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SPIN NETWORKS IN QUANTUM GRAVITY

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Abstract. This is a review of one of the approaches to unify Quantum Mechanics and the theory of General Relativity. Starting from the pioneer work of Regge and Penrose, other scientists have constructed state sum models, as Feymann path integrals, that are topological invariant on the triangulated Riemannian surfaces, and that become the Hilbert-Einstein action in the continuous limit.

1. Introduction

In this review we will present the main ideas of the spin foam approach. This line of research in quantum gravity has attracted a great deal of attention and been explored by many physicists and mathematicians.

However, we would first point out there are three main lines of research in quantum gravity denoted as "canonical", "covariant" and "sum over histories" [20].

The canonical line of research is a theory in which the Hilbert space carries a representation of the quantum operators corresponding to the full metric without background metric being fixed. It can be considered as a quantum field theory on a differentiable manifold. The basis of the Hilbert space are cylindrical functions defined on a graph (Wilson loops) depending on Ashtehar variables [21]. Very important results of this approach were the discrete eigenvalues for the area and volume operators.

The covariant line of research is the attempt to build the theory as a quantum field theory of the fluctuations of the metric over a flat Minkowski space, or some other background metric space. The theory has been proved to be renormalizable and finite order by order [1].

The sum over histories line of research uses the Feymann path integral to quantize the Einstein Hilbert action. A duality exists between this model and group field theories. The sum over spin foam can be generated as the Feymann perturbative expansion of the group field theories. Each space-time appears as the Feymann graph of the auxiliary groups field theory [2]. Our presentation concentrates on this third line of research, namely, the spin network and the spin foam models, from an historical point of view.

2. Regge Calculus

The Regge's paper [18] was a pioneer work in the discretization of General Relativity (GR), motivated by the need to avoid coordinates, because the physical prediction of the theory was coordinate independent. For that reason he discretized a continuous manifold by n-simplices, glued together by identification of their $(n-1)$-simplices. The curvature lies on the $(n-2)$-dimensional subspaces, known as hinges or bones. For pedagogical reasons we take a triangulation of a two-dimensional surface. When a collection of triangle meeting at a vertex is flattened there will be a gap or deficit angle ϵ, indicating the presence of curvature. Using the Gauss-Bonet formula we can calculate the excess angle by $\epsilon = KA$, where K is the curvature at that vertex and A is the area of the triangles around the vertex. If the number of vertices increases we can take $K = \epsilon\rho$, where ρ is the density of vertices in the triangulation (equal to the number of vertices by unit area). This method is easily enlarged to higher dimensions.

In order to connect with GR we translate into the triangulated surface (the skeleton) the Hilbert-Einstein action $\mathcal{L} = (1/8\pi)\int K\sqrt{-g}\mathrm{d}^4x$ where K is the scalar curvature in four-dimensions. The discrete version for a four-dimensional skeleton is given in terms of the deficit angle in each bone where the curvature K is calculated and some measure function L is defined

$$\mathcal{L} = \sum_{n=1}^{N} \epsilon_n L_n$$

here the summation extends to all the bones in the skeleton. In the continuous case Einstein's equations are derived from a stationary action, varying $\mathcal{L}$ with respect to the metric. In the discrete version one derives the action with respect to the edge lengths, because in the simplicial decomposition all the properties can be derived from these edges. Using Schlaefli differential identity one finds

$$\delta\mathcal{L} = \frac{1}{8\pi}\sum_{n=1}^{N} \varepsilon_n \delta L_n = 0 \quad \Rightarrow \quad \sum_{n=1}^{N} \varepsilon_n \frac{\partial L_n}{\partial l_p} = 0$$

which is the discrete version of Einstein's equations [19].

