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FREDHOLM ANALYTIC OPERATOR FAMILIES AND PERTURBATION OF RESONANCES

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Abstract. The purpose of this contribution is to display an outlook of a certain approach that enables one to study various spectral perturbation phenomena such as perturbation of eigenvalues and resonances (scattering poles) from the general viewpoint. Some applications of this elaborated technique are presented as well.

1. Introduction

Let us consider a Hilbert space $\mathcal{H}$ and an operator function $K(\lambda, \alpha)$ which is analytic in two variables and takes its values in the class of compact operators. The variable λ stands for the "spectral parameter", while the second variable α is treated as a "perturbation parameter". For a fixed $\alpha = \alpha_0$ the point $\lambda = \lambda_0$ is called singular for the operator family $K(\lambda, \alpha_0)$ if $\ker(K(\lambda_0, \alpha_0) - I) \neq \{0\}$. The problem in question is to study the analytic properties of the **singular points** $\lambda(\alpha)$ as functions of the parameter α.

Below the notation $K(\lambda) = K(\lambda, \alpha_0)$ will be used. Let $n = \dim\ker(K(\lambda_0) - I)$ and $\{\varphi_1^{(0)}, \ldots, \varphi_n^{(0)}\}$ be a basis of the subspace $\ker(K(\lambda_0) - I)$. It may happens that for a given eigenvector $\varphi_j^{(0)}$ there exists a chain of adjoint vectors $\{\varphi_j^{(1)}, \ldots, \varphi_j^{(m_j-1)}\}$ of maximal length $(m_j - 1)$ such that

$$\begin{gathered}
K(\lambda_0)\varphi_j^{(1)} + K'(\lambda_0)\varphi_j^{(0)} = \varphi_j^{(1)} \\
K(\lambda_0)\varphi_j^{(2)} + K'(\lambda_0)\varphi_j^{(1)} + \frac{1}{2}K''(\lambda_0)\varphi_j^{(0)} = \varphi_j^{(2)} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
K(\lambda_0)\varphi_j^{(m_j-1)} + \ldots + \frac{1}{(m_j-1)!}K^{(m_j-1)}(\lambda_0)\varphi_j^{(0)} = \varphi_j^{(m_j-1)}.
\end{gathered}$$

The resolvent $(K(\lambda) - I)^{-1}$ is known (see [1]) to have a pole of order $m = \max_{1 \leqslant j \leqslant n} m_j$ at the point $\lambda = \lambda_0$ so that the principle part of its Laurent expansion in the neighbourhood of λ_0 is a finite dimensional operator of the form

$$\sum_{j=1}^{n} \left\{ \frac{(\cdot, \psi_j^{(0)})\varphi_j^{(0)}}{(\lambda - \lambda_0)^{m_j}} + \frac{(\cdot, \psi_j^{(1)})\varphi_j^{(0)} + (\cdot, \psi_j^{(0)})\varphi_j^{(1)}}{(\lambda - \lambda_0)^{m_j - 1}} \right.$$
$$\left. + \ldots + \frac{(\cdot, \psi_j^{(m_j-1)})\varphi_j^{(0)} + \ldots + (\cdot, \psi_j^{(0)})\varphi_j^{(m_j-1)}}{\lambda - \lambda_0} \right\}$$

where $\{\psi_j^{(0)}, \psi_j^{(1)}, \ldots, \psi_j^{m_j-1)}\}$ is a certain chain of eigen- and associated adjoint vectors of the operator function $K^*(\lambda)$ corresponding to its singular point $\lambda = \lambda_0$.

To study the "perturbation problem" by the analytic properties of singular points $\lambda(\alpha)$ of Fredholm operator families an approach will be used that goes back to Poincare (see [2]). In what follows we shall restrict ourselves to two important and essentially different cases, namely i) the case of a single "Jordan chain", i.e., $n = 1$, and ii) the "semi-simple" singular point, i.e., $m = n$. The general "mixed" case will be considered elsewhere.

