



## TWISTOR SPACES AND COMPACT MANIFOLDS ADMITTING BOTH KÄHLER AND NON-KÄHLER STRUCTURES\*

LJUDMILA KAMENOVA

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**Abstract.** In this expository paper we review some twistor techniques and recall the problem of finding compact differentiable manifolds that can carry both Kähler and non-Kähler complex structures. Such examples were constructed independently by Atiyah, Blanchard and Calabi in the 1950's. In the 1980's Tsanov gave an example of a simply connected manifold that admits both Kähler and non-Kähler complex structures - the twistor space of a  $K3$  surface. Here we show that the quaternion twistor space of a hyperkähler manifold has the same property.

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### 1. Introduction

In this paper we discuss a couple of classical approaches to twistor theory. Roughly speaking, the twistor space  $Z(M)$  is a family of (almost) complex structures on an orientable Riemannian manifold  $(M, g)$  compatible with the given metric  $g$  and the orientation. We are going to apply twistor techniques towards the problem of constructing simply connected compact manifolds that carry both Kähler and non-Kähler complex structures.

Atiyah's idea behind his examples in [1] was to consider the set of all complex structures  $M_n$  on the real torus  $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$  coming from the complex vector space structures on  $\mathbb{R}^{2n}$ . The space  $M_n$  is a complex manifold which is differentially a product of an algebraic variety and the torus  $T^{2n}$ , and therefore admits a Kähler structure. On the other hand, there exists a "twisted" complex structure on  $M_n$  which is non-Kähler. This rationale works in many other cases, and in particular, one can produce simply connected examples of similar nature. In [22] Tsanov showed that the twistor space of a  $K3$  surface is a simply connected 6-dimensional compact manifold which carries both Kähler and non-Kähler complex structures. Here we give examples of twistor spaces of hyperkähler manifolds and show that they also carry both Kähler and non-Kähler complex structures.

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\*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.  
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Twistor theory was originally introduced by Penrose [20] and studied by Atiyah, Hitchin and Singer [2] in four-dimensional Riemannian geometry. Given a four-dimensional manifold  $M$  with a conformal structure, the twistor transform associates to  $M$  the projective bundle of anti-self-dual spinors, which is a  $\mathbb{S}^2$ -bundle over  $M$ . In higher dimensions the notion of twistor space was generalized by Bérard-Bergery and Ochiai in [4]. Given a  $2n$ -dimensional oriented manifold with a conformal structure, the twistor space  $Z(M)$  parametrizes almost complex structures on  $M$  compatible with the orientation and the conformal structure. For quaternionic Kähler and hyperkähler manifolds there is an alternative twistor generalization introduced by Salamon [21] and independently by Bérard-Bergery (see [5, Theorem 14.9]). For a quaternionic Kähler manifold  $M$  (i.e., a manifold with holonomy group contained in  $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ ) there is a natural  $\mathbb{S}^2$ -bundle  $Z_0(M)$  over  $M$  of complex quaternionic structures on  $M$ , the “quaternion twistor space”. The quaternion twistor construction applied to hyperkähler manifolds gives examples of simply connected manifolds that carry both Kähler and non-Kähler complex structures. The idea behind this result is that on one hand, the quaternion twistor space of a hyperkähler manifold is diffeomorphic to a product of Kähler manifolds, and on the other hand, the twistor complex structure is not Kähler.

**Theorem 1.** *Let  $M$  be a hyperkähler manifold of real dimension  $4n$ . Then the quaternion twistor space  $Z_0(M)$  is a simply connected compact manifold that carries both Kähler and non-Kähler complex structures.*

## 2. Basics on Twistor Geometry

Fix a scalar product and an orientation on the  $2n$ -dimensional real space  $\mathbb{R}^{2n}$ . Denote the type-DIII compact hermitian symmetric space  $\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)}$  by  $\Gamma_n$ . Notice that, as a hermitian symmetric space,  $\Gamma_n$  is a complex Kähler manifold. The space  $\Gamma_n$  can be identified with the set of complex structures on  $\mathbb{R}^{2n}$ , compatible with the metric and the given orientation. Indeed, if  $J \in \mathrm{SO}(2n)$  with  $J^2 = -1$  is a complex structure on  $\mathbb{R}^{2n}$ , then the group  $\mathrm{SO}(2n)$  acts on  $J$  by conjugation. The isotropy group is  $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ . The real dimension of  $\Gamma_n$  is  $n(n-1)$ .