3. The Ponzano-Regge Model

Some years later Ponzano and Regge [17] made use of $\{6j\}$ symbols attached to the tetrahedra decomposition in order to calculate the state sum and were able to connect it to the Feymann integral corresponding to the Hilbert-Einstein action. The $\{6j\}$ Wigner symbols are a generalization of the Clebsch-Gordan coefficients that appear in the coupling of two angular momenta $J = J_1 + J_2$. The new basis is given in term of the old basis

$$|j_1 j_2 j m\rangle = \sum \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle |j_1 j_2 m_1 m_2\rangle.$$

If we couple J with a new angular momentum J_3 we have two possibilities

$$(J_1 + J_2) + J_3 = J \quad \text{or} \quad J_1 + (J_2 + J_3) = J.$$

In the first case the new basis is given (in obvious notation)

$$|j_1 j_2 j_3 j_{12} j m\rangle = \sum \langle j_1 j_2 j_3 m_1 m_2 m_3 | j_1 j_2 j_3 j_{12} j m\rangle |j_1 j_2 j_3 m_1 m_2 m_3\rangle.$$

In the second case the new basis is given by

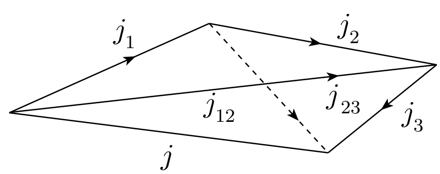
$$|j_1 j_2 j_3 j_{23} j m\rangle = \sum \langle j_1 j_2 j_3 m_1 m_2 m_3 | j_1 j_2 j_3 j_{23} j m\rangle |j_1 j_2 j_3 m_1 m_2 m_3\rangle.$$

The transformating matrix between the two bases is given precisely by the $\{6j\}$ symbols, namely

$$U(j_{12} j_{23}) = (-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}.$$

Given a tetrahedra decomposition of a three-dimensional surface we can attach a $\{6j\}$ symbol to each tetrahedra, the edges of which are equal in length to the numerical values of the angular momenta appearing in the $6j$-symbol.

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} \Longleftrightarrow$$



This choice is consistent with the inequalities

$$j_{12} < j_1 + j_2 \quad \text{and} \quad j_{23} < j_2 + j_3$$

and the equalities

$$j_1 + j_2 + j_{12} \subset N, \quad j_2 + j_3 + j_{23} \subset N.$$

The $\{6j\}$ symbols are proportional to the Racah polynomials [14]

$$(-1)^{j_1+j_2+j_3}\sqrt{(2j_{12}+1)(2j_{23}+1)}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\} = \frac{\sqrt{\rho(x)}}{d_n} U_n^{(\alpha,\beta)}(x,a,b).$$

From this equality and the assymptotic properties of Racah polynomials one can derive a very important limit

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\} \underset{j_i\to\infty}{\longrightarrow} \frac{1}{\sqrt{12\pi V}} \cos\left\{\sum_{i=1}^{6}\left(j_i + \frac{1}{2}\right)\vartheta_i + \frac{\pi}{4}\right\}$$

where V is the volume of the tetrahedra and ϑ_i the exterior dihedral angle adjoint to the edge j_i. In order to see the connection between $\{6j\}$ symbols and quantum gravity we take a tetrahedra decomposition and external edges l_i of the bounding surface and internal edges x_i. Then Ponzano and Regge construct the state sum as follows

$$S(l_i) = \sum_{x_i} \prod_{\text{tetrahedra}} \{6j\}(-1)^X \prod_{\text{edges}} (2x_i + 1)$$

where X is a phase factor. Substituting the $\{6j\}$ symbols by their assymptotic values and the function cosine by the Euler expression we arrive at

$$S(l_i) = \sum_{x_i} \prod_{i=1} (2x_i + 1) \exp\left\{ \mathrm{i} \left[\left(\sum_{\text{tetrahedra}\, k} \vartheta_i^k \right) - \pi p_i + 2\pi \right] x_i \,. \right.$$

We may replace the summation with an integral. Then the most important contribution comes from the stationary phase, that is to say when one has $\sum\limits_{\text{tetrahedra}} (\pi - \vartheta_i^k) = 2\pi$.