2. Perturbation of Resonances

Consider in $\mathcal{H} = L^2(B), B \subset \mathbb{R}^3$, an operator family

$$K(\lambda, \alpha) = (V + \alpha W)R_0(\lambda)$$

where $R_0(\lambda)$ is an integral operator with the kernel $\dfrac{\exp(-\mathrm{i}\lambda|x-y|)}{4\pi|x-y|}$, while $V + \alpha W$ is a multiplication by a $C_0^\infty(\mathbb{R}^3)$ function of variable x and the ball B is chosen to contain the supports of $V(x)$ and $W(x)$.

It is known (see [3]) that the mappings $K(\lambda, \alpha) : \mathcal{H} \to \mathcal{H}$ form an entire family of compact operators. Hence by the analytic Fredholm theorem the resolvent $R_V(\lambda) = R_0(\lambda)\big(I + VR_0(\lambda)\big)$ for the Schrödinger operator $-\Delta + V(x)$ can be continued from the lower to the upper half-plane as a meromorphic operator function with finite rank residues.

The poles of the analytic continuation of $R_V(\lambda) = \big(-\Delta + V - \lambda^2\big)^{-1}$ are called **resonances** (scattering poles) related to the Schrödinger operator $-\Delta + V(x)$. If $\lambda = \lambda_0$ is a resonance then the equation

$$\big(-\Delta + V(x) - \lambda_0^2\big)\varphi = 0$$

has a solution $\varphi_0(x)$ called **resonance eigenstate** possessing the expansion

$$\varphi_0(x) \;=\; \mathrm{e}^{-\mathrm{i}\lambda_0 r} r^{-1} \sum_{k=0}^{\infty} h_k^{(0)}(\theta) r^{-k}, \qquad \theta = x/r$$

valid for large $r = |x|$. With the eigenstate φ_0 one can associate the "Jordan chain" of adjoint resonance states $\{\varphi_1, \ldots, \varphi_{m-1}\}$ such that (see [4])

$$\big(-\Delta + V(x) - \lambda_0^2\big)\varphi_s = \varphi_{s-1}, \qquad s = 1, \ldots, m-1$$

and for which the asymptotic representations are valid

$$\varphi_s(x) = \mathrm{e}^{-\mathrm{i}\lambda_0 r} r^{s-1} \sum_{k=0}^{\infty} h_k^{(s)}(\theta) r^{-k}, \qquad r \to \infty.$$

The **multiplicity** of a resonance λ_0 is defined as the dimension of the subspace spanned by the ranges of all the singular terms in the Laurent series for the resolvent $R_V(\lambda)$ at $\lambda = \lambda_0$. A multiple resonance with no adjoint states is naturally treated as a *semi-simple* one. According to the definition, a resonance of the Schrödinger operator $-\Delta + V$ is a singular point of the corresponding operator function $-VR_0(\lambda)$ and vice versa.

The perturbation problem for the resonances of the Schrödinger operator has been stated and originally studied (in the case of a simple resonance) in [5]. It was shown in [6] that all the resonances as a rule are simple, i.e., the multiplicity is unstable with respect to generic perturbations of a potential. Nevertheless the question about analytic description of the corresponding perturbation phenomena remained opened. Our approach gives the following answers to this question:

i) *Suppose that λ_0 is a resonance of multiplicity m for the Schrödinger operator $-\Delta+V(x)$ and there exists a unique (up to a constant factor) resonance eigenstate $\varphi_0(x)$. If $A = \int W(x)\varphi_0^2(x)\mathrm{d}x \neq 0$ then for sufficiently small α the operator $-\Delta + V(x) + \alpha W(x)$ has m simple resonances $\lambda_j(\alpha)$ in the neighbourhood of λ_0 and, moreover*

$$\lambda_j(\alpha) - \lambda_0 \sim \alpha^{1/m} \exp\left(\frac{2\pi\mathrm{i}}{m} j\right) A^{1/m}.$$

ii) *Let λ_0 be a semi-simple resonance of multiplicity n for the operator $-\Delta + V(x)$ and the basis $\{\varphi_1, \ldots, \varphi_n\}$ of the corresponding resonance eigenstate subspace is chosen to satisfy the following condition*

$$\int V(x)\Big(\int \mathrm{e}^{-\mathrm{i}\lambda_0|x-y|} V(y)\varphi_j(y)\mathrm{d}y\Big)\varphi_i(x)\mathrm{d}x \;=\; \delta_{ij}.$$

