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WIGNER FUNCTIONS AND WEYL OPERATORS ON THE EUCLIDEAN MOTION GROUP

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Abstract. The Wigner distribution function is one of the pillars of the phase space formulation of quantum mechanics. Its original formulation may be cast in terms of the unitary representations of the Weyl - Heisenberg group. Following the construction proposed by Wolf and coworkers in constructing the Wigner functions for general Lie groups using the irreducible unitary representations of the groups, we develop here the Wigner functions and Weyl operators on the Euclidean motion group of rank three. We give complete derivations and proofs of their important properties.

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Contents

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1. Introduction

Ever since E. Wigner introduced [21] the function

$$W_\psi(q,p) = \frac{1}{h^n}\int_{\mathbb{R}^n} \psi(q+x)^*\psi(q-x)\mathrm{e}^{2\mathrm{i}p\cdot x/\hbar}\mathrm{d}x$$

now named after him, in his study of quantum corrections to systems in thermodynamic equilibrium, the phase space formulation of quantum mechanics steadily gained the attention of physicists and mathematicians, as well as scientists from various fields, in theoretical investigations and practical applications. Together with the Weyl transform or Weyl quantization, the Wigner distribution function is a pillar of phase space quantum mechanics [8]. The following list collects the well-known properties of the Wigner distribution, one involving the Weyl quantization, which is given by

$$\mathcal{W}_\sigma\psi(x) = \frac{1}{h^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}p\cdot(x-y)}\sigma\left(\frac{x+y}{2},p\right)\psi(p)\,\mathrm{d}p\,\mathrm{d}y.$$

The Weyl quantization is the most recognizable among the many quantization schemes assigning to a phase space function $\sigma(q,p)$ a Hilbert space operator $\mathcal{W}_\sigma$. If ψ is a normalized wavefunction and * is complex conjugation, then (for one degree of freedom)

a) $\int_{-\infty}^{\infty} W_\psi(q,p)\,\mathrm{d}p = |\psi(q)|^2$

b) $\int_{-\infty}^{\infty} W_\psi(q,p)\,\mathrm{d}q = |\widetilde{\psi(p)}|^2$, where $\widetilde{\psi}$ is the Fourier transform of the wavefunction

c) $\int_{-\infty}^{\infty} W_\psi(q,p)\,\mathrm{d}q\,\mathrm{d}p = 1$

d) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi(q,p)W_\phi(q,p)\,\mathrm{d}q\,\mathrm{d}p = |\left\langle\psi|\phi\right\rangle|^2$

e) $\left\langle\psi|\hat{A}\psi\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(q,p)W_\psi(q,p)\,\mathrm{d}q\,\mathrm{d}p$, where $\hat{A}$ is the Hilbert space operator corresponding to the symbol $a(a,p)$ under Weyl quantization, that is, $\mathcal{W}_a = \hat{A}$.

The properties a), b), and c) strongly suggests that $W_\psi(q,p)$ may very well be a probability distribution, but this is true precisely for Gaussians only [15]. That the Wigner function attains negative values in its domain is seen from property d). If the wavefunctions are orthogonal, $\langle\psi,\phi\rangle = 0$, then it is not possible for the factors inside the integrals on the left hand side of d) to be both nonnegative. Thus, the Wigner function is not a true probability distribution and is termed a quasiprobability distribution. Property e) not only indicates the intimate relationship between the Wigner function and the Weyl quantization, it also provides the computation of expected values of observables in quantum theory in a classical like manner. The reader may consult [13] for more extensive discussions on the Wigner functions, while [12] is a seminal work on the rigorous treatment of Weyl quantization.

The applications of the Wigner functions, in its many different formulations and generalizations, are varied and numerous. In [9], the Wigner function was introduced in optics in a beam description of wavefields. This was also done in [20], as well as an attempt to resolve coherence of light coming from apparently incoherent point sources. In [4], the Wigner function was again used in optics as representations of signals in the spatial and frequency domains simultaneously, termed double Wigner distribution by the author. An important property of the Wigner representation of optical systems is its "close affinity with geometric optics and its independence on the coherence of signals" [4].

In the papers [3, 22], the authors introduced the Weyl operator and the Wigner function on Lie groups, besides the standard one on the phase space orbit of the Weyl-Heisenberg group. The definition adopted in these works takes advantage of the crucial role of the unitary, irreducible representations of Lie groups appearing as symmetries of quantum systems. The Wigner function is the Fourier transform of the matrix elements of the Schrödinger representation. On the other hand, Weyl quantization is the integral form of the Schrödinger representation in composition with the inverse Fourier transform. It is property e) in the above list that ties the two threads together, and is quite natural, as they both arise from the irreducible representations of the Weyl-Heisenberg group, motivating the definitions of the Weyl operator and Wigner function on general groups. This role of the representations of the Weyl-Heisenberg group is merely the beginning of a whole philosophy of using group theory in geometry and quantum mechanics, and the sought after geometrization of quantum mechanics. For example, the role of group representations in quantization has a highly developed and quite technical, but nevertheless geometric, formulation in the Kirillov-Kostant-Souriau theory of geometric quantization [17].

In [22], the Wigner functions for position, momentum and wavelength were constructed for paraxial polychromatic optical systems using the full Weyl-Heisenberg group. In [3], Wigner distributions for finite data sets arising from finite optical systems were constructed based on the dynamical group $\mathrm{SU}(2)$. Wigner functions were assigned as classical observables corresponding to operators of position, momentum and waveguide mode. In [2], Wigner functions were computed on the

Euclidean motion group $\mathrm{E}(2)$ of the plane in the study of Helmholtz and polychromatic wavefields, as well as infinite discrete data sets. Attention were given to a couple of coordinatization of the group, as well as different, but isometric, representation spaces, whose physical contents are distinct.

We also point out that phase space methods have had a profound impact on signal analysis via the so-called Wigner-Ville distribution [7, 19], one of the time-frequency distributions used in this vast research area. The review [7] makes the assertion that a unified approach to time-frequency distributions is achieved by a sort of generalized Wigner function (equation 4.1 in [7]).