Introducing this value in the state sum we obtain [19]

$$S(l_i) = \int \prod_{x_i} (2x_i + 1) \exp(\mathrm{i} \sum j_l \epsilon_l)\, \mathrm{d}x_i$$

where $\epsilon_\ell = 2\pi - \sum\limits_k (\pi - \theta_l^k)$.

The summation $\sum j_l \epsilon_l$ approaches the Hilbert-Einstein action that was given in the Regge calculus, therefore, in the limit the state sum strongly resembles the

Feymann summation over all possible histories with Lagrangian density $\mathcal{L} = (1/8\pi) \int R\sqrt{-g}\mathrm{d}^4x$ namely

$$S = \int \mathrm{d}\mu(x_i)\mathrm{e}^{\mathrm{i}\mathcal{L}}.$$

4. Penrose's Spin Networks

Penrose was interested in the interpretation of space-time [16] by purely combinatorial properties of some elementary units that are connected among themselves by some interactions that follow the angular momentum quantum rules, and form a network of elementary units with assigned spins. Soon it was realized that the spin network was analog to simplicial quantum gravity, in particular the Ponzano-Regge model [12]. His networks had trivalent vertices and the edges were labeled with spin, satisfying the standard conditions at the vertices. The model was generalized to any group different from the rotation group. Formally a spin network is a triple (γ, ρ, ι) where

i) γ is a graph with a finite set of edges e and a finite set of vertices v

ii) to each edge e we attach an irreducible representation of a group G, ρ_e

iii) to each vertex v we attach an intertwiner.

When we take the dual of an spin network we obtain a triangulated figure, which, after embedding in a three-dimensional manifold becomes the triangulation of Regge calculus.

5. The Turaev-Viro State-sum Invariant

They defined a state sum for triangulated three-manifold (as in the Ponzano-Regge model) that was independent as the triangulation and finite [22]. For this purpose they assign a value from the set $I_r = (0, 1/2, 1,\ (r-2)/2)$, integer, to each edge of the triangulation, subject to the condition that the "coloring" of the three edges forming a triangle should satisfy the triangle inequalities and their sum should be an integer less than or equal to $r - 2$. Define the quantum object

$$|M|_\phi = \omega^{-2\rho} \prod_{\mathrm{tetra}\, k} |T_k^\varphi| \prod_{\mathrm{edge}\, j} \omega_j^2$$

where φ is an admissible coloring of the edge j

$$\omega_j = (-\mathrm{i})^j [2j+1]^{1/2}, \qquad \omega^2 = \sum_{j \in I_r} \omega_j^4$$

and $|T_k^{\varphi}|$ is the quantum $6j$-symbol corresponding to the tetrahedron k with coloring φ, such that

$$\left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| = (-1)^{i+j+k+l+m+n} \left\{ \begin{array}{ccc} [i] & [j] & [k] \\ [l] & [m] & [n] \end{array} \right\}$$

where $[n]$ is the quantum number satisfying $[n] \rightarrow n$. Summing $|M|_{\phi}$ over all admissible coloring we obtain an expresion in the limit $q \rightarrow 1$ or $r \rightarrow \infty$ that becomes identical to the Ponzano-Regge state sum. Turaev and Viro proved that their expression is manifold invariant (or independent of triangulation) under Alexander moves, and also finite.