**Definition 2.** *Let  $M$  be an oriented Riemannian manifold of dimension  $2n$ . The holonomy group of  $M$  is contained in  $\mathrm{SO}(n)$ . Denote by  $P$  the associated principal  $\mathrm{SO}(2n)$ -bundle of orthonormal linear frames on  $M$ . The fiber bundle  $Z(M) = P \times_{\mathrm{SO}(2n)} \Gamma_n \rightarrow M$  is called the twistor bundle or twistor space of  $M$ . Equivalently,  $Z(M)$  can be identified with  $P/\mathrm{U}(n)$ .*

The dimension of  $Z(M)$  is even: indeed,  $\dim_{\mathbb{R}} Z(M) = \dim_{\mathbb{R}} M + \dim_{\mathbb{R}} \Gamma_n = 2n + n(n-1) = n(n+1)$ . The twistor space  $Z(M)$  has a “tautological” almost complex structure  $J$  which we describe here. Denote by  $p : Z(M) \rightarrow M$  the projection. For a point  $z \in Z(M)$ , let its image be  $x = p(z) \in M$ . The fiber of  $p$  over  $x$  is  $p^{-1}(x) = \Gamma_n$ , and therefore  $z \in \Gamma_n$  can be considered as a complex structure  $I_z$  on the tangent space  $T_x M$ . The Riemannian connection of  $M$  determines a splitting of the tangent space  $T_z Z(M) = V_z \oplus H_z$  into vertical and horizontal parts. The vertical part  $V_z$  is identified with the tangent space to the fiber  $p^{-1}(x) = \Gamma_n$  at  $z$ , and it has an integrable almost complex structure  $K$ . The connection of  $M$  defines an isomorphism  $H_z \cong T_x M$  and we can consider  $I_z$  as a complex structure on  $H_z$ .

**Definition 3.** *The almost complex structure  $J$  on  $Z(M)$  is defined by  $J_z = K \oplus I_z : V_z \oplus H_z \rightarrow V_z \oplus H_z$ . As  $z$  varies in  $p^{-1}(x) = \Gamma_n$ , the projection of  $J_z$  on the horizontal part  $H_z \cong T_x M = \mathbb{R}^{2n}$  varies in the space of complex structures compatible with the metric and orientation.*

When  $\dim_{\mathbb{R}}(M) = 4$ , the fiber  $\Gamma_2 = \mathbb{S}^2$  and the generalized twistor definition coincides with the classical definition which was first introduced for Riemannian 4-manifolds. Twistor spaces can also be defined in terms of spinors. In our notations  $Z(M)$  coincides with the projectivized bundle  $\mathbb{P}(PS^+)$  of positive pure spinors on  $M$  [15, Proposition 9.8].

**Proposition 4.** *Let  $M$  be an oriented Riemannian manifold of dimension  $2n$ . Then the orthogonal orientation preserving almost complex structures on  $M$  are in one-to-one correspondence with the sections of the twistor space  $Z(M) = \mathbb{P}(PS^+)$ . In particular,  $M$  is Kähler if and only if there is a parallel cross section of  $Z(M)$ .*

From the definition of the almost complex structure  $J$  on  $Z(M)$  it is clear that the integrability of  $J$  depends only on the Riemannian metric  $g$  on  $M$ . When  $\dim_{\mathbb{R}}(M) > 4$ , the Weyl tensor  $W$  is irreducible, however when  $\dim_{\mathbb{R}}(M) > 4$ , the Weyl tensor splits into two parts: self-dual and anti-self-dual  $W = W_+ \oplus W_-$ .

**Theorem 5.** *Let  $M$  be an oriented Riemannian manifold of dimension  $2n$ . The almost complex structure  $J$  on  $Z(M)$  is integrable if and only if:*

- $M$  is anti-self-dual, i.e., the self-dual part of the Weyl tensor vanishes:  $W_+ = 0$ , when  $\dim_{\mathbb{R}}(M) = 4$  [2, Theorem 4.1]
- $M$  is conformally flat (i.e.,  $W = 0$ ) when  $\dim_{\mathbb{R}}(M) > 4$  [4, Section 3].

Another important question in complex geometry is to determine when a given manifold is Kähler.