The very recent appearances of the review and tutorial articles [10, 23] clearly points to the rapid and increasing importance of phase space methods, in general, and Wigner functions, in particular, in physics, mainly in optics, signal and information analysis, in quantum electronics, and even entanglement.

On a more theoretical vein, Wigner functions on the two-dimensional wavelet group were considered [1], motivated by problems in two-dimensional image analysis. Meanwhile, the work of Kastrup [16] point to the crucial role again of the group $\mathrm{E}(2)$ in the quantization of the angle and angular momentum pair.

Quantization here refers to the Weyl quantization and its generalizations. The so-called symbols, which are the Wigner transforms of the Weyl operators, multiply via the Moyal star-product which corresponds to product of operators [11, 18]. The Moyal product was the motivation for deformation quantization of symplectic manifolds [5] whose aim is to arrive at an autonomous phase space formulation of quantum mechanics. See the relatively recent article [14] and the references therein regarding deformation quantization and its applications to quantum physics.

Wavelets, localization operators, and coherent states are other important topics for which the twin concepts of Wigner function and Weyl quantization are the guiding ideas.

In the recent article [23], the importance of $\mathrm{E}(3)$ in optics is emphasized and we feel that concrete and exact computations of the Wigner functions and Weyl operators is a significant contribution towards further understanding its role in other areas, both applied and theoretical. This work then is a contribution to one of the basic tasks in phase space quantum mechanics. We will construct the Weyl operators and Wigner functions on the Euclidean motion group $\mathrm{E}(3)$, of rank three and prove their important properties. We employ the method introduced in [3, 23], by considering the Weyl operator as a sort of Fourier transform of the group elements, where the group elements are parametrized by the unitary irreducible representations of the group. In a future paper, we will construct the star-product on the Euclidean motion groups and consider its important properties.

In Section 2, we discuss the main ingredients in constructing the Wigner operator and functions on general Lie groups. Section 3 revisits the Euclidean motion group of rank N, paying attention to $\mathrm{E}(3)$. This includes the discussion of its Lie algebra, Haar measure, the outline of the unitary irreducible representation and its matrix elements. Section 4 shows the construction of the Weyl operator and Wigner function on $\mathrm{E}(3)$. Here, special functions play a prominent role, as they do in the representation theory of the group itself. Properties of $\mathrm{E}(3)$ Wigner function are presented in Section 5. Section 6 ends the article with some words about Wigner function on the Euclidean motion group and its possible extension to higher dimensions.

2. The Weyl Operator and the Wigner Function

From the work of Ali, Atakishiyev, Chumakov and Wolf [2, 3, 22], an operator associated to each element of the $N-$dimensional Lie group G was proposed, called variously the Wigner operator or the Weyl operator. We briefly recall their definition, using also their notations.

Consider a Lie group G generated by the Lie algebra elements $X_1, ..., X_N$. These are simply some convenient choice of basis elements of the Lie algebra of G. The Weyl operator on G, corresponding to a vector $\mathbf{x} \in \mathbb{R}^n$, denoted by $\mathcal{W}^G$, is given by

$$\mathcal{W}^G(\mathbf{x}) = \int_G \exp(-\mathrm{i}\gamma \cdot \mathbf{x}) g[\gamma] \mathrm{d}g[\gamma] = \int_G \exp\left(\mathrm{i}\left(\sum_{k=1}^{d} \gamma_k (X_k - x_k)\right)\right) \mathrm{d}g[\gamma] \tag{1}$$

where $\gamma \cdot \mathbf{x} = \sum_{k=1}^{d} \gamma_k \mathbf{x}_k$ and $g[\gamma]$ is the element of the group parametrized by so-called polar coordinates, which means that the group is of exponential type. This means that

$$g[\gamma] = \exp(\mathrm{i}\gamma \cdot X) = \exp\left(\mathrm{i} \sum_{k=1}^{d} \gamma_k X_k\right).$$

The element $g[0]$ is the identity of the group and $g[\gamma]^{-1} = g[-\gamma]$ is the inverse of $g[\gamma]$. Here, $\gamma \in G \subset \mathbb{R}^N$ indicates the parametrization of the manifold G and $\mathrm{d}g[\gamma]$ is the invariant Haar measure.

In the following, we will compute the expression of the Weyl operators for the Euclidean motion group $\mathrm{E}(3)$ in a certain coordinate system for the group and with respect to the representation Hilbert space $\mathcal{H}$ of the unitary irreducible representations of $\mathrm{E}(3)$. The Wigner functions associated to this group are defined as the matrix elements of the Weyl operators

$$W^G(\phi,\psi|\mathbf{x}) = \langle\phi|\mathcal{W}^G(\mathbf{x})\psi\rangle = \frac{1}{(2\pi)^n}\int \overline{\phi}(q)\mathcal{W}^G(\mathbf{x})\psi(q)\mathrm{d}q.$$

For general Lie groups, this integral expression is also the definition of Wigner functions [2, 3, 23]. The operator $\mathcal{W}^G(\mathbf{x})$ is a sort of Fourier transform of the group elements. For such a definition to make sense, $g[\gamma]$ is actually not used, but its parametrization by the unitary irreducible representations. We may view the formula for $\mathcal{W}^G(\mathbf{x})$ as an ansatz, ignoring thereby the technical issues on Banach space-valued integrals, where it becomes concrete and meaningful when computed for particular Lie groups appearing as dynamical groups of physical systems.

