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SMOOTH EXTENSIONS AND SPACES OF SMOOTH AND HOLOMORPHIC MAPPINGS

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Abstract. In this paper we present a new notion of a smooth manifold with corners and relate it to the commonly used concept in the literature. We also introduce complex manifolds with corners show that if M is a compact (respectively, complex) manifold with corners and K is a smooth (respectively, complex) Lie group, then $C^\infty(M, K)$ (respectively, $\mathcal{O}(M, K)$) is a smooth (respectively, complex) Lie group.

1. Introduction

We introduce the notion of a smooth manifold with corners, which is an extension of the existing notion of smooth manifolds with corners or boundary for the finite-dimensional case (cf. [6] or [7, Ch. 2]). The notation presented here is the appropriate notion for a treatment of mapping spaces and Whitney's extension theorem [9] implies that for finite-dimensional smooth manifolds our definition coincides with the one given in [6]. We give an alternative proof of a similar statement by elementary methods from real Analysis (cf. also [4, Theorem 22.17] and [4, Proposition 24.10]).

We also introduce complex manifolds with corners and derive several properties of the spaces $C^\infty(M, K)$ and $\mathcal{O}(M, K)$. Eventually it turns out that these mapping spaces are smooth (respectively, complex) Lie groups. This is in particular interesting since it seems to be the only way to put a complex or even smooth structure onto spaces of holomorphic mappings since the Open Mapping Theorem implies that in the case of a closed compact manifolds all holomorphic maps are constant. With the results of this paper a Lie theoretic treatment of groups like $\mathcal{O}(M, K)$ for non-compact M becomes possible as a projective limit of Lie groups.

2. Notions of Differential Calculus

In this section we present the elementary notions of differential calculus on locally convex spaces and for not necessarily open domains.

Definition 1. *Let E and F be a locally convex spaces and $U \subseteq E$ be open. Then $f : U \to F$ is called continuously differentiable or C^1 if it is continuous, for each $v \in E$ the differential quotient*

$$\mathrm{d}f(x).v := \lim_{h \to 0} \frac{1}{h}(f(x + hv) - f(x))$$

exists and the map $\mathrm{d}f : U \times E \to F$ is continuous. If $n > 1$ we recursively define

$$\mathrm{d}^n f(x).(v_1, \ldots, v_n) := \lim_{h \to 0} \frac{1}{h} \left(\mathrm{d}^{n-1} f(x + h).(v_1, \ldots, v_{n-1}) - \mathrm{d}^{n-1} f(x).(v_1, \ldots, v_n)\right)$$

and say that f is C^n if $\mathrm{d}^k f : U \times E^k \to F$ exists for all $k = 1, \ldots, n$ and is continuous. We say that f is C^∞ or smooth if it is C^n for all $n \in \mathbb{N}$. The corresponding sets of maps will be denoted by $C^1(U, E)$, $C^n(U, E)$ and $C^\infty(U, E)$. This is the notion of differentiability used in [8] *and* [2] *and it is equivalent to the one used in* [3] (cf. [2, Lemma 1.14]).

If E and F are complex vector spaces, then f is called holomorphic if it is C^1 and the map $\mathrm{d}f(x) : E \to F$ is complex linear map for all $x \in U$ (cf. [8, p.1027]). *We denote the set of all holomorphic maps by $\mathcal{O}(U, F)$.*

Definition 2. *From the above definition it is clear what the notions of a smooth (respectively, complex) Lie group is, i.e. a group which is a smooth (respectively, complex) manifold modelled on a locally convex (respectively, complex) space such that the group operations are smooth (respectively, holomorphic).*

Definition 3. *Let E and F be locally convex spaces, and let $U \subseteq E$ be a set with dense interior. We say that a map $f : U \to F$ is C^1 if it is continuous, $f_{\text{int}} := f|_{\text{int}(U)}$ is C^1 and the map $\mathrm{d}(f_{\text{int}})$ extends to a continuous map on $U \times E$, which is called the differential $\mathrm{d}f$ of f. If $n > 1$ we inductively define f to be C^n if if is C^1 and $\mathrm{d}f$ is C^{n-1} for $n > 1$. We say that f is C^∞ or smooth if f is C^n for all $n \in \mathbb{N}_0$.*