**Theorem 6.** *The twistor space  $Z(M)$  of a compact oriented Riemannian manifold  $M$  is Kähler if and only if:*

- $M$  is conformally equivalent to the complex projective space  $\mathbb{C}P^2$  or to the sphere  $\mathbb{S}^4$  when  $\dim_{\mathbb{R}}(M) = 4$  [10, Theorem 6.1]
- $M$  is conformally equivalent to the sphere  $\mathbb{S}^{2n}$  when  $\dim_{\mathbb{R}}(M) = 2n > 4$  see [3].

Notice that the sections of  $p : Z(M) \rightarrow M$  represent almost complex structures on  $M$  compatible with the Riemannian metric and the orientation. If the section is holomorphic, it represents an integrable almost complex structure.

**Theorem 7 (Michelsohn [19]).** *Let  $M$  be an oriented Riemannian manifold of even dimension with an almost complex structure  $I$  determined by a projective spinor field  $s \in \Gamma(Z(M))$ . Then  $I$  is integrable if and only if the section  $s$  is holomorphic.*

We conclude this section with some interesting examples of twistor spaces and fibers of the twistor projection in small dimensions.

**Example 8.** *Consider the sphere  $\mathbb{S}^{2n}$  as the conformal compactification of  $\mathbb{R}^{2n}$ . Then from the topology of fiber bundles it is clear that  $Z(\mathbb{S}^{2n}) = \Gamma_{n+1}$ . This implies that there are no complex structures on  $\mathbb{S}^6$  compatible with the standard metric. Indeed, if  $s : \mathbb{S}^6 \rightarrow Z(\mathbb{S}^6) = \Gamma_4$  is a section representing such a complex structure, then  $s(\mathbb{S}^6) \subset \Gamma_4$  would be a complex submanifold by Theorem 7. This would imply that  $s(\mathbb{S}^6)$  is Kähler which is impossible since  $H^2(\mathbb{S}^6, \mathbb{R}) = 0$  and there could not be any Kähler  $(1, 1)$  classes on  $\mathbb{S}^6$ .*

**Example 9.** *Due to a lot of special isomorphisms between symmetric spaces of small dimensions, we have the following isomorphisms*

- $\Gamma_1$  is a point, because  $\text{SO}(2) \cong \text{U}(1)$
- $\Gamma_2 = \mathbb{S}^2 = \mathbb{C}P^1$ , because  $\frac{\text{SO}(4)}{\text{U}(2)} \cong \frac{\text{U}(2)}{\text{U}(1)}$
- $\Gamma_3 = \mathbb{C}P^3$ , because  $\frac{\text{SO}(6)}{\text{U}(3)} \cong \frac{\text{U}(4)}{\text{U}(1) \times \text{U}(3)}$
- $\Gamma_4 = Q_6$  is a complex 6-dimensional quadric, because  $\frac{\text{SO}(8)}{\text{U}(4)} \cong \frac{\text{SO}(8)}{\text{SO}(2) \times \text{SO}(6)}$ .

### 3. Examples of Twistor Spaces of Surfaces

Let  $M$  be a 4-dimensional oriented Riemannian manifold. By the construction of the almost complex structure  $J = K \oplus I$  on the twistor space  $Z(M)$ , the fibers of the twistor map  $p : Z(M) \rightarrow M$  are holomorphic rational curves in  $Z(M)$ , called “twistor lines”. The normal bundle of each twistor line is  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ . From the results in [2] we get the following universal property of twistor spaces.

**Theorem 10.** *Let  $M$  be an anti-self-dual 4-manifold, i.e.,  $W_+ = 0$ . Then the twistor space  $Z(M)$  is a complex manifold that admits a “real structure”, i.e., an anti-holomorphic involution  $\iota : Z(M) \rightarrow Z(M)$ . The restriction of the involution  $\iota$  on each twistor line  $\mathbb{C}\mathbb{P}^1 = \mathbb{S}^2$  is the antipodal map. Conversely, the holomorphic data above is sufficient to define a twistor space. More precisely, if  $Z$  is a 3-dimensional complex manifold with an antiholomorphic involution  $\iota$ , foliated by rational curves with normal bundles  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ , such that  $\iota$  restricted to a fiber of the foliation  $\mathbb{C}\mathbb{P}^1$  is the antipodal map, then  $Z$  is a twistor space of an anti-self-dual 4-manifold.*

Here are the first examples where a twistor-type (almost) complex structure was explored even before twistor spaces were defined.

