the submanifolds $j_0(M')$ and $j_1(M')$.*

The binding of a contact open book decomposition is a contact submanifold and hence admits an open book structure itself. Furthermore, it has trivial normal bundle. Given two contact open books with contactomorphic bindings, we can thus perform the contact fibre connected sum along their bindings, and, topologically, also along the push-offs B_i^+ of the bindings. We want to show that this topological construction can be adapted to the contact scenario.

In fact, the push-off is a contact submanifold contact isotopic to the binding. Thus, we can form the contact fibre connected sum along the push-off rather than along the binding itself when performing the contact binding sum.

Proposition 6.4. *The push-off B^+ of the binding B of a contact open book (B, π) is a contact submanifold contact isotopic to the binding.*

Proof. We first show that the push-off B^+ of the binding B is a contact submanifold of (M, ξ). The binding B is a contact submanifold, so in a neighbourhood $B \times D^2$ (as described in Section 4) we can assume our contact form α to be

$$\alpha = h_1\alpha_B + h_2 d\theta,$$

where α_B is a contact form on B and $(h_1(r), h_2(r))$ a Lutz pair. To prove that the push-off B^+, which arises as the image of the embedding $g\colon\ B \to M$ (see Definition 4.1), is a contact submanifold of M, we have to show that $g^*\alpha$ is a contact form on B. We will first verify this condition away from the binding B' of B. Here, g sends an element $[x', \theta']$ in the mapping torus part of the open book of B to $([x', \theta'], c, \theta') \in B \times D^2$ for constant $c > 0$. So we have

$$g^*\alpha = h_1(c)\alpha_B + h_2(c)d\theta'$$

and $dg^*\alpha = h_1(c)d\alpha_B$. Hence,

$$g^*\alpha \wedge (dg^*\alpha)^{n-1} = (h_1(c))^n\alpha_B \wedge (d\alpha_B)^{n-1} + (h_1(c))^{n-1}h_2(c)d\theta' \wedge (d\alpha_B)^{n-1}.$$

Observe that $(h_1(c))^{n-1}h_2(c) > 0$. Furthermore, since $d\theta'(R_{\alpha_B}) > 0$, the forms $\alpha_B \wedge (d\alpha_B)^{n-1}$ and $(h_1(c))^{n-1}h_2(c)d\theta' \wedge (d\alpha_B)^{n-1}$ induce the same orientation, i.e. $g^*\alpha$ is indeed a contact form away from the binding.

Now we inspect the situation near the binding B' of B, i.e. we can work in a neighbourhood $B' \times D^2 \times D^2$ (as described in Section 4). The contact form α_B of the binding can be assumed to be of the form

$$\alpha_B = g_1\alpha_{B'} + g_2 d\theta',$$

where $\alpha_{B'}$ is a contact form on B' and $(g_1(r'), g_2(r'))$ a Lutz pair. Thus, we have

$$\alpha = h_1g_1\alpha_{B'} + h_1g_2d\theta' + h_2d\theta.$$

Recall that the defining embedding g for the push-off is given by

$$g(b', r', \theta') = (b', f(r'), \theta', h(r'), \theta')$$

in this neighbourhood.

We compute

$$g^*\alpha = \lambda\alpha_{B'} + \mu d\theta'$$

with

$$\lambda(r') = (h_1 \circ h)(g_1 \circ f)(r')$$

and

$$\mu(r') = \big((h_1 \circ h)(g_2 \circ f) + h_2 \circ h\big)(r').$$

So we have

$$g^*\alpha = \lambda'\alpha_{B'} + \lambda d\alpha_{B'} + \mu' d\theta',$$

$$(dg^*\alpha)^{n-1} = (n-1)\lambda^{n-2}(d\alpha_{B'})^{n-2} \wedge (\lambda' dr' \wedge \alpha_{B'} + \mu' dr' \wedge d\theta')$$

and

$$(g^*\alpha) \wedge (dg^*\alpha)^{n-1} = \frac{1}{r'}(n-1)\lambda^{n-2}(\lambda\mu' - \lambda'\mu)(\alpha_{B'} \wedge (d\alpha_{B'})^{n-2} \wedge r'dr' \wedge d\theta').$$