Remark 4. *Since* $\text{int}(U \times E^{n-1}) = \text{int}(U) \times E^{n-1}$ *we have for $n = 1$ that* $(\mathrm{d}f)_{\text{int}} = \mathrm{d}(f_{\text{int}})$ *and we inductively obtain* $(\mathrm{d}^n f)_{\text{int}} = \mathrm{d}^n(f_{\text{int}})$. *Hence the* *higher differentials* $\mathrm{d}^n f$ *are defined to be the continuous extensions of the differentials* $\mathrm{d}^n(f_{\text{int}})$ *and thus we have that a map* $f : U \to F$ *is smooth if and only if* $\mathrm{d}^n(f_{\text{int}})$ *has a continuous extension* $\mathrm{d}^n f$ *to* $U \times E^{n-1}$ *for all* $n \in \mathbb{N}$.

Remark 5. *Let* $f : U_1 \to U_2$ *and* $g : U_2 \to F$ *with* $f(\text{int}(U_1)) \subseteq \text{int}(U_2)$ *are* C^1, *then the chain rule for locally convex spaces* [2, Proposition 1.15] *and* $(g \circ f)_{\text{int}} = g_{\text{int}} \circ f_{\text{int}}$ *imply that* $g \circ f : U_1 \to F$ *is* C^1 *and its differential is given by* $\mathrm{d}(g \circ f)(x).v = dg(f(x)).\mathrm{d}f(x).v$. *In particular,* $g \circ f$ *is smooth if* g *and* f *are so.*

Definition 6. (cf. [6] *for the finite-dimensional case*) *Let* E *be a locally convex space,* $\lambda_1, \ldots, \lambda_n$ *be continuous functionals and* $E^+ := \bigcap_{k=1}^n \lambda_k^{-1}(\mathbb{R}_0^+)$. *If* M *is a Hausdorff space, then a collection* $(U_i, \varphi_i)_{i \in I}$ *of homeomorphisms* $\varphi_i : U_i \to \varphi(U_i)$ *onto open subsets* $\varphi_i(U_i)$ *of* E^+ *(called charts) defines a differential structure on* M *of codimension* n *if* $\cup_{i \in I} U_i = M$ *and for each pair of charts* φ_i *and* φ_j *with* $U_i \cap U_j \neq \emptyset$ *the coordinate change*

$$\varphi_i\left(U_i \cap U_j\right) \ni x \mapsto \varphi_j\left(\varphi_i^{-1}(x)\right) \in \varphi_j(U_i \cap U_j)$$

is smooth in the sense of Definition 3. Furthermore, M *together with a differential structure* $(U_i, \varphi_i)_{i \in I}$ *is called a smooth manifold with corners of codimension* n.

Remark 7. *Note that the previous definition of a smooth manifold with corners coincides for* $E = \mathbb{R}^n$ *with the one given in* [6] *and in the case of codimension one and a Banach space* E *with the definition of a manifold with boundary in* [5], *but our notion of smoothness differs. In both cases a map* f *defined on a non-open subset* $U \subseteq E$ *is said to be smooth if for each point* $x \in U$ *there exists an open neighbourhood* $V_x \subseteq E$ *of* x *and a smooth map* f_x *defined on* V_x *with* $f = f_x$ *on* $U \cap V_x$. *However, it will turn out that for finite-dimensional smooth manifolds with corners the two notions coincide.*

Lemma 8. *If* M *is a smooth manifold with corners modelled on the locally convex space* E *and* φ_i *and* φ_j *are two charts with non-empty intersection, i.e.,* $U_i \cap U_j \neq \emptyset$, *then* $\varphi_j \circ \varphi_i^{-1}(\text{int}(\varphi_i(U_i \cap U_j))) \subseteq \text{int}(\varphi_j(U_i \cap U_j))$.

Proof: Denote by $\alpha : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$, $x \mapsto \varphi_j(\varphi_i^{-1}(x))$ and $\beta = \alpha^{-1}$ the corresponding coordinate changes. We claim that $\mathrm{d}\alpha(x) : E \to E$ is an isomorphism if $x \in \text{int}(\varphi_i(U_i \cap U_j))$. Since β maps a neighbourhood W_x of $\alpha(x)$ into $\text{int}(\varphi_i(U_i \cap U_j))$, we have $\mathrm{d}\alpha(\beta(y)).\big(\mathrm{d}\beta(y).v\big) = v$ for $v \in E$ and $y \in \text{int}(W_x)$ (cf. Remark 5). Since $(y, v) \mapsto \mathrm{d}\alpha(\beta(y)).\big(\mathrm{d}\beta(y).v\big)$ is continuous and $\text{int}(W_x)$ is dense in W_x, $\mathrm{d}\beta(\alpha(x))$ is a continuous inverse of $\mathrm{d}\alpha(x)$.