Provided that the eigenvalues $\{\mu_j\}$ *of the matrix* $\Big(\int W(x)\varphi_j(x)\varphi_k(x)\mathrm{d}x\Big)$ *are all different, the operator* $-\Delta + V(x) + \alpha W(x)$ *for* α *sufficiently small has* (*in the neighbourhood of* λ_0) *just* n *simple resonances* $\lambda_j(\alpha)$ *so that* $\lambda_j(0) = \lambda_0$ *and* $\lambda_j'(0) = -4\pi \mathrm{i}\mu_j$.

The perturbation of a semi-simple resonance has been investigated in [7] by a different method.

3. Single Jordan Chain

To deal with the case of a single Jordan chain $\{\varphi_0, \varphi_1, \ldots, \varphi_{m-1}\}$ of the operator family $K(\lambda) = K(\lambda, \alpha_0)$ corresponding to the singular point $\lambda = \lambda_0$ it is useful to introduce the one-dimensional projection

$$P \;=\; (\cdot, \Psi)\Phi$$

where the vectors $\Phi = K'(\lambda_0)\varphi_{m-1} + \frac{1}{2}K''(\lambda_0)\varphi_{m-2} + \ldots + \frac{1}{m!}K^{(m)}(\lambda_0)\varphi_0$ and $\Psi = K^{*\prime}(\lambda_0)\psi_{m-1} + \frac{1}{2}K^{*\prime\prime}(\lambda_0)\psi_{m-2} + \ldots + \frac{1}{m!}K^{*(m)}(\lambda_0)\psi_0$ satisfy the normalization condition

$$(\varphi_0, \Psi) \;=\; (\Phi, \psi_0) \;=\; 1.$$

It can be seen easily that the operator $(K(\lambda_0, \alpha_0) + P - I)$ is invertible in $\mathcal{H}$ and that the same holds true for the operator $(K(\lambda, \alpha) + P - I)$ provided that λ and α are close enough to λ_0 and α_0 respectively.

Due to this fact it makes sense to consider (cf. [2]) the following equation

$$\big((K(\lambda, \alpha) + P - I)^{-1}\Phi, \Psi\big) = 1 \tag{$*$}$$

which determines all singular points $\lambda = \lambda(\alpha)$ of the operator function $K(\lambda, \alpha)$ in a certain neighbourhood of λ_0. Indeed, given a solution $\lambda = \lambda(\alpha)$ of the equation $(*)$ one has

$$\varphi = (K(\lambda, \alpha) + P - I)^{-1}\Phi \in \ker(K(\lambda, \alpha) - I)$$

since $(\varphi, \Psi) = 1$ by virtue of $(*)$ and hence $P\varphi = \Phi$. The singular points of $K(\lambda, \alpha)$, i.e., the solutions to the equation $(*)$, that are subject to initial condition $\lambda(\alpha_0) = \lambda_0$, prove to be the branches of one or perhaps several multivalued analytic functions with branching point $\alpha = \alpha_0$ of the order not greater than m. Under additional condition $\big(K_\alpha'(\lambda_0, \alpha_0)\varphi_0, \psi_0\big) \neq 0$ this order is exactly equal to m.