3. The Euclidean Motion Group $\mathrm{E}(3)$

We first present the rank n Euclidean motion group $\mathrm{E}(n)$. The semidirect product of $\mathbb{R}^n$ with the special orthogonal group $\mathrm{SO}(n)$ is called the special Euclidean motion group of rank n. In symbols, $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$. Elements of $\mathrm{E}(n)$ are denoted by $g = (\mathbf{a}, A)$ or $g(a_1, a_2, \cdots, a_n; A)$, where $A \in \mathrm{SO}(n)$ and $\mathbf{a} \in \mathbb{R}^n$. If $g = (\mathbf{a}, A)$ and $h = (\mathbf{r}, R) \in \mathrm{E}(n)$, the group law is given by $g \circ h = (\mathbf{a} + A\mathbf{r}, AR)$ and $g^{-1} = (-A^T\mathbf{a}, A^T)$.

We may also write any element of $\mathrm{E}(n)$ as an $(n+1)\times(n+1)$ homogeneous transformation matrix of the form

$$M(g) = \begin{bmatrix} A & \mathbf{a} \\ 0 & 1 \end{bmatrix}.$$

The product of two matrices is given by $M(g)M(h) = M(g\circ h)$ and its inverse is $M(g^{-1}) = [M(g)]^{-1}$. The elements of $\mathrm{E}(n)$ can also be written as $(\mathbf{a}, A) = (\mathbf{a}, I)\cdot(0, A)$. The action of $\mathrm{E}(n)$ on $\mathbb{R}^n$ is the rotation A followed by translation by the vector $\mathbf{a}$ and has the expression $(\mathbf{a}, A)\cdot\mathbf{x} = A\mathbf{x} + \mathbf{a}$. The Lie algebra $\mathfrak{e}(n)$ of $\mathrm{E}(n)$ is the semidirect product $\mathfrak{e}(n) = \mathbb{R}^n \rtimes \mathfrak{so}(n)$. Basis elements of $\mathfrak{e}(n)$, called the generators of $\mathrm{E}(n)$, are denoted by $X_k, k = 1, 2, \cdots, \dfrac{n(n+1)}{2}\cdot$ The elements $\mathfrak{g} = (\mathbf{v}, K)$ of $\mathfrak{e}(n)$ is written as the matrix

$$M(\mathfrak{g}) = \begin{bmatrix} K & \mathbf{v} \\ 0 & 0 \end{bmatrix}.$$

The exponential map $\exp : \mathfrak{e}(n) \to \mathrm{E}(n)$ will be considered concretely only for $n = 3$.

3.1. The Euclidean Motion Group of Rank Three

The Euclidean motion group of rank three consists of the matrices of the form

$$\mathrm{E}(3) = \left\{ \begin{bmatrix} A & \mathbf{a} \\ 0 & 1 \end{bmatrix} \in \mathrm{Mat}(4, \mathbb{R});\ A \in \mathrm{SO}(3), \mathbf{a} \in \mathbb{R}^3 \right\}.$$

First, consider the special orthogonal group of dimension three, $\mathrm{SO}(3)$. Its Lie algebra $\mathfrak{so}(3)$ is spanned by the following matrices

$$R_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad R_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad R_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that if we use one-parameter subgroups generated by the basis elements R_1, R_2, and R_3 of $\mathfrak{so}(3)$ we have

$$\exp\ R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad \exp\ R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\exp\ R_z(\eta) = \begin{bmatrix} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \eta \leq \pi$ are angles of rotation about the x, y and z axis, respectively. The commutation relations of these basis elements are

$$[R_x, R_y] = R_z, \qquad [R_z, R_x] = R_y, \qquad [R_y, R_z] = Rx.$$

The Euler angles, i.e., (α, β, η) are well-known parametrization of rotation group. The group elements of $\mathrm{SO}(3)$ are generated by three successive rotation about independent axes. The three most commonly used parametrizations are ZXZ, ZYZ, and ZYX Euler angles. In this article, we use the ZXZ Euler angles as parametrization of the elements of $\mathrm{SO}(3)$.

Considering the exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ and the basis elements R_x, R_y and R_z of $\mathfrak{so}(3)$, we obtain the matrix representation of $A(\alpha, \beta, \eta) \in \mathrm{SO}(3)$ given by

$$A(\alpha, \beta, \eta) = \mathrm{e}^{R_z(\alpha)} \mathrm{e}^{R_x(\beta)} \mathrm{e}^{R_z(\eta)}.$$

The Lie algebra $\mathfrak{e}(3)$ of $\mathrm{E}(3)$ is defined as the semidirect product of the Lie algebra of $\mathbb{R}^3$ with the Lie algebra of $\mathrm{SO}(3)$ where elements are identified with matrices of size four, where

$$\mathfrak{e}(3) = \left\{ \begin{bmatrix} K & \mathbf{v} \\ 0 & 0 \end{bmatrix} \in \mathrm{Mat}(4, \mathbb{R});\ K \in \mathfrak{so}(3), \mathbf{v} \in \mathbb{R}^3 \right\}.$$

The above Lie algebra is spanned by matrices $R_i, T_i, i = 1, 2, 3$, where R_1, R_2, R_3 span $\mathfrak{so}(3)$ and T_1, T_2, T_3 span $\mathbb{R}^3$. The Lie brackets are as follows

$$[R_i, R_j] = (-1)^{k+1} R_k \text{ for } i < j, \qquad [T_i, R_j] = (-1)^{k+1} T_k \quad \text{for} \quad i < j$$

$$[T_i, R_j] = (-1)^k T_k \text{ for } i > j, \quad [T_i, T_j] = [T_i, R_i] = 0 \quad \text{for} \quad i, j, k = 1, 2, 3.$$