6. The Three-dimensional Boulatov Model

The Ponzano-Regge state sum and the Turaev-Viro model are defined over three-dimensional manifold. To enlarge the model to four dimensions it was necessary to increase the Wigner symbols to $3nj$. The key to this approach was given by Boulatov [9] by the use of topological lattice gauge theories, taking group elements as variables (matrix models). The basic object is the set of real functions of three variables $\phi(x, y, z)$ (where $x, y, z \in \mathrm{SU}(2)$) invariant under simultaneous right shift of all variables by $u \in \mathrm{SU}(2)$ and also by cyclic permutation of x, y, z. This function ϕ can be expanded, by the Peter-Weyl theorem, in terms of representations of $\mathrm{SU}(2)$ and 3j-symbols. An action of interest can be constructed with those functions as follows

$$\begin{aligned} S =& \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \phi^2(x, y, z) \\ & - \frac{\lambda}{4!} \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \mathrm{d}u \mathrm{d}v \mathrm{d}w \phi(x, y, z) \phi(x, u, v) \phi(y, v, w) \phi(z, w, u). \end{aligned}$$

If we attach the variable to the edges, the first term (the kinematical term) represents two glued triangles and the second one (the interacting term) four triangles forming a tetrahedron. Substituting the Fourier expansion of function ϕ, and integrating out group variables we obtain an action depending on the Fourier coefficients and 6j-symbols. From this result we calculate the partition function as a

Feymann path integral with respect to the Fourier coefficients

$$Z = \int D\phi \mathrm{e}^{-S} = \sum_{\{C\}} \lambda^{N_3} \sum_{j} \prod_{l} (2j_e + 1) \prod_{T} \{6j\}$$

where the products extend to all tetrahedra T, all edges l, and the summation extends to all the representations $\{j\}$, all the simplicial complexes $\{C\}$ and N_3 is the number of tetrahedra in complex C. This partition function is equivalent, up to renormalization, to the Ponzano-Regge state sum applied to a triangulation of three-dimensional manifold. The underlying mathematical structure is a topological lattice gauge theory, it has the advantage that is topological invariant. In order to prove it, Boulatov used the Alexander moves, by which one complex, and the corresponding partition function is topological invariant.

7. The Four-dimensional Ooguri's Model

The three-dimensional Boulatov model paved the way for Ooguri's model in four dimensions. [15] Let ϕ be a real valued function of four variables $\phi(g_1, g_2, g_3, g_4)$ on $G(g_i \in G)$ a compact group. For simplicity we take $G = \mathrm{SU}(2)$. We require ϕ to be invariant under the right action of G and by cyclic permutation of these variables. Following the Peter-Weyl theorem, we can expand ϕ in terms of these representations and the 3j-symbols. We define the action

$$\begin{aligned} S =& \frac{1}{2} \int \prod \mathrm{d}g_i \phi^2(g_1 g_2 g_3 g_4) + \frac{\lambda}{5!} \int \prod_{i=1}^{10} \mathrm{d}g_i \phi(g_1 g_2 g_3 g_4) \\ & \times \phi(g_4 g_5 g_6 g_7) \phi(g_7 g_3 g_8 g_9) \phi(g_9 g_6 g_2 g_{10}) \phi(g_{10} g_8 g_5 g_1) \end{aligned}$$

where the first term (the kinematical term) represents the coupling of a tetrahedrum with itself because each element g_i is associated with each face of the tetrahedrum, and the second term (the interacting term) represents gluing faces of five tetrahedra to make a four-simplex. Substituting the Fourier expansion into the action we can integrate out the group variable, and then the action can be used to calculate a partition function as a Feyman path integral with respect to this action

$$Z = \int DM \mathrm{e}^{-S(M)} = \sum_{C} \lambda^{N_4} \sum_{\{j\}} \prod_{t} (2j_t + 1) \prod_{T} \{6j\} \prod_{S} \{15j\}$$

where the integral is defined in terms of the Fourier coefficients M, appearing in the action and in the measure, the first sum is over all complexes C (four-dimensional combinatorial manifolds), $N_4(C)$ is the number of four-simplices in

C, the second summation is over all irreducible representations as $\mathrm{SU}(2)$ with angular momentum $j; t, T$ and S are the triangles, tetrahedra and four-simpleces respectively appearing in the complex. Ooguri also proved that the partition function is a topological invariant under the Alexander moves. As in the Boulatov model two complexes are combinatorially equivalent if and only if they are connected by a sequence of transformations called the Alexander moves.

