JGSP **48** (2018) 33–45

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

AN EXAMINATION OF PERPENDICULAR INTERSECTIONS OF BFRS AND MFRS IN E^3

ŞEYDA KILIÇOĞLU AND SÜLEYMAN ŞENYURT

Communicated by Ivaïlo M. Mladenov

Abstract. We already have defined and found the parametric equations of Frenet ruled surfaces which are called Bertrandian Frenet Ruled Surfaces (BFRS) and Mannheim Frenet Ruled Surfaces (MFRS) of a curve α, in terms of the Frenet apparatus. In this paper, we find a matrix which gives us all sixteen positions of normal vector fields of eight BFRS and MFRS in terms of the Frenet apparatus. Further using the orthogonality conditions of the eight normal vector fields, we give perpendicular intersection curves of the eight BFRS and MFRS.

MSC: 53A04, 53A05
Keywords: Involute curves, Mannheim curves, ruled surface

1. Introduction

The surface-surface intersection (SSI) problems can be cast as three types: parametric-parametric, implicit-implicit, parametric-implicit. The SSI is called transversal if the normal vectors of the surfaces are linearly independent or The SSI is called tangential if the normal vectors of the surfaces are linearly dependent at the intersecting points. In transversal intersection problems, the tangent vector of the intersection curve can be found easily by the vector product of the normal vectors of the surfaces. Because of this, there are many studies related to the transversal intersection problems in the literature on differential geometry. There also are some studies about tangential intersection curve and its properties. Some of these studies are mentioned below. Wu, Alessio and Costa [16], using only the normal vectors of two regular surfaces, present an algorithm to compute the local geometric properties of the transversal intersection curve. Tangential intersection of two surfaces are examined in [1]. We have already try to derive a surface based on the other surface by using the similar method to derive curves based on the other curves which is very interesting subject in geometry. The involute-evolute curves, Bertrand curves are such kind of curves. We produce a new ruled surface based on the other ruled surface which are called involute $\tilde{D}$-scroll that were examined in [15]. In this paper we consider the following four special ruled surfaces associated to a space curve

α with $\kappa \neq 0$. They are called Frenet ruled surface, because their generators are the Frenet vector fields of the curve. The quantities $\{T, N, B, D, \kappa, \tau\}$ present the Frenet-Serret apparatus of the curve α. Here

$$\tilde{D}(s) = \frac{\tau(s)}{\kappa(s)} T\left(s\right) + B\left(s\right) \tag{1}$$

is the modified Darboux vector field of α [6]. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. Frenet ruled surface is one which can be generated by the motion of a Frenet vector of any curve in E^3. To illustrate the current situation, we bring here the famous example of Graves (see [3]), the so called the B-scroll. Here tangent, normal, binormal and Darboux ruled surfaces of any curve are named Frenet Ruled Surfaces (FRS) of the curve α. Some results concerning FRS according to their normal vector fields can be found in [7]. They have the following equations

Definition 1. *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equations*

$$\begin{aligned} \varphi_T\left(s, u_1\right) &= \alpha\left(s\right) + u_1 T\left(s\right) \\ \varphi_N\left(s, u_2\right) &= \alpha\left(s\right) + u_2 N\left(s\right) \\ \varphi_B\left(s, u_3\right) &= \alpha\left(s\right) + u_3 B\left(s\right) \\ \varphi_{\tilde{D}}\left(s, u_4\right) &= \alpha\left(s\right) + u_4 \tilde{D}(s) \end{aligned} \tag{2}$$

are the parametrization of tangent ruled surface, normal ruled surface, binormal ruled surface, and Darboux ruled surface.

Theorem 2. *The normal vector fields of Frenet ruled surfaces along the curve α, can be expressed by the following matrix*

$$[\eta] = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3}$$

where

$$a = \frac{-u_2\tau}{\sqrt{\left(u_2\tau\right)^2 + \left(1 - u_2\kappa\right)^2}}, \qquad c = \frac{-u_3\tau}{\sqrt{\left(u_3\tau\right)^2 + 1}}$$
$$b = \frac{1 - u_2\kappa}{\sqrt{\left(u_2\tau\right)^2 + \left(1 - u_2\kappa\right)^2}}, \qquad d = \frac{-1}{\sqrt{\left(u_3\tau\right)^2 + 1}}.$$

