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NEW PROPERTIES OF EUCLIDEAN KILLING TENSORS OF RANK TWO

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Abstract. Due to the importance of Killing tensors of rank two in providing quadratic first integrals we point out several algebraic and geometrical features of this class of Killing tensor fields for the two-dimensional Euclidean metric.

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A symmetric tensor field on a Riemannian manifold is called *a Killing tensor field* if the symmetric part of its covariant derivative is equal to zero. There exists a well-known bijection between Killing tensor fields and conserved quantities of the geodesic flow which depend polynomially on the momentum variables. In particular, Killing tensors of rank (or valence) two yields quadratic first integrals and we discuss some aspects of this process in Crasmareanu [7] from a dynamical point of view. Some classes of physical examples associated with the Euclidean 2D metric are provided in Crasmareanu and Baleanu [8].

The present paper returns to the Euclidean plane geometry $\mathbb{E}^2$ and its purpose is to derive other algebraic and geometrical properties of the generators of real vector space $\mathcal{K}^2(\mathbb{E}^2)$ of Killing tensors of rank two.

In Boccaletti and Pucacco [2, p. 195] is given the general expression of an element $A^{(2)} \in \mathcal{K}^2(\mathbb{E}^2)$

$$A^{(2)}(x,y) = aM + bL_1 + cL_2 + eE_1 + dE_2 + gE_3$$

with a, b, c, d, e, g arbitrary real numbers and

$$\begin{aligned} M(x,y) =& \frac{1}{2}\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}, \quad L_1(x,y) = \frac{1}{2}\begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ L_2(x,y) =& \frac{1}{2}\begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{1}$$

So, the dimension of $\mathcal{K}^2(\mathbb{E}^2)$ is six and a general formula for this dimension appears in Chanu, Degiovanni and McLenaghan [5].

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Property 1. Fix $c \in \mathbb{R}$ and for $\alpha = 1, 2, 3$ let Γ^c_α be the conic associated to the symmetric matrix E_α

$$\Gamma^c_\alpha : (x, y) \cdot E_\alpha \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c.$$

Hence $\Gamma^c_1 : x^2 = c$, $\Gamma^c_2 : y^2 = c$ and $\Gamma^c_3 : xy = c$. For $c = 0$ we have: $\Gamma^0_1 = Oy$, $\Gamma^0_2 = Ox$ and Γ^0_3 is the union of the axes Ox and Oy. For $c < 0$ we have only the hyperbola Γ^c_3 while for $c > 0$ the first conics are pairs of parallel lines and Γ^c_3 is again an equilateral hyperbola. □

Property 2. As usually, fix the complex number $z = x + \mathrm{i}y$ with its associated conjugate $\bar{z} = x - \mathrm{i}y$; an useful generalization for both classical and quantum mechanics is provided by hypercomplex numbers from [10]. Since $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{\mathrm{i}}{2}(\bar{z} - z)$ we have the complex variant of matrices from relations (1)-(2)

$$8M(z, \bar{z}) = \begin{pmatrix} -(z-\bar{z})^2 & \mathrm{i}(z^2 - \bar{z}^2) \\ \mathrm{i}(z^2 - \bar{z}^2) & (z+\bar{z})^2 \end{pmatrix}, \quad \mathrm{tr}M(z, \bar{z}) = \frac{|z|^2}{2}$$
$$4L_1(z, \bar{z}) = \begin{pmatrix} 0 & \mathrm{i}(z-\bar{z}) \\ \mathrm{i}(z-\bar{z}) & 2(z+\bar{z}) \end{pmatrix}, \quad 4L_2(z, \bar{z}) = \begin{pmatrix} 2\mathrm{i}(\bar{z}-z) & -(z+\bar{z}) \\ -(z+\bar{z}) & 0 \end{pmatrix}.$$

Denoting a generic $A^{(2)} \in \mathcal{K}^2(\mathbb{E}^2)$ as $(A_{ab}(x, y))_{a,b=1,2}$ the associated quadratic first integrals are