8. The Barrett-Crane Model

A more abstract approach was taken by Barrett and Crane [5] generalizing Penrose's spin networks to four dimensions. The novelty of this model consists in the association of representations of $\mathrm{SO}(4)$ group with the faces of the tetrahedra, instead of the edges. They decompose a triangulation of a four-dimensional manifold into four-simplices, the geometrical properties of which are characterized in terms of bivectors. A geometric four-simplex in Euclidean space is given by the embedding of an ordered set of 5 points in $R^4(0, x, y, z, t)$ which is required to be non-degenerate (the points should not lie in any hyperplane). Each triangle in it determines a bivector constructed out of the vectors for the edges. Barrett and Crane proved that classically, a geometric four-simplex in Euclidean space is completely characterized (up to parallel translation and inversion through the origin) by a set of 10 bivectors b_i, each corresponding to a triangle in the four-simplex and satisfying the following properties:

i) the bivector changes sign if the orientation of the triangle is changed

ii) each bivector is simple, i.e., is given by the wedge product of two vectors for the edges

iii) if two triangles share a common edge, the sum of the two bivectors is simple

iv) the sum (considering orientation) of the 4 bivectors corresponding to the faces of a tetrahedron is zero

v) for six triangles sharing the same vertex, the six corresponding bivectors are linearly independent

vi) the bivectors (thought of as operators) corresponding to triangles meeting at a vertex of a tetrahedron satisfy Tr $b_1[b_2, b_3] > 0$, i.e., the tetrahedron has non-zero volume.

Then Barrett and Crane define the quantum four-simplex with the help of bivectors thought of as elements of the Lie algebra $\mathrm{SO}(4)$, associating a representation with each triangle and a tensor to each tetrahedron. The representations chosen should satisfy the following conditions corresponding to the geometrical ones:

i) different orientations of a triangle correspond to dual representations

ii) the representations of the triangles are "simple" representations of $\mathrm{SO}(4)$, i.e., $j_1 = j_2$

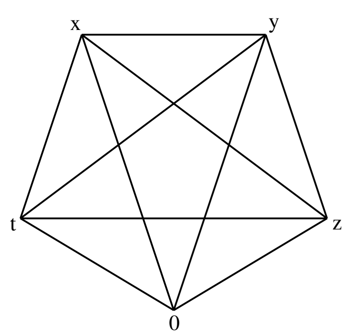
iii) given two triangles, if we decompose the pair of representations into its Clebsch-Gordan series, the tensor for the tetrahedron is decomposed into summands which are non-zero only for simple representations

iv) the tensor for the tetrahedron is invariant under $\mathrm{SO}(4)$.

Now it is easy to construct an amplitude for the quantum four-simplex. The graph for a relativistic spin network is the one-complex, dual to the boundary of the four-simplex, having five four-valent vertices (corresponding to the five tetrahedra), with each of the ten edges connecting two different vertices (corresponding to the ten triangles of the four-simplex each shared by two tetrahedra). Now we associate with each triangle (the dual of which is an edge) a simple representation of the algebra $\mathrm{SO}(4)$ and to each tetrahedra (the dual of which is a vertex) we associate an intertwiner; and to a four-simplex the product of the five intertwiner with the indices suitably contracted, and the sum for all possible representations. The proposed state sum suitable for quantum gravity for a given triangulation (decomposed into four-simplices) is

$$Z_{BC} = \sum_J \prod_{\text{triangles}} A_{\text{triangle}} \prod_{\text{tetrahedra}} A_{\text{tetraheder}} \prod_{\text{four}-\text{simplices}} A_{\text{simplex}}$$

where the sum extends to all possible values of the representations J.