1.1. Bertrandian Frenet Ruled Surfaces (BFRS)

Let $\alpha(s)$ and $\alpha_1(s_1)$ be two curves which are parametrized by arc-length parameters s and s_1, respectively. Furthermore, let the sets $\{T, N, B\}$ and $\{T_1, N_1, B_1\}$ denote the Frenet frames of α and α_1, respectively. Two curves $\{\alpha, \alpha_1\}$ are called Bertrand pair curves if they have common principal normal lines, i.e., $N = N_1$ [4,10]. Then, $\{N(s), N_1(s)\}$ are linearly dependent, and thus we have $\langle T_1, N\rangle = 0$. Sometimes α_1 is called Bertrand mate of the curve α. If the curve α_1 is Bertrand mate of α, then we may write

$$\alpha_1(s) = \alpha(s) + \lambda N(s).$$

In the Euclidean three-space E^3, if the curve α_1 is Bertrand mate of α, then we have $\langle T_1(s), T(s)\rangle = \cos\mu = \text{constant}$. Bertrand curves have the following fundamental properties which are given with more details in [5], and [14]. Also $\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that $\lambda\kappa + \beta\tau = 1$ for any $s \in I$. This is called offset property of Bertrand curves. The converse assertion is also true and

$$\frac{\lambda}{\beta} = \frac{1}{\beta\kappa} - \frac{\tau}{\kappa}.$$

The following theorem says that we can write the Frenet apparatus of the Bertrand mate α_1 based on the Frenet apparatus of the curve α (see [14]).

Theorem 3. *The Frenet vectors of the Bertrand mate α_1 can be expressed via those of the curve α as*

$$T_1 = \frac{\beta T + \lambda B}{\sqrt{\lambda^2 + \beta^2}}, \quad N_1 = N, \quad B_1 = \frac{-\lambda T + \beta B}{\sqrt{\lambda^2 + \beta^2}}, \quad \tilde{D}_1 = \frac{\kappa\sqrt{\lambda^2 + \beta^2}}{(\beta\kappa - \lambda\tau)}\tilde{D}.$$

Also, the first and the second curvatures of the Bertrand mate α_1 are given by

$$\kappa_1 = \frac{\beta\kappa - \lambda\tau}{(\lambda^2 + \beta^2)\,\tau}, \qquad \tau_1 = \frac{1}{(\lambda^2 + \beta^2)\,\tau} \tag{4}$$

where $\tau\tau_1 = \dfrac{1}{(\lambda^2 + \beta^2)}$ is a non-negative constant. Due to this theorem we have $\kappa_1 + \gamma\tau_1 = 0$.

We produce the FRS of the Bertrad mate α_1 of the curve α. Further we write their parametric equations in terms of the Frenet apparatus of α. Hence they are called collectively Bertrandian Frenet Ruled Surfaces (BFRS) of the curve α.

Definition 4. *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equations*

$$\begin{aligned}
\varphi_{T_1}(s,u_1) &= \alpha+\lambda N+u_1\frac{\beta T+\lambda B}{\sqrt{\lambda^2+\beta^2}}\\
\varphi_{N_1}(s,u_2) &= \alpha+(\lambda+u_2)N\\
\varphi_{B_1}(s,u_3) &= \alpha+\lambda N+u_3\left(\frac{-\lambda T+\beta B}{\sqrt{\lambda^2+\beta^2}}\right)\\
\varphi_{\tilde{D}_1}(s,u_4) &= \alpha+\lambda N+u_4\frac{\kappa\sqrt{\lambda^2+\beta^2}}{(\beta\kappa-\lambda\tau)}\tilde{D}
\end{aligned} \tag{5}$$

are the parametrization of the ruled surface which are called Bertrandian Tangent Ruled Surface (BTRS)*, Bertrandian Normal Ruled Surface* (BNRS)*, Bertrandian Binormal Ruled Surface* (BBRS)*, Bertrandian Darboux Ruled Surface* (BDRS)*, respectively.*

Theorem 5. *The normal vector fields $[\sigma]$ of* BFRS *along the Bertran mate α_1, which is given by the equation $[\sigma]=[\mathrm{A}_1]\cdot[\mathrm{V}_1]$ can be expressed by the following matrix* [9]

$$[\sigma]=\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\end{bmatrix}=\begin{bmatrix}0&0&-1\\a_1&0&b_1\\c_1&d_1&0\\0&-1&0\end{bmatrix}\cdot\begin{bmatrix}T_1\\N_1\\B_1\end{bmatrix} \tag{6}$$

where

$$a_1=\frac{-u_2\tau_1}{\sqrt{(u_2\tau_1)^2+(1-u_2\kappa_1)^2}},\qquad c_1=\frac{-u_3\tau_1}{\sqrt{(u_3\tau_1)^2+1}}$$
$$b_1=\frac{1-u_2\kappa_1}{\sqrt{(u_2\tau_1)^2+(1-u_2\kappa_1)^2}},\qquad d_1=\frac{-1}{\sqrt{(u_3\tau_1)^2+1}}.$$