Now suppose $x \in \text{int}(\varphi_i(U_i \cap U_j))$ and $\alpha(x) \notin \text{int}(\varphi_j(U_i \cap U_j))$. Then $\lambda_i(\alpha(x)) = 0$ for some $i \in \{1, \ldots, n\}$ and thus there exists a $v \in E$ such that $\alpha(x) + tv \in \varphi_j(U_i \cap U_j)$ for $t \in [0,1]$ and $\alpha(x) + tv \notin \varphi_j(U_i \cap U_j)$ for $t \in [-1, 0)$. But then $v \notin \text{im}(\mathrm{d}\alpha(x))$, contradicting the surjectivity of $\mathrm{d}\alpha(x)$. ■

Definition 9. *The preceding Lemma shows that the points of* $\text{int}(E_+)$ *are invariant under coordinate changes and thus the interior* $\text{int}(M) = \bigcup_{i \in I} \varphi_i^{-1}(\text{int}(E_+))$ *is an intrinsic property of* M. *We denote by* $\partial M := M \backslash \text{int}(M)$ *the boundary of* M.

A map $f : M \to N$ *between smooth manifolds with corners is said to be* C^n *(respectively, smooth) if* $f(\text{int}(M)) \subseteq \text{int}(N)$ *and the corresponding coordinate representation*

$$\varphi_i(U_i \cap f^{-1}(U_j)) \ni x \mapsto \varphi_j\left(f\left(\varphi_i^{-1}(x)\right)\right) \in \varphi_j(U_j)$$

is C^n *(smooth) for each pair* φ_i *and* φ_j *of charts on* M *and* N. *We again denote the corresponding spaces of mappings by* $C^n(M, N)$ $(C^\infty(M, N))$.

Definition 10. *If* M *is a smooth manifold with corners and differentiable structure* $(U_i, \varphi_i)_{i \in I}$, *which is modelled on the locally convex space* E, *then the tangent space in* $m \in M$ *is defined to be* $T_m M := (E \times I_m) / \sim$, *where* $I_m := \{i \in I; m \in U_i\}$ *and* $(x, i) \sim \left(\mathrm{d}\left(\varphi_j \circ \varphi_i^{-1}\right)(\varphi_i(m)).x, j\right)$. *The set* $TM := \cup_{m \in M}\{m\} \times T_m M$ *is called the tangent bundle of* M.

Proposition 11. *The tangent bundle* TM *is a smooth manifold with corners and the map* $\pi : TM \to M$, $(m, [x, i]) \mapsto m$ *is smooth.*

Proof: This can be shown exactly as in the non-boundary case. ■

Lemma 12. *If* M *and* N *are two smooth manifolds with corners, then the map* $f : M \to N$ *is* C^1 *if* $f(\text{int}(M)) \subseteq \text{int}(N)$, $f_{\text{int}} := f|_{\text{int}(M)}$ *is* C^1 *and* $Tf_{\text{int}} : T(\text{int}(M)) \to T(\text{int}(N)) \subseteq TN$ *extends continuously to* TM. *If, in addition,* f *is* C^n *for* $n \geq 2$, *then the map*

$$Tf : TM \to TN, \qquad (m, [x, i]) \mapsto \left(f(m), \left[\mathrm{d}\left(\varphi_j \circ f \circ \varphi_i^{-1}\right)(\varphi_i(m)).x, j\right]\right)$$

is well-defined and C^{n-1}.

Definition 13. *If* M *is a smooth manifold with corners, then for* $n \in \mathbb{N}_0$ *the higher tangent bundles* $T^n M$ *are the inductively defined smooth manifolds with corners* $T^0 M := M$ *and* $T^n M := T\left(T^{n-1} M\right)$. *If* N *is a smooth manifold with* *corners and $f : M \to N$ is C^n map, then the higher tangent maps $T^m f : T^m M \to T^m N$ are the maps defined inductively by $T^0 f := f$ and $T^m f := T(T^{m-1} f)$ if $1 < m$.*

3. Extensions of Smooth Maps

This section draws on a suggestion by H. Glöckner and was inspired by [1, Ch. IV]. We relate the notions introduced in Definition 3 to the usual notion of differentiability on a non-open subset $U \subseteq \mathbb{R}^n$ (cf. Remark 7).

