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EXAMPLES OF AUTOMORPHISM GROUPS OF IND-VARIETIES OF GENERALIZED FLAGS*

IVAN PENKOV

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Abstract. We compute the automorphism groups of finite and cofinite ind-grassmannians, as well as of the ind-variety of the maximal flags indexed by $\mathbb{Z}_{>0}$. We pay special attention to differences with the case of ordinary flag varieties.

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1. Introduction

The flag varieties of the classical Lie groups are central objects of study both in geometry and representation theory. In a sense, they are a hub for many directions of research in both fields. Several different infinite-dimensional analogues of the ordinary flag varieties have been studied in the literature, one such analogue being the ind-varieties of generalized flags introduced in [1] and further investigated in [2–5]; see also the survey [6]. The latter ind-varieties are direct limits of classical flag varieties and are homogeneous ind-spaces for the simple ind-groups $\mathrm{SL}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{Sp}(\infty)$. Without doubt, some of these ind-varieties, in particular the ind-grassmannians, have been known long before the paper [1].

A natural question of obvious importance is the question of finding the automorphism groups of the ind-varieties of generalized flags. The purpose of the present paper is to initiate a discussion in this direction and to point out some differences with the case of ordinary flag varieties - see Section 4.

2. Automorphisms of Finite and Cofinite Ind-Grassmannians

The base field is $\mathbb{C}$. Let V be a fixed countable-dimensional complex vector space. We fix a basis $E = \{e_1, \dots, e_n, \dots\}$ of V and set $V_n := \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_n\}$. Then $V = \cup_n V_n$. Fix $k \in \mathbb{Z}_{>0}$. By definition, $\mathrm{Gr}(k, V)$ is the set of all k-dimensional

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subspaces in V and has an obvious ind-variety structure

$$\operatorname{Gr}(k,V)=\varinjlim \operatorname{Gr}(k,V_n).$$

The projective ind-space $\mathbb{P}(V)$ equals $\operatorname{Gr}(1,V)$. Note that the basis E plays no role in this construction. We think of the ind-varieties $\operatorname{Gr}(k,V)$ for $k\in\mathbb{Z}_{>0}$ as the "finite ind-grassmannians."

The basis E plays a role when defining the "cofinite" ind-grassmannians. Fix a subspace $W\subset V$ of finite codimension in V and such that $E\cap W$ is a basis of W. Let $\operatorname{Gr}(W,E,V)$ be the set of all subspaces $W'\subset V$ which have the same codimension in V as W and in addition contain almost all elements of E. Then $\operatorname{Gr}(W,E,V)$ has the following ind-variety structure

$$\operatorname{Gr}(W,E,V)=\varinjlim \operatorname{Gr}(\operatorname{codim}_V W,\bar{V}_n)$$

where $\{\bar{V}_n\}$ is any set of finite-dimensional spaces with the properties that $\bar{V}_n\supset \operatorname{span}\{E\backslash\{E\cap W\}\}$, $\dim\bar{V}_n=n>\operatorname{codim}_V W$, $E\cap\bar{V}_n$ is a basis of $\bar{V}_n$, and $\cup\bar{V}_n=V$. The map identifying the direct limit of $\operatorname{Gr}(\operatorname{codim}_V W,\bar{V}_n)$ with $\operatorname{Gr}(W,E,V)$ is

$$W''\mapsto W''\oplus\operatorname{span}\{E\backslash(E\cap\bar{V}_n)\}$$

for $W''\in\operatorname{Gr}(\operatorname{codim}_V W,\bar{V}_n)$.

It is clear that the ind-varieties $\operatorname{Gr}(W,E,V)$ and $\operatorname{Gr}(k,V)$ are isomorphic: the isomorphism is given by

$$\operatorname{Gr}(W,E,V)\ni W'\to \operatorname{Ann}W'\subset V_*:=\operatorname{span}\{E^*\} \tag{1}$$

where $E^*=\{e_1^*,e_2^*,\dots\}$ is the system of linear functionals dual to the basis E, i.e., $e_i^*(e_j)=\delta_{ij}$. The map (1) is an obvious analogue of finite-dimensional duality. Therefore the automorphism groups $\operatorname{Aut}\operatorname{Gr}(k,V)$ and $\operatorname{Aut}\operatorname{Gr}(W,E,V)$ for $\operatorname{codim}_W V=k$ are isomorphic, and by an automorphism we mean of course an automorphism of ind-varieties.

The following result should in principle be known. We present a proof which shows a connection with the work [2].