It remains to show that the term $\lambda\mu' - \lambda'\mu$ is positive. A calculation shows

$$\begin{aligned} \lambda\mu' - \lambda'\mu &= \underbrace{(h_1 \circ h)^2\big((g_1 \circ f)(g_2 \circ f)' - (g_1 \circ f)'(g_2 \circ f)\big)}_{=:A} \\ &+ \underbrace{(h_1 \circ h)\big((h_2 \circ h)'(g_1 \circ f) - (g_1 \circ f)'(h_2 \circ h)\big)}_{=:B} \\ &+ \underbrace{\big(-(h_1 \circ h)'(g_1 \circ f)(h_2 \circ h)\big)}_{=:C}. \end{aligned}$$

Observe that all three summands A, B, and C are non-negative. It thus suffices to show that at least one of them is positive. Assume $C = 0$. Then either $r' = 0$ or $h' = 0$. We will first deal with the case $h' = 0$. This happens exactly where $h \equiv c$. But on this set, both f and its derivative f' are positive and therefore

$$A = (h_1 \circ h(c))^2 f'\big((g_1 g_2' - g_1' g_2) \circ f\big) > 0,$$

because (g_1, g_2) is a Lutz pair, i.e. in particular $(g_1 g_2' - g_1' g_2) > 0$. Now consider $r' = 0$. For small r' we have $f \equiv 0$ and thus $g_1 \circ f \equiv 1$ and $g_2 \circ f \equiv 0$. Then $\lambda\mu' - \lambda'\mu$ reduces to

$$(h_1 \circ h)(h_2 \circ h)' - (h_1 \circ h)'(h_2 \circ h) = h'\big((h_1 h_2' - h_1' h_2) \circ h\big).$$

As h' is positive for small r' and (h_1, h_2) is a Lutz pair, this is positive. Hence, $(g^*\alpha) \wedge (dg^*\alpha)^{n-1}$ is a volume form and so $(g^*\alpha)$ a contact form, i.e. the push-off B^+ is indeed a contact submanifold.

Observe that by varying c and the ε_i in the definition of the push-off, we get parametrised versions of the functions f and h yielding a homotopy from f to the identity and from h to the constant zero function, which in turn gives a topological isotopy betweeen the push-off and the binding. A similar calculation to the one above then shows that the push-off B^+ is furthermore contact isotopic to the binding B. □

A first step of showing that the topological binding sum construction along the push-off can be made compatible with the underlying contact structures is to show that the push-off can be realised as a suitable abstract contact nested open book. To this end, we first show that there exists a monodromy vector field tangent to the push-off.

Proposition 6.5. *There exists a monodromy vector field (in the contact geometric sense defined above) tangent to the push-off.*

Proof. Without loss of generality, we can assume that the ambient contact manifold $(M, \ker\alpha)$ with contact open book (B, π) arises from an abstract contact open book $(\Sigma, d\lambda, \phi)$ by a generalised Thurston–Winkelnkemper construction. The push-off B^+ of the binding B is contained in a trivial neighbourhood $B \times D^2$, on which the monodromy ϕ restricts to the identity and the contact form α is given by $\alpha = h_1\alpha_B + h_2 d\theta$, where (h_1, h_2) is a Lutz pair and α_B a contact form on B with the induced contact structure. Hence, we can furthermore assume that $\phi = \mathrm{id}_\Sigma$, which implies that ∂_θ is a monodromy vector field.

By construction, the intersection of the push-off B^+ with the fibres of π is a cylinder over the binding B' of the compatible open book decomposition with page Σ' used in the construction of the push-off, i.e. for some small constant $k > 0$, we have

$$B^+ \cap \pi^{-1}(\theta) \cap \{r \leq \varepsilon\} = B' \times (0, k] \subset B \times D^2.$$

We will now work in $M \setminus (B \times D^2_\varepsilon) \cong \Sigma \times S^1$ with $\varepsilon < k$ (i.e. we will work in a trivial mapping torus).

Observe that $B^+ \cap (\Sigma \times \{\theta\})$ is a codimension 2 symplectic submanifold with trivial normal bundle of $(\Sigma \times \{\theta\}, d\alpha|_{\Sigma\times\{\theta\}})$. Also, the symplectic structure on $\Sigma \times \{\theta\}$ induced by $d\alpha$ is independent of θ. Thus, we get a fibre-wise symplectic projection

$$p\colon\ \Sigma \times S^1 \to \Sigma.$$

This defines a family of symplectic submanifolds

$$\Sigma'_t := p(B^+ \cap (\Sigma \times \{t\}))$$

of Σ which all coincide near the boundary.