**Example 11.** *Consider the 4-dimensional real torus  $M = T^4 = \mathbb{R}^4/\mathbb{Z}^4$ . The twistor space  $Z(T^4)$  of the torus is the quotient of  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  by a corresponding lattice action. In [6] Blanchard considered this example and showed that  $Z(T^4)$  is a compact complex manifold admitting a holomorphic fibration to  $\mathbb{C}\mathbb{P}^1$  whose fibers are complex tori. In [7] Calabi also explored 6-dimensional examples of oriented Riemannian manifolds embedded into the Cayley space, which are diffeomorphic to a Kähler manifold but do not admit a Kähler metric. In [1] Atiyah considered fiber spaces of higher-dimensional complex tori that arise as twistor spaces. On one hand, Atiyah established the non-Kähler property of these fiber spaces, and on the other hand, he showed that they are diffeomorphic to the product of two complex Kähler manifolds (namely, the given complex torus and a symmetric space of type DIII).*

Atiyah was interested in these examples, because they illustrate the existence of Kähler and non-Kähler complex structures on the same differentiable manifold. In [1] he asked if a simply connected compact differentiable manifold can carry both Kähler and non-Kähler structures. The following example answers Atiyah’s question.

**Example 12.** In [22] Tsanov noticed that the twistor space  $Z(S)$  of a K3 surface  $S$  is simply connected and admits both Kähler and non-Kähler complex structures. Indeed, the quaternions  $\mathbb{H}$  act as parallel endomorphisms on the tangent bundle  $TS$ . Fix the standard basis  $\{I, J, K\}$  of  $\mathbb{H}$ . This gives a trivialization of  $Z(S)$ , i.e.,  $Z(S)$  is diffeomorphic to  $S \times \mathbb{C}\mathbb{P}^1$ . Notice that since every K3 surface is Kähler,  $S \times \mathbb{C}\mathbb{P}^1$  admits a Kähler product structure, however  $Z(S)$  endowed with the twistor complex structure is not Kähler by Hitchin's result (Theorem 6).

For any anti-self-dual complex surface  $S$  we can consider  $Z(S)$  as the projectivized bundle  $\mathbb{P}(PS^+)$  of positive pure spinors on  $S$  [15, Proposition 9.8]. For the tautological complex structure  $J$  on  $Z(S)$ , the projection  $p : (Z(S), J) \rightarrow S$  is not a holomorphic map in general. However, there exists a complex structure  $J'$  on  $Z(S)$  such that  $p : (Z(S), J') \rightarrow S$  becomes a holomorphic  $\mathbb{C}\mathbb{P}^1$ -bundle. Tsanov proved that  $(Z(S), J)$  and  $(Z(S), J')$  are not deformation equivalent to each other, and therefore the moduli space of complex structures on the complex manifold  $Z(S)$  is not connected. The method used in [22] is an explicit computation of the first Chern classes of  $(Z(S), J)$  and  $(Z(S), J')$ . Two complex structures on a 3-dimensional complex manifold are homotopic if and only if their first Chern classes coincide. Tsanov shows that there is no diffeomorphism  $\phi$  of  $Z(S)$  such that  $\phi^*$  sends  $c_1(Z(S), J)$  to  $c_1(Z(S), J')$ .

**Example 13.** Let  $\Sigma_g$  be a Riemannian surface of genus  $g \geq 2$ . Kato [13] considers the twistor space  $Z(S)$  of the ruled surface  $S = \mathbb{C}\mathbb{P}^1 \times \Sigma_g$ . By Theorem 6,  $Z(S)$  is non-Kähler. Kato showed that there are no non-constant meromorphic functions on  $Z(S)$ . A classical result by Catanese states that the existence of a non-constant holomorphic map from a compact Kähler manifold to a compact Riemannian surface of genus  $g \geq 2$  is determined by its topology. Kato's example shows that the Kähler assumption is essential and cannot be removed from Catanese's result. In [12] we described explicitly the complex structures of  $S$  in terms of holomorphic sections  $S \rightarrow Z(S)$ . Kato's example fits into the theory of scalar-flat Kähler surfaces, which are explored in the key papers [16] of LeBrun and [14] of Kim, LeBrun and Pontecorvo.