1.2. Mannheim Frenet Ruled Surfaces (MFRS)

Mannheim curve was firstly defined by Mannheim in 1878. A curve is called a Mannheim curve if and only if the expression $\kappa/\left(\kappa^2+\tau^2\right)$ is a nonzero constant, where κ is the curvature and τ is the torsion. Mannheim curve was redefined. If the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then the first curve is called Mannheim curve, and the second curve is called Mannheim partner curve by Liu and Wang [11]. Let $\alpha_2: I\to E^3$ be some differentiable curve of class C^2. Let $T_2(s_2), N_2(s_2), B_2(s_2)$ be the

Frenet frame of the curve α_2. If the principal normal vector N of the curve α is linearly dependent with the binormal vector B_2 of the curve α_2, then the pair $\{\alpha, \alpha_2\}$ is said to be Mannheim pair, α is called a Mannheim curve, α_2 is called Mannheim partner curve of α where $(T, T_2) = \cos\theta$ and besides the equality $\kappa/\left(\kappa^2+\tau^2\right) = \text{constant}$ is known as the offset property. In [13] Mannheim offsets of ruled surfaces are defined and characterized. Since N and B are linearly dependent their equation can be rewritten for some function λ as

$$\alpha_2(s) = \alpha(s) - \lambda_2 N(s) \tag{7}$$

where $\lambda_2 = -\kappa/\left(\kappa^2+\tau^2\right)$. Frenet-Serret apparatus of Mannheim partner curve α_2, based on the Frenet-Serret vectors of Mannheim curve α are

$$\begin{gathered} T_2 = \cos\theta\, T - \sin\theta\, B, \qquad N_2 = \sin\theta\, T + \cos\theta\, B, \qquad B_2 = N \\ \tilde{D}_2(s) = \frac{\kappa}{\lambda_2\tau}\frac{\cos^2\theta}{\dot{\theta}}T + N - \frac{\kappa}{\lambda_2\tau}\frac{\cos\theta\cdot\sin\theta}{\dot{\theta}}B \end{gathered} \tag{8}$$

where $\tilde{D}_2$ is the modified Darboux vector of Mannheim partner α_2 of a Mannheim curve α, based on the Frenet apparatus of Mannheim curve α. The curvature and the torsion satisfy the following equalities

$$\kappa_2 = -\frac{\mathrm{d}\theta}{\mathrm{d}s_1} = \frac{\dot{\theta}}{\cos\theta}, \qquad \tau_2 = \frac{\kappa}{\lambda_2\tau} \tag{9}$$

and we have used dot to denote the derivative with respect to the arc length parameter of the curve α. Also $\dfrac{\mathrm{d}s}{\mathrm{d}s_2} = \dfrac{1}{\cos\theta}$, where $|\lambda_2|$ is the distance between the curves α and α_1. For more details see [12]. Also we can write $\dfrac{\mathrm{d}s}{\mathrm{d}s_2} = \dfrac{1}{\sqrt{1+\lambda_2\tau}}$. One can give also the tangent, normal, binormal and the Darboux Frenet ruled surfaces of the Mannheim partner α_2 of curve α. Further we write their parametric equations in terms of the Frenet apparatus of the Mannheim curve α and they are called hereafter MFRS. For more details see [8].

Definition 6. *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equations*

$$\begin{aligned} \varphi_{T_2}(s, w_1) &= \alpha + w_1\cos\theta T - \lambda N - w_1\sin\theta\, B \\ \varphi_{N_2}(s, w_2) &= \alpha + w_2\sin\theta\, T - \lambda N + w_2\cos\theta\, B \\ \varphi_{B_2}(s, w_3) &= \alpha + w_3 N - \lambda N \end{aligned} \tag{10}$$

$$l\varphi_{\tilde{D}_2}(s, w_4) = \alpha + w_4\frac{\kappa\cos^2\theta}{\lambda\tau\dot{\theta}}T + (w_4-\lambda)N - w_4\frac{\kappa\cos^2\theta}{\lambda\tau\dot{\theta}}B$$

are the parametrization of the ruled surfaces MTRS, MNRS, MBRS, *and* MDRS. *For more details see* [8].

Theorem 7. *The normal vector fields* $[\psi]$ *of* MFRS *along the Mannheim partner curve* α_2 *can be expressed by the following matrix*

$$[\psi] = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a_2 & 0 & b_2 \\ c_2 & d_2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} T_2 \\ N_2 \\ B_2 \end{bmatrix} \tag{11}$$

where

$$a_2 = \frac{-w_2\tau_2}{\sqrt{(w_2\tau_2)^2 + (1 - w_2\kappa_2)^2}}, \qquad c_2 = \frac{-w_3\tau_2}{\sqrt{(w_3\tau_2)^2 + 1}}$$
$$b_2 = \frac{1 - w_2\kappa_2}{\sqrt{(w_2\tau_2)^2 + (1 - w_2\kappa_2)^2}}, \qquad d_2 = \frac{-1}{\sqrt{(w_3\tau_2)^2 + 1}}.$$

For details see [8].