In order to know the representation attached to each triangle of the tesselation, we take the unitary representation of $\mathrm{SO}(4)$ in terms of Euler angles, i.e.,

$$U(\varphi,\theta,\tau,\alpha,\beta,\gamma) = R_3(\varphi)R_2(\theta)S_3(\tau)R_3(\alpha)R_2(\beta)R_3(\gamma)$$

where R_2 is the rotation matrix in the (x_1x_3) plane, R_3 the rotation matrix in the (x_1x_2) plane and S_3 the rotation "boost" in the (x_3x_4) plane. In the angular momentum basis, the action of S_3 is as follows

$$S_3(\tau)\psi_{jm} = \sum_{j'} d^{j_1j_2}_{j'jm}(\tau)\psi_{j'm}$$

where

$$d^{j_1j_2}_{j'jm}(\tau) = \sum_{m_1+m_2=m} \langle j_1j_2m_1m_2 \,|.\, jm\rangle \mathrm{e}^{-\mathrm{i}(m_1-m_2)\tau}\langle j_1j_2m_1m_2 \,|.\, j'm\rangle$$

is the Biedenharn-Dolginov function [8].

9. Evaluation of the State Sum for the Four-dimensional Spin Network

In order to evaluate the state sum for a particular triangulation of the total R^4 space by four-simplices, we assign an element $h_k \in \mathrm{SU}(2)$ to each tetrahedrum of the four-simplex $(k = 1,2,3,4,5)$ and a representation ρ_{kl} of $\mathrm{SO}(4)$ to each triangle shared by two tetrahedra. From this triangulation we obtain a two-complex by the dual graph where one vertex corresponds to a tetrahedrum and an edge corresponds to a triangle, with the ends of the edges identified with the vertices. Then we attach a representation of $\mathrm{SU}(2)$, $\rho(h_k)$ and $\rho(h_l)$ to the vertices k and l and contract both representations along the edges $(k,l) \equiv e$, giving

$$\mathrm{Tr}\rho(h_k)\rho(h_l^{-1}) = \mathrm{Tr}\rho_{kl}(h_kh_l^{-1})$$

where ρ_{kl} is the representation of $\mathrm{SO}(4)$ corresponding to the product $h_kh_l^{-1}$, the left and right components of the $\mathrm{SO}(4)$ group. The state sum for the two-dimensional complex (the Feymann graph of the model) is obtained by taking the product for all the edges of the graph and integrating for all the copies of $\mathrm{SU}(2)$

$$I = \int_{h\in\mathrm{SU}(2)^5} \prod \mathrm{Tr}\rho_{kl}(h_kh_l^{-1})$$

Due to the trace condition this expression is invariant under left and right multiplication of some elements of $\mathrm{SU}(2)$ [3].

For the representation ρ_{kl} we choose the spherical function with respect to the identity representation. Given a completely irreducible representation of the group $G : g \to T_g$ on the space R, we define the spherical function with respect to the finite irreducible representation of the subgroup K

$$f_k(g) = \mathrm{Tr}\{E^k T_g\}$$

where E^k is a projector of R onto the space R_k of K.

We take for $G \equiv \mathrm{SO}(4)$ the simple representation $(j_1 = j_2)$ and for the subgroup $\mathrm{SU}(2)$ the identity representation $k = 0$. Since f_k is invariant under K we can restrict the unitary representations to those of the boost $S_3(\tau)$. With the help of the Biedenharn-Dolginov function it can be proved

$$f_0(\tau) = \mathrm{Tr}\{E^0 S_3(\tau)\} = \frac{\sin(2j_1+1)\tau}{\sin\tau}.$$