The rotation matrices R_1, R_2 and R_3 correspond to rotations about the x, y and z axes, respectively, while T_1, T_2, T_3 correspond to the translations along the x, y, and z axes. Using the ZXZ Euler angles, we can explicitly write the elements of $\mathrm{E}(3)$ in the form $g(\mathbf{a}, A) = g(\mathbf{a}, I)g(0, A)$, where $g(\mathbf{a}, I)$ represents the group elements of the translation group $\mathbb{R}^3$ and $g(0, A)$ represents the group elements of the special orthogonal group of dimension three, $\mathrm{SO}(3)$. Using the exponential map $\exp : \mathfrak{e}(3) \to \mathrm{E}(3)$, the elements of $\mathrm{E}(3)$ can be written in matrix form as follows

$$\begin{aligned} & g(\mathbf{a}, A) = g(\mathbf{a}, I)g(0, A) \\ & = \exp[(a_1T_1 + a_2T_2 + a_3T_3)] \exp R_3(\alpha) \exp R_1(\beta) \exp R_3(\eta) \\ & = \begin{bmatrix} \cos\eta\cos\alpha - \sin\eta\sin\alpha\cos\beta & -\sin\eta\cos\alpha - \cos\eta\sin\alpha\cos\beta & \sin\beta\sin\alpha & a_1 \\ \cos\eta\sin\alpha + \sin\eta\cos\beta\cos\beta & -\sin\eta\sin\alpha + \cos\eta\cos\alpha\cos\beta & -\sin\beta\cos\alpha & a_2 \\ \sin\beta\sin\eta & \sin\beta\cos\eta & \cos\beta & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

3.2. The Haar Measure

Generally, the volume element with respect to which we integrate over the motion group is of the form

$$\mathrm{d}(g(q_1, q_2, \cdots, q_n)) = |\det(\mathcal{J})| \mathrm{d}q_1 \mathrm{d}q_2 \cdots \mathrm{d}q_n$$

where q_i are the parameters used to describe the motion group and $\det(\mathcal{J})$ is the product of Jacobian determinants for the rotation and translations. In the case of $\mathrm{E}(3)$, the volume element is the product of the volume elements of $\mathbb{R}^3$ and for $\mathrm{SO}(3)$.

It is well known that for the left Jacobian matrix and the right Jacobian matrix for ZXZ Euler angles

$$|\det(\mathcal{J}_L(A(\alpha, \beta, \eta)))| = |-\sin\beta| = |\det(\mathcal{J}_R(A(\alpha, \beta, \eta)))|. \qquad (2)$$

Therefore, $\mathrm{SO}(3)$ is unimodular. Thus, volume element of $\mathrm{SO}(3)$ using equation (2) is of the form

$$\mathrm{d}g(A(\alpha, \beta, \eta)) = \sin\beta \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\eta. \qquad (3)$$

Next, for the volume element $\mathrm{d}\mathbf{a} = \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3$ of $\mathbb{R}^3$ written in spherical coordinates

$$\mathbf{a} = (a_1, a_2, a_3) = (a\cos\phi\sin\theta, a\sin\phi sin\theta, a\cos\theta)$$

we have

$$\mathrm{d}(g(a_1, a_2, a_3)) = a^2\sin\theta\mathrm{d}a\mathrm{d}\phi\mathrm{d}\theta \quad (4)$$

where $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$, ϕ and θ are the azimuthal and polar angles, respectively.

Finally, the volume element of $\mathrm{E}(3)$ using (3) and (4) is

$$\mathrm{d}(g(a_1, a_2, a_3; \alpha, \beta, \eta)) = a^2\sin\theta\sin\beta\mathrm{d}a\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\eta.$$

The unimodularity of the groups $\mathrm{SO}(3)$ and $\mathbb{R}^3$ implies that $\mathrm{E}(3)$ itself is unimodular.

3.3. The Unitary Irreducible Representation and Its Matrix Elements

We discuss here the unitary irreducible representations (UIRs) of $\mathrm{E}(3)$ which are essential in the formulation of Wigner distribution function on $\mathrm{E}(3)$.

The representation space of the unitary irreducible representations of $\mathrm{E}(3)$ is the space $L^2(S^2)$ of square integrable functions $\varphi : S^2 \to \mathbb{C}$ on the two-sphere. The inner product in this space is given by

$$\langle\varphi_1|\varphi_2\rangle = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \overline{\varphi_1}(\mathbf{p})\varphi_2(\mathbf{p})\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi$$

where $\mathbf{p} = (p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$ and $p > 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$.

Given that $A \in \mathrm{SO}(3)$, the representation of $\mathrm{SO}(3)$ acting on the function $\varphi \in L^2(S^2)$ is expressed as

$$(U^s(A)\varphi)(\mathbf{p}) = \Delta_s(R_{\mathbf{p}}^{-1}AR_{A^{-1}\mathbf{p}})\varphi(A^{-1}\mathbf{p}) \quad (5)$$

where Δ_s are the helicity representations of $H_{\hat{u}} \cong \mathrm{SO}(2)$. The helicity representation of $H_{\hat{u}} \cong \mathrm{SO}(2)$ is written in the form

$$\Delta_s : \phi \mapsto \mathrm{e}^{\mathrm{i}s\phi}, \qquad 0 \leq \phi \leq 2\pi$$

where $s = 0, \pm1, \pm2, \cdots$, ϕ is the angle of rotation of the matrix $R_u^{-1}AR_{A^{-1}u}$ and Δ_s are representations that form the Fourier basis for $S^1 \cong \mathrm{SO}(2) \cong H_{\hat{u}}$.

Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$. The translation operator acts on $\varphi(\mathbf{p})$ as

$$(U^s(\mathbf{a}, I)\phi)(\mathbf{p}) = \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}}\phi(\mathbf{p}). \quad (6)$$

We can write the unitary irreducible representations of $\mathrm{E}(n)$ as the product of the UIRs of the translation and rotation groups [6]. That is,

$$U_g^s = U^s(\mathbf{a}, A) = U^s(\mathbf{a}, I) \cdot U^s(0, A).$$

Using equations (5) and (6), the UIRs of $\mathrm{E}(3)$ are the operators U_g^s, $g = (\mathbf{a}, A) \in \mathrm{E}(3)$, acting on functions $\varphi \in L^2(S^2)$, given by

$$\begin{aligned}(U_g^s\varphi)(\mathbf{p}) &= [U^s(\mathbf{a}, I)(U^s(0, A)\varphi)](\mathbf{p}) \\ &= \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}}[U^s(0, A)\varphi](\mathbf{p}) = \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}}\Delta_s(R_{\mathbf{p}}^{-1}AR_{A^{-1}\mathbf{p}})\varphi(A^{-1}\mathbf{p}).\end{aligned} \tag{7}$$