2. Perpendicular BFRS and MFRS

In this section, using a matrix the sixteen positions of normal vector fields of eight BFRS and MFRS are examined. Further some interesting results are given, with simple matrices product and equality. The product matrix of unit normal vector fields $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\psi_1, \psi_2, \psi_3, \psi_4$ of BFRS and MFRS, respectively, along the curve α is

$$[\sigma] \cdot [\psi]^{\mathbf{t}} = \begin{bmatrix} \langle\sigma_1, \psi_1\rangle & \langle\sigma_1, \psi_2\rangle & \langle\sigma_1, \psi_3\rangle & \langle\sigma_1, \psi_4\rangle \\ \langle\sigma_2, \psi_1\rangle & \langle\sigma_2, \psi_2\rangle & \langle\sigma_2, \psi_3\rangle & \langle\sigma_2, \psi_4\rangle \\ \langle\sigma_3, \psi_1\rangle & \langle\sigma_3, \psi_2\rangle & \langle\sigma_3, \psi_3\rangle & \langle\sigma_3, \psi_4\rangle \\ \langle\sigma_4, \psi_1\rangle & \langle\sigma_4, \psi_2\rangle & \langle\sigma_4, \psi_3\rangle & \langle\sigma_4, \psi_4\rangle \end{bmatrix}. \tag{12}$$

Proof: It is trivial from product of the matrices. ■

Theorem 8. *The product of Frenet vector fields of the* $Bertrand\ mate$ α_1 *and* $Mannheimm\ partner$ α_2 *has the following matrix form*

$$\begin{bmatrix} T_1 \\ N_1 \\ B_1 \end{bmatrix} \cdot \begin{bmatrix} T_2 & N_2 & B_2 \end{bmatrix} = \frac{1}{m} \cdot \begin{bmatrix} \beta\cos\theta - \lambda\sin\theta & \lambda\cos\theta + \beta\sin\theta & 0 \\ 0 & 0 & m \\ -\lambda\cos\theta - \beta\sin\theta & \beta\cos\theta - \lambda\sin\theta & 0 \end{bmatrix} \tag{13}$$

where $m = \sqrt{\kappa^2 + \tau^2} \neq 0.$

Proof: Since

$$
\begin{aligned}
[V_1] \cdot [V_2]^{\mathbf{t}} &= \begin{bmatrix} T_1 \\ N_1 \\ B_1 \end{bmatrix} \cdot \begin{bmatrix} T_2 & N_2 & B_2 \end{bmatrix} \\
&= \frac{1}{\sqrt{\lambda^2+\beta^2}} \cdot \begin{bmatrix} \beta\cos\theta - \lambda\sin\theta & \lambda\cos\theta + \beta\sin\theta & 0 \\ 0 & 0 & \sqrt{\lambda^2+\beta^2} \\ -\lambda\cos\theta - \beta\sin\theta & \beta\cos\theta - \lambda\sin\theta & 0 \end{bmatrix}
\end{aligned}
$$

we have the proof. ■

Theorem 9. *The product matrix $m\,[\sigma]\,[\psi]^{\mathbf{t}}$ of the unit normal vector fields of* BFRS *and* MFRS*, along the curve α is*

$$
\begin{bmatrix}
0 & a_2(\lambda c\theta + \beta s\theta) & \begin{matrix} c_2(\lambda c\theta + \beta s\theta) \\ +d_2(-\beta c\theta + \lambda s\theta) \end{matrix} & \beta c\theta - \lambda s\theta \\
0 & a_2 \begin{bmatrix} a_1(\beta c\theta - \lambda s\theta) \\ -b_1(\lambda c\theta + \beta s\theta) \end{bmatrix} & \begin{matrix} c_2 \begin{bmatrix} a_1(\beta c\theta - \lambda s\theta) \\ -b_1(\lambda c\theta + \beta s\theta) \end{bmatrix} \\ +d_2 \begin{bmatrix} a_1(\lambda c\theta + \beta s\theta) \\ +b_1(\beta c\theta - \lambda s\theta) \end{bmatrix} \end{matrix} & \begin{bmatrix} -a_1(\lambda c\theta + \beta s\theta) \\ -b_1(\beta c\theta - \lambda s\theta) \end{bmatrix} \\
-md_1 & a_2 c_1(\beta c\theta - \lambda s\theta) & c_1 \begin{bmatrix} c_2(\beta c\theta - \lambda s\theta) \\ +d_2(\lambda c\theta + \beta s\theta) \end{bmatrix} & -c_1(\lambda c\theta + \beta s\theta) \\
m & -mb_2 & 0 & 0
\end{bmatrix} \tag{14}
$$

where for brevity we have introduced $c\theta \equiv \cos\theta$, $s\theta \equiv \sin\theta$ and $m = \sqrt{\kappa^2+\tau^2}$.