Theorem 1. *Suppose that* $\dim\ker(K(\lambda_0) - I) = 1$ *and* $(K(\lambda) - I)^{-1}$ *has a pole of order* m *at the point* $\lambda = \lambda_0$*. If* $(K'_\alpha(\lambda_0, \alpha_0)\varphi_0, \psi_0) \neq 0$ *for eigenvectors* $\varphi_0 \in \ker(K(\lambda_0, \alpha_0) - I)$ *and* $\psi_0 \in \ker(K^*(\lambda_0, \alpha_0) - I)$ *then for* α *close enough to* α_0 *the singular points of* $K(\lambda, \alpha)$ *from sufficiently small neighbourhood of* λ_0 *are represented by the Puiseux power expansions in variable* $(\alpha - \alpha_0)^{1/m}$*. All these singular points are necessarily simple and can be enumerated in such a way that*

$$\lambda_s(\alpha) = \lambda_0 + \lambda_s^{(1)}(\alpha - \alpha_0)^{1/m} + \ldots, \qquad s = 0, 1, \ldots, m-1$$

where $\lambda_s^{(1)} = \big(-\,(K'_\alpha(\lambda_0, \alpha_0)\varphi_0, \psi_0)\big)^{1/m} \mathrm{e}^{2\mathrm{i}\pi s/m}$.

To outline the proof let us introduce the scalar function

$$\begin{aligned} F(\lambda, \alpha) :=& \big((K(\lambda, \alpha) + P - I)^{-1}\Phi, \Psi\big) \\ =& \big((K(\lambda, \alpha) - I)^{-1}\big(I + P(K(\lambda, \alpha) - I)^{-1}\big)^{-1}\Phi, \Psi\big) \end{aligned}$$

where $\big(I + P(K(\lambda, \alpha) - I)^{-1}\big)^{-1}\Phi = \big\{1 + \big((K(\lambda, \alpha) - I)^{-1}\Phi, \Psi\big)\big\}^{-1}\Phi$ and hence

$$F(\lambda, \alpha) \;=\; \frac{\big((K(\lambda, \alpha) - I)^{-1}\Phi, \Psi\big)}{1 + \big((K(\lambda, \alpha) - I)^{-1}\Phi, \Psi\big)}.$$

Since the resolvent $(K(\lambda) - I)^{-1}$ has the pole of the order m at the point $\lambda = \lambda_0$ so does $\big((K(\lambda) - I)^{-1}\Phi, \Psi\big)$ and it follows that

$$F(\lambda_0, \alpha_0) = 1, \quad F'_\lambda(\lambda_0, \alpha_0) = \ldots = F_\lambda^{(m-1)}(\lambda_0, \alpha_0) = 0, \quad F_\lambda^{(m)}(\lambda_0, \alpha_0) \neq 0.$$

Therefore by the implicit function theorem solutions $\lambda = \lambda(\alpha)$ of the equation $(*)$ satisfying initial condition $\lambda(\alpha_0) = \lambda_0$ are given by the values of analytic functions which have isolated algebraic branching point at $\alpha = \alpha_0$

$$\lambda(\alpha) \;=\; \lambda_0 + \sum_{n=1}^{\infty} \lambda^{(n)}(\alpha - \alpha_0)^{n/p}, \qquad 0 < p \leqslant m.$$

Additional condition $F'_\alpha(\lambda_0, \alpha_0) = -(K'_\alpha(\lambda_0, \alpha_0)\varphi_0, \psi_0) \neq 0$ guarantees that the equation $(*)$ has a unique solution

$$\alpha(\lambda) \;=\; \alpha_0 + \frac{1}{F'_\alpha(\lambda_0, \alpha_0)}(\lambda - \lambda_0)^m + \ldots$$

where $\alpha'(\lambda_0) = \ldots = \alpha^{(m-1)}(\lambda_0) = 0$ since $F_\lambda^{(s)}(\lambda_0, \alpha_0) = 0, 1 \leqslant s \leqslant m-1$. It readily follows now that $p = m$ and $\lambda^{(1)} = (F'_\alpha(\lambda_0, \alpha_0))^{1/m}$.