Definition 14. *If M is a smooth manifold with corners and F is a locally convex space, then we endow $C^\infty(M, F)$ with the topology making the canonical map*

$$C^\infty(M, F) \hookrightarrow \prod_{n \in \mathbb{N}_0} C(T^n M, F), \qquad f \mapsto \mathrm{d}^n f$$

a topological embedding, where $\mathrm{d}^n f := \mathrm{pr}_{2^n} \circ T^n f$ (note $T^n F \cong F^{2^n}$) and $C(T^n M, F)$ is endowed with the topology of compact convergence (cf. [3, Definition 3.1]).

Remark 15. *This defines a locally convex vector topology on $C^\infty(M, F)$. Furthermore, if M is a second countable finite-dimensional smooth manifold with corners and F is a Fréchet space, then $C^\infty(M, F)$ is a Fréchet space since then $C^\infty(\mathrm{int}(M), F)$ and $C(M, F)$ are Fréchet spaces. Note that this is not immediate if one uses the notion of smoothness on M from* [6] *or* [5].

Proposition 16. *If E, F are Fréchet spaces, $U_1 \subseteq E$ and $U_2 \subseteq \mathbb{R}^n$ have dense interior, then we have a linear isomorphism*

$$^\wedge : C^\infty(U_1 \times U_2, F) \to C^\infty(U_1, C^\infty(U_2, F)), \qquad f^\wedge(x)(y) = f(x, y).$$

Proof: First we check that $f^\wedge$ actually is an element of $C^\infty(U_1, C^\infty(U_2, F))$. Since for Fréchet spaces, the notion of differentiability from Definition 1 and the one used in the convenient setting coincide, [4, Lemma 3.12] implies that $f^\wedge(x)|_{\mathrm{int}(U_2)} \in C^\infty(\mathrm{int}(U_2), F)$ if $x \in \mathrm{int}(U_1)$. Since $\mathrm{d}^n f$ extends continuously to the boundary, so does $\mathrm{d}^n(f^\wedge(x))$. So $f^\wedge|_{\mathrm{int}(U_1)}$ defines a map to $C^\infty(U_2, F)$ which is continuous since $C(X \times Y, Z) \cong C(X, C(Y, Z))$ if Y is locally compact. Next we show that we can extend it to a continuous map on U_1. If $x \in \partial U_1 \cap U_1$, then there exists a sequence of points $(x_i)_{i \in \mathbb{N}}$ in $\mathrm{int}(U_1)$ with $x_i \to x$ and thus $(\mathrm{d}^n(f^\wedge(x_i)))_{i \in \mathbb{N}}$ is a Cauchy sequence in $C(T^n U_2, F)$ since $\mathrm{d}^n f$ is continuous.

Since the space $C^\infty(U_2,F)$ is complete, the sequence $(f^\wedge(x_i))_{i\in\mathbb{N}}$ converges to some $f^\wedge(x)\in C^\infty(U_2,F)$, and this extends $f^\wedge|_{\mathrm{int}(U_1)}$ continuously. Since inclusion $C^\infty(U_2,F)\hookrightarrow C(U_2,F)$ is continuous and continuous extensions are unique we know that this extension is actually given by $f^\wedge$. With Remark 4, the smoothness of $f^\wedge$ follows in the same way as the continuity. It is immediate that $^\wedge$ is linear and injective, and surjectivity follows directly from $C(X\times Y,Z)\cong C(X,C(Y,Z))$. ■

Lemma 17. *If E is a locally convex space and $(f_n)_{n\in\mathbb{N}_0}$ is such sequence in $C^1(\mathbb{R},E)$ that $(f'_n)_{n\in\mathbb{N}_0}$ converges uniformly on compact subsets to some element $\bar{f}\in C(\mathbb{R},E)$, then (f_n) converges to some $f\in C^1(\mathbb{R},E)$ with $f'=\bar{f}$.*

Proof: This can be proved as in the case $E=\mathbb{R}$ (cf. [1, Proposition IV.1.7]). ■

Lemma 18. *Let F be a Fréchet space. If $(v_n)_{n\in\mathbb{N}_0}$ is an arbitrary sequence in F, then there exists an $f\in C^\infty(\mathbb{R},F)$ such that $f^{(n)}(0)=v_n$ for all $n\in\mathbb{N}_0$.*