Auroux's version of Banyaga's isotopy extension theorem (see [2, Proposition 4], cf. [20, Theorem 3.19] and the proof of Corollary 6.6 below) for symplectic submanifolds then yields a symplectic isotopy

$$\phi_t\colon\ \Sigma \to \Sigma$$

with $\phi_t(\Sigma'_0) = \Sigma'_t$ in such a way that it is equal to the identity in a neighbourhood U_1 of $\partial\Sigma$ and outside a bigger neighbourhood U_2 of the boundary.

Differentiating ϕ_t yields a time-dependent vector field X_t on Σ, which can be assumed to coincide for $t = 0$ and $t = 2\pi$ (it extends a vector field along the submanifold Σ' with that property). Thus, X_t lifts to a vector field X on $\Sigma \times S^1$ with $d\theta(X) = 1$, whose projection to each fibre $\Sigma \times \{\theta\}$ is symplectic. Furthermore, the vector field X is equal to ∂_θ inside $U_1 \times S^1$ and outside $U_2 \times S^1$. To simplify notation, we set $V := (U_2 \setminus U_1) \times S^1$.

Now given any monodromy vector field Y which is equal to ∂_θ on $U_2 \times S^1$, we can replace Y by X over V to get a vector field $\widetilde{Y}$, which is tangent to the push-off by construction. We claim that $\widetilde{Y}$ is a monodromy vector field.

Indeed, we have $d\theta(\widetilde{Y}) = 1$ and near the binding $\widetilde{Y}$ equals ∂_θ. Furthermore, the Lie derivative of $d\alpha$ with respect to $\widetilde{Y}$ coincides with $\mathcal{L}_Y d\alpha$ outside of V and with $\mathcal{L}_X d\alpha$ on V. Hence, as Y is a monodromy vector field and X is symplectic on pages, the restriction of $\mathcal{L}_{\widetilde{Y}} d\alpha$ to any page vanishes, which means that $\widetilde{Y}$ is a monodromy vector field tangent to the push-off B^+. □

Corollary 6.6. *The push-off is an abstract nested open book of an abstract open book description with page Σ and monodromy ϕ, where ϕ restricts to*

$$\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$$

for a symplectomorphism $\phi' \colon \Sigma' \to \Sigma'$ in a neighbourhood $\Sigma' \times D^2$ of Σ' given by the symplectic normal bundle.

Proof. The main ingredient in the proof of Proposition 6.5 is Auroux's version of Banyaga's isotopy extension theorem [2, Proposition 4] (or rather its proof) which states that for a familiy W_t ($t \in [0, 1]$) of symplectic submanifolds of a symplectic manifold X, there exist symplectomorphisms $\phi_t \colon X \to X$ (depending continuously on t) such that $\phi_0 = \mathrm{id}$ and $\phi_t(W_0) = W_t$.

By the symplectic neighbourhood theorem (see [20, Theorem 3.30]) a neighbourhood of Σ' in Σ can be written as $\Sigma' \times D^2$ with split symplectic form. A smooth family of symplectic submanifolds can be assumed to arise as the image of an isotopy. The first step in proving Auroux's theorem is to extend this isotopy to an open neighbourhood. In our case, the symplectic neighbourhood theorem allows us to do this in a trivial way. Applying Auroux's construction to this trivial extension yields an isotopy which is invariant in fibre direction, i.e. the time-2π map of the resulting isotopy restricts to a map of the form $\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$ on the neighbourhood of Σ'. Hence, we have the corollary. □

Remark 6.7. Observe that by requiring the monodromy to be trivial in fibre direction, the resulting abstract nested open book is also a framed nested abstract open book in the sense of Section 4.2. The monodromy does not correspond to the Chinese burn Ψ but to the concatenation $\mathcal{D} \circ \Psi$ with the twist map $\mathcal{D}$, which was used to turn a nested open book into a framed nested open book in the topological setting and exactly ensured triviality in fibre direction.

Having described the push-off as an abstract nested open book, it is natural to perform a fibre sum construction of the ambient abstract open books. This will then yield a contact structure adapted to the resulting natural open book decomposition. However, it is a priori unclear whether this contact structure is indeed the contact structure resulting from the contact fibre connected sum.