Remark 2. *It is appropriate to compare the assertion of* Theorem 1 *with the fact concerning perturbation of a multiple eigenvalue in the finite-dimensional case. Given a Jordan block J associated with the chain $\{\varphi_0, \dots, \varphi_{m-1}\}$ of eigen- and adjoint vectors one can ask when $J + A$ is diagonal, i.e., the eigenvalue of J (of multiplicity m) splits into m different and hence simple eigenvalues under such a perturbation. The sufficient condition for this is as follows $(A\varphi_0, \varphi_{m-1}) \neq 0$ (see e.g.* [8]*); surely it coincides with that from the* Theorem 1 *since $\varphi_{m-1} = \psi_0$ in this case.*

4. Semi-simple Singular Point

Consider now the case of a semi-simple singular point $\lambda = \lambda_0$ of the operator family $K(\lambda) = K(\lambda, \alpha_0)$. Perturbation of a semi-simple isolated eigenvalue of finite multiplicity has been studied in detail (see e.g. [9]). Theorem 2 below can be regarded as an analogue (and a generalization as well) of Theorem 2.3 from [9] within our setting.

Starting the investigation of a semi-simple singular point $\lambda = \lambda_0$ of multiplicity n let $\{\varphi_1, \dots, \varphi_n\}$ be a certain basis of the eigen-subspace $\ker(K(\lambda_0) - I)$ and let us introduce the finite-dimensional operator

$$P \;=\; \sum_{j=1}^{n} (\cdot, \Psi_j)\Phi_j$$

where $\Phi_j = K'_\lambda(\lambda_0)\varphi_j$ and $\Psi_j = K^{*\prime}_\lambda(\lambda_0)\psi_j$ while the basis $\{\psi_1, \dots, \psi_n\}$ of the subspace $\ker(K^*(\lambda_0) - I)$ is chosen in such a way that $(\Phi_i, \psi_j) = (\varphi_i, \Psi_j) = \delta_{ij}$.

One can verify the invertibility of the operator $(K(\lambda, \alpha) + P - I)$ in $\mathcal{H}$ provided that λ and α are close enough to λ_0 and α_0 respectively. For such λ and α the scalar functions

$$A_{ij}(\lambda, \alpha) := \big((K(\lambda, \alpha) + P - I)^{-1}\Phi_i, \Psi_j\big)$$

are well defined and $\big(\delta_{ij} - A_{ij}(\lambda, \alpha)\big)$ is a matrix representation of the operator $\big(I + P\big(K(\lambda, \alpha) - I\big)^{-1}\big)^{-1}$ with respect to the basis $\{\Phi_i\}$. Due to this fact a relationship

$$F(\lambda, \alpha) := \det\big(\delta_{ij} - A_{ij}(\lambda, \alpha)\big) = 0 \qquad (**)$$

determines all singular points $\lambda = \lambda(\alpha)$ of the operator function $K(\lambda, \alpha)$ in the neighbourhood of λ_0. Thus the problem concerning the singular points of the operator function $K(\lambda, \alpha)$ is reduced to the study of solutions to the equation $(**)$.

Theorem 3. *Suppose that* $\dim\ker\big(K(\lambda_0)-I\big)=n$ *and that the resolvent* $\big(K(\lambda)-I\big)^{-1}$ *has a simple pole at the point* $\lambda=\lambda_0$. *Let* $\{\varphi_1,\dots,\varphi_n\}$ *and* $\{\psi_1,\dots,\psi_n\}$ *are the bases of subspaces* $\ker\big(K(\lambda_0)-I\big)$ *and* $\ker\big(K^*(\lambda_0)-I\big)$ *respectively satisfying the normalization condition* $\big(K'_\lambda(\lambda_0,\alpha_0)\varphi_i,\psi_j\big)=\delta_{ij}$. *If all the eigenvalues* $\{\mu_j\}$ *of the matrix* $\big(K'_\alpha(\lambda_0,\alpha_0)\varphi_i,\psi_j\big)$ *are different then for* α *sufficiently close to* α_0 *there exist* (*in a certain neighbourhood of* λ_0) *just* n *simple singular points* $\lambda_j(\alpha)$ *of the operator family* $K(\lambda,\alpha)$ *one can enumerate in such a way that*

$$\lambda_j(\alpha)=\lambda_0-\mu_j(\alpha-\alpha_0)+\dots,\qquad j=1,\dots,n.$$

The sketch of the proof presented below shows that this approach can be applied to a rather general situation. Since $\big(K(\lambda_0)+P-I\big)^{-1}\Phi_i=\varphi_i$ one has