$$\mathcal{F}_{A^{(2)}}(x, y, \dot{x}, \dot{y}) = A_{ab}\dot{x}^a\dot{x}^b.$$

In complex coordinates it results

$$\begin{aligned}
32\mathcal{F}_M(z, \bar{z}, \dot{z}, \dot{\bar{z}}) &= 2(\bar{z}^2 - z^2)(\dot{\bar{z}}^2 - \dot{z}^2) - (\bar{z} - z)^2(\dot{z} + \dot{\bar{z}})^2 - (z + \bar{z})^2(\dot{\bar{z}} - \dot{z})^2 \\
&= 32\left(\frac{y^2}{2}\dot{x}^2 - xy\dot{x}\dot{y} + \frac{x^2}{2}\dot{y}^2\right) \\
8\mathcal{F}_{L_1}(z, \bar{z}, \dot{z}, \dot{\bar{z}}) &= (z - \bar{z})(\dot{z}^2 - \dot{\bar{z}}^2) - (z + \bar{z})(\dot{\bar{z}} - \dot{z})^2 = 8(-y\dot{x}\dot{y} + x\dot{y}^2) \\
8\mathrm{i}\mathcal{F}_{L_2}(z, \bar{z}, \dot{z}, \dot{\bar{z}}) &= (z + \bar{z})(\dot{\bar{z}}^2 - \dot{z}^2) + (z - \bar{z})(\dot{z} + \dot{\bar{z}})^2 = 8\mathrm{i}(y\dot{x}^2 - x\dot{x}\dot{y}) \\
4\mathcal{F}_{E_1}(\dot{z}, \dot{\bar{z}}) &= (\dot{z} + \dot{\bar{z}})^2 = 4\dot{x}^2, \quad 4\mathcal{F}_{E_2}(\dot{z}, \dot{\bar{z}}) = (\dot{z} - \dot{\bar{z}})^2 = 4\dot{y}^2 \\
4\mathrm{i}\mathcal{F}_{E_3}(\dot{z}, \dot{\bar{z}}) &= \dot{z}^2 - \dot{\bar{z}}^2 = 4\mathrm{i}\dot{x}\dot{y}.
\end{aligned}$$

We point out that the Chapter 5 of the book Calin, Chang and Greiner [4] deals with complex Hamiltonian mechanics. □

Property 3. Following the idea of Property 1 we remark that E_3 generates a hyperbolic metric g_h through

$$g_h = (\mathrm{d}x, \mathrm{d}y) \cdot E_3 \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \mathrm{d}x\mathrm{d}y.$$

With the transformation of coordinates: $x = u + v$, $y = u - v$ we arrive at the Lorentz-Minkowski metric $g_h = \mathrm{d}u^2 - \mathrm{d}v^2$. In a similar manner

$$(\mathrm{d}x, \mathrm{d}y) \cdot 2M \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = (x\mathrm{d}y - y\mathrm{d}x)^2$$

and the right-hand-side expression recalls the classical differential one-form

$$\omega = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}$$

which is closed but not exact on the punctured plane $\mathbb{R}^2 \setminus \{O(0,0)\}$. But the restriction of ω to, say, the right half-plane $x > 0$ is an exact form. $\square$

Property 4. Let us turn now to polar coordinates (r, φ) on the punctured plane. With $x = r\cos\varphi$ and $y = r\sin\varphi$ we have

$$\frac{2}{r^2}M(r,\varphi) = \begin{pmatrix} \sin^2\varphi & -\sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi & \cos^2\varphi \end{pmatrix}, \qquad \mathrm{tr}M(r,\varphi) = \frac{r^2}{2}\cdot$$

Therefore

$$\frac{4}{r^2}M(r,\varphi) = I_2 - S(2\varphi), \qquad S(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \in \mathrm{O}^-(2).$$

Here I_2 is the unit 2×2-matrix and $\mathrm{O}^-(2)$ is the subset of orthogonal matrices of order two having the determinant (-1), Crasmareanu and Plugariu [9, p. 37]. Also

$$\frac{2}{r}L_1(r,\varphi) = \begin{pmatrix} 0 & -\sin\varphi \\ -\sin\varphi & 2\cos\varphi \end{pmatrix}, \quad \frac{2}{r}L_2(r,\varphi) = \begin{pmatrix} 2\sin\varphi & -\cos\varphi \\ -\cos\varphi & 0 \end{pmatrix}.$$