We will first show that the symplectic fibre connected sum (cf. [20, Section 7.2]) of exact symplectic manifolds is exact under suitable conditions.

Remark 6.8. Note that the symplectic fibre sum can be adapted to work in a relative setting. Let W be a symplectic manifold with contact type boundary and let X be a codimension 2 symplectic submanifold with trivial normal bundle and contact type boundary which is a contact submanifold of ∂X. Let $\partial W \times (-\epsilon, 0]$ be a collar neighbourhood of ∂W given by a Liouville field transverse to ∂W and suppose that

$$X \cap (\partial W \times (-\epsilon, 0]) = \partial X \times (-\epsilon, 0].$$

The symplectic fibre sum can be performed in this setting with the effect on the boundary being a contact fibre connected sum.

The following technical lemma will be useful to interpolate between Liouville forms which agree on a symplectic submanifold.

Lemma 6.9. *Let M be a manifold and let λ_0 and λ_1 be two 1-forms on $M \times D^2$ such that $d\lambda_0 = d\lambda_1$ and $\lambda_0|_{T(M\times\{0\})} = \lambda_1|_{T(M\times\{0\})}$. Then $\lambda_1 - \lambda_0$ is exact.* □

Proposition 6.10. *Let $(W_i, \omega_i = d\lambda_i)$ $(i = 0, 1)$, be exact symplectic manifolds with symplectomorphic submanifolds $X_i \subset W_i$ of codimension 2 whose normal bundles are trivial. Assume furthermore that the restriction $\lambda|_{TX_i}$ of the Liouville forms to these submanifolds coincide through the above identification. Then the symplectic fibre sum of the W_i along the X_i' is again exact symplectic.*

Proof. We will drop the indices in the first part of the proof and work in W_0 and W_1 separately. The submanifold $X \subset W$ is symplectic with symplectic form

$$\omega' := \omega|_{TX} = (d\lambda)|_{TX} = d(\lambda|_{TX}),$$

i.e. X is exact symplectic and a Liouville form is given by $\lambda' := \lambda|_{TX}$. As X is of codimension 2 and has trivial normal bundle, the symplectic neighbourhood theorem (see [20, Theorem 3.30]) allows us to write a neighbourhood of X in W as $X \times D^2$ with symplectic form given as $\omega = \omega' + s ds \wedge d\vartheta$, where s and ϑ are polar coordinates on the D^2-factor. In these coordinates one (local) primitive of ω is given by $\lambda' + 1/2s^2 d\vartheta$. Hence, by Lemma 6.9, we have

$$\lambda|_{X\times D^2} = \lambda' + \frac{1}{2}s^2 d\vartheta + dh$$

for an appropriate function h.

Now by assumption, the restriction of the Liouville forms to the symplectic submanifolds X_i agree, so in the coordinates adapted to the symplectic normal bundle as above they are $\lambda_i = \lambda' + 1/2s^2 d\vartheta + dh_i$.

It follows that the symplectic fibre sum (cf. [20, Section 7.2]) along the symplectic submanifolds X_i is again exact symplectic. The Liouville form can be chosen to coincide with the original ones outside the area of identification and is given by

$$\lambda' + \frac{1}{2}s^2 d\vartheta + d((1-g)h_0 + gh_1)$$

on the annulus of identification. Here g is a function on the annulus equal to 1 near one boundary and equal to 0 near the other boundary component. □

We can now apply the proposition to the setting of abstract open books, which gives a contact version of the topological fibre sum of nested open books described in Remark 4.6.

Corollary 6.11. *Let $\Sigma_i' \subset (\Sigma_i, d\lambda_i, \phi_i)$, $i = 0, 1$, be two contact abstract open books with trivial normal bundle. Let $\psi\colon \nu(\Sigma_0') \to \nu(\Sigma_1')$ be a symplectomorphism of neighbourhoods with $\psi(\Sigma_0') = \Sigma_1'$ satisfying $\psi \circ \phi_0 = \phi_1 \circ \psi$ and $(\psi^*\lambda_1)|_{T\Sigma_0'} = \lambda_0|_{T\Sigma_0'}$. Then the fibre connected sum of the Σ_i along the Σ_i' with respect to ψ yields again an abstract open book.*

In particular, the symplectic fibre sum of two abstract open books along the push-offs of their contactomorphic bindings yields again an abstract open book. Hence, the topological binding sum along two contactomorphic binding components carries a contact structure which is adapted to the natural open book structure and coincides with the original structures outside a neighbourhood of the push-offs of the respective binding components.