Proposition 1. $\operatorname{Aut}\operatorname{Gr}(k,V)=\operatorname{PGL}(V)$ *where* $\operatorname{GL}(V)$ *denotes the group of all invertible linear operators on* V *and* $\operatorname{PGL}(V):=\operatorname{GL}(V)/\mathbb{C}_{\mathrm{mult}}\mathrm{Id}$ (*where* $\mathbb{C}_{\mathrm{mult}}$ *is the multiplicative group of* $\mathbb{C}$).

Proof: An automorphism $\phi:\operatorname{Gr}(k,V)\to\operatorname{Gr}(k,V)$ induces embeddings $\phi_n:\operatorname{Gr}(k,V_n)\hookrightarrow\operatorname{Gr}(k,V_{N(n)})$ for appropriate $N(n)\geq n$. These embeddings are linear in the sense that $\phi_n^*(\mathcal{O}_{\mathrm{Gr}(k,V_{N(n)})}(1))$ is isomorphic to $\mathcal{O}_{\mathrm{Gr}(k,V_n)}(1)$, where by $\mathcal{O}.(1)$ we denote the positive generator of the respective Picard group. According to Theorem 1 in [2], ϕ_n is one of the following

i) an embedding induced by the choice of an n-dimensional subspace $W_n \subset V_{N(n)}$ for some $N(n) \geq n$

ii) an embedding factoring through a linearly embedded projective space $\mathbb{P}^{M(n)} \subset \mathrm{Gr}(k, V_{N(n)})$ for some $M(n) < N(n)$.

If $k > 2$, the option ii) may hold only for finitely many n as the contrary implies that the image of ϕ_n is contained in a projective ind-subspace

$$\mathbb{P} := \varinjlim \mathbb{P}^{M(n)} \subset \mathrm{Gr}(k, V).$$

Then, since $\mathbb{P}$ is not isomorphic to $\mathrm{Gr}(k, V)$ by Theorem 2 in [2], the image of ϕ_n would necessarily be a proper ind-subvariety of $\mathrm{Gr}(k, V)$, which is a contradiction.

For $k = 1$, options i) and ii) are the same, and therefore without loss of generality we can now assume that for our fixed k option i) holds for all n. The embeddings $\phi_n : \mathrm{Gr}(k, V_n) \hookrightarrow \mathrm{Gr}(k, V_{N(n)})$ determine injective linear operators $\tilde{\phi}_n : V_n \to V_{N(n)}$. Moreover, the operators $\tilde{\phi}_n$ are defined up to multiplicative constants which can be chosen so that $\tilde{\phi}_n|_{V_{n-1}} = \tilde{\phi}_{n-1}$ for any n. Therefore, we obtain a well-defined linear operator

$$\tilde{\phi} : V = \varinjlim V_n \to V = \varinjlim V_{N(n)}$$

which induces our automorphism ϕ. Since ϕ is invertible, $\tilde{\phi}$ is also invertible, and since $\tilde{\phi}$ depends on a multiplicative constant, we conclude that ϕ determines a unique element $\bar{\phi} \in \mathrm{PGL}(V)$.

In this way we have constructed an injective homomorphism

$$\mathrm{Aut}\,\mathrm{Gr}(k, V) \to \mathrm{PGL}(V), \qquad \phi \mapsto \bar{\phi}.$$

The inverse homomorphism

$$\mathrm{PGL}(V) \to \mathrm{Aut}\,\mathrm{Gr}(k, V)$$

is obvious because of the natural action of $\mathrm{PGL}(V)$ on $\mathrm{Gr}(k, V)$. The statement follows. ■

3. Ind-Variety of Maximal Ascending Flags

We now consider a particular ind-variety of maximal generalized flags, in fact the simplest case of maximal generalized flags. Let V and E be as above. Define $\mathrm{Fl}(F_E, E, V)$ as the set of all infinite chains F'_E of subspaces of V

$$0 \subset (F'_E)^1 \subset \cdots \subset (F'_E)^k \subset \dots$$

where $\dim(F'_E)^k = k$ and $(F'_E)^n = F^n_E := \mathrm{span}\{e_1, \dots, e_n\}$ for large enough n. This set has an obvious structure of ind-variety as

$$\mathrm{Fl}(F_E, E, V) = \varinjlim \mathrm{Fl}(F^n_E)$$

where $\mathrm{Fl}(F^n_E)$ stands for the variety of maximal flags in the finite-dimensional vector space F^n_E.

Denote by $\mathrm{GL}(E, V)$ the subgroup of $\mathrm{GL}(V)$ of automorphisms of V which keep all but finitely many elements of E fixed. The elements of $\mathrm{GL}(E, V)$ are the E-*finitary* automorphisms of V.