With this formula it is still possible to give a geometrical interpretation of the probability amplitude encompassed in the trace. In fact the spin dependent factor appearing in the exponential of the spherical function

$$\mathrm{e}^{\mathrm{i}(2j_{kl}+1)\tau_{kl}}$$

corresponding to the two tetrahedra k, l intersecting the triangle kl, can be interpreted as the product of the angle between the two vectors h_k, h_l, perpendicular to the triangle, and the area $A_{kl} = 2j_{kl} + 1$ of the intersecting triangle, j_{kl} being the spin corresponding to the representation ρ_{kl} associated to the triangle k, l. Substituting this value in the state sum, we obtain

$$I = \prod_{h\in\mathrm{SU}(2)} \frac{1}{\sin\tau_{kl}} \exp(\mathrm{i} \sum_{\text{triangle } kl} A_{kl}\tau_{kl})$$

where the product extends to all tetrahedra with the vector h perpendicular to the subspace where the tetrahedra is embedded, and summation is extended to all the triangle k, l intersected by two tetrahedra k and l. The exponential term corresponds to the Regge action, that in the assymptotic limit becomes the Hilbert-Einstein action [6].

Because we are interested in the physical and mathematical properties of the Barrett-Crane model, we mention some recent work on this model combined with the matrix model approach of Boulatov and Ooguri [11]. In this work the two-dimensional quantum space-time emerges as a Feymann graph, in the manner of the four-dimensional matrix models. In this way a spin foam model is connected to the Feyman diagram of quantum gravity.

10. The Lorentzian Spin Foam Model

Now we apply the same technique to calculate the state sum invariant under the Lorentz group that we have used in the case of the $\mathrm{SO}(4)$ group for the Barrett-Crane model.

The unitary irreducible representation of the $\mathrm{SL}(2,\mathbb{C})$ group for the principal series is given by the formula [7]

$$(T_g^{[m,\rho]}\psi)(z) = (\beta z+\delta)^{m+i\rho-2}(\beta z+\delta)^{-m}\psi(\frac{\alpha z+\gamma}{\beta z+\delta})$$

where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, m, integer, ρ real and $\psi(z) \in L^2(\mathbb{C})$. The numbers m, ρ determine the eigenvalues of the representation

$$C_1 = -\frac{m^2-\rho^2-4}{2}, \qquad C_2 = m\rho.$$

In order to calculate the state sum we need the spherical functions of the irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. These are given in terms of the Biedenharn-Dolginov function that corresponds to the boost operator

$$d^{[m,\rho]}_{JJ'M}(\tau) = \int\limits_{-\infty}^{\infty} d^{-1}_{J-M} p^{(M-m,M+m))}_{J-M}(\lambda,\rho)\mathrm{e}^{-\mathrm{i}\tau\lambda} \times\ d^{-1}_{J'-M} p^{(M-m,M+m))}_{J'-M}(\lambda,\rho)\omega(\lambda)\mathrm{d}\lambda$$

where J, M are the angular momentum eigenvalues, d_n is a normalization constant, and $p_n^{(\alpha,\beta)}$ are the Hahn polynomials of imaginary argument [14]. Given the unitary representation T_g of the group $\mathrm{SL}(2,\mathbb{C})$ and the identity representation of $\mathrm{SU}(2)$, the spherical function is defined as in the case of $\mathrm{SO}(4)$

$$f_0(\tau) = \mathrm{Tr}\{E^0 T_g\} = d^{0,\rho}_{000}(\tau) = \frac{1}{\rho}\frac{\sin\rho\tau}{\mathrm{sh}\tau}$$

where the last step has been calculated with the residue theorem.

11. A SO(3,1) Invariant for the State Sum of Spin Foam Model

As in the case of Euclidean $\mathrm{SO}(4)$ invariant model, we take a non degenerate finite triangulation of a four-manifold. We consider the four-simplices in the homogeneous space $\mathrm{SO}(3,1)/\mathrm{SO}(3) \sim H_3$, the hyperboloid $\{x; x.x = 1, x^0 > 0\}$

and define the bivectors b on each face of the four-simplex, that can be space-like, null or timelike $\langle b, b\rangle > 0, = 0$ or < 0, respectively [4].

In order to quantize the bivectors, we take the isomorphism

$$b = *L, \qquad b^{ab} = \frac{1}{2}\varepsilon^{abcd} L^e_d g_{ec}$$

with g a Minkowski metric.