Proof. We have to show that the symplectic fibre sum is again an exact symplectic manifold with a Liouville vector field pointing outwards at the boundary and that the original monodromies give rise to a monodromy on the fibre sum. The latter is ensured by the condition $\psi \circ \phi_0 = \phi_1 \circ \psi$. Exactness follows almost immediately from the preceding proposition. Observe that Σ_i' being contact abstract nested open books (cf. Definition 6.2) allows us to perform a relative version of the symplectic fibre connected sum as described in Remark 6.8. So by the proposition the fibre sum yields an exact symplectic manifold with boundary with Liouville field still pointing outwards.

Note that the description of the push-off as an abstract open book as in Corollary 6.6 fulfils the hypothesis of the first part of this corollary (the Liouville forms can be assumed to agree as the push-offs live in trivial neighbourhoods of contactomorphic binding components). Performing a generalised Thurston–Winkelnkemper construction on the resulting abstract open book then yields the second part of this corollary. □

6.1. Naturality of the contact structure. The preceding corollary ensures the existence of *a* contact structure adapted to the resulting open book structure on the fibre sum. It does not tell us however, that the contact structure from the *contact* fibre connected sum is adapted to this open book. This is what we want to show in the following. The problem is that the fibres of the symplectic normal bundle to the push-off B^+ are not tangent to the pages, i.e. the operation of the contact fibre connected sum, which uses these fibres, does not fit nicely to the open book structure. We are going to manipulate the abstract open book (without changing the underlying contact manifold) in such a way that the symplectic normal fibres of the push-off are tangent to the pages of the open book and thus guaranteeing compatibility of open book structure and fibre sum.

Lemma 6.12. *Let X be a codimension 2 symplectic submanifold of an exact symplectic manifold $(W, \omega = d\lambda)$ and suppose that the normal bundle of X is trivial. Then there is a Liouville form $\widetilde{\lambda}$ such that the corresponding Liouville vector field is tangent to X.*

Proof. The submanifold $X \subset W$ is symplectic with symplectic form

$$\omega' := \omega|_{TX} = (d\lambda)|_{TX} = d(\lambda|_{TX}),$$

i.e. X is exact symplectic and a Liouville form is given by $\lambda' := \lambda|_{TX}$. As X is of codimension 2 and has trivial normal bundle, the symplectic neighbourhood theorem (see [20, Theorem 3.30]) allows us to write a neighbourhood of X in W as $X \times D^2$ with symplectic form given as $\omega = \omega' + s ds \wedge d\vartheta$, where s and ϑ are polar coordinates on the D^2-factor. In these coordinates one (local) primitive of ω is given by $\widetilde{\lambda} := \lambda' + 1/2 s^2 d\vartheta$. Observe that the restriction of both λ and $\widetilde{\lambda}$ to X equals λ'. Hence, by Lemma 6.9, we have

$$\lambda|_{X \times D^2} = \widetilde{\lambda} + dh$$

for an appropriate function h. Consider a function $\widetilde{h}$ on W which is equal to h near X and vanishes outside a neighbourhood of X, and denote its Hamilton vector field by X_h. If Y is the Liouville vector field corresponding to the Liouville form λ, then the sum $Y + X_h$ is again Liouville. The associated Liouville form restricts to $\widetilde{\lambda}$ near X. In particular, the Liouville vector field is tangent to X. □

Remark 6.13. Also note that the contact structures on the open book obtained by the generalised Thurston–Winkelnkemper construction performed with two Liouville forms λ and $\lambda + dh$ which coincide near the boundary are isotopic. Indeed, a family of contact structures is given by using $\lambda + d(sh)$ for $s \in [0, 1]$ and Gray stability can be applied.

Proposition 6.14. *Let (W, ω) be an exact symplectic manifold and $X \subset W$ a symplectic submanifold of codimension 2 with trivial symplectic normal bundle. Let λ_t $(t \in \mathbb{R})$ be a smooth family of Liouville forms such that the corresponding Liouville vector fields Y_t are tangent to X and such that the 1-form $\frac{d}{dt}\lambda_t$ vanishes on the fibres of the symplectic normal bundle of X.*

Then the fibres of the conformal symplectic normal bundle of the contact submanifold $X \times \mathbb{R} \subset (W \times \mathbb{R}, \alpha = \lambda_t + dt)$ coincide with the fibres of the symplectic normal bundle of $X \times \{t\}$ in $W \times \{t\}$. In particular, they are tangent to the slices $W \times \{t\}$.