The representations are unitary, that is, $\left\langle U_g^s\varphi_1 | U_g^s\varphi_2\right\rangle = \langle\varphi_1|\varphi_2\rangle$. The construction of the matrix elements $U^s_{l',m';l,m}(\mathbf{a}, A; p)$ of the UIRs of $E(3)$ are found in [6], as do many other details. The basis functions of the irreducible representations of $\mathrm{SO}(3)$ in equation (5), enumerated by the integers l and m, are expressed in the form

$$h^l_{m,s}(\mathbf{u}(\Theta, \Phi)) = \mathcal{Q}^l_{s,m}(\cos\Theta)\mathrm{e}^{\mathrm{i}(m+s)\Phi} \tag{8}$$

where $\mathcal{Q}^l_{s,m}(\cos\Theta) = (-1)^{l-s}\sqrt{\frac{2l+1}{4\pi}}P^l_{sm}(\cos\Theta)$, $P^l_{sm}(\cos\Theta)$ are the generalized Legendre functions and $l \geq |s|$; $l \geq |m|$.

The unitary operators for $\mathrm{SO}(3)$ acts on the basis functions $h^l_{ms}(\mathbf{u})$ as

$$(U^s(0, A)h^l_{ms})(\mathbf{u}) = \sum_{n=-l}^{l} \tilde{U}^l_{nm}(A)h^l_{ns}(\mathbf{u}) \tag{9}$$

where the matrix elements $\tilde{U}^l_{nm}(A)$ are expressed in terms of the special functions [6]

$$\tilde{U}^l_{nm}(g(\alpha, \beta, \eta)) = (-1)^{n-m}\mathrm{e}^{-\mathrm{i}m\alpha}P^l_{mn}(\cos\beta)\mathrm{e}^{-\mathrm{i}n\eta}$$

with α, β and η are ZXZ Euler angles of rotation.

On the other hand, the translation matrix elements are given by

$$\begin{aligned}\left\langle h^{l'}_{m's}|U^s(\mathbf{a}, I)h^l_{ms}\right\rangle = &\int_{\Theta=0}^{\pi}\int_{\Phi=0}^{2\pi} \mathcal{Q}^{l'}_{s,m'}(\cos\Theta)\mathrm{e}^{-\mathrm{i}(m'+s)\Phi}\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}} \\ &\times\mathcal{Q}^l_{s,m}(\cos\Theta)\mathrm{e}^{\mathrm{i}(m+s)\Phi}\sin\Theta\mathrm{d}\Theta\mathrm{d}\Phi =: [l', m'|p, s|l, m](\mathbf{a}).\end{aligned} \tag{10}$$

The closed form of the expression $[l', m'|p, s|l, m](\mathbf{a})$ is

$$\begin{aligned}[l', m'|p, s|l, m](\mathbf{a}) =&(4\pi)^{\frac{1}{2}} \sum_{k=|l'-l|}^{l'+l} \mathrm{i}^k\sqrt{\frac{(2l'+1)(2k+1)}{2l+1}}j_k(pa)C(k, 0; l', s|l, s) \\ &\times C(k, m-m'; l', m'|l, m)Y_k^{m-m'}(\Theta, \Phi)\end{aligned}$$

where $a = ||\mathbf{a}||, p = ||\mathbf{p}||, C(k, m-m'; l', m'|l, m)$ are the Clebsch-Gordan coefficients, $j_k(pa)$ are the half-integer Bessel functions, $Y_k^{m-m'}(\Theta, \Phi)$ are the spherical harmonics and Θ and Φ are the polar and azimuthal angles of **a**, respectively.

Applying the inner product of functions on $L^2(S^2)$, equations (8), (9) and (10), we obtain the matrix elements $U^s_{l',m';l,m}(\mathbf{a}, A; p)$ of the unitary representation in (7) of $\mathrm{E}(3)$ as follows

$$\begin{aligned} U^s_{l',m';l,m}(\mathbf{a}, A; p) &= \left\langle h^{l'}_{m's'} | U^s(\mathbf{a}, I) U^s(0, A) h^l_{ms} \right\rangle \\ &= \sum_{j=-l}^{l} \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \overline{h}^{l'}_{m's'}(\mathbf{u}) \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}} \tilde{U}^l_{jm}(A) h^l_{js}(\mathbf{u}) \sin\Theta \mathrm{d}\Theta \mathrm{d}\Phi \\ &= \sum_{j=-l}^{l} \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \mathcal{Q}^{l'}_{s,m'}(\cos\Theta) \mathrm{e}^{-\mathrm{i}(m'+s)\Phi} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}} \tilde{U}^l_{jm}(A) \\ &\quad \times \mathcal{Q}^l_{s,j}(\cos\Theta) \mathrm{e}^{\mathrm{i}(j+s)\Phi} \sin\Theta \mathrm{d}\Theta \mathrm{d}\Phi \\ &= \sum_{j=-l}^{l} [l', m'|p, s|l, j](\mathbf{a}) \tilde{U}^l_{jm}(A). \end{aligned} \tag{11}$$