The condition for b to be a simple bivector $\langle b, *b\rangle = 0$, gives $C_2 = \langle L, *L\rangle = \vec{J}\cdot\vec{K} = m\rho = 0$.

We have two cases:

1) $\rho = 0, \quad C_1 = \langle L, L\rangle = \vec{J}^2 - \vec{K}^2 = m^2 - 1 > 0$; L, space-like, b time-like
2) $m = 0, \quad C_1 = \vec{J}^2 - \vec{K}^2 = -\rho^2 - 1 < 0$; L, time like, b space like (remember, the Hodge operator $*$ changes the signature).

In case 2) b is space-like, $\langle b, b\rangle > 0$. Expanding this expression in terms of space like vectors, x, y

$$\begin{aligned} b_{\mu\nu}b^{\mu\nu} &= (x_\mu y_\nu - x_\nu y_\mu)(x^\mu y^\nu - x^\nu y^\mu) = \|x\|^2\|y\|^2 - \|x\|^2\|y\|^2\cos^2\eta(x,y) \\ &= \|x\|^2\|y\|^2\sin^2\eta(x,y) \end{aligned}$$

where $\eta(x, y)$ is the Lorentzian space-like angle between x and y. This result gives a geometric interpretation between the parameter ρ and the area expanded by the bivector $b = x \wedge y$, namely, $\langle b, b\rangle = \text{area}^2\{x, y\} = \langle *L, *L\rangle \cong \rho^2$. (This result is equivalent to that obtained in the Euclidean case where the area of the triangle expanded by the bivector was proportional to the value $(2j + 1)$, j being the spin of the representation).

In order to construct the Lorentz invariant state sum we take a non-degenerate finite triangulation in four-dimensional simplices in such a way that all three-dimensional and two-dimensional subsimplices have space-like edge vectors which span space-like subspace. We attach to each two-dimensional face a simple irreducible representation of $\mathrm{SO}(3, 1)$ characterized by the parameters $[0, \rho]$.

The state sum is given by the expression [10]

$$Z = \int_{\rho=0}^{\infty} \mathrm{d}\rho \prod_{\text{triang}} \rho^2 \prod_{\text{tetra}} \Theta_4(\rho'_1, \cdots, \rho'_4) \prod_{4-\text{simplex}} I_{10}(\rho''_1, \cdots, \rho''_{10})$$

where ρ refers to all the faces in the triangulation, ρ' corresponds to the simple irreducible representation attached to 4 triangles in the tetrahedra and ρ'' corresponds to the simple irreducible representation attached to the 10 triangles in the four-simplices. The functions Θ_4 and I_{10} are defined as traces of recombination diagrams for the simple representations. The traces are explicitly given as multiple integrals on the upper sheet H of the two-sheeted hyperboloid in Minkowski space. For the integrand we take the spherical function

$$f_p(x,y) = \frac{1}{\rho}\frac{\sin \rho\tau(x,y)}{\sin h\tau(x,y)}$$

where $\tau(x,y)$ is the hyperbolic distance between x and y.

The trace of a recombination diagram is given by a multiple integral of products of spherical functions. For a tetrahedrum we have

$$\Theta_4(\rho_1',\cdots,\rho_4') = \frac{1}{2\pi^2}\int\limits_H f_{\rho 1}(x,y)\cdots f_{\rho 4}(x,y)\mathrm{d}y$$

where we have dropped one integral for the sake of normalization without losing Lorentz symmetry.

For a four-simplex we have

$$I_{10}(\rho_1',\cdots,\rho_4') = \frac{1}{2\pi^2}\int\limits_{H^4}\prod_{i<j<1,5} f_{\rho_{ij}}(x_i,x_j)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4.$$

The last four equations defines the state sum completely, that has been proved to be finite [6].

The assymptotic properties of the spherical functions are related to the Einstein-Hilbert action giving a connection of the model with the theory of the general relativity [13].

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