Proof. The proof is a straight-forward calculation in a symplectic neighbourhood $X \times D^2$ of X. See [5] for details. □

We will only need the proposition in a special case during a generalised Thurston–Winkelnkemper construction. However, note that it also holds for a family

$$\lambda_t = (1 - \mu(t))\lambda + \mu(t)\widetilde{\lambda}$$

interpolating two Liouville forms with tangent Liouville vector field provided that their difference is exact with a primitive function only depending on Σ'-directions. This can then be used when working with Giroux domains (cf. Section 6.2.3) instead of the Thurston–Winkelnkemper construction.

We can now show that the binding sum construction can be made compatible with the underlying contact structures and thus prove Theorem 6.1.

Proof of Theorem 6.1. By Corollary 6.6 the push-off is an abstract nested open book of an abstract open book description with page Σ and monodromy ϕ, where ϕ restricts to

$$\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$$

for a symplectomorphism $\phi' \colon \Sigma' \to \Sigma'$ in a neighbourhood $\Sigma' \times D^2$ of Σ' given by the symplectic normal bundle. Furthermore, we can assume that the Liouville form restricts to $\lambda = \lambda' + 1/2s^2 d\vartheta$ in this neighbourhood by Lemma 6.12 and its proof and Remark 6.13.

Note that the monodromy ϕ will not necessarily be exact symplectic but according to [10, Lemma 7.3.4] it is isotopic through symplectomorphisms equal to the identity near the boundary to an exact symplectomorphism. The idea of the proof is to define a vector field X on Σ by the condition $i_X\omega = \lambda - \phi^*\lambda$ and checking that precomposing ϕ with the time-1 flow of X is an exact symplectomorphism with the desired properties. Now in our situation, observe that the vector field X is tangent to Σ', and moreover, projects to zero under the natural projection $\Sigma' \times D^2 \to D^2$ also in a neighbourhood $\Sigma' \times D^2$ as above. As a consequence, the restriction of the resulting monodromy (still denoted by ϕ by abuse of notation) will still be of the form

$$\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$$

but now for an *exact* symplectomorphism $\phi' \colon \Sigma' \to \Sigma'$. In particular, we have $\phi^*\lambda - \lambda = dh$

for a function h which only depends on the Σ'-directions on $\Sigma' \times D^2$.

If we form the generalised mapping torus $\Sigma_h(\phi)$ and equip it with the contact form $\lambda + dt$, Proposition 6.14 tells us that the fibres of the conformal symplectic normal bundle of the push-off are tangent to the slices $\Sigma \times \{t\}$ and are given by the D^2-direction of $\Sigma' \times D^2 \subset \Sigma$. We now want to show that the fibres of the conformal symplectic normal bundle are then also tangent to the pages in the genuine mapping torus. For this, it is enough to observe that the diffeomorphism between the generalised and the genuine mapping tori maps a point (p, t) to a point $(p, \widetilde{t})$, where $\widetilde{t}$ only depends on t and the value of the function h at p. Thus, as in a symplectic neighbourhood $\Sigma' \times D^2$ the function h only depends on Σ', we have that for a point $p' \in \Sigma'$ and fixed t (p', s, ϑ, t) is mapped to $(p', s, \vartheta, \widetilde{t})$ independent of (s, ϑ). Hence, the fibres of the conformal symplectic normal bundle are tangent to the pages. Also, the resulting Reeb vector field is tangent to the push-off, as it is a multiple of the monodromy vector field outside a neighbourhood of the binding. Furthermore, by construction it is adapted to the open book structure, i.e. it is transverse to the pages. Thus, denoting the restriction of the contact form α to the push-off B^+ by α_{B^+}, the α restricts to $\alpha_{B^+} + s^2 d\vartheta$ on a tubular neighbourhood $B^+ \times D^2$ (with polar coordinates (s, ϑ)) given by the conformal symplectic normal bundle of B^+. The fibres of the symplectic normal bundle being tangent to the pages means that the contact fibre connected sum along B^+ has a natural open book structure. The contact form for the resulting contact structure used in the contact fibre connected sum construction is of the form $\widetilde{\alpha} = \alpha_{B^+} + f(s)d\vartheta$ for an appropriate function f and coincides with α near $B^+ \times \partial D^2$ (see [10, proof of Theorem 7.4.3]). In particular, the Reeb vector field of $\widetilde{\alpha}$ is still transverse to the fibres of the open book fibration. Hence, the resulting contact structure in the contact fibre connected sum is adapted to the resulting open book structure. □

6.2. Examples in the contact setting. We conclude the paper with some applications and examples of the binding sum construction in the contact setting.