The quadratic first integrals are

$$\begin{aligned}
\frac{2}{r^2}\mathcal{F}_M(r,\varphi,\dot r,\dot\varphi) &= \sin^2\varphi(\dot r\cos\varphi - r\sin\varphi\dot\varphi)^2 \\
&\quad - \sin(2\varphi)(\dot r\cos\varphi - r\sin\varphi\dot\varphi)(\dot r\sin\varphi + r\cos\varphi\dot\varphi) \\
&\quad + \cos^2\varphi(\dot r\sin\varphi + r\cos\varphi\dot\varphi)^2 \\
\frac{1}{r}\mathcal{F}_{L_1}(r,\varphi,\dot r,\dot\varphi) &= \cos\varphi(\dot r\sin\varphi + r\cos\varphi\dot\varphi)^2 \\
&\quad - \sin\varphi(\dot r\cos\varphi - r\sin\varphi\dot\varphi)(\dot r\sin\varphi + r\cos\varphi\dot\varphi) \\
\frac{1}{r}\mathcal{F}_{L_2}(r,\varphi,\dot r,\dot\varphi) &= \sin\varphi(\dot r\cos\varphi - r\sin\varphi\dot\varphi)^2 \\
&\quad - \cos\varphi(\dot r\cos\varphi - r\sin\varphi\dot\varphi)(\dot r\sin\varphi + r\cos\varphi\dot\varphi) \\
\mathcal{F}_{E_1}(r,\varphi,\dot r,\dot\varphi) &= (\dot r\cos\varphi - r\sin\varphi\dot\varphi)^2, \quad \mathcal{F}_{E_2}(r,\varphi,\dot r,\dot\varphi) = (\dot r\sin\varphi + r\cos\varphi\dot\varphi)^2 \\
\mathcal{F}_{E_3}(r,\varphi,\dot r,\dot\varphi) &= (\dot r\cos\varphi - r\sin\varphi\dot\varphi)(\dot r\sin\varphi + r\cos\varphi\dot\varphi).
\end{aligned}$$

□

Property 5. With polar coordinates is easy to compute the exponential of M on the punctured plane. Indeed

$$M^2 = (x^2+y^2)M = r^2 M, \qquad M^3 = r^4 M, \qquad M^4 = r^6 M$$

and hence

$$\begin{aligned}\exp(M)(x,y) &= \frac{1}{x^2+y^2}\begin{pmatrix} x^2+y^2 e^{\frac{x^2+y^2}{2}} & xy(1-e^{\frac{x^2+y^2}{2}}) \\ xy(1-e^{\frac{x^2+y^2}{2}}) & y^2+x^2 e^{\frac{x^2+y^2}{2}}\end{pmatrix} \\ &= \begin{pmatrix} \cos^2\varphi + e^{\frac{r^2}{2}}\sin^2\varphi & (1-e^{\frac{r^2}{2}})\sin\varphi\cos\varphi \\ (1-e^{\frac{r^2}{2}})\sin\varphi\cos\varphi & \sin^2\varphi + e^{\frac{r^2}{2}}\cos^2\varphi\end{pmatrix}.\end{aligned}$$

The matrix E_3 has the eigenvalues $\pm\frac{1}{2}$ and hence is diagonalizable and

$$\exp(E_3) = \begin{pmatrix} \cosh\frac{1}{2} & \sinh\frac{1}{2} \\ \sinh\frac{1}{2} & \cosh\frac{1}{2}\end{pmatrix}.$$

Also: $E_1E_2 = E_2E_1 = O_2 =$ the null matrix and $E_1+E_2 = I_2$, $E_1^2 = E_1$, $E_2^2 = E_2$ which means that E_1, E_2 are complementary and commuting projectors. Indeed, E_1 represents the projection of the plane $\mathbb{E}^2$ on the real axis $Ox = \{(x,0)\in\mathbb{R}^2\}$ while E_2 represents the projection on the imaginary axis $Oy = \{(0,y)\in\mathbb{R}^2\}$. The matrix E_3 represents the linear operator: $(x,y)\to\frac{1}{2}(y,x)$ i.e. the half of the axial symmetry with respect to the first bisectrix $B_1 : y = x$. □

Property 6. For the given six Killing matrices (1)-(2) there are $C_6^2 = 15$ Lie brackets

$$[E_1,E_2] = O_2, \qquad [E_2,E_3] = [E_3,E_1] = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} =: \frac{1}{2}J$$

and the matrix J represents the clockwise rotation in plane. Also

$$\begin{aligned}
[M,E_1] &= -\frac{xy}{2}J = -[M,E_2], \qquad [L_1,E_1] = -\frac{y}{2}J = -[L_1,E_2] = [L_2,E_3] \\
[L_2,E_1] &= -\frac{x}{2}J = -[L_2,E_2] = -[L_1,E_3] \\
[M,E_3] &= \frac{x^2-y^2}{2}J = \frac{z^2+\bar{z}^2}{4}J = \frac{r^2\cos(2\varphi)}{2}J \\
[L_1,L_2] &= -\frac{x^2+y^2}{2}J = -\frac{|z|^2}{2}J = -\frac{r^2}{2}J \\
[M,L_1] &= \frac{y(x^2+y^2)}{4}J, \quad [M,L_2] = -\frac{x(x^2+y^2)}{4}J.
\end{aligned}$$