Proof: Let $[\sigma] = [A_1] \cdot [V_1]$ and $[\psi] = [A_2] \cdot [V_2]$

$$
[\sigma] \cdot [\psi]^{\mathbf{t}} = [A_1] \cdot [V_1] \cdot ([A_2] \cdot [V_2])^{\mathbf{t}} = [A_1] \cdot \left([V_1] \cdot [V_2]^{\mathbf{t}}\right) \cdot [A_2]^{\mathbf{t}}
$$

and using (13) we have

$$
[\sigma] \cdot [\psi]^{\mathbf{t}} = \frac{1}{m} [A_1] \cdot \begin{bmatrix} \beta\cos\theta - \lambda\sin\theta & \lambda\cos\theta + \beta\sin\theta & 0 \\ 0 & 0 & m \\ -(\lambda\cos\theta + \beta\sin\theta) & \beta\cos\theta - \lambda\sin\theta & 0 \end{bmatrix} \cdot [A_2]^t .
$$

Hence

$$
[\sigma]\cdot[\psi]^{\mathbf{t}}=\frac{1}{m}\begin{bmatrix}0 & 0 & -1\\ A_1 & 0 & b_1\\ c_1 & d_1 & 0\\ 0 & -1 & 0\end{bmatrix}\cdot\begin{bmatrix}\beta c\theta-\lambda s\theta & \lambda c\theta+\beta s\theta & 0\\ 0 & 0 & m\\ -(\lambda c\theta+\beta s\theta) & \beta c\theta-\lambda s\theta & 0\end{bmatrix}\cdot[A_2]^{\mathbf{t}}
$$

$$
=\frac{1}{m}\begin{bmatrix}(\lambda c\theta+\beta s\theta) & (\lambda s\theta-\beta c\theta) & 0\\ \begin{array}{c}A_1(\beta c\theta-\lambda s\theta)\\ -b_1(\lambda c\theta+\beta s\theta)\end{array} & \begin{array}{c}A_1(\lambda c\theta+\beta s\theta)\\ +b_1(\beta c\theta-\lambda s\theta)\end{array} & 0\\ c_1(\beta c\theta-\lambda s\theta) & c_1(\lambda c\theta+\beta s\theta) & md_1\\ 0 & 0 & -m\end{bmatrix}\cdot\begin{bmatrix}0 & a_2 & c_2 & 0\\ 0 & 0 & d_2 & -1\\ -1 & b_2 & 0 & 0\end{bmatrix}
$$

and this product give us the result. ■

In Euclidean three-space, the position of two surface, can be examined by the position of their unit normal vector fields $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\psi_1, \psi_2, \psi_3, \psi_4$. We can examine the sixteen positions of eight surfaces, basically, according to the position of their unit normal vector fields in a matrix. Since the equality of the matrices (12) and (14), we have sixteen interesting results according to the normal vector fields given in following theorems.

Theorem 10. *There are four pairs of Frenet ruled surface which are perpendicular, these are* BTRS, MTRS*, of the curve α and* BNRS, MTRS, BDRS, MB.

Theorem 11. *There are four pairs perpendicular surfaces which are* BTRS, MTRS, BNRS, MTRS, BDRS, MBRS *and* BDRS, MDRS.

Proof: According to equality of the matrices we can say that

$$
\langle\sigma_1, \psi_1\rangle = \langle\sigma_2, \psi_1\rangle = \langle\sigma_4, \psi_3\rangle = \langle\sigma_4, \psi_4\rangle = 0 \tag{15}
$$

and therefore their normal vector fields are perpendicular to each other. ■

Theorem 12. *Bertrand Tangent Ruled Surface and Mannheim Normal Ruled Surface of the curve α have perpendicular normal vector fields, since $w_2\tau_2 \neq 0$,* $\tan\theta = -\dfrac{\lambda}{\beta}$.