$$A_{ij}(\lambda_0,\alpha_0)=(\varphi_i,\Psi_j)=\delta_{ij}$$

and hence $F(\lambda_0,\alpha_0)=0$. Taking into account that $\big(K^*(\lambda_0)+P^*-I\big)^{-1}\Psi_j=\psi_j$ we similarly get

$$\begin{aligned}(A_{ij})'_\lambda&(\lambda_0,\alpha_0)\\ &=-\big(K'_\lambda(\lambda_0)\big(K(\lambda_0)+P-I\big)^{-1}\Phi_i,\big(K^*(\lambda_0)+P^*-I\big)^{-1}\Psi_j\big)\\ &=-(\Phi_i,\psi_j)=-\delta_{ij}.\end{aligned}$$

By virtue of the determinants differentiation rule it follows that

$$F'_\lambda(\lambda_0,\alpha_0)=\dots=F^{(n-1)}_\lambda(\lambda_0,\alpha_0)=0,\qquad F^{(n)}_\lambda(\lambda_0,\alpha_0)\neq 0.$$

Now according to the implicit function theorem some small neighbourhood of λ_0 contains $p\leqslant n$ roots $\lambda_k(\lambda)$ of the equation $(**)$ for α sufficiently close to α_0 and for a certain $q\in\mathbb{N}$ the Puiseux expansions are valid

$$\lambda_k(\alpha)=\lambda_0+\lambda_k^{(1)}(\alpha-\alpha_0)^{1/q}+\lambda_k^{(2)}(\alpha-\alpha_0)^{2/q}+\dots.$$

Substituting this expression into $F(\lambda,\alpha)$ and setting the coefficients at the successive (fractional) powers of $(\alpha-\alpha_0)$ equal to zero we come (by the induction arguments) to the conclusion that $\lambda_k^{(s)}=0$, $s=1,\dots,q-1$, and so

$$\lambda_k(\alpha)=\lambda_0+\lambda_k^{(q)}(\alpha-\alpha_0)+\dots.$$

Further on, one can verify that

$$A_{ij}(\lambda_k(\alpha),\alpha)=\delta_{ij}+A_{ij}^{(k)}(\lambda_0,\alpha_0)(\alpha-\alpha_0)+\dots$$

where the coefficients $A_{ij}^{(k)}(\lambda_0,\alpha_0)$ are given by $\{\lambda_k^{(q)}\delta_{ij}+(K'_\alpha(\lambda_0,\alpha_0)\varphi_i,\psi_j)\}$ due to the normalization condition $(K'_\lambda(\lambda_0,\alpha_0)\varphi_i,\psi_j)=\delta_{ij}$. Hence the determinant $F(\lambda_k(\alpha),\alpha)$ takes the form

$$F(\lambda_k(\alpha),\alpha)=(-1)^n\det\big(A_{ij}^{(k)}(\lambda_0,\alpha_0)\big)(\alpha-\alpha_0)^n+\ldots$$

and therefore $\det\big(A_{ij}^{(k)}(\lambda_0,\alpha_0)\big)=0$, i.e., $-\lambda_k^{(q)}$ is an eigenvalue of the matrix $(K'_\alpha(\lambda_0,\alpha_0)\varphi_i,\psi_j)$. Since all the eigenvalues of this matrix are assumed to be different it follows that $p=n$ and the proof is complete.

Remark 4. *The eigenvector $\varphi_k(\alpha)\in\ker\big(K(\lambda_k(\alpha),\alpha)-I\big)$ has the limit*

$$\lim_{\alpha\to\alpha_0}\varphi_k(\alpha) \;=\; \sum_{j=1}^{n}c_j^{(0)}\varphi_j$$

where $\{c_1^{(0)},\ldots,c_n^{(0)}\}$ is an eigenvector of the matrix $\big(K'_\alpha(\lambda_0,\alpha_0)\varphi_i,\psi_j\big)$ corresponding to the eigenvalue $-\lambda'_k(\alpha_0)$.

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