4. The Wigner Function on $\mathrm{E}(3)$

We follow the construction of equation (1). The Weyl operator on $\mathrm{E}(3)$ is an operator-valued function on vectors **x**, with $\mathbf{x} = (t_1, t_2, t_3, r_1, r_2, r_3) \in \mathbb{R}^6$ being the classical variables (c-numbers) corresponding to the generators R_1, R_2, R_3, T_1, T_2, T_3 of $\mathrm{E}(3)$, given by

$$\begin{aligned} \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x}) &= \int_{\mathrm{E}(3)} \exp[-\mathrm{i}(a_1 t_1 + a_2 t_2 + a_3 t_3 + \alpha r_1 + \beta r_2 + \eta r_3] \\ &\quad \times g(a_1, a_2, a_3, \alpha, \beta, \eta) \mathrm{d}g[\gamma] \\ &= \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \exp[-\mathrm{i}(\mathbf{x}\cdot\gamma)] U^s(\mathbf{a}, A) \mathrm{d}\mathbf{a}\, \mathrm{d}A \end{aligned}$$

where $\mathrm{d}\mathbf{a}\mathrm{d}A = \sin\beta \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\eta$ and $\mathbf{x}\cdot\gamma = \sum_k \mathbf{x}_k \gamma_k$. Since the UIRs of $\mathrm{E}(3)$ act on the basis functions $h^l_{m,s}$ on $L^2(S^2)$, we obtain the Wigner matrix, $\mathcal{W}^{\mathrm{E}(3)}_{l',m';l,m}(\mathbf{x})$, by getting the inner product of the basis functions with the Wigner operator $\mathcal{W}^{\mathrm{E}(3)}(\mathbf{x})$ as follows

$$\begin{aligned}\mathcal{W}^{\mathrm{E}(3)}_{l',m';l,m}(\mathbf{x}) &= \left\langle h^{l'}_{m',s} | \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x}) h^{l}_{m,s} \right\rangle \\ &= \sum_{j=-l}^{l} \int_{\alpha=0}^{2\pi} \int_{\beta=0}^{\pi} \int_{\eta=0}^{2\pi} \int_{a_1\in\mathbb{R}} \int_{a_3\in\mathbb{R}} \int_{a_3\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\gamma} [l', m'|p, s|l, j](\mathbf{a}) \\ &\quad \times \tilde{U}^{l}_{jm}(A) \sin\beta \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\eta \\ &= \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\gamma} U^{s}_{l',m';l,m}(\mathbf{a}, A; p) \sin\beta \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\eta\end{aligned} \tag{12}$$

where $U^{s}_{l',m';l,m}$ are the matrix elements of the UIRs of $\mathrm{E}(3)$ in equation (11) expressed as integrals of special functions.

The $\mathrm{E}(3)$ Wigner function is then obtained by taking the inner product of the basis functions $\phi_1, \phi_2 \in L^2(S^2)$ with the Wigner matrix in (12). Applying the definition of the inner product of functions in Hilbert space $\mathcal{H} = L^2(S^2)$, we have

$$\begin{aligned}& W^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) = \left\langle \phi_1 | \mathcal{W}^{\mathrm{E}(3)}_{l',m';l,m}(\mathbf{x}) \phi_2 \right\rangle \\ &= \sum_{j=-l}^{l} \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \int_{\alpha=-\pi}^{\pi} \int_{\beta=0}^{\pi} \int_{\eta=-\pi}^{\pi} \int_{a_1\in\mathbb{R}} \int_{a_3\in\mathbb{R}} \int_{a_3\in\mathbb{R}} \overline{\phi_1}(\mathbf{p}) \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\gamma} \\ &\quad \times [l', m'|p, s|l, j](\mathbf{a}) \tilde{U}^{l}_{jm}(A) \phi_2(\mathbf{p}) \sin\beta \sin\Theta \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\eta \mathrm{d}\Theta \mathrm{d}\Phi \\ &= \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \overline{\phi_1}(\mathbf{p}) \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\gamma} U^{s}_{l',m';l,m}(\mathbf{a}, A; p) \\ &\quad \times \sin\beta \sin\Theta \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\eta \mathrm{d}\Theta \mathrm{d}\Phi.\end{aligned} \tag{13}$$

When $\phi_1 = \phi_2$, the Wigner function of $\mathrm{E}(3)$ becomes $W^{\mathrm{E}(3)}(\phi_1|\mathbf{x})$, which is called the expected value of the Weyl operator in the basis function ϕ_1.

5. Properties of $\mathrm{E}(3)$ Wigner Function

The properties of the Wigner function on the Weyl - Heisenberg [23] group are also observed in the constructed Wigner functions on $\mathrm{E}(3)$. We have the following properties:

A. **Sesquilinearity and Reality Condition of** $\mathrm{E}(3)$ **Wigner Functions**
Given the functions ϕ_2, ϕ_2 and, ϕ_3 in $\mathcal{H}$ with a and b in $\mathbb{R}$, we have

$$\begin{aligned}W^{\mathrm{E}(3)}(\phi_1, a\phi_2 + b\phi_3|\mathbf{x}) &= \left\langle \phi_1 | \mathcal{W}_{l',m';l,m}(\mathbf{x})[a\phi_2 + b\phi_3] \right\rangle \\ &= \left\langle \phi_1 | a\mathcal{W}_{l',m';l,m}(\mathbf{x})\phi_2 \right\rangle + \left\langle \phi_1 | b\mathcal{W}_{l',m';l,m}(\mathbf{x})\phi_3 \right\rangle \\ &= aW^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) + bW^{\mathrm{E}(3)}(\phi_1, \phi_3|\mathbf{x}).\end{aligned}$$

Also, because the Wigner operator is self-adjoint in the Hilbert space [3], i.e., $\overline{\mathcal{W}}^{\mathrm{E}(3)}(\mathbf{x}) = \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x})$, we have

$$\begin{aligned}W^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) &= \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \overline{\phi_1}(\mathbf{p}) \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x}) \phi_2(\mathbf{p}) \sin\Theta \mathrm{d}\Phi \mathrm{d}\Theta \\ &= \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \overline{\phi_1}(\mathbf{p}) \;\; \overline{\mathcal{W}}^{\mathrm{E}(3)}(\mathbf{x}) \;\; \overline{\overline{\phi_2}}(\mathbf{p}) \sin\Theta \mathrm{d}\Phi \mathrm{d}\Theta \\ &= \overline{\int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \overline{\phi_2}(\mathbf{p}) \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x}) \phi_1(\mathbf{p}) \sin\Theta \mathrm{d}\Phi \mathrm{d}\Theta} \\ &= \overline{\left\langle \phi_2, \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x}) \phi_1 \right\rangle} = \overline{W}^{\mathrm{E}(3)}(\phi_2, \phi_1|\mathbf{x}).\end{aligned}$$