Proposition 2.

$$\mathrm{Aut}\,\mathrm{Fl}(F_E, E, V) = P(\mathrm{GL}(E, V) \cdot B_E)$$

where $B_E \subset \mathrm{GL}(V)$ *is the stabilizer of the chain* F_E *in the group* $\mathrm{GL}(V)$ *and* $\mathrm{GL}(E, V) \cdot B_E$ *is the subgroup of* $\mathrm{GL}(V)$ *generated by* $\mathrm{GL}(E, V)$ *and* B_E.

We start with a lemma.

Lemma 3. *Fix* $k \geq 2$. *Let* $\psi_{k-1}, \psi_k : V \to V$ *be invertible linear operators such that* $\psi_{k-1}(W_{k-1}) \subset \psi_k(W_k)$ *for any pair of subspaces* $W_{k-1} \subset W_k$ *of* V *with* $\dim W_{k-1} = k - 1$, $\dim W_k = k$. *Then* $\psi_{k-1} = c\psi_k$ *for some* $0 \neq c \in \mathbb{C}$.

Proof: Assume the contrary. Let v be a vector in V such that the space $Z := \mathrm{span}_{\mathbb{C}}\{\psi_{k-1}(v), \psi_k(v)\}$ has dimension two. Extend v to a basis $v = v_1, v_2, \dots$ of V. Then, setting $W_k = \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_k\}$ and $W_{k-1} = \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_{k-1}\}$, we see that the condition $\psi_{k-1}(W_{k-1}) \subset \psi_k(W_k)$ implies $Z \subset \psi_k(W_k)$. Similarly, setting $W'_k = \mathrm{span}_{\mathbb{C}}\{v_1, v_{k+1}, v_{k+2} \dots, v_{2k-1}\}$ and $W'_{k-1} = \mathrm{span}_{\mathbb{C}}\{v_1, v_{k+1}, v_{k+2} \dots, v_{2k-2}\}$ we have $Z \subset \psi_k(W'_k)$. However clearly

$$\dim(W_k \cap W'_k) = 1$$

hence the dimension of the intersection $\psi_k(W_k) \cap \psi_k(W'_k)$ must also be one due to the invertibility of ψ_k. Contradiction. ■

Proof of Proposition 2: We first embed $A := \operatorname{Aut}\operatorname{Fl}(F_E, E, V)$ into the group $\operatorname{PGL}(V)$. For this we consider the obvious embedding

$$A \hookrightarrow \Pi_{i=1}^{\infty} \operatorname{Aut}\operatorname{Gr}(i, V)$$

arising from the diagram of surjective morphisms of ind-varieties

$$\begin{array}{ccccc} & & \operatorname{Fl}(F_E, E, V) & & \\ & \swarrow & \downarrow & \searrow & \\ \mathbb{P}(V) = \operatorname{Gr}(1, V) & & \operatorname{Gr}(2, V) \quad \dots & & \operatorname{Gr}(k, V) \dots \end{array} .$$

By Proposition 1, the groups $\operatorname{Aut}\operatorname{Gr}(k, V)$ are isomorphic to $\operatorname{PGL}(V)$ for all $k \in \mathbb{Z}_{>0}$. Moreover, it is clear that the homomorphism $A \to \Pi_k \operatorname{PGL}(V)$ is injective as the ind-varieties $\operatorname{Gr}(k, V)$ are pairwise nonisomorphic for $k \geq 1$ [2] (this argument is false in the finite-dimensional case). It is also clear that this homomorphism factors through the diagonal of $\Pi_k \operatorname{PGL}(V)$ since Lemma 1 shows that an automorphism from A induces necessarily the same element in $\operatorname{PGL}(V)$ via any projection $\operatorname{Fl}(F_E, E, V) \to \operatorname{Gr}(k, V)$.

It remains to determine which elements of the group $\operatorname{PGL}(V)$ arise as images of elements of A. It is clear that this image contains both $\operatorname{PGL}(E, V)$ and PB_E as each of these groups acts faithfully on $\operatorname{Fl}(F_E, E, V)$. Indeed, the fact that $\operatorname{PGL}(E, V)$ acts on $\operatorname{Fl}(F_E, E, V)$ is clear. To see that PB_E acts on $\operatorname{Fl}(F_E, E, V)$ one notices that for any $F'_E \in \operatorname{Fl}(F_E, E, V)$ and any $\gamma \in PB_E$, the flag $\gamma(F'_E)$ differs from F_E only in finitely many positions, hence is a point on $\operatorname{Fl}(F_E, E, V)$.