Proof: Since $\langle \sigma_1, \psi_2 \rangle = a_2 (\lambda \cos\theta + \beta \sin\theta)$ and using the orthogonality condition

$$\tan\theta = -\frac{\lambda}{\beta}$$

with $w_2\tau_2 \neq 0$ we have the proof. ■

Theorem 13. *Bertrand Tangent Ruled Surface and Mannheim Binormal Ruled Surface of curve α have perpendicular normal vector fields, if*

$$\tan\theta = \frac{\lambda_2\tau\beta - w_3\kappa\lambda}{w_3\kappa\beta + \lambda_2\tau\lambda}. \tag{16}$$

Proof: Since $\langle \sigma_1, \psi_3 \rangle = c_2 (\lambda \cos\theta + \beta \sin\theta) + d_2 (-\beta \cos\theta + \lambda \sin\theta)$ and under the orthogonality condition

$$c_2 (\lambda \cos\theta + \beta \sin\theta) + d_2 (-\beta \cos\theta + \lambda \sin\theta) = 0$$

we have

$$\tan\theta = \frac{\beta - w_3\tau_2\lambda}{\lambda + w_3\tau_2\beta}.$$

This completes the proof. ■

Theorem 14. *Bertrand Tangent Ruled Surface and Mannheim Darboux Ruled Surface of curve have perpendicular normal vector fields, if* $\tan\theta = \frac{\beta}{\lambda}$.

Proof: Since $\langle \sigma_1, \psi_4 \rangle = \beta \cos\theta - \lambda \sin\theta$ and under the orthogonality condition $\lambda \sin\theta - \beta \cos\theta = 0$, we get the proof. ■

Theorem 15. BNRS *and* MNRS *of Bertrand curve α have perpendicular normal vector fields along under the condition*

$$\tan\theta = \frac{u_2\tau_1\beta + (1 - u_2\kappa_1)\lambda}{u_2\tau_1\lambda - (1 - u_2\kappa_1)\beta}. \tag{17}$$

Proof: Since $\langle \sigma_2, \psi_2 \rangle = a_2 [A_1 (\beta \cos\theta - \lambda \sin\theta) - b_1 (\lambda \cos\theta + \beta \sin\theta)]$ and

$$A_1 (\beta \cos\theta - \lambda \sin\theta) - b_1 (\lambda \cos\theta + \beta \sin\theta) = 0$$

and under the orthogonality condition we get

$$\tan\theta = \frac{A_1\beta - b_1\lambda}{A_1\lambda + b_1\beta} = \frac{u_2\tau_1\beta + (1 - u_2\kappa_1)\lambda}{u_2\tau_1\lambda - (1 - u_2\kappa_1)\beta}$$

and therefore the proof. ■

Theorem 16. *Bertrand Normal Ruled Surface and Mannheim Binormal Ruled Surface have perpendicular normal vector fields for the value*

$$\frac{\beta\cos\theta-\lambda\sin\theta}{\lambda\cos\theta+\beta\sin\theta}=\frac{b_1c_2-A_1}{A_1c_2-b_1} \tag{18}$$

or

$$\tan\theta=\frac{w_3\tau_2\left(\left(-1+u_2\kappa_1\right)\lambda-u_2\tau_1\beta\right)-\left(u_2\tau_1\lambda-\left(1-u_2\kappa_1\right)\beta\right)\sqrt{\left(w_3\tau_2\right)^2+1}}{w_3\tau_2\left(-u_2\tau_1\lambda+\left(1-u_2\kappa_1\right)\beta\right)+\left(\left(1-u_2\kappa_1\right)\lambda+u_2\tau_1\beta\right)\sqrt{\left(w_3\tau_2\right)^2+1}}.$$

Proof: Since

$$\begin{aligned}\langle\sigma_2,\psi_3\rangle &= c_2\left[A_1\left(\beta\cos\theta-\lambda\sin\theta\right)-b_1\left(\lambda\cos\theta+\beta\sin\theta\right)\right]\\ &\quad+d_2\left[A_1\left(\lambda\cos\theta+\beta\sin\theta\right)+b_1\left(\beta\cos\theta-\lambda\sin\theta\right)\right]\end{aligned}$$

and under the orthogonality condition

$$\begin{aligned}&\Big(A_1c_2\beta\cos\theta-\lambda A_1c_2\sin\theta\Big)-b_1c_2\Big(\lambda\cos\theta+\beta\sin\theta\Big)\\ &\qquad-A_1\Big(\lambda\cos\theta+\beta\sin\theta\Big)-b_1\Big(\beta\cos\theta-\lambda\sin\theta\Big)=0\end{aligned}$$