For $\phi_1 = \phi_2$, the reality condition for $W^{\mathrm{E}(3)}(\phi_1|\mathbf{x})$ is given by

$$W^{\mathrm{E}(3)}(\phi_1|\mathbf{x}) = \overline{W}^{\mathrm{E}(3)}(\phi_1, |\mathbf{x}).$$

B. **Covariance of** $\mathrm{E}(3)$ **Wigner Functions**
Let $\mathrm{Ad}(g)$ be the adjoint representation of $\mathrm{E}(3)$ and ϕ_1, ϕ_2 be functions in $L^2(S^2)$. The invariance of the Haar integral

$$\int_G f(g) \mathrm{d}g = \int_G f(gg_0) \mathrm{d}g$$

gives the Weyl operator the property

$$\mathcal{W}^G(\mathrm{Ad}(g_0)\mathbf{x}) = g_0^{-1} \mathcal{W}^G(\mathbf{x}) g_0.$$

If $U^s_{g_0}$ is the group representation in the Hilbert space $\mathcal{H}$, the unitary operator satisfies $U^s_{g_0^{-1}} = (U^s_{g_0})^\dagger$, so

$$\mathcal{W}^{\mathrm{E}(3)}(\mathrm{Ad}(g_0)\mathbf{x}) = U^\dagger(g_0, p) \mathcal{W}^{\mathrm{E}(3)}(\mathbf{x}) U(g_0, p).$$

It follows that

$$\begin{aligned}W^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathrm{Ad}(\mathrm{g}_0)\mathbf{x}) &= \left\langle \phi_1 | U^\dagger(g_0, p) W^{\mathrm{E}(3)}(\mathbf{x}) U(g_0, p) \phi_2 \right\rangle \\ &= W^{\mathrm{E}(3)}(U(g_0, p)\phi_1, U(g_0, p)\phi_2|\mathbf{x})\end{aligned}$$

thus, satisfying the covariant property.

C. **Moyal Identity for $\mathrm{E}(3)$ Wigner Functions**

To show the Moyal identity, we let $\mathbf{x} = (t_1, t_2, t_3, r_1, r_2, r_3) \in \mathbb{R}^6$. Consider the orthogonality relation of $\mathrm{E}(3)$ matrix elements given by

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathrm{SO}(3)} \overline{U}^{s_1}_{l_1m_1;j_1k_1}(\mathbf{a}, A; p_1) U^{s_2}_{l_2m_2;j_2k_2}(\mathbf{a}, A; p_2)\mathrm{d}\mathbf{a}\mathrm{d}A \\ &\quad = (2\pi)^2 \delta_{l_1l_2}\delta_{j_1j_2}\delta_{k_1k_2}\delta_{s_1s_2}\frac{\delta(p_1 - p_2)}{(p_2)^2}.\end{aligned} \tag{14}$$

Applying properties of delta functions on $\mathbb{R}^n$, namely,

$$\int_{\mathbb{R}^n} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}}\mathrm{d}^n\mathbf{r} = (2\pi)^n\delta((\mathbf{p} - \mathbf{p}'), \quad \int_{\mathbb{R}^n} f(\mathbf{r})\delta(\mathbf{x} - \mathbf{r})\mathrm{d}^n\mathbf{r} = f(\mathbf{x})$$

we have

$$\begin{aligned}&\int_{\mathbb{R}^6} \overline{W}^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) W^{\mathrm{E}(3)}(\psi_1, \psi_2|\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \langle\psi_1|\phi_1\rangle \langle\phi_2|\psi_2\rangle \int_{\mathrm{E}(3)}\int_{\mathrm{E}(3)}\int_{\mathbb{R}^6} \mathrm{e}^{\mathrm{i}[\mathbf{x}\cdot(\gamma_1-\gamma_2)]}\overline{U}^{s_1}_{l_1m_1;j_1k_1}(\mathbf{a}_1, A_1; p_1) \\ &\quad \times U^{s_2}_{l_2m_2;j_2k_2}(\mathbf{a}_2, A_2; p_2)\mathrm{d}^6\mathbf{x}\,\mathrm{d}a_1\,\mathrm{d}A_1\mathrm{d}a_2\mathrm{d}A_2.\end{aligned}$$

Evaluating the **x**- integral, we get

$$\begin{aligned}&\int_{\mathbb{R}^6} \overline{W}^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) W^{\mathrm{E}(3)}(\psi_1, \psi_2|\mathbf{x})\mathrm{d}\mathbf{x} \\ &=(2\pi)^6 \langle\psi_1|\phi_1\rangle \langle\phi_2|\psi_2\rangle \int_{\mathrm{E}(3)}\int_{\mathrm{E}(3)} \delta(\gamma_1 - \gamma_2)\overline{U}^{s_1}_{l_1m_1;j_1k_1}(\mathbf{a}_1, A_1; p_1) \\ &\quad \times U^{s_2}_{l_2m_2;j_2k_2}(\mathbf{a}_2, A_2; p_2)\mathrm{d}a_1\mathrm{d}A_1\mathrm{d}a_2\mathrm{d}A_2.\end{aligned}$$

Note that $g(\gamma_1) = g(\mathbf{a}_1, A_1)$ and $g(\gamma_2) = g(\mathbf{a}_2, A_2)$. Then evaluating γ_1-integral, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^6} \overline{W}^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) W^{\mathrm{E}(3)}(\psi_1, \psi_2|\mathbf{x})\mathrm{d}\mathbf{x} \\ &=(2\pi)^6 \langle\psi_1|\phi_1\rangle \langle\phi_2|\psi_2\rangle \int_{\mathrm{E}(3)} \overline{U}^{s_1}_{l_1m_1;j_1k_1}(\mathbf{a}_2, A_2; p_1) \\ &\quad \times U^{s_2}_{l_2m_2;j_2k_2}(\mathbf{a}_2, A_2; p_2)\mathrm{d}a_2\mathrm{d}A_2.\end{aligned}$$