The Lie brackets with J are as follows

$$[E_1, J] = -2E_3, \quad [E_2, J] = 2E_3, \quad [E_3, J] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E_1 - E_2$$

and therefore the data $\{E_1, E_2, E_3, J\}$ is a Lie algebra. We point out that the well-known Pauli matrices (following the site [1]) can be expressed within this Lie algebra

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2E_3, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}J, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E_1 - E_2.$$

Also

$$\begin{aligned} [M, J] &= \frac{1}{2}\begin{pmatrix} -2xy & x^2 - y^2 \\ x^2 - y^2 & 2xy \end{pmatrix} = \frac{r^2}{2}\begin{pmatrix} -\sin(2\varphi) & \cos(2\varphi) \\ \cos(2\varphi) & \sin(2\varphi) \end{pmatrix} \\ &= xy(E_2 - E_1) + (x^2 - y^2)E_3 \\ [L_1, J] &= \begin{pmatrix} -y & x \\ x & y \end{pmatrix} = y(E_2 - E_1) + 2xE_3 \\ [L_2, J] &= -\begin{pmatrix} x & y \\ y & x \end{pmatrix} = -xI_2 - 2yE_3. \end{aligned}$$

□

Property 7. In addition to the Lie product on the real algebra $\mathrm{Mat}(2, \mathbb{R})$ of real 2×2 matrices there exists the Jordan product, Crasmareanu [6, p. 28]

$$[A, B]_{1,1} := AB + BA.$$

The 16+5 Jordan brackets of our matrices are

$$\begin{aligned} [E_1, E_2]_{1,1} &= O_2, \quad [E_1, E_3]_{1,1} = [E_2, E_3]_{1,1} = E_3, \quad [M, E_1]_{1,1} = y^2E_1 - xyE_3 \\ [M, E_2]_{1,1} &= x^2E_2 - xyE_3, \quad [M, E_3]_{1,1} = -\frac{xy}{2}I_2 + \frac{x^2 + y^2}{2}E_3 \\ [L_1, E_1]_{1,1} &= -yE_3, \quad [E_1, J]_{1,1} = [E_2, J]_{1,1} = J \\ [L_1, E_2]_{1,1} &= -yE_3 + 2xE_2, \quad [L_1, E_3]_{1,1} = -\frac{y}{2}I_2 + xE_3 \\ [L_2, E_1]_{1,1} &= 2yE_1 - xE_3, \quad [E_3, J]_{1,1} = O_2 \\ [L_2, E_2]_{1,1} &= -xE_3, \quad [L_2, E_3]_{1,1} = -\frac{x}{2}I_2 + yE_3 \\ [M, L_1]_{1,1} &= x^3E_2 + \frac{xy^2}{2}I_2 - \frac{y^3 + 3x^2y}{2}E_3 \\ [M, L_2]_{1,1} &= y^3E_1 + \frac{x^2y}{2}I_2 - \frac{x^3 + 3xy^2}{2}E_3, \quad [L_1, L_2]_{1,1} = \frac{xy}{2}I_2 - (x^2 + y^2)E_3 \\ [M, J]_{1,1} &= \frac{x^2 + y^2}{2}J, \qquad [L_1, J]_{1,1} = 2xE_3, \qquad [L_2, J]_{1,1} = yJ. \end{aligned}$$

□

Property 8. In $\mathbb{E}^2$ in addition to Cartesian and polar coordinates there are other two *separable coordinates systems*, Boskoff, Crasmareanu and Pişcoran [3, pp.143-144]. These are as follows

i) parabolic coordinates: $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$. Then

$$8M(u,v) = \begin{pmatrix} 4u^2v^2 & 2uv(v^2-u^2) \\ 2uv(v^2-u^2) & (u^2-v^2)^2 \end{pmatrix}, \quad \mathrm{tr}M(u,v) = \frac{(u^2+v^2)^2}{4}.$$

ii) elliptic coordinates: $x^2 = c^2(u-1)(v-1)$, $y^2 = -c^2uv$. Hence

$$\frac{2}{c^2}M(u,v) = \begin{pmatrix} -uv & \sqrt{uv(1-u)(v-1)} \\ \sqrt{uv(1-u)(v-1)} & (u-1)(v-1) \end{pmatrix}$$

$$\mathrm{tr}M(u,v) = \frac{c^2}{2}(1-u-v) \geq 0.$$

□

Conclusions

As main conclusion of this work one should point out the richness of the Euclidean two-dimensional setting seeing from a Killing tensor fields point of view. A lot of new properties, of both algebraic and geometrical nature, supports this landscape. It remains as future project to study similar frameworks, such as spheres or tori or other natural manifolds.

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