## 4. Twistor Spaces of Hyperkähler Manifolds

**Definition 14.** A  $4n$ -dimensional Riemannian manifold is called hyperkähler if its holonomy group is contained in the compact symplectic group  $\mathrm{Sp}(n)$ .

Since  $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$  is a subgroup of  $\mathrm{SU}(2n)$ , by Berger's classification, every hyperkähler manifold is Kähler with zero Ricci curvature.

**Definition 15.** *A  $4n$ -dimensional Riemannian manifold is called quaternion Kähler if its holonomy group is contained in  $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ .*

On the other hand,  $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$  is not a subgroup of  $\mathrm{U}(n)$ , and therefore a general quaternion Kähler manifold is not Kähler. If  $n \geq 2$ , any quaternion Kähler manifold is Einstein.

From now on we assume that  $M$  is a hyperkähler manifold of dimension  $\dim_{\mathbb{R}} M = 4n$ . Then the imaginary quaternions act on  $TM$  as parallel endomorphisms. Fix the standard basis  $\{I, J, K\}$  of the imaginary quaternions. Both Salamon [21] and Bérard-Bergery [5, Theorem 14.9 and Definition 14.67] independently introduced an alternative notion of a twistor space for quaternion Kähler manifolds. Let  $E$  be the 3-dimensional vector subbundle of  $\mathrm{End}(TM)$  spanned by  $\{I, J, K\}$ . The vector bundle  $E$  carries a natural Euclidean structure, with respect to which  $\{I, J, K\}$  is an orthonormal basis.

**Definition 16.** *The unit-sphere subbundle  $Z_0(M)$  of  $E$  is the quaternion twistor space of  $M$ .*

Salamon [21] proved that the quaternion twistor space  $Z_0(M)$  of  $M$  admits a natural complex structure such that the fibers of the projection  $\pi : Z_0(M) \rightarrow M$  are holomorphic rational curves called “twistor lines”. Notice that the complex structures  $\{I, J, K\}$  give a trivialization of  $Z_0(M)$ , and therefore  $Z_0(M)$  is diffeomorphic to  $M \times \mathbb{S}^2 \cong M \times \mathbb{C}\mathbb{P}^1$ .

We can also describe the quaternion twistor space  $Z_0(M)$  of a hyperkähler manifold  $M$  as follows. We set  $Z_0(M) = \mathbb{C}\mathbb{P}^1 \times M$ , where  $\mathbb{C}\mathbb{P}^1$  is identified with the unit sphere  $\mathbb{S}^2 \subset \mathbb{H}$  of complex structures on  $M$ . We define the “tautological” (almost) complex structure  $J'$  on  $Z_0(M)$  in the same way as in the case of Riemannian 4-folds. Denote by  $\pi : Z_0(M) \rightarrow M$  the projection. Every point  $z \in Z_0(M)$  is of the form  $z = (\alpha, \pi(z))$ , where  $\alpha \in \mathbb{C}\mathbb{P}^1$ . Let  $K$  be the natural complex structure of  $\mathbb{C}\mathbb{P}^1$ . Then we define the “tautological” almost complex structure  $J'$  on  $T_z Z_0(M)$  by  $J'_z = (K, \alpha)$ , where  $\alpha \in \mathbb{C}\mathbb{P}^1$  is considered as a complex structure on  $T_{\pi(z)} M$ . This defines an integrable almost complex structure  $J'$  on  $Z_0(M)$  by Salamon [21] and  $Z_0(M)$  becomes a  $(2n+1)$ -dimensional complex manifold. The quaternion twistor space  $Z_0(M)$  is an almost complex submanifold (and a subbundle) of the “big” twistor space  $Z(M)$ .

As Joyce notes in [11], the quaternion twistor space  $Z_0(M)$  of a hyperkähler manifold  $M$  satisfies very similar properties to the 4-dimensional Riemannian case. The projection  $p : (Z_0(M), J') \rightarrow \mathbb{C}\mathbb{P}^1$  is holomorphic and  $p^{-1}(\alpha)$  is isomorphic to the complex manifold  $(M, \alpha)$  for any  $\alpha \in \mathbb{C}\mathbb{P}^1$ . There is an antiholomorphic symplectic involution  $\iota : Z_0(M) \rightarrow Z_0(M)$  defined by  $\iota(\alpha, x) = (-\alpha, x)$ . Consider

the projection  $\pi : Z_0(M) \rightarrow M$ . For every point  $x \in M$  the fiber over  $x$  is called a *twistor line* and it is a holomorphic rational curve (see [21]) with normal bundle  $\mathcal{O}(1)^{\oplus 2n}$ . The twistor lines are preserved by the involution  $\iota$  and the restriction of  $\iota$  on a twistor line coincides with the antipodal map. As in Theorem 10, the holomorphic data above is sufficient to show that a given  $(2n + 1)$ -dimensional complex manifold with this data is biholomorphic to a quaternion twistor space of a hyper-complex manifold equipped with a pseudo-hyperkähler metric [11, Theorem 7.1.4].