Applying the orthogonality relation in equation (14), we obtain the Moyal identity for $\mathrm{E}(3)$ Wigner functions as

$$\begin{aligned}&\int_{\mathbb{R}^6} \overline{W}^{\mathrm{E}(3)}(\phi_1, \phi_2|\mathbf{x}) W^{\mathrm{E}(3)}(\psi_1, \psi_2|\mathbf{x})\mathrm{d}\mathbf{x} \\ &=(2\pi)^5 \delta_{l_1l_2}\delta_{j_1j_2}\delta_{k_1k_2}\delta_{s_1s_2}\frac{\delta(p_1 - p_2)}{(p_2)^2} \langle\psi_1|\phi_1\rangle \langle\phi_2|\psi_2\rangle\,.\end{aligned}$$

D. **Marginality Condition of** $\mathrm{E}(3)$ **Wigner Function**

The well-known projection or marginality conditions are satisfied by the original Wigner function on the Weyl - Heisenberg group. A similar relation when we project or integrate over the translation parameters t^* or over the rotation parameters r^* gives the marginality condition for $\mathrm{E}(3)$ Wigner functions.

To show this, set $\gamma = (\alpha^*, a^*)$ where $\alpha^* = (\alpha, \beta, \eta)$ and $a^* = (a_1, a_2, a_3)$. First, we project or integrate over the translation parameters t^*, which gives

$$\begin{aligned} M(\phi_1, \phi_2|r^*) &:= \int_{\mathbb{R}^3} W^{\mathrm{E}(3)}(\phi_1, \phi_2|r^*, t^*)\mathrm{d}t^* \\ &= \langle\phi_1|\phi_2\rangle \int_{\mathrm{E}(3)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}r^*\cdot\alpha^*}\mathrm{e}^{-\mathrm{i}t^*\cdot a^*} U^s_{l'm';l,m}(\mathbf{a}, A; p)\mathrm{d}t^*\mathrm{d}\mathbf{a}\mathrm{d}A \qquad (15) \\ &= \langle\phi_1|\phi_2\rangle \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}r^*\cdot\alpha^*}\mathrm{e}^{-\mathrm{i}t^*\cdot a^*} U^s(\mathbf{a}, I)U^s(0, A)\mathrm{d}t^*\mathrm{d}\mathbf{a}\mathrm{d}A. \end{aligned}$$

Note that using the Dirac delta functions, we have the following equations

$$\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}t^*\cdot a^*}\mathrm{d}t^* = (2\pi)^3\delta(\mathbf{a}),\ \ (2\pi)^3 \int_{\mathbb{R}^3} \delta(\mathbf{a})U^s(\mathbf{a}, I)\mathrm{d}\mathbf{a} = (2\pi)^3. \qquad (16)$$

Here, we set $\mathbf{a} = 0$, which implies that $U^s(0, I) = I$ (the identity matrix). If we let

$$\mathcal{R}(r^*) = \int_{\mathrm{SO}(3)} \mathrm{e}^{-\mathrm{i}r^*\cdot\alpha^*}U^s(0, A)\mathrm{d}A$$

then applying equations (16) in equation (15) we obtain

$$M(\phi_1, \phi_2|r^*) = (2\pi)^3 \langle\phi_1|\phi_2\rangle \mathcal{R}(r^*).$$

Similarly, if we integrate over the rotation parameters r^* we get

$$M(\phi_1, \phi_2|t^*) = (2\pi)^3 \langle\phi_1|\phi_2\rangle \mathcal{R}(t^*)$$

where $\mathcal{R}(t^*) = \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}t^*\cdot a^*}U^s(\mathbf{a})\mathrm{d}\mathbf{a}$.

E. **Traciality of** $\mathrm{E}(3)$ **Wigner Functions**

Traciality condition is satisfied when we both integrate over the translation and rotation parameters t^* and r^* as follows

$$\begin{aligned} T(\phi_1, \phi_2) &:= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W^{\mathrm{E}(3)}(\phi_1, \phi_2|r^*, t^*)\mathrm{d}r^*\mathrm{d}t^* \\ &= \langle\phi_1|\phi_2\rangle \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}r^*\cdot\alpha^*}\mathrm{e}^{-\mathrm{i}t^*\cdot a^*}U^s(\mathbf{a}, I) \\ &\quad \times U^s(0, A)\mathrm{d}\mathbf{a}\,\mathrm{d}A\,\mathrm{d}r^*\mathrm{d}t^*. \end{aligned}$$

Evaluating the r^* and t^* integrals using Dirac δ functions, we have

$$T(\phi_1, \phi_2) = \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \delta(\mathbf{a})\delta(A)U^s(\mathbf{a}, I)U^s(0, A)\mathrm{d}\mathbf{a}\mathrm{d}A.$$

Then applying the **a** and A - integrals, we obtain

$$T(\phi_1, \phi_2) = (2\pi)^6 \langle \phi_1 | \phi_2 \rangle .$$

6. Summary and Conclusion

In this paper, we have constructed the Wigner distributions for a particular Lie group, the Euclidean motion group of rank three. Because of its numerous applications in physical problems [23], it is thus very natural that the Wigner functions be computed for various Lie groups which appear as symmetry groups of quantum systems, and whose coadjoint orbits are the phase spaces of the quantum system. Since we constructed $\mathrm{E}(3)$ Wigner functions, this provides us the first step in formulating the $\star$-product of functions on $\mathrm{E}(3)$ and on its orbits. This will be considered elsewhere. The constructions considered here should also lead to the general form of the Wigner functions and Weyl operators on the rank n Euclidean motion group $\mathrm{E}(n)$.

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