Proof: (cf. [1, Proposition IV.4.5] for the case $F=\mathbb{R}$). Let $\zeta\in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\mathrm{supp}(\zeta)\subseteq[-1,1]$ and $\zeta(x)=1$ if $-\frac{1}{2}\le x\le\frac{1}{2}$ and put $\xi(x):=x\,\zeta(x)$. Then $\mathrm{supp}(\xi)\subseteq[-1,1]$ and $\xi|_{[-\frac{1}{2},\frac{1}{2}]}=\mathrm{id}_{[-\frac{1}{2},\frac{1}{2}]}$. Since ξ^k is compactly supported, there exists for each $n\in\mathbb{N}$ an element $M_{n,k}\in\mathbb{R}$ such that $|\left(\xi^k\right)^{(n)}(x)|\le M_{n,k}$ for all $x\in\mathbb{R}$. Now let $(p_m)_{m\in\mathbb{N}}$ be a sequence of seminorms defining the topology on F with $p_1\le p_2\le\ldots$. We now choose $c_k>1$ such that $p_k(v_k)c_k^{n-k}M_{n,k}<2^{-k}$ if $n<k$. Note that this is possible since there are only finitely many inequalities for each k. Set $f_m:=\sum_{k=0}^m v_k\left(c_k^{-1}\xi(c_k\,\cdot\,)\right)^k$. We show that $f:=\lim_{m\to\infty}f_m$ has the desired properties. If $\varepsilon>0$ and $\ell\in\mathbb{N}$ we let $m_{\varepsilon,\ell}>\ell$ be such that $2^{-m_{\varepsilon,\ell}}<\varepsilon$. Thus

$$p_\ell(f_m^{(n)}-f_{m_{\varepsilon,\ell}}^{(n)})=p_\ell\Big(\sum_{k=1+m_{\varepsilon,\ell}}^{m}v_kc_k^{-k}(\xi(c_k\,\cdot\,)^k)^{(n)}\Big)$$
$$\le\sum_{k=1+m_{\varepsilon,\ell}}^{m}p_k(v_k)c_k^{n-k}M_{n,k}\le 2^{-m_{\varepsilon,\ell}}<\varepsilon$$

for all $m>m_{\varepsilon,\ell}$ and $n<\ell$. It follows for $n<\ell$ that the sequence $(f_m^{(n)})_{m\in\mathbb{N}}$ converges uniformly to some $f^n\in C^\infty(\mathbb{R},F)$ and the preceding lemma implies $(f^{n-1})'=f^n$, whence $f^{(n)}=f^n$. Since ℓ was chosen arbitrarily, f is smooth. We may interchange differentiation and the limit by the preceding lemma and since $c_k\xi(c_k\,\cdot\,)$ equals the identity on a zero neighbourhood, we have $f^{(n)}(0)=\left(\lim_{m\to\infty}f_m^{(n)}\right)(0)=\lim_{m\to\infty}\left(f_m^{(n)}(0)\right)=v_n$. ■

Corollary 19. *If F is a Fréchet space, then for each $f \in C^{\infty}([0,1], F)$ there exists an $\bar{f} \in C^{\infty}(\mathbb{R}, F)$ with $\bar{f}\big|_{[0,1]} = f$.*

Theorem 20. *If F is a Fréchet space and $f \in C^{\infty}([0,1]^n, F)$, then there exists an $\bar{f} \in C^{\infty}(\mathbb{R}^n, F)$ with $\bar{f}\big|_{[0,1]^n} = f$.*

Proof: This is a direct consequence of Proposition 16 and Corollary 19. ■

Corollary 21. *If $U \subseteq (\mathbb{R}^n)^+$ is open, F a Fréchet space and $f : U \to F$ is smooth in the sense of Definition 3, then there exists an open subset $\widetilde{U} \subseteq \mathbb{R}^n$, with $U \subseteq \widetilde{U}$, such that for each $f \in C^{\infty}(U, F)$ there exists an $\widetilde{f} \in C^{\infty}(\widetilde{U}, F)$ with $\widetilde{f}\Big|_U = f$.*

4. Spaces of Mappings

In this section we prove several results on mapping spaces like $C^{\infty}(M, K)$ or $\mathcal{O}(M, K)$. Since many proofs carry over from case of closed compact manifolds, we provide here only the necessary changes and extensions to the statements in [3, pp. 366-375].