hence

$$\begin{aligned}\tan\theta &= \frac{\left(A_1c_2-b_1\right)\beta-\left(b_1c_2+A_1\right)\lambda}{\left(A_1c_2-b_1\right)\lambda+\left(b_1c_2+A_1\right)\beta}\\ &= \frac{w_3\tau_2\left(\left(-1+u_2\kappa_1\right)\lambda-u_2\tau_1\beta\right)-\left(u_2\tau_1\lambda-\left(1-u_2\kappa_1\right)\beta\right)\sqrt{\left(w_3\tau_2\right)^2+1}}{w_3\tau_2\left(-u_2\tau_1\lambda+\left(1-u_2\kappa_1\right)\beta\right)+\left(\left(1-u_2\kappa_1\right)\lambda+u_2\tau_1\beta\right)\sqrt{\left(w_3\tau_2\right)^2+1}}\end{aligned}$$

and this completes the proof. ■

Theorem 17. *Bertrand Normal Ruled Surface and Mannheim Darboux Ruled Surface have perpendicular normal vector fields if*

$$\tan\theta=\frac{\left(1-u_2\kappa_1\right)\beta-u_2\tau_1\lambda}{u_2\tau_1\beta+\left(1-u_2\kappa_1\right)\lambda}. \tag{19}$$

Proof: Since $\langle\sigma_2,\psi_4\rangle=-A_1\left(\lambda\cos\theta+\beta\sin\theta\right)-b_1\left(\beta\cos\theta-\lambda\sin\theta\right)$ and under the orthogonality condition

$$\lambda A_1\cos\theta+A_1\beta\sin\theta+\left(b_1\beta\cos\theta-b_1\lambda\sin\theta\right)=0$$

we have

$$\tan\theta=\frac{\lambda A_1+b_1\beta}{b_1\lambda-A_1\beta}.$$

■

Theorem 18. BBRS *and* MTRS *of the curve* α *have not perpendicular normal vector fields except when* $w_3 = 0$ *or* $\tau_2 = 0$.

Proof: Since $\langle \sigma_3, \psi_1 \rangle = -md_1$ and under the orthogonality condition it follows that $w_3\tau_2 \neq 0$ it is trivial. ∎

Theorem 19. *Bertrand Binormal Ruled Surface and Mannheim Normal Ruled Surface of curve* α *have perpendicular normal vector fields if* $\tan\theta = \frac{\beta}{\lambda}$.

Proof: Since $\langle \sigma_3, \psi_2 \rangle = a_2c_1 (\beta\cos\theta - \lambda\sin\theta)$ and $w_2\tau_2 \neq 0$ under the orthogonality condition $\beta\cos\theta - \lambda\sin\theta = 0$. ∎

Theorem 20. *Bertrand Binormal Ruled Surface and Mannheim Binormal Ruled Surface have perpendicular normal vector fields when*

$$\tan\theta = \frac{w_3\tau_2\beta + \lambda\sqrt{(w_3\tau_2)^2 + 1}}{w_3\tau_2\lambda - \beta\sqrt{(w_3\tau_2)^2 + 1}}. \tag{20}$$

Proof: Since $\langle \sigma_3, \psi_3 \rangle = c_2c_1 (\beta\cos\theta - \lambda\sin\theta) + d_2c_1 (\lambda\cos\theta + \beta\sin\theta)$ and $u_3\tau_1 \neq 0$, under the orthogonality condition

$$c_2c_1 (\beta\cos\theta - \lambda\sin\theta) + d_2c_1 (\lambda\cos\theta + \beta\sin\theta) = 0$$

we have

$$\tan\theta = \frac{c_2\beta - \lambda}{c_2\lambda + \beta}$$

and this completes the proof. ∎

Theorem 21. *Bertrand Binormal Ruled Surface and Mannheim Darboux Ruled Surface of curve* α *have perpendicular normal vector fields if* $\tan\theta = -\frac{\lambda}{\beta}$.

Proof: Since $\langle \sigma_3, \psi_4 \rangle = -c_1 (\lambda\cos\theta + \beta\sin\theta)$ and $d_1 \neq 0$, under the condition that we have $\lambda\cos\theta + \beta\sin\theta = 0$ the statement follows. ∎

Theorem 22. *Bertrand Darboux Ruled Surface and Mannheim Normal Ruled Surface of Bertrand curve* α *have not perpendicular normal vector fields.*

Proof: Since $\langle \sigma_4, \psi_1 \rangle = m = \sqrt{\kappa^2 + \tau^2} \neq 0$ and under the orthogonality condition m is always $\neq 0$. ∎

Theorem 23. *Bertrand Darboux Ruled Surface and Mannheim Binormal Ruled Surface of curve α have not perpendicular normal vector fields.*

Proof: Since $\langle \sigma_4, \psi_2 \rangle = -mb_2$ and under the orthogonality condition 0. Hence $m = \sqrt{\kappa^2 + \tau^2} \neq 0$. ■

Corollary 24. *The perpendicular conditions and curves of the eight Frenet ruled surfaces are given in the* Table 1.