**Theorem 17.** *Let  $M$  be a hyperkähler manifold of real dimension  $4n$ . Then the quaternion twistor space  $Z_0(M)$  is a simply connected compact manifold that carries both Kähler and non-Kähler complex structures.*

**Proof:** From the definition of  $Z_0(M)$  it is clear that the quaternion twistor space is diffeomorphic to the product of complex manifolds  $M \times \mathbb{C}\mathbb{P}^1$ , which admits a product Kähler structure. We have that  $\pi_1(Z_0(M)) = \pi_1(M) \times \pi_1(\mathbb{S}^2) = 0$ , i.e., the quaternion twistor space is simply connected.

Now consider  $Z_0(M)$  together with its twistor complex structure  $J'$ . For a compact hyperkähler manifold  $M$  Fujiki [8] showed that the general fiber of the twistor family  $p : (Z_0(M), J') \rightarrow \mathbb{C}\mathbb{P}^1$  does not contain neither effective divisors nor curves, and therefore the general fiber is not projective.

Notice that  $H^0(Z_0(M), \Omega_{Z_0(M)}^i) = 0$  for  $i > 0$ . Let's first show this for  $i = 1$ . If we assume there is a section  $\sigma \in H^0(Z_0(M), \Omega_{Z_0(M)}^1)$ , then  $\sigma$  defines a linear map  $TZ_0(M)|_{\mathbb{C}\mathbb{P}^1} \rightarrow \mathbb{C}$ , where  $\mathbb{C}\mathbb{P}^1$  is a twistor line of the projection  $\pi : Z_0(M) \rightarrow M$ . Since the normal bundle of a twistor line is  $\mathcal{O}(1)^{\oplus 2n}$  by [21], we have the following splitting:  $TZ_0(M)|_{\mathbb{C}\mathbb{P}^1} = T\mathbb{C}\mathbb{P}^1 \oplus N_{\mathbb{C}\mathbb{P}^1/Z_0(M)} = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2n}$  for every twistor line, i.e., for every fiber of  $\pi$ . But since the dual  $\mathcal{O}(-1)$  of the hyperplane bundle doesn't have non-trivial sections, it follows that  $\sigma = 0$ . Since we can express  $\Lambda^i TZ_0(M)|_{\mathbb{C}\mathbb{P}^1}$  as a direct sum of tensor powers of  $\mathcal{O}(1)$ , we can use the same argument for  $i > 1$ .

Assume that  $Z_0(M)$  is Kähler, then  $H^0(Z_0(M), \Omega_{Z_0(M)}^2) = 0$  and  $H^2(Z_0(M)) = H^{1,1}(Z_0(M)) \neq 0$  and  $H^{1,1}(Z_0(M))$  would contain rational classes, hence  $Z_0(M)$  is projective by the Kodaira embedding theorem. However, every closed analytic subvariety of a projective variety is projective. Then the fibers of the twistor family  $p : Z_0(M) \rightarrow \mathbb{C}\mathbb{P}^1$  would be smooth projective varieties, which contradicts Fujiki's result. Therefore, the quaternion twistor space  $Z_0(M)$  together with the tautological twistor complex structure is not Kähler, and the underlying simply connected differentiable manifold  $M \times \mathbb{C}\mathbb{P}^1$  carries both Kähler and non-Kähler complex structures. ■