6.2.1. $S^4 \times S^1$**.** Let $M = M_0 \sqcup M_1$ with M_i the five-dimensional sphere S^5 with standard contact structure and compatible open book decomposition $(\Sigma_i = D^4, \mathrm{id})$. Then the binding has two components B_i, both a standard 3-sphere that we can equip with the compatible open book decomposition $(\Sigma_i' = D^2, \mathrm{id})$. We have $M_i = D^4 \times S^1 \cup S^3 \times D^2$, so performing the binding sum on M along the B_i yields $S^4 \times S^1$. Note that the framing of the binding is unique up to homotopy because B is simply-connected. By Theorem 5.1, the binding sum $S^4 \times S^1$ has a natural open book decomposition obtained by forming the sum along the push-off of the binding.

Pushing a page $\Sigma_i' = D^2$ of the binding open book into the page $\Sigma_i = D^4$ and then removing a neighbourhood of Σ_i', topologically turns Σ_i into a copy of $D^3 \times S^1$. The resulting page is then obtained by identifying two copies of $D^3 \times S^1$ along a neighbourhood of $\{*\} \times S^1 \subset \partial D^3 \times S^1$, i.e. it is $D^3 \times S^1$ as well. The new binding is the contact binding sum of the B_i with respect to the specified open book decomposition. Hence, the new binding has an open book decomposition with page an annulus and – applying the formula from [15] for the page framing – monodromy isotopic to the identity. This means that the resulting binding is $S^2 \times S^1$ with standard contact structure. This has a unique symplectic filling up to blow-up and symplectic equivalence (cf. [23, Theorem 4.2]), so the resulting page is indeed symplectomorphic to $D^3 \times S^1$ arising as the 4-ball with a 1-handle attached.

6.2.2. **Mori's class of examples.** In [22] Mori constructs an infinite family of contact structures with compatible open book decompositions on $S^4 \times S^1$ via a fibre connected sum construction. Topologically, it is the sum of two copies of S^5 along 3-spheres as described in the previous example. However, contact forms and open books are altered such that the 3-spheres are no longer the bindings of the open book decompositions but nested open books. We will briefly explain his construction and refer the reader to the original article [22] for details.

Let $S^5 = \{r_1^2 + r_2^2 + r_3^2 = 1\}$ be the unit sphere in $\mathbb{C}^3$ with standard contact structure as defined by $\alpha = \sum r_i^2 d\varphi_i$ and open book fibration $\pi\colon\ S^5 \setminus B \to S^1$ with $B := \{r_1 = 0\}$. For p a positive integer, Mori constructs a function f equal to a positive constant near 0 such that the 1-form $\alpha_p := f(r_1)\alpha$ is contact. Furthermore, there is a function g vanishing around 0 such that the argument of the complex-valued function

$$w_p = g(r_1)r_1^p \mathrm{e}^{ip\varphi_1} + r_2 r_3 \mathrm{e}^{i(\varphi_1+\varphi_2)}$$

is an open book fibration $\pi_p\colon\ S^5 \setminus B_p \to S^1$ supporting $\ker \alpha_p$, where B_p is the vanishing set of w_p. Note that the restriction of π_p to B induces the standard open book with annular pages of S^3. Also, the contact structure on B induced by α_p is up to scaling with a positive constant the standard contact structure of S^3. Hence, the original binding B is realised as a contact nested open book of (S^5, α_p, π_p). One can show that the binding B_p of this new open book is diffeomorphic to the lens space $L(p, p-1)$. Clearly, for $p > 1$, B and B_p are not diffeomorphic, i.e. B is an explicit example of a nested open book which is not a push-off of the binding.