Definition 22. *If E and F are locally convex complex vector spaces and $U \subseteq E$ has dense interior, then a smooth map $f : U \to F$ is called holomorphic if f_{int} is holomorphic, i.e., that each map $\mathrm{d}f_{\text{int}}(x) : E \to F$ is complex linear* (cf. [8, p. 1027]). *We denote the space of all holomorphic functions on U by $\mathcal{O}(U, F)$.*

Remark 23. *Note that in the above setting $\mathrm{d}f(x)$ is complex linear for all $x \in U$ due to the continuity of the extension of $\mathrm{d}f_{\text{int}}$.*

Definition 24. *A smooth manifold with corners is called a complex manifold with corners if it is modelled on a complex vector space E and the coordinate changes in Definition* 6 *are holomorphic. A smooth map $f : M \to N$ between complex manifolds with corners is said to be holomorphic if and for each pair of charts on M and N the corresponding coordinate representation is holomorphic* (cf. Definition 9). *We denote the space of holomorphic mappings from M to N by $\mathcal{O}(M, N)$.*

Remark 25. *If M is a complex manifold with corners and F is a locally convex complex vector space, then $\mathcal{O}(M, F)$ is a closed subspace of $C^{\infty}(M, F)$ since the* *requirement on* $\mathrm{d}f(x)$ *being complex linear is a closed condition as an equational requirement on* $\mathrm{d}f(x)$ *in the topology defined in* Definition 14.

Proposition 26. a) *If* M *is a compact smooth manifold with corners,* E *and* F *are locally convex spaces,* $U \subseteq E$ *is open and* $f : M \times U \to F$ *is smooth, then the mapping* $f_\sharp : C^\infty(M, U) \to C^\infty(M, F)$, $\gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$ *is smooth.*

b) *If, in addition,* E *and* F *are complex vector spaces and* $f(m) : U \to F$ *is holomorphic for all* $m \in M$, *then* $f_\sharp$ *is holomorphic.*

Proof: a) Since the Lemmas referred to in [3, Proposition 3.10] carry over to the case of manifolds with corners in exactly the same way, we obtain the smoothness of $f_\sharp$ as in *loc.cit.*

b) The formula $\mathrm{d}(f_\sharp) = (\mathrm{d}_2 f)_\sharp$ derived in [3, Proposition 3.10] shows that $\mathrm{d}(f_\sharp)$ is complex linear. ■

Corollary 27. *If* M *is a compact smooth manifold with corners,* E *and* F *are locally convex spaces,* $U \subseteq E$ *are open and* $f : U \to F$ *is smooth (respectively, holomorphic), then the push forward* $f_* : C^\infty(M, U) \to C^\infty(M, F)$, $\gamma \mapsto f \circ \gamma$ *is a smooth (respectively, holomorphic) map.*

Theorem 28. *Let* M *be a compact smooth manifold with corners,* K *be a Lie group and let* $\varphi : W \to \varphi(W) \subseteq \mathfrak{k} := L(K)$ *be a chart of* K *around* e *with* $\varphi(e) = 0$. *Furthermore let* $\varphi_* : C^\infty(M, W) \to C^\infty(M, \mathfrak{k})$, $\gamma \mapsto \varphi \circ \gamma$.

a) *If* M *and* K *are smooth, then* φ_* *induces a smooth manifold structure on* $C^\infty(M, K)$, *turning it into a smooth Lie group w.r.t. pointwise operations.*

b) *If* M *is smooth and* K *is complex, then* φ_* *induces a complex manifold structure on* $C^\infty(M, K)$, *turning it into a complex Lie group w.r.t. pointwise operations.*

c) *If* M *and* K *are complex, then the restriction of* φ_* *to* $\mathcal{O}(M, W)$ *induces a complex manifold structure on* $\mathcal{O}(M, K)$, *turning it into a complex Lie group w.r.t. pointwise operations, modelled on* $\mathcal{O}(M, \mathfrak{k})$.

Proof: Using Corollary 27 and Proposition 26, the proof of the smooth case in [3, 3.2] also yields a). Since Proposition 26 also implies that the group operations are holomorphic, b) is now immediate. Using the same argument as in a), we deduce c) since φ_* maps $\mathcal{O}(M, W)$ bijectively to $\mathcal{O}(M, \varphi(W))$, which is open in $\mathcal{O}(M, \mathfrak{k})$. ■

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