Table 1. The perpendicular intersection conditions and curves of the eight Frenet ruled surfaces.

$\langle\, , \rangle$	ψ_1	ψ_2	ψ_3	ψ_4
		$\tan\theta$		
σ_1	0	$-\frac{\lambda}{\beta}$	$\frac{w_3\kappa\lambda-\beta\lambda_2\tau}{-w_3\kappa\beta-\lambda\lambda_2\tau}$	$\frac{\beta}{\lambda}$
σ_2	0	$\frac{(1-u_2\kappa_1)\lambda+u_2\tau_1\beta}{u_2\tau_1\lambda-(1-u_2\kappa_1)\beta}$	$\frac{-(b_1c_2+A_1)\lambda+(A_1c_2-b_1)\beta}{(A_1c_2-b_1)\lambda+(b_1c_2+A_1)\beta}$	$\frac{-u_2\tau_1\lambda+(1-u_2\kappa_1)\beta}{(1-u_2\kappa_1)\lambda+u_2\tau_1\beta}$
σ_3	$\neq 0$	$\frac{\beta}{\lambda}$	$\frac{-u_3\kappa\beta-\lambda\lambda_2\tau}{-u_3\kappa\lambda+\beta\lambda_2\tau}$	$-\frac{\lambda}{\beta}$
σ_4	$\neq 0$	$\neq 0$	0	0

References

[1] Düldül B. and Çalişkan M., *On the Geodesic Torsion of Tangential Intersection Curve of Two Surfaces in R^3*, Acta Math. Univ. Comenianae **2** (2013) 177-189.

[2] Ergüt M., Körpinar T. and Turhan E., *On Normal Ruled Surfaces of General Helices In The Sol Space Sol^3*, TWMS J. Pure Appl. Math. **4** (2013) 125-130.

[3] Graves L., *Codimension One Isometric Immersions Between Lorentz Spaces*, Trans. Amer. Math. Soc. **252** (1979) 367-392.

[4] Hacisalihoğlu H., *Differential Geometry* (in Turkish), İnönü Univ., Malatya 1994.

[5] Hoschek J., *Liniengeometrie*, BI-Hochschulskripte, Zurich 1971.

[6] Izumiya S. and Takeuchi N., *Special Curves and Ruled Surfaces*, Beitrage zur Algebra und Geometrie **44** (2003) 203-212.

[7] Kiliçoğlu S., *Some Results on Frenet Ruled Surfaces Along the Evolute-Involute Curves Based on Normal Vector Fields in E^3*, Geom. Integrability & Quantization **17** (2016) 296-308.

[8] Kiliçoğlu Ş., *An Examination on the Mannheim Frenet Ruled Surface Based on Normal Vector Fields in E^3*, Konuralp J. Math. **4** (2016) 223-229.

[9] Kiliçoğlu Ş., Şenyurt S. and Hacisalihoglu H., *An Examination on the Positions of Frenet Ruled Surfaces Along Bertrand Pairs and According to Their Normal Vector Fields in E^3*, Appl. Math. Sci. **9** (2015) 7095-7103.

[10] Lipschutz M., *Differential Geometry*, Schaum's Outlines, McGraw-Hill, New York 1969.

[11] Liu H. and Wang F., *Mannheim Partner Curves in 3-Space*, J. Geom. **88** (2008) 120-126.

[12] Orbay K. and Kasap E., *On Mannheim Partner Curves in E^3*, Int. J. Phys. Sci. **4** (2009) 261-264.

[13] Orbay K., Kasap E. and Aydemir İ., *Mannheim Offsets of Ruled Surfaces*, Mathematical Problems in Engineering 2009, pp 1-9.

[14] Schief W., *On the Integrability of Bertrand Curves and Razzaboni Surfaces*, J. Geom. Phys. **45** (2003) 130-150.

[15] Şenyurt S. and Kiliçoğlu Ş., *On the Differential Geometric Elements of the $Involute\ \tilde{D}\ Scroll$*, Adv. Appl. Clifford Algebras **25** (2015) 977-988.

[16] Wu T., Alessio O. and Costa S., *On Estimating Local Geometric Properties of Intersection Curves*, In: Proceedings of Sibgrapi 2000, pp 152-159.

Şeyda Kiliçoğlu
Faculty of Education
Department of Mathematics
Başkent University, Ankara, TURKEY
E-mail address: `seyda@baskent.edu.tr`

Süleyman Şenyurt
Faculty of Arts and Sciences
Department of Mathematics
Ordu University, Ordu, TURKEY
E-mail address: `senyurtsuleyman@hotmail.com`