Mori then performs the contact fibre connected sum of two copies of S^5 equipped with contact forms α_{p_i} and open books π_{p_i} along their common nested open books B. As the open book fibrations agree in a neighbourhood of the 3-spheres B in both copies, this immediately gives rise to an open book structure supporting the resulting contact structure on $S^4 \times S^1$. Furthermore, the procedure does not only yield an infinite family of contact structures and leaf-wise symplectic foliations but also to an infinite family of so-called *ϵ-τ-confoliations* (see [22]).

6.2.3. **Open books with Giroux torsion.** Let M be a closed oriented manifold admitting a *Liouville pair* (α_+, α_-), i.e. a pair consisting of a positive contact form α_+ and a negative contact form α_- such that $1/2(e^{-s}\alpha_- + e^s\alpha_+)$ is a positively oriented Liouville form on $\mathbb{R}\times M$ (with s denoting the coordinate on the $\mathbb{R}$-factor). Then the 1-form

$$\lambda_{GT} = \frac{1+\cos s}{2}\alpha_+ + \frac{1-\cos s}{2}\alpha_- + \sin s dt$$

defines a positive contact structure on $\mathbb{R}\times S^1 \times M$ (s and t denote the respective coordinates on the first two factors) (see [18, Proposition 8.1]). With this model, Massot, Niederkrüger and Wendl [18] define a **Giroux $2k\pi$-torsion domain** as $([0, 2k\pi] \times S^1 \times M, \lambda_{GT})$. Just as in the 3-dimensional setting this higher-dimensional version of Giroux torsion is a filling obstruction in the sense that a contact manifold admitting a contact embedding of a Giroux 2π-torsion domain is not strongly fillable (see [18, Corollary 8.2]). Observe that a Giroux 2π-torsion domain with boundary blown down (cf. [18, Section 4]) is the binding sum of two copies of the open book with page $([0,\pi] \times M, \beta)$ and trivial monodromy, where

$$\beta = \frac{1}{2}(e^{-s}\alpha_- + e^s\alpha_+),$$

along $\{\pi\} \times M$. Given any contact open book (Σ, ϕ) with Σ having two boundary components contactomorphic to M, Theorem 6.1 yields an open book decomposition of the binding sum $(\Sigma, \phi) \# ([0,\pi] \times M, \mathrm{id}) \# ([0,\pi] \times M, \mathrm{id})$, which is a manifold admitting an embedding of a Giroux 2π-torsion domain modelled on M.

6.2.4. **Fibrations over the circle.** Theorems 5.1 and 6.1 yield a contact open book decomposition of certain bundles over the circle with fibres being convex hypersurfaces.

Recall that an oriented hypersurface S in a contact manifold is called **convex** (in the sense of Giroux [11]) if there is a contact vector field transverse to S. A neighbourhood of the hypersurface can then be identified with $S \times \mathbb{R}$ such that the contact structure is $\mathbb{R}$-invariant, i.e. there is a contact form of type $\beta + u\,dt$ with β a 1-form on S and $u\colon S \to \mathbb{R}$ a function such that $(d\beta)^{n-1} \wedge (u\,d\beta + n\beta \wedge du)$ is a volume form on S (here $2n$ is the dimension of S). Conversely, given a triple (S, β, u) consisting of a $(2n)$-dimensional closed manifold S, a 1-form β on S and a function $u\colon S \to \mathbb{R}$ satisfying the above conditions, the 1-form $\beta + u\,dt$ defines an $\mathbb{R}$- or S^1-invariant contact form on $S \times \mathbb{R}$ or $S \times S^1$, respectively. Observe that S with the zero set of u removed is an exact symplectic manifold with Liouville form β/u.

Now given a triple (S, β, u) as above and a diffeomorphism ϕ of S restricting to the identity near $\Gamma := \{u = 0\}$ and to symplectomorphisms $\phi_\pm$ on the interior of $S_\pm := \{\pm u \geq 0\}$, the S-bundle M over S^1 with monodromy ϕ carries a natural contact structure, such that each fibre defines a convex surface modelled by (S, β, u). In particular, the fibration admits a contact vector field transverse to the fibres, which is tangent to the contact structure exactly over Γ. Observe that M is equal to the binding sum of the open books (S_+, ϕ_+) and $(-S_-, \phi_-^{-1})$. Thus, Theorem 5.1 yields an open book description of M, which is adapted to the contact structure by Theorem 6.1.

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