On the other hand, it is clear that the image $\bar{\phi} \in \operatorname{PGL}(V)$ of $\phi \in A$ is contained in $P(\operatorname{GL}(E, V) \cdot B_E)$. Indeed the composition $\psi \circ \bar{\phi}$ with a suitable element of $\operatorname{PGL}(E, V)$ will fix the point F_E on $\operatorname{Fl}(F_E, E, V)$. This means that $\psi \circ \bar{\phi} \in PB_E$. Therefore the image of A in $\operatorname{PGL}(V)$ is contained in $P(\operatorname{GL}(E, V) \cdot B_E)$, and we are done.

■

4. Discussion

First, Proposition 1 can be generalized to ind-varieties of the form $\operatorname{Fl}(F, E, V)$ where F is a finite chain consisting only of finite-dimensional subspaces of V, or only of subspaces of finite codimension of V. The precise definition of the ind-varieties $\operatorname{Fl}(F, E, V)$ is given in [1]. In these cases, the respective automorphism groups are always isomorphic to $\mathrm{PGL}(V)$, however in the case of finite codimension there is a natural isomorphism with $\mathrm{PGL}(V_*)$.

We now point out some differences with the case of ordinary flag varieties. A first obvious difference is the following. Despite the general fact that $\mathrm{Gr}(k,V) = \mathrm{PGL}(E,V)/P_k$, where P_k is the stabilizer in $\mathrm{PGL}(E,V)$ of a k-dimensional subspace of V, the automorphism group of $\mathrm{Gr}(k,V)$ is much larger than $\mathrm{PGL}(E,V)$. Therefore $\mathrm{Gr}(k,V)$ is a quotient of any subgroup G satisfying $\mathrm{PGL}(E,V) \subset G \subset \mathrm{PGL}(V)$, and there is quite a variety of such subgroups. Similar comments apply to the other examples we consider.

Next, we note that the automorphism group of an ind-grassmannian is in general not naturally embedded into $\mathrm{PGL}(V)$. Indeed, the case of the cofinite ind-grassmannian $\mathrm{Gr}(W,E,V)$ shows that the natural isomorphism $\mathrm{Aut}\,\mathrm{Gr}(W,E,V) = \mathrm{PGL}(V_*)$ does not embed $\mathrm{Aut}\,\mathrm{Gr}(W,E,V)$ into $\mathrm{PGL}(V)$ by duality, but only embeds $\mathrm{Aut}\,\mathrm{Gr}(W,E,V)$ into the much larger group $\mathrm{PGL}((V_*)^*)$ in a way that its image does not keep the subspace $V \subset (V_*)^*$ invariant. This is clearly an infinite-dimensional phenomenon.

Finally, recall that the group of automorphisms of a finite-dimensional grassmannian is naturally a subgroup of the automorphism group of the corresponding full flag variety. More precisely, the former group is the connected component of unity of the latter group. This note shows that the situation in the infinite-dimensional case essentially different: indeed, the injection $\mathrm{Aut}\,\mathrm{Fl}(F_E,E,V) \hookrightarrow \mathrm{Aut}\,\mathrm{Gr}(k,V)$ constructed in the proof of Proposition 2 is proper.

We hope that the above differences motivate a more detailed future study of the automorphism groups of arbitrary ind-varieties of generalized flags.

5. In Memoriam

The topic of this paper is very close to Vasil's interests and expertise, and for sure I would have discussed it with him if he were still alive.

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References

[1] Dimitrov I. and Penkov I., *Ind-Varieties of Generalized Flags as Homogeneous Spaces for Classical Ind-Groups*, IMRN **2004** (2004) 2935–2953.

[2] Penkov I. and Tikhomirov A., *Linear Ind-Grassmannians*, Pure Appl. Math. Quarterly **10** (2014) 289–323.

[3] Ignatyev M., Penkov I. and Wolf J., *Real Group Orbits on Flag Ind-Varieties of* $\mathrm{SL}(\infty, \mathbb{C})$. In: Lie Theory and Its Applications in Physics, Springer Proceedings in Mathematics and Statistics, V. Dobrev (Ed), Springer, Heidelberg 2016, pp. 111–135.

[4] Fresse L. and Penkov I., *Schubert Decompositions for Ind-Varieties of Generalized Flags*, Asian J. Math. **21** (2017) 599–630.

[5] Fresse L. and Penkov I., *Orbit Duality in Ind-Varieties of Maximal Generalized Flags*, Trans. Moscow Math. Soc. **78** (2017) 131–160.

[6] Ignatyev M. and Penkov I., *Ind-Varieties of Generalized Flags*: *A Survey of Results*, arXiv:1701.08478.

Ivan Penkov
Jacobs University Bremen
Campus Ring 1, 28759 Bremen
GERMANY
E-mail address: i.penkov@jacobs-university.de