**Remark 18.** *Claude LeBrun pointed out an alternative argument that the complex structure  $J'$  on the “small” twistor space  $Z_0(M)$  is non-Kähler for a hyperkähler manifold  $M$  of complex dimension  $2n$ . Let  $p : Z_0(M) \rightarrow \mathbb{C}\mathbb{P}^1$  be the holomorphic twistor map. Hitchin’s argument [10] on the action of the real structure on  $H^{1,1}(Z_0(M))$  shows that if  $J'$  is a Kähler structure, then the anti-canonical line bundle  $K^*$  is ample. In the hyperkähler case,  $K^* = p^*\mathcal{O}(2n + 2)$  is a pull-back from the base  $\mathbb{C}\mathbb{P}^1$  via the projection  $p : (Z_0(M), J') \rightarrow \mathbb{C}\mathbb{P}^1$ , and hence it is trivial on every fiber of  $p : Z_0(M) \rightarrow \mathbb{C}\mathbb{P}^1$ . Therefore, its sections are pull-backs, too. Then the linear system of any power of  $K^*$  collapses every fiber of  $Z_0(M) \rightarrow \mathbb{C}\mathbb{P}^1$ . This contradicts ampleness of  $K^*$ . By the same argument, the algebraic dimension of  $Z_0(M)$  is  $a(Z_0(M)) = 1$  for a hyperkähler manifold  $M$ .*

If  $M$  is a hyperkähler manifold, then there exists a universal deformation space  $\mathcal{U} \rightarrow \text{Def}(M)$  of  $M$ , where the base  $\text{Def}(M)$  is smooth. The quaternion twistor space  $p : Z_0(M) \rightarrow \mathbb{C}\mathbb{P}^1$  induces a non-trivial map  $\mathbb{C}\mathbb{P}^1 \rightarrow \text{Def}(M)$ . In [9] Fujiki showed that the corresponding 1-dimensional subspace of the Zariski tangent space  $H^1(M, T^{1,0}M) \cong H^{1,1}(M)$  of  $\text{Def}(M)$  is spanned by the Kähler class of the given hyperkähler metric on  $M$ , which coincides with the Kodaira-Spencer class.

**Example 19.** *In [18] Claude LeBrun gives the following example. Let  $M$  be a hyperkähler manifold of complex dimension  $2n$  and let  $f : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  be a ramified cover of degree  $k$ . Consider the pull-back  $\hat{Z}_0(M) = f^*Z_0(M)$  of the map  $p : Z_0(M) \rightarrow \mathbb{C}\mathbb{P}^1$  via  $f$  and let  $\hat{p} = f^*p$  be the associated holomorphic submersion. Then  $\hat{Z}_0(M)$  is a complex  $(2n + 1)$ -manifold with canonical line bundle  $K = \hat{p}^*\mathcal{O}(-2nk - 2)$ . LeBrun noticed that once we take a ramified cover of the base  $\mathbb{C}\mathbb{P}^1$  of the twistor space, then the deformation space of  $\hat{Z}_0(M)$  is the deformation space of the corresponding rational curve in the Teichmüller space, which grows as the degree  $k$  of the cover grows. The spaces of type  $\hat{Z}_0(M)$  can also be constructed if one takes any rational curve in the Teichmüller space and restricts the universal fibration  $\mathcal{U}$  to this curve. The methods used in the proof of our Theorem 17 also show that  $\hat{Z}_0(M)$  is non-Kähler. LeBrun’s example is related to his earlier work [17], where he shows that if  $S$  is any complex surface with even Betti number  $b_1$ , and if  $M = S \# k(-\mathbb{C}\mathbb{P}^2)$  is its blow-up at a large number  $k \gg 0$  of points, then the twistor space  $Z(M)$  admits both Kähler and non-Kähler complex structures, and moreover, they have different Chern numbers.*

**Remark 20.** *Let  $M$  be a hyperkähler manifold of real dimension  $4n$ . When  $n > 1$ , the twistor space  $Z(M)$  is integrable if and only if  $M$  is flat, i.e.,  $M$  is a torus, because the vanishing of the Weyl tensor (as required by Theorem 5) for a Ricci-flat manifold implies the vanishing of the entire curvature tensor. However, it is*

still an interesting question to answer when the almost complex manifold  $Z(M)$  is diffeomorphic to a product  $M \times \Gamma_{2n}$ . In particular, it would be interesting to know if there is a relation that the characteristic classes of a hyperkähler complex  $2n$ -fold  $M$  have to satisfy so that  $Z(M)$  is a trivial bundle.

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Ljudmila Kamenova

Department of Mathematics, Room 3-115

Stony Brook University

Stony Brook, NY 11794-3651, USA

*E-mail address:* kamenova@math.